THE NUCLEATION MODEL OF STRINGS
AND THE HAGEDORN PHASE TRANSITION

Fedele Lizzi

and

Ikuo Senda

1990 MIRAMARE - TRIESTE
International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

THE NUCLEATION MODEL OF STRINGS
AND THE HAGEDORN PHASE TRANSITION *

Fedele Lizzi and Ikuo Senda
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

In this paper we discuss a model of interacting strings at finite densities based on nucleation theory, the study of formation of droplets in a supersaturated gas, the analogy being between drops of various sizes and strings with various excitation number. The interaction of the strings is considered to be the usual merging and splitting. We do not assume equilibrium a priori but find equilibrium configurations of strings as a result of their dynamics. We study these configurations as we change the energy density, and find the presence of two phases. A low density 'gas' phase, in which the energy is in strings in the fundamental or the first few excited levels, and a high density 'liquid' phase in which the number of strings is low, all the energy being carried by few very excited strings. For the gas phase we also discuss the thermodynamics of the system.

MIRAMARE – TRIESTE
July 1990

* Submitted for publication.
1 Introduction

In this paper we try to give an answer to the following question: what happens to a gas of strings at finite density if the number of strings is not kept constant and the strings are allowed to interact via the usual merging and splitting mechanism? The reason to ask this question lies in work Hagedorn did more than a quarter of a century ago, even before the first advent of string theories. Hagedorn [1] noticed that in a theory with an exponentially increasing density of states (such as a string theory, or more generally a dual model) the thermal partition function diverges at a temperature equal (in the appropriate units) to the inverse of the rate of the exponential increase of states.

In particular for a string theory the density of states is asymptotically:

\[ \rho(m) \propto m^{-(d+1)}e^{\beta m} \]  

In the following we will consider closed string theory, for which \( \beta_H = 2\pi\sqrt{2} \) for the bosonic string, and \( 2\pi \) and \( (\sqrt{2}+1)\pi \) for the type II and heterotic string respectively. The quantity \( d \) is the number of large scale space dimensions and we have neglected an unimportant proportionality constant. The information on the compactified small scale dimensions is included in the form of (1.1) [2]. The thermal partition function is:

\[ Z(\beta) = \sum_i e^{-\beta E_i} \propto \sum_m m^{d-(d+1)}e^{(\beta m - \beta)m} \]  

where the \( \sum \) is intended over all possible states. It is easy to see that \( Z \) diverges for the inverse temperature \( \beta < \beta_H \), that is for temperatures larger than \( T_H = 1/\beta_H \). Something must happen at \( T_H \), we will leave here aside the question whether this is a limiting temperature [1] or the indication of a phase transition [3,4]. We notice however that the canonical ensemble partition function assumes the presence of a thermal reservoir, with which the ideal gas of strings would be in equilibrium, and the concept itself of thermal reservoir may not be an useful one for physics at Planckian or near Planckian temperatures.

Thus one is motivated to treat the system of strings as an isolated system. The standard way to study isolated system is provided by the method of microcanonical ensemble, [5,6,7] where the investigation is of the system in equilibrium with total energy \( E \), number of particles \( N \), volume \( V \) and possibly some other conserved quantities. One of the interesting results obtained, using this method in the string
case, is that the most favored configuration of \( N \) strings is the one where there is one highly excited string, which carries most of energy of the system, and the others \( \sim (N-1) \) strings are at the lowest mass level. In this way of treating statistical mechanics, the contact with thermodynamics is made via Boltzmann's relation: \( S_m = k_b \ln W \), where \( W \) is the number of partition and \( S_m \) is the entropy, in the microcanonical sense. In the usual case, such as ideal gas, we can take \( S_m = S_t \), where \( S_t \) is the entropy in the thermodynamical sense. Then using the standard relation of thermodynamics, we can find a temperature of the system as a function of \( E, V, N \), namely \( T^{-1} = \left( \frac{2E}{\mathcal{E}} \right)_v \). In the case of string theory, the matter is very different. The formal definition of the temperature using the entropy obtained in microcanonical ensemble cannot be used at high energy density region comparable with Planckian scale. A possible answer to this problem is that the relation \( S_t = S_m \), which is valid when the thermal equilibrium of the system is ensured, does not hold anymore at the high energy region of string theory. Therefore, we have to reconsider our starting point where we postulate the equilibrium of the system with a set of conserved quantities \( E, N, V \).

Common to both the canonical and microcanonical approach is the assumption of equilibrium and the fact that the strings are considered an ideal gas of noninteracting strings (apart from the weak interaction needed to reach equilibrium). This is not a safe assumption in the case of strings at the beginning of the universe, where those considerations supposedly have some physical relevance. In ref. [2] Salomonson and Skagerstam have proposed a model (the heretic string model) in which strings are considered macroscopic interacting objects, with a length and an interaction probability proportional to their length. They then did a microcanonical-like analysis of the equilibrium configurations of the system. Where there is an overlap their results are in broad agreement with the ones of the model presented here, although the details and the quantitative analysis differ somewhat. In particular they also find the presence of two phases, dominated by short and long strings respectively.

The model we present in the next section does not assume equilibrium at all, it considers in fact the gas of strings as a dynamical system evolving in time, with the number of strings at a particular energy as the variable. The analogy and the tool being nucleation theory (the formation and evolution of droplets in a supersaturated

\[ ^1 \text{And in fact if one insists in doing so with the usual definition of temperature } T = \left( \frac{\mathcal{E}}{2E} \right)^{-1}, \text{ one would find that } T > T_H. \]
gas). The equilibrium configurations (if they exist) are simply the configurations for which all the time derivatives vanish. The changes in the number of strings are given by the interaction laws of fundamental strings (decay rates and cross sections) and the density of states. In section 3 we discuss analytical solutions for particular cases, here we obtain (under some suitable approximations) an exact solution to the problem of equilibrium configurations in the low energy density region as well as evidence of a phase transition when the energy density increases. In section 4 the numerical solution for the general case. In this section we also discuss the general stability and equilibrium properties and the evidence for a phase transition from a 'gas' phase to a 'liquid' of string theory. We then make connections with the usual statistical and thermodynamical quantities in section 5, here we also argue that the transition we found corresponds to the Hagedorn transition and also discuss the equation of state of the system. Some of the details of the fitting to thermodynamical laws of the equilibrium configurations are collected in an appendix. Finally in section 6 we discuss the results and draw some conclusions. Some of the results of this paper have appeared in a shortened version in [8].
2 The Nucleation Model

The model we would like to introduce in this section is inspired by nucleation theory, [9] that is the study of the behaviour of a supersaturated gas. In this case droplets, that is bound states of the molecules, will form, and those droplets will merge and split, thus giving rise to larger or smaller droplets. The analogy here is between the various levels of excitation of the strings and the droplets of various sizes. We will therefore consider a gas of interacting strings, the interaction being the decay of one string in two strings, or the recombination of two strings to merge into one, more massive string.

Consider a system with total energy $E$, and divide $E$ into $m$ bins of equal size $\Delta E = E/m$, with $E_i$ approximatively the energy of the $i^{th}$ bin. As is customary for this kind of calculations, $\Delta E$ must be large enough to contain a substantial number of levels, and small enough for the approximation to be a good one. We are interested in the distribution of the number of strings with respect to the energy, that is in the number of strings in each bin, how they will change with time, and their equilibrium configurations. Let $N_i$ be the number of strings which have individual energy in the $i^{th}$ bin, so that,

$$E \approx \sum_{i=1}^{m} N_i E_i \quad .$$

(2.1)

The approximation will of course improve as $m$ increases. Another important quantity is the energy distribution, represented by the concentration $c_i$, the fraction of the energy carried by strings with energy in the $i^{th}$ bin:

$$c_i \approx \frac{N_i E_i}{E} \quad , \quad \sum_{i=1}^{m} c_i = 1 \quad .$$

(2.2)

The various $N_i$'s (and $c_i$'s) will change in time due to the interaction of strings that we have described above. We will first consider the change of $N_i$ with respect
to the time $t$. In our model four terms will contribute to the time derivative $\dot{N}_i$:

$$
\dot{N}_i = -\lambda_i N_i + \sum_{j=i+1}^m B_{ij} \lambda_j N_j - \sum_{j=1}^{m-i} \phi_{ij} N_i N_j + \frac{1}{2} \sum_{j=1}^{i-1} \phi_{ji} N_j N_{i-j}
$$

- some strings will decay with decay rate $\lambda_i$
- some will come into bin $i$ from bin $j$ as decay product, with $B_{ij}$ branching ratio
- some will recombine, thus falling out of the $i$th bin
- some will come into bin $i$ from lower bins as product of recombinations.

In the third term the upper limit of the sum is such that the formation of string with energy larger than $E$ is forbidden, this because we have assumed a system with finite total energy.

In the above equation, which 'per se' is common to many nucleation models, the string theory input lies in the quantities $\lambda_i$, $B_{ij}$ and $\phi_{ij}$. $\lambda_i$ is the average decay rate for a string in bin $i$, $\phi_{ij}$ the total interaction rate for two strings in bins $i$ and $j$ to form a string in bin $i+j$, and $B_{ij}$ the branching ratio for the decay of a string in bin $j$ into strings in bin $i$ and $(j-i)$.

Before calculating these quantities we will make here an important assumption, we will assume that most of the energy of the strings is in the form of rest mass rather than kinetic energy:

$$
m_i \simeq E_i \simeq i \Delta E
$$

where we are assuming that our energy unit $\Delta E$ is measured in the unit of Planck mass.

The assumption in eq. (2.4) enables us to consider the quantity $N_i$ to have a single subscript instead of being labelled by a vector, and in the following it will simplify the evaluation of $\lambda_i$, $B_{ij}$ and $\phi_{ij}$. This assumption is of course an useful approximation, but is justified by the analysis of the phase space volume of a state with total energy $\epsilon$, which is divided into rest mass $m$ and kinetic energy $\epsilon_k$, namely $\epsilon^2 = m^2 + \epsilon_k^2$. The probability that this state is realised is proportional to the phase space volume of this state and, in the present case, it is a product of the mass density $\rho(m)$ and the momentum phase space volume. Owing to the crucial feature of string theories, the exponential growth of states, the phase space volume of heavy states
is large compared with a state with large kinetic energy and small mass. Therefore, highly excited heavy states have a much larger probabilities to be realized than those with corresponding large kinetic energy. Therefore, among the strings with energy \( \epsilon \), states with \( \epsilon \approx m \) will dominate, this justifies the approximation (2.4).

With this assumption we now pass to the evaluation of the three quantities of interest, \( \lambda_i \) has been calculated in [10], its values

\[
\lambda_i \approx g^3 E_i = g^3 \Delta E \quad i \neq 1 ,
\]

where \( g \) is the string coupling constant in \( d \) spatial dimensions. The decay rate is proportional to the mass (energy) of the string. Not surprisingly more excited strings have shorter lifetimes. We will also consider \( \lambda_1 = 0 \), since strings in the first bin cannot decay anywhere else.

As for \( \phi_{ij} \), this is, in the average, the total unpolarized interaction rate for the merging of two highly excited strings. We have considered this problem in a recent paper [11] and found that, with our approximations,

\[
\phi_{ij} = \frac{1}{V} \frac{\pi}{16} g^3 E_i E_j = \frac{\Delta E^2}{V} \frac{\pi}{16} g^3 i j .
\]

Again the result is what one expects, the more the string is excited, the larger is its cross section.

Next we consider the branching ratio \( B_{ij} \), it will be related to the phase space volume of the strings. In our approximation this volume for a string in the \( i^{th} \) bin is \( \rho(m) \sim \rho(E) \propto E_i^{-(d+1)} \epsilon^{d} \Delta E \),

\[
B_{ij} = \frac{\rho(E_i) \rho(E_{j-i})}{\frac{1}{2} \sum_{k=1}^{j-1} \rho(E_k) \rho(E_{j-k})} = \frac{(i(j-i))^{-(d+1)}}{\frac{1}{2} \sum_{k=1}^{j-1} (k(j-k))^{-(d+1)}} .
\]

In this expression combinatorial factors have been taken into account. In the above, the factor \( \epsilon^{d} \Delta E^{-(d+1)} \) cancels between numerator and denominator. All dependence on \( \beta_H \) is lost, and the exponential density no longer appears explicitly. Actually the exponential density still plays a crucial role, albeit a hidden one. Without this exponential density, eq. (2.6) would be very different, moreover it is again the exponential increase in density which justifies our approximation for the dependence of decay rate and cross section from the energies rather than masses.
In (2.3) we have made another approximations, we have considered only processes of the kind (i) (j) \( \rightarrow \) (i + j) and not, for example (i) (1) \( \rightarrow \) i which, due to the finiteness of the bins, are in principle also possible. Of course when \( \Delta E \) is small those processes will not be a dominant contribution. We have also ignored four or more bodies reactions, they are taken partly into account as two step on shell processes however. And the list of approximations of course does not stop here, we could include fermions, or winding modes etc. In the conclusions we will briefly comment on possible ways to improve the present model.

With our definitions of (2.5), (2.6) and (2.7), equation (2.3) now becomes:

\[
\dot{N}_i = g^2 \Delta E \left( -N_i (1 - \delta_{ii}) + \sum_{j=i+1}^m B_{ij} N_j \right) + g^2 \pi \frac{\Delta E^2}{V} \left( -\sum_{j=1}^{m-i} N_i N_{ij} + \frac{1}{2} \sum_{j=1}^{i-1} N_{i-j} N_j (i-j) \right). \tag{2.8}
\]

To further simplify the system, we consider the quantity \( c_i \), rescale the time to \( \tau = g^2 t \Delta E \), and consider a quantity which is proportional to the energy density: \( \Lambda = \pi E/16V \), in this case

\[
F_i \equiv \frac{dc_i}{d\tau} = i \left\{ -c_i (1 - \delta_{ii}) + \sum_{j=i+1}^m B_{ij} c_j + \Lambda \left( -\sum_{j=1}^{m-i} c_j + \frac{1}{2} \sum_{j=1}^{i-1} c_{i-j} \right) \right\}. \tag{2.9}
\]

This system of equations is our model for the behaviour of interacting strings at finite densities\(^1\), in \( d \) spatial dimensions, with \( d \) appearing explicitly in \( B_{ij} \).

We will mostly study the equilibrium configurations, that is the solution of the system with \( F_i = 0 \), in this case we are dealing with a system of \( m \) algebraic equations. The important fact is that the equations are non linear, so that a number of things can in principle happen. To start with, it is not guaranteed that there are solutions at all, in particular we are of course interested in solutions belonging to the 'physical region', that is with \( \sum_i c_i = 1 \), with all \( c_i \geq 0 \). Solutions can then be unique or not, and stable or not against small perturbations. This last point can be checked by calculating the eigenvalues of the jacobian

\[
J_{ij} = dF_i/dc_j.
\]

\(^1\)In the following we will often refer to \( \Lambda \) as energy density, although there is the proportionality factor \( \pi/16 \).
The negative eigenvalues correspond to stable directions (determined by the eigenvectors), while a positive eigenvalue signals that a small variation in the direction of the relative eigenvalue would lead to a difference from the solution growing with time and thus to an instability. Imaginary eigenvalues are also of interest, in this case the solution (if stable) will reach stability only after oscillation in the plane of the corresponding eigenvalues. Also in principle possible are closed orbit and even more exotic occurrences such as strange attractors or even chaos.

To facilitate the analysis let us define some 'global' quantities which will enable us to visualize better the properties of the solution to \( F_i = 0 \). Of interest is the quantity \( N \), which is proportional to the total number of strings

\[
N = \sum_{i=1}^{m} N_i = m \sum_{i=1}^{m} c_i .
\]

This quantity \( N \) is always \( 1 \leq N \leq m \). Useful are also the mean value of the energy and of its square which, up to a constant are:

\[
< E > = \sum_{i=1}^{m} \frac{E_i N_i}{N} = \frac{E}{N} ,
\]

and

\[
< E^2 > = \sum_{i+1}^{m} \frac{E_i^2 N_i}{N} = \frac{\Delta E \cdot E}{N} \sum_{i=1}^{m} i c_i .
\]

In the following we will analyse the model mostly in terms of these quantities, properly normalized to compare the solution for different number of bins.
3 Exact Solution in Particular Cases

In this section we discuss two versions of our model which we were able to solve exactly using analytical methods. They are particular cases, but we will be able to infer from them some features which will be of interest in the general case, which we will solve numerically in the next sections.

3.1 Exact Solution for the Limit of Infinite Energy and Evaporation-Like Decay

This subsection is devoted to the analytic solution of our model in eq. (2.9) in an idealized situation. The remarkable point is that the equilibrium configuration of this idealized model is solved exactly in the region of the energy density of the system smaller than the critical value, \( \Lambda < \Lambda_c = 0.5 \).

The modification of the model is a simple one. We will consider that the decay of the string can only happen for evaporation, that is by the emission of a string in the first bin. This means to replace the branching ratio in our equation eq. (2.7) by,

\[
B_{i,j} = \delta_{i,1} + \delta_{i,j-1} \quad (3.1)
\]

This branching ratio means that the string in the \( j^{th} \) bin decays into the strings which are in the 1st and \( (j-1)^{th} \) bins\(^a\). This modification of the branching ratio is considered to be an idealization of our original expression in eq. (2.7), which has a sharp peak at the decay mode: \( j^{th} \rightarrow 1^{st} + (j-1)^{th} \). Therefore, this simplification should not drastically alter the qualitative features of our model\(^b\). In order to be able to solve analytically our model in this case we have also taken the limit \( E \rightarrow \infty \) and \( V \rightarrow \infty \), \( \Lambda \) fixed. Automatically then \( m \rightarrow \infty \) and the sums in the second and third term of (2.9) have infinity as upper limit. Energy conservation still requires that the sum of the \( c_i \)'s is 1. It is also necessary to require that \( \lim_{i \rightarrow \infty} c_i \rightarrow 0 \) sufficiently fast so that the upper limit of the sum in the third term of (2.9) can be taken to be \( \infty \) for all \( i \).

\(^a\)This branching ratio makes the decay part of our model the same as the classical nucleation model, namely Becker-Döring theory [9].

\(^b\)Since it is also possible to consider that the branching ratio in eq.(3.1) is formally obtained from eq. (2.7) in the limit \( d \rightarrow \infty \). When we will refer to the \( d = \infty \) case in the following we will always mean in short-hand notation that we are using the branching ratio (3.1).
The equations which give the equilibrium configuration of the energy concentrations are considerably simplified in the present case:

\[ 0 = (1 - \Delta)c_i + c_{i+1} - \frac{1}{2} \Delta \sum_{j=1}^{i-1} c_j c_{i-j}, \quad i > 1. \]  

(3.2)

In order to solve these equations, we will define the generating function of the \( c_i \)'s,

\[ f(x) \equiv \sum_{i=1}^{\infty} c_i x^i, \]

where \( x \) is a c-number. Multiplying \( x^i \) in the equations (3.2) and summing over \( i \), we obtain the equation in terms of the generating function:

\[ 0 = x - (1 + \Delta) f(x) + x^{-1} f(x) + \frac{1}{2} \Delta (f(x))^2. \]  

(3.3)

The requirement of the energy conservation, which now is \( f(1) = 1 \), provides us with a solution for \( c_1 \):

\[ c_1 = 1 - \frac{1}{2} \Delta. \]

Inserting this result into eq. (3.3), we obtain an equation for the generating function,

\[ z - (1 - \frac{1}{2} \Delta) - (1 + \Delta) f(x) + x^{-1} f(x) + \frac{1}{2} \Delta (f(x))^2 = 0. \]  

(3.4)

Remarkably this equation is purely algebraic. To solve this equation with respect to \( f(x) \) is straightforward. By imposing the regularity of \( f(x) \) at \( x = 0 \), we find the result:

\[ f(x) = \frac{1}{\Delta} \left\{ 1 + \Delta - x^{-1} + [1 - x^{-1} \sqrt{1 - 2Ax}] \right\}. \]  

(3.5)

Once we have obtained the generating function, the procedure to derive the generic expression for the energy concentration \( c_i \) is not difficult,

\[ c_i = \frac{1}{i!} \frac{\partial^i}{\partial x^i} f(x) \bigg|_{x=0}, \]

\[ = \frac{(2i-3)!!}{(i+1)!} \Delta^{i-1} ((i+1) - (2i-1)\Delta), \]

\[ = \gamma_{i-1} \Delta^{i-1} - \gamma_i \Delta^i, \]  

(3.6)

\[ \gamma_{i-1} \equiv \frac{(2i-3)!!}{i!}. \]
Let us derive the asymptotic form of \( c_i \) for large bin number, \( i \gg 1 \). With the use of Stirling’s formula, we obtain the following results:

\[
\gamma_{i-1} \sim \frac{1}{\sqrt{\pi}} i^{-3/2} 2^{i-1} |1 + O(1/i)|,
\]

\[
c_i \sim \frac{1}{\sqrt{\pi}} i^{-3/2} (2\Delta)^{i-1} (1 - 2\Delta),
\]

\[2\Delta < 1, \quad \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{3}{2} \ln i - i \cdot \ln \frac{1}{2\Delta} \right\}.
\]

In the literatures of the nucleation theory, the asymptotic form of \( c_i \) is referred as the scaling form. The power of \( i \) in \( c_i \), namely the factor \( i^{-3/2} \), is related to the universality of the scaling form. In terms of the number of strings in \( i^{th} \) bin, the above asymptotic form reads \( N_i \sim i^{-3/2}(2\Delta)^i \). The power behavior \( i^{-3/2} \) in the asymptotic form is universal for a large class of nucleation models [9]. We will see similar behavior of \( c_i \) for the general case in the coming sections.

From a quick analysis of equations (3.7) we see immediately that 1/2 is a critical value for \( \Delta \), in fact for \( \Delta < 1/2 \) all \( c_i \)'s are positive which is a physical requirement. For \( \Delta > 1/2 \) and \( i \) large enough \( c_i \) becomes negative, moreover for these values of \( \Delta \) the \( \gamma_i \)'s diverge. In other words for \( \Delta > 1/2 \) energy conservation cannot be implemented because the sum of the \( c_i \)'s diverges to \(-\infty\) for \( \Delta > 0.5 \),

\[
\sum_{i=1}^{\infty} c_i = \lim_{m \to \infty} \sum_{i=1}^{m} c_i \approx \lim_{m \to \infty} \left( 1 - \frac{1}{\sqrt{\pi}} m^{-3/2}(2\Delta)^m \right).
\]

There is no static physical solution for high values of \( \Delta \) in this version of the model. Therefore, it is appropriate to call \( \Delta = 0.5 \) the critical value of the energy density in our idealised model in eq. (3.2).

Since we know the exact form of the energy concentration in eq. (3.6), we can calculate some physically relevant quantities exactly for \( \Delta \) smaller than 0.5. Let us first derive the expression for \( \sum c_i/i \) and \( \sum ic_i \), which appear in the number of strings and energy density:

\[
\sum_{i=1}^{\infty} \frac{c_i}{i} = -1 + 2(1 + \frac{1}{\Delta}) \ln \left( 1 + \sqrt{1 - 2\Delta} \right) + \frac{3}{\Delta} (1 - \sqrt{1 - 2\Delta}),
\]

\[
\sum_{i=1}^{\infty} ic_i = \left. \frac{\partial f(x)}{\partial x} \right|_{x=1} = \frac{1}{\Delta} (1 - \sqrt{1 - 2\Delta}), \quad (3.8)
\]

where we assume that \( m \) and \( E \) are finite, however large enough that the assumptions made are valid.
The number of strings defined in (2.10) becomes
\[
N = m \left\{ -1 + 2 \left(1 + \frac{1}{A}\right) \ln \left(\frac{1 + \sqrt{1 - 2A}}{2}\right) + \frac{3}{A} \left(1 - \sqrt{1 - 2A}\right) \right\}. \tag{3.9}
\]

The asymptotic form of the number of strings as \(A \to 0.5\) is given by,
\[
N \sim 0.841 + (6.455)e,
\]
where we defined \(A = 0.5 - \epsilon\) for a small positive number \(\epsilon\). The energy density of the system then is obtained as,
\[
\frac{\sqrt{<E^2>}}{<E>} = \left\{ \sum_i i c_i \sum_j c_j - 1 \right\}^{1/2}
\]
\[
= \left[ \frac{1}{A} (1 - \sqrt{1 - 2A}) (-1 + 2(1 + \frac{1}{A})\ln\frac{1 + \sqrt{1 - 2A}}{2}
+ \frac{3}{A^2} (1 - \sqrt{1 - 2A})^2 - 1 \right]^{1/2}.
\]

We have seen the behaviour of our idealized model in the region of \(A\) smaller than 0.5. The natural question to ask is then what is happening in the region \(A\) greater than 0.5. The lack of a stable solution, associated with the failure of the conservation of energy (\(\sum c_i = 1\)), can be interpreted as the formation of an infinitely long string\(^6\). It seems appropriate to expect a phase transition in this case. In the next section, we will solve the full model, with finite energy, numerically and find a similar behaviour with a phase transition, which this time will be with most of the energy carried by a single string (or a few strings). Furthermore, we will find that the results obtained in this section describe the low energy phase (gas phase) of the full model quite well, and in fact even critical value of \(A\) will not be greatly affected.

### 3.2 The Three Bins Case

In this subsection, we will perform an extensive study of the model (2.9) in the case of \(m = 3\). Although this case is not a relevant one to discuss the physical implications of our model, it nevertheless possesses some general features of the

\[^6\text{In fact, in nucleation theory, the same phenomenon is understood as a formation of infinitely large clusters.}\]
generic number of bins case. The importance of it is that in this case the model reduces to a system of two equations in two unknown, and can therefore be solved exactly with the help of some useful mathematical tools, which are available only for non-linear systems of two equations. In this sense, this section is separated form the others, and the reader can skip it if he is willing to trust the results of the numerical calculations without the help of the rigorous results available in this case.

When the number of bins is equal to three the branching ratios have a simple form and are independent of \( d \), namely \( B_{12} = 2, B_{13} = B_{23} = 1 \), then our model in eq. (2.9) becomes

\[
\begin{align*}
\frac{\partial c_1}{\partial \tau} &= 1 - c_1 + c_2 - \Delta(c_1^2 + c_1 c_2), \\
\frac{\partial c_2}{\partial \tau} &= 2 - 2c_1 - 4c_2 + \Delta(c_1^2 - 2c_1 c_2), \\
c_1 + c_2 + c_3 &= 1,
\end{align*}
\]  

(3.10)

where we do not write the differential equation for \( c_3 \) explicitly, because \( c_3 \) is obtained by solving the constraint of the energy conservation. Therefore, we can consider the above equations as the ones for two-dimensional non-linear flow in the region \( \{ \phi | 0 < c_1 + c_2 < 1, 0 < c_1, 0 < c_2 \} \),

\[
\frac{\partial c_1}{\partial \tau} = f(c_1, c_2), \quad \frac{\partial c_2}{\partial \tau} = g(c_1, c_2),
\]  

(3.11)

where \( f \) and \( g \) are given by the right hand side of eq. (3.10). The set of functions \((f, g)\) in eq. (3.11) is considered as a vector field on the \((c_1, c_2)\)-plane.

In the mathematical literature there are many useful tools on two-dimensional non-linear flow. In the following, we will review some of mathematical notations, which is available to investigate our \( m = 3 \) model (3.10). Let us first define two-dimensional flow \( \phi_t(x), x \in D \), where \( t \) is the time variable and \( D \) is the two-dimensional region on which the flow \( \phi_t \) acts. The evolution of the flow \( \phi_t \) is described by the differential equations,

\[
(\dot{x}, \dot{y}) = (f(x, y), g(x, y)) ,
\]  

(3.12)

where \( x, y \) are two-dimensional coordinate \((x, y) \in D\) and the dot as usual means the derivative with respect to time. In general, the functions \( f, g \) are non-linear ones of \( x \) and \( y \).
An important notion in the description of the asymptotic behavior of the flow is the one of nonwandering set. A point \( p \in \mathcal{D} \) is called nonwandering for the flow \( \phi_t \) if for any neighborhood \( U \) of \( p \), there exist arbitrary large time \( t \) such that \( \phi_t(U) \cap U \neq \emptyset \). The nonwandering set is a set of all such points. Nonwandering sets are classified into three classes: (i) Fixed points; (ii) closed orbits, and (iii) the union of fixed points and the trajectories connecting them. In the class (i), the fixed points of the flow are defined by the vanishing of the vector field: \( (f,g) = 0 \). In the follows, we will use "equilibrium point" in the same meaning of "fixed point". There are several kinds of fixed points namely sources, sinks, centres and a saddle points. The closed orbits in class (ii) are defined as periodic solutions to the differential equation for which there exists \( 0 < T < \infty \) such that \( (x(t),y(t)) = (x(t+T),y(t+T)) \).

A useful notion to identify non-wandering sets is the index of a closed curve \( C \). Let \( (f,g) \) be the two-dimensional vector field and \( C \) a closed curve not passing through any fixed points. The index of \( C \) with respect to the vector field is defined as follows: Letting the point \( P \) traverse \( C \) anticlockwise, along the curve the vector \( (f,g) \) rotates continuously and at the moment of returning to the original position the vector field must have rotated through the angle \( 2\pi k \) for some integer \( k \). The number \( k \) is called a index of the closed curve \( C \). The index of \( C \) is closely related to the character of the fixed points inside the curve \( C \): (i) The index of a sink, a source and a centre is +1. (ii) The index of a saddle point is −1. (iii) The index of a closed orbit is +1. If there is no fixed point inside \( C \), the index is 0. It can be proven [13] that the index of a closed curve \( C \) is equal to the sum of the indices of the fixed points within \( C \).

A powerful theorem concerning closed orbits is Bendixson's criterion, which says: If on a simply connected region \( D \subseteq \mathbb{R}^2 \) the expression \( \partial f/\partial x + \partial g/\partial y \) is not identically zero and does not change sign. Then the system of the flow does not have closed orbits lying entirely in \( D \). This theorem is a simple consequence of Green's theorem (see ref. [13] for the proof).

Now, we will apply the above tools to our problem in the \( m = 3 \) case. Our vector field \( (f,g) \) is given by,

\[
\begin{align*}
f(c_1, c_2) &= 1 - c_1 + c_2 - A(c_1^3 + c_1) \\
g(c_1, c_2) &= 2 - 2c_1 - 4c_2 + A(c_1^2 - 2c_1) 
\end{align*}
\]

and the domain on which this flow acts is \( D = \{0 \leq c_1 + c_2 \leq 1, 0 \leq c_1, 0 \leq c_2\} \).
First let us apply Bendixson's theorem. Because,
\[ \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -5 - 4A c_1 - A c_2 < 0 , \]
we can immediately see that our system does not have closed orbit in \( D \).

Next let us turn to the property of fixed points in our model. To this purpose, we need to know the number of the fixed points in \( D \). The equations to obtain the fixed points is given by

\[ \bar{c}_1 : A^2 \bar{c}_1^2 + A \bar{c}_1^2 + 2 \bar{c}_1 - 2 = 0 , \]
\[ \bar{c}_2 : \bar{c}_2 = \frac{1 - \bar{c}_1 - A \bar{c}_1^2}{A \bar{c}_1 - 1} , \]
\[ \bar{c}_3 : \bar{c}_3 = 1 - \bar{c}_1 - \bar{c}_2 , \]

where bars on \( c \)'s are intended to indicate fixed points. \( \bar{c}_1 \) is given as a solution to the cubic equation and \( \bar{c}_2, \bar{c}_3 \) in terms of \( \bar{c}_1 \). It is not difficult to see that there is only one solution to the above cubic equation in \( D \). The solutions for \( 10^{-4} < A < 10^4 \) are shown in figure 1, the points are values of \( A \) equally spaced on a logarithmic scale.

We can now ask ourselves the nature of the stability of the solutions we found. It is easily found that the vector field \((f, g)\) always points inward along the boundary.
of the domain $D$. Therefore, the index of the closed curve along the boundary of $D$ is 1 and the fixed point inside $D$ is a sink. This result is also obtained by solving the eigenvalue problem of the linearized flow equation in eq. (3.11) in the vicinity of the fixed point, where we always find negative eigenvalues. This property of the vector field to point inward on the boundary of $D$ ensure that $D$ is a trapping set, that is if the initial configuration of $c_1$ and $c_2$ lies inside $D$, then the flow will not bring the values of the $c_i$'s outside it, which would of course show an inconsistency of the model.

From the above discussion, we found that our model of $m = 3$ case has only one stable fixed point (or equilibrium configuration) in our domain. The flow diagram of a particular value of $A$ ($A = 1$), is given in figure 2. The qualitative features of the flow diagram do not change for different values of $A$. The system always reaches a unique equilibrium point independently on the initial condition. Later we will discuss our model for a generic $m$. However, the stability properties appear to be the same of the simple case considered here. We always find one stable equilibrium configuration (sink fixed point) in the physical region. In this sense, we can say that the simplest $m = 3$ model has the general stability features of the model.
4 The Phase Transition

In this section we discuss the solutions of (2.9) for a generic number of bins. Unfortunately for \( m > 3 \) it is impossible (at least for us) to solve exactly the system of non-linear equations, and we were therefore forced to resort to numerical methods. We have looked for the solutions of (2.9) with \( dc/dr = 0 \) using two different methods, the Newton-Raphson method, and the Powell method, of course they both gave the same results. For problems of computer memory we were forced to limit the number of bins \( m \) to 500. As a rule the accuracy was of the order of \( 10^{-12} \), that is, a set of values for \( c_i \) was considered a solution if all the \( dc_i/dr \)'s were smaller than \( 10^{-12} \), which gives an error (which can be evaluated) on the \( c_i \)'s of approximatively the same order of magnitude.

With our algorithms (as with any algorithm) it is not possible to say if the found solution is unique, they are routines which search for a zero of the system starting from an initial ansatz, and can say nothing on the possible existence of other solutions. We have however searched with a variety of initial ansatze and found always one unique solution. This solution appears stable, that is the jacobian defined in section 2 has only negative eigenvalues. Some of the eigenvalues were however imaginary for large values of \( m \). We have also time evolved \( c_i \) for a variety of values of the density \( \Lambda \) and of \( m \), again starting from various initial configurations, and found always the configuration to converge to the one found with the above mentioned methods. This gives us some confidence in asserting that the exact results we found for the \( m = 3 \) case, that there is always only one unique solution in the physical region, and that this last region is a trapping set, generalises to an arbitrary number of bins. In the first subsection we discuss the equilibrium (stationary) configurations of the \( c_i \)'s and show numerical evidence for a phase transition. In the second subsection we instead discuss the stability of the equilibrium configurations.

4.1 The Phase Transition

An equilibrium configuration for a particular value of \( \Lambda \) is given by a vector with components \( c_i \). As it is difficult to visualise the whole vector, and see how it changes with \( \Lambda \), we have found it useful to define the quantities \( N_{\text{nor}} \) and \( F_{\text{nor}} \), which we will analyse in detail. They are normalized values of the total number of string and the
fluctuations of the energy around the mean. Their definition is as follows:

\[
N_{\text{nor}} = \frac{\sum_{i=1}^{m} c_i - \frac{1}{m}}{1 - \frac{1}{m}} \alpha \frac{1}{m} \left(\sum_{i=1}^{m} N_i - 1\right) \tag{4.1}
\]

\[
F_{\text{nor}} = \left[ \frac{\left(\sum_{i=1}^{m} c_i^2\right) \left(\sum_{i=1}^{m} c_i^2\right) - 1}{\left(\frac{1}{2} + \frac{1}{2m}\right) \left(\frac{1}{2} + \frac{m}{2}\right)} - 1 \right] \alpha \frac{\sqrt{\langle E^2 \rangle - \langle E \rangle^2}}{\langle E \rangle} \tag{4.2}
\]

We have normalized \(N_{\text{nor}}\) and \(F_{\text{nor}}\) in such a way that \(N_{\text{nor}} = 1\), \(F_{\text{nor}} = 0\) for \(c_i = \delta_{1,i}\) and \(N_{\text{nor}} = 0\), \(F_{\text{nor}} = 0\) for \(c_i = \delta_{m,i}\) (all energy in the first or last bin respectively), moreover \(F_{\text{nor}} = 1\) for a configuration in which the energy is divided in equal parts between the first and last bin, \(c_1 = c_m = \frac{1}{2}\). This to enable comparisons among the different equilibrium solutions for different \(m\)'s.

The behaviour of \(N_{\text{nor}}\) and \(F_{\text{nor}}\) vs. \(\Lambda\) for three spatial dimensions is summarised in figures 3 and 4 for various values of \(m\). Those figures are the main result of the model. They show the presence of a discontinuity point at a value of \(\Lambda\) close to 1/2. For small densities all strings are in the lowest bin, in the massless or first few
excited states, all the energy is carried by massless or nearly massless states. This is indicated by a value of $N_{nor}$ nearly unity as can be seen in figure 3. When the density $A$ approaches a critical value, the quantity $N_{nor}$ decreases and sharply goes to zero. It is very important to notice that the step behaviour of figure 1 becomes sharper as $m$ increases. The behaviour of $F_{nor}$ in figure 4 is analogous, $F_{nor}$ jumps from the 0 value (all strings in the first bin) to values of order one (strings more or less equally distributed in the first and last bins), to quickly become again small, this time with most of the energies in the last bins. The rate of decrease of $F_{nor}$ past its maximum is however rather slower, than its abrupt increase. The qualitative behaviour of $N_{nor}$ and $F_{nor}$ is similar for the various values of the number of dimensions, as it can be seen from figure 5 and 6**.

The interpretation of these results is that there are two phases of string theory, in the low density phase (which we will call the 'gas' phase) all the strings are in the massless or first few excited states, and (as we will see in more detail later)

**In these figures $d = -1$ and $d = \infty$ indicate the two cases in which the branching ratio is equal for all channels regardless of the energy (the $d = -1$ case), and the case in which the branching ratio is the evaporation-like one considered in section 5.
Figure 5: The behaviour of $N_{nor}$ for $m = 500$ for various choices of $d$.

Figure 6: The behaviour of $F_{nor}$ for $m = 500$ for various choices of $d$. 
they can be well approximated by an ideal gas of particles, and have a Boltzmann
distribution to a good approximation. When the density is of order one, abruptly
there is the passage to a different phase, which we have called the 'liquid' phase.
In this phase the strings have coalesced into one or few extremely excited strings.
Although our numerical method is not sophisticated enough to establish the order
of the transition, it appears to be of first order\(^{11}\). It is quite possible that in this
phase the degrees of freedom of the theory would probably have to be changed for
a consistent description.

4.2 Stability of the Model

Even insofar the stability of the model is concerned the results of the \(m = 3\) case
appear to be confirmed. As a test of the stability of the model we have calculated
the eigenvalues of the jacobian of the equilibrium solution. The presence of
eigenvalues with positive real part would indicate an unstable saddle point, while
complex eigenvalues (with negative real part) indicate a focus, a stable point which
is reached only after oscillations in the plane defined by the couple of eigenvectors
corresponding to the complex eigenvalue. We have not found positive eigenvalues
(larger than our machine accuracy). We found instead frequently complex eigen-
values, but never purely imaginary (an indication of a nearby closed orbit). The
number, value and behaviour with the density of the eigenvalues depend strongly
on the number of dimensions, but the qualitative features do not change.

We have also actually time evolved the \(c_i\)'s starting from various initial ansatze.
As an example the behavior of the configuration in the \(N - F\) plane is indicated in
figure 7 in the \(d = 3, m = 30\) case. In this figure the initial configurations where
chosen with all energy in one bin, the 3\(^{rd}\), 5\(^{th}\), 10\(^{th}\), 15\(^{th}\) and 30\(^{th}\). Again, due
to the numerical nature of this investigation, it is impossible to prove beyond any
doubt that there are no close orbits, but the similarity with the exact \(m = 3\) case
indicate this quite convincingly.

\(^{11}\)In the heretic model [2] a similar transition from short to long strings has a continuously varying
number of strings [18].
Figure 7: The time evolution of the configuration for various initial conditions.
5 Equilibrium Configurations and Thermodynamics

In this section we discuss the relations between the equilibrium configurations of our model and the results one would expect from a traditional thermodynamical or statistical mechanics treatment. The utility of our model is mostly for a region of some orders of magnitude around $A \sim 1$, when the behaviour of the $c_i$'s is smooth enough, the thermodynamical considerations are moreover usually limited to a region $A \lesssim 1$. In the first subsection we will discuss a best fit of the behaviour of the equilibrium configurations for various $m$, $d$ and $A$ to some ansatz suggested by thermodynamics. Some of the details of the fitting are collected in an appendix. In this subsection we also discuss the independence of the behaviour of the gas phase on the number of dimensions. This provides a sort of universality of this phase which reinforces the result of section 3.1. In the following subsection we interpret the result and discuss the thermodynamics of the system. At the end, we will try to make connection with the ordinary nucleation theory.

5.1 Fitting of the Equilibrium Configurations

This subsection is devoted to the preparation of the discussion on the thermal properties of our model. In order to discuss the thermodynamical properties of equilibrium configurations in our model, it is not convenient to deal directly with the numerical results. Thus, we will find an analytical function which reflects the behavior of $\{c_i\}$ best; namely we will do the fitting of our numerically obtained equilibrium configurations to a certain analytical function. This procedure of fitting is important not only for its convenience but also for revealing the nature of the phase transition we found in the previous section.

Our ansatz to this function is of the form:

$$c_i \sim \exp\{u_1 \ln(i) + u_2 \cdot i\} ,$$

where $u_1$ and $u_2$ are the parameters to be determined by the fit. There are three motivations for the choice of this ansatz. The first (and most naive one) is the analysis of the equilibrium distribution one can almost do by just looking at them. In figures 8 and 9 the equilibrium distribution of $c_i$ are shown on the case of $d = 3$ and $m = 250$, but for various $A$'s. Notice that the $c_i$'s are plotted on a logarithmic scale. The appearance of the linear tails for large bins suggests the form: $c_i \sim e^{u_1 i}$.
Figure 8: $c_i$ as a function of the bin (Energy) in the gas phase.

Figure 9: $c_i$ as a function of the bin (Energy) in the liquid phase.
The second reason is the correspondence with the Maxwell-Boltzmann distribution. In the gas phase, $\Lambda \lesssim 0.5$, the concept of the string gas in the thermal equilibrium seems to be available. Then, one can expect that the equilibrium configurations $c_i$ is given, to a good approximation, by that of Maxwell-Boltzmann,

$$c_i \sim \rho(E_i) e^{-\beta E_i} \sim e^{-\alpha \ln E_i - (\beta - \beta_B) E_i}, \quad (5.1)$$

where $\alpha$ is a constant, which possibly related to the number of spatial dimensions.

The third reason comes from the results of the subsection 3.1, where we have investigated the simplification of evaporation-type branching ratio. In that subsection, the asymptotic form of $c_i$ is given by, $c_i \sim \exp\left\{-(3/2)\ln(i) + \ln(2\Lambda) \cdot i\right\}$, (see eq. (3.7)).

The second and third motivations for our ansatz are available only in the low density region. However, this ansatz works reasonably well for both low and high density regions and we found it helpful to understand the property of equilibrium configurations in these two different regions.

The details of the fitting are in the appendix, and here we just collect the results. In figs. 10 and 11 the obtained values of $u_1$ and $u_2$ are given in the case $d = 3, 9$ and $m = 100$ as functions of the energy density, $\Lambda$. The results for the other number of the spatial dimensions are qualitatively the same as these two examples.

From these figures, one can read several features of the equilibrium configurations, which we will investigate in detail.

1. The behavior of the equilibrium distribution divides the region of the energy density, $\Lambda$, into three regions; the first is the low energy region (gas phase, $\Lambda \lesssim 0.5$), then there is an intermediate region and the finally the high energy region (liquid phase, $10^3 \lesssim \Lambda$). At the point of $\Lambda$, where two different regions meet, $u_1$ and $u_2$ have discontinuities. It should be kept in mind that the presence of an intermediate region is one of the features of a first order phase transitions.

2. The equilibrium configurations for different values of the spatial dimensions, $d$, have a similar behavior in the gas phase, but are quite different in the liquid phase. The main difference in the liquid phase comes from the value of $u_1$.

---

1We also tried the other ansätze like $c_i \sim \exp(m^2 + ci)$ and found that the one we have chosen better.
Figure 10: The fitting parameters $u_1$ and $u_2$ for $d = 3$ and $m = 100$.

Figure 11: The fitting parameters $u_1$ and $u_2$ for $d = 9$ and $m = 100$. 

27
3. The value $u_2$ is proportional to $\ln 2A$, $u_2 \propto \ln 2A$, in both gas and liquid phase. The proportionality constant is however very different between two phases.

In the following of this subsection, we will discuss the appropriateness of above three statements. Let us first discuss the dependence of the fitted values $u_1$ and $u_2$ on the number of spatial dimensions using the $m = 100$ results.

For the value of $u_1$, figs. 10 and 11 suggest that it is independent of spatial dimensions in the gas phase and it goes to $-(d + 1)$ in the liquid phase. In fig. 12, the fitted values of $u_1$ for various spatial dimensions are shown. From this figure, we draw a result that $u_1 \sim -2.4$ in the gas phase independently of the dimensions. In the appendix, we discuss the asymptotic form of numerically evaluated configurations and find that $u_1$ tends to a value close to the prediction of the idealised model in subsection 3.1. Therefore, our result on $u_1$ is that $-5/2 \leq u_1 \leq -3/2$ in the gas phase and $u_1 \sim -(d + 1)$ in the liquid phase.

Next let us turn to $u_2$. As we have mentioned above, the fitted value $u_2$ in figs 10, 11 depends on the energy density in the way proportional to $\ln 2A$. This is
Figure 13: The proportionality constant \( \alpha \) vs. \( d \). Since in the liquid phase it is quite difficult to get appropriate values of \( \alpha \) for higher dimensions, we have listed those for \( d \leq 15 \).

Indeed the case for the other values of spatial dimensions. Therefore we define

\[
\omega_j = \alpha \ln(2\Lambda)
\]

The proportionality constant \( \alpha \) can be seen from fig. 15 in both phases for various \( d \). In the gas phase, the value of \( \alpha \) is independent on \( d \), and \( \alpha \sim 1 \). Notice that \( \omega_j = \ln(2\Lambda) \) is the prediction in subsection 3.1. The same seems to hold in the liquid phase too, in this case however, for much smaller value of the proportionality constant, \( \alpha \approx 0.01 \sim \frac{1}{m} \).

The above observation suggests the following conclusion: In the gas phase, the equilibrium configurations is universal with respect to the number of spatial dimensions. In the liquid phase, the behavior of the equilibrium configurations depends on the number of the spatial dimension. The dependence on the number of the spatial dimensions in the liquid phase is supposed to be a consequence of the form of the branching ratio in the model.

In order to show the universality in the gas phase independently on the fit-
Figure 14: The differences of the equilibrium configurations for various $d$.

We have evaluated a quantity $\sum_i |\ln c_i(d) - \ln c_i(d')|$, where $\{c_i(d)\}$ is a equilibrium configurations obtained for the number of spatial dimensions $d$. The result of $\sum_i |\ln c_i(d=1) - \ln c_i(d=3)|$ and $\sum_i |\ln c_i(d=3) - \ln c_i(d=0)|$ are shown in fig. 14 for various $\Lambda$ in case $m = 100$. This result indicates that in the gas phase the difference of the equilibrium configurations obtained for different $d$'s is quite small. On the other hand, in the liquid phase it is large. Moreover, in the liquid phase, the difference of the two different spatial dimensions is given approximately,

$$\sum_i |\ln c_i(d_1) - \ln c_i(d_2)| \sim |d_1 - d_2| \sum_i \ln(i),$$

which again justifies our ansätz.

Another important property of the equilibrium configurations is its scaling property with respect to the total number of bins $m$. In the above discussion, we concentrated mainly on the case of $m = 100$. However, one might be interested in the dependence of the form $c_i \sim \exp\{u_1 \ln(i) + u_2 \cdot i\}$ on the total number of bins, $m$. For the behaviour of $u_1$, it does not depend on the number of bins in both gas and liquid phase. In fig. 15 the dependence of the proportionality constant $\alpha$, defined in $u_2 = \alpha \ln(2\Lambda)$, on the number bins exhibited in the cases $d = 3$ and $d = 9$. 
Figure 15: The behaviour of \( \alpha \) in the gas phase for various values of \( m \).

One can see that the proportionality constant \( \alpha \) is independent of the number of bins. This is what one expected from the result of subsection 3.1, where the energy separation between subsequent bins, \( \Delta E \), is not set by the model itself. The situation is somehow the same in the full model before the formation of long strings which carry most energy of the system. Therefore, in the gas phase the statement \( \Delta E = E/m \) does not have any definite meaning. Moreover, physically speaking, we should consider that in the region \( \Lambda < 0.5 \) the energy separations of bins \( \Delta E \) is equal to the Planck mass independently of the number of bins in the model, namely \( \Delta E \sim (\alpha^*)^{-1/2} \). Thus, contents of first bin in gas phase is considered to be massless states. On the other hand, in the liquid phase, \( \alpha \) depend on the number of bins. From fig. 16, one can read \( \alpha \sim 1/m \), where \( m \) is the number of bins. This behavior of \( \alpha \) with respect to \( m \) is the one which satisfies the requirement of the energy conservation for the fixed total energy,

\[
c_{i|m} = c_{m-1}|z_m + c_m|z_m
\]

Due to the physical requirement, which comes from the validity of the approximation, \( \Delta E \) is constrained to \( \Delta E \geq m_0 \), where \( m_0 \) is the energy scale of our interest, namely Planck mass in the fundamental string theory and pion mass in QCD dual string theory.

31
where \( \{c_i\}_{m} \) is the equilibrium configurations in the case of the total number of bins \( m \). This behavior of \( u_3 \) is independent on the number of spatial dimensions.

One of the interesting facts we have noticed concerning the dependence of equilibrium configurations on the number of bin \( m \) is the tendency which the intermediate region gets smaller with increase of \( m \). Due to numerical nature of our analysis, we can not tell if it vanishes in \( m \rightarrow \infty \) limit. At least we can say the transition becomes sharper as the number of bins increases.

We conclude this subsection with the main results of the fitting. The equilibrium distribution of our model has distinctly different behaviors in each region, namely the gas and the liquid phase, with an intermediate region which connects the two. This suggests the phase transition we found in our model is of the first order. In the gas phase, the behavior of the equilibrium configurations is universal with respect to the number of spatial dimensions. In the liquid phase, however, it depends on the number of spatial dimensions. The form of the energy concentrations are given by

\[
A \leq 0.5 \quad c_i \sim i^{-\alpha} e^{-i3\ln(1/2\Lambda)} \quad 3/2 \leq \alpha \leq 5/2 \quad ,
\]

(5.2)
The idealized model in subsection 3.1 describes the equilibrium properties of the gas phase quite well. Following these results, the discussion on the thermodynamical property of our model will be developed in the next subsection.

5.2 Thermodynamical Implications of the Model

In this subsection, we will discuss the thermodynamical implications of our model on the basis of the results obtained in the previous sections. We will argue that our model has a critical temperature which is in fact the Hagedorn temperature, and make some connections with the usual thermodynamics of strings. We will then obtain an equation of state and briefly comment on the singularity in the thermal potential.

In our model of interacting strings, the equilibrium distribution is determined by its dynamics and the equilibrium configurations are not necessarily interpreted in the sense of string gas model. Therefore, the procedure to determine the temperature is not clear and we cannot make any definite statement except somewhat naive qualitative discussions. As we have discussed at the beginning of the section, in the gas phase, the notion of the string gas is available to a good approximation. Therefore, it is possible to identify temperature in our system in the gas phase. Let us concentrate on the gas phase for a while. Naively one would proceed as follows: in the gas phase, the distribution is considered to be given by that of Maxwell-Boltzmann eq. (5.1), while our model gives the equilibrium distribution in eq. (5.2). It is then reasonable to identify

\[ \beta - \beta_H \simeq \frac{1}{\Delta E} \ln \left( \frac{1}{2\Delta} \right) , 2\Delta \leq 1 \quad . \]  

At \( \Delta \sim 1/2 \), which is the critical value in our model, the temperature of the system equals the Hagedorn temperature. In figs. 8 and 9, the configuration at \( \Delta \simeq 0.545 \) is shown, where the linear asymptotic behavior at its tail vanishes and it behaves \( c_1 \sim \frac{1}{\sqrt{\theta}} \). 3/2 \( \leq \theta \leq 5/2 \). It is important to notice that this \( \Delta \) corresponds to the points where the curves in fig. 5 and 6 have flex points. Once we are convinced by this identification of the temperature, we have a relation between energy density and temperature,

\[ \frac{E}{V} \sim e^{-\frac{\beta}{\beta_H}}, \quad T < T_H \quad . \]
where $\mu$ is a constant related to the energy scale of the model, $\mu \propto 1/\Delta E$.

As for the high energy region (liquid phase), we do not have any simple way to identify the temperature of our system, we are nor even sure such a quantity can be defined at all. As we have mentioned before, one should probably change the physical degrees of freedom of the theory in the high energy liquid phase. Therefore we cannot identify the results of equation (5.3) with a Maxwell-Boltzmann distribution eq. (5.1), for which the physical degrees of freedom are fluctuations around the vacuum of the gas phase$^1$.

But in the high density region our model is not complete as a model of fundamental strings since, for example, we are neglecting winding modes of strings and the effect of gravity, which are important for very high densities. Therefore it is dangerous to discuss the significance of eq. (5.3) for the thermodynamics of strings here.

The equilibrium distribution obtained in our model (eq(5.2)) is different from the one obtained by thermodynamical methods. The main difference is the power $\alpha$ in eq. (5.2). In our model the value of $\alpha$ is universal with respect to the spatial dimensions in the gas phase, but the one derived by thermodynamical methods depends on the spatial dimensions. The dependence on the spatial dimension in the thermodynamical methods appears partly from the integration over the momentum, while we are neglecting kinetic energies in our model. In the very low density region, $\Lambda \ll 1$, the system consists mainly of low energy states. In this situation the contribution of the kinetic energy is important and our model would need modifications. However for values of $\Lambda$ of the order of the critical density, where there are enough energetic strings, our approximation is correct. It might be an interesting problem to modify our model so that the kinetic energy of strings are incorporated and see how this power behavior will be changed.

The second subject we discuss in this subsection is the equation of state for the gas phase in our model. In the same sense as that in the kinetic theory of gas, the partial pressure, $p$, of a string of energy $\varepsilon$ in d-dimensional space of volume $V$ is given by,

$$ p = \frac{1}{V} \frac{2}{d} \langle \frac{\varepsilon^2}{2m} \rangle, \quad (5.5) $$

$^1$A naive identification of eq. (5.3) to a Maxwell-Boltzmann distribution would give even negative temperature.
where \( < \ldots >_{\epsilon} \) is the mean value of the kinetic energy of a string with energy \( \epsilon \) and the non-relativistic approximation is employed because of the assumption of highly excited string dominance, which is justified by the following discussion. The volume \( V_{\epsilon} \) in eq. (5.5) means the one after van der Waals correction, namely the effective spatial volume where strings can move. The mean value of the kinetic energy is given by,

\[
< \frac{p^2}{2m} >_{\epsilon} = \frac{\int \rho(m) dm \frac{d^d \epsilon}{(2\pi)^d} \delta(\epsilon - \sqrt{m^2 + p^2}) \frac{p^2}{2m}}{\int \rho(m) dm \frac{d^d \epsilon}{(2\pi)^d} \delta(\epsilon - \sqrt{m^2 + p^2})},
\]

where \( \rho(m) \) is as usual the mass level density of string. Using the asymptotic form of \( \rho(m) \) given in eq. (1.1), these integrals can be evaluated by saddle-points methods, which gives the result:

\[
< \frac{p^2}{2m} >_{\epsilon} = \frac{1}{2 \beta_H} \frac{(d + 1)^{d+1}}{(d - 1)^{d/2}} \epsilon^{-1} \left( 1 + O\left( \frac{1}{\epsilon \cdot \beta_H} \right) \right),
\]

here we are assuming \( \epsilon \cdot \beta_H \gg 1 \). Notice that the leading term in the mean kinetic energy does not depend on its energy \( \epsilon \), which means that the kinetic energy of string does not increase even if its energy is increased. This is the highly excited string dominance, namely most of energy of the string is carried by its rest mass. This result is not changed by the relativistic treatment. The total pressure, represented by \( P \), is the sum of all partial pressures,

\[
P V_{\epsilon} = \sum_{i} \frac{2}{d} < \frac{p^2}{2m} >_{\epsilon_i} N_i
\]

\[
\sim \frac{S(d, \beta_H)}{d} \sum_{i} N_i.
\]

\[
S(d, \beta_H) = \frac{1}{d \cdot \beta_H} \frac{(d + 1)^{d+1}}{(d - 1)^{d/2}} \epsilon^{-1}.
\]

This is the equation of state in our model. In terms of energy concentration \( \epsilon_i \), the equation of state becomes,

\[
P V_{\epsilon} = S(d, \beta_H) m \sum_i \frac{\epsilon_i}{N} = S(d, \beta_H) N.
\]

The crude approximation of \( V_{\epsilon} \) is \( V_{\epsilon} = V - E \) in Planckian units, because a string of energy \( E_i \) is considered to occupy the spatial volume \( E_{\epsilon_i} \). In fig. 17, the behaviour of the pressure as the system is adiabatically compressed is given for \( m = 300, d = 3 \) and some fixed total energy \( E \). The quantity \( P' \) shown is the one appearing in

\[
P(\Delta) = \frac{S 16}{E \pi} \frac{1}{1 - \frac{4\Delta}{E}} \equiv \frac{S 16}{E \pi} P'.
\]

35
Due to our approximation on $V_n$, the pressure blows up at $A \sim 1/5$. However, this blow up does not seem to be a physical one. In order to discuss the equation of state in liquid phase, we have to find correct physical degree of the system to describe it.

It is interesting to compare eq. (5.7) with the corresponding quantity in the grand canonical ensemble. There, the equation of state is given by:

$$PV = kT \ln \mathbb{P}(\xi, V, T),$$

(5.8)

where $\mathbb{P}(\xi, V, T)$ is a grand partition function with chemical potential $\xi$, volume $V$ and temperature $T$. At this point, the correspondences to $\xi$ and $T$ are not clear.

This observation on the correspondence between eq. (5.7) and eq. (5.8) leads to further correspondence between eq. (5.7) in our model and the cluster (virial) expansion of strongly interacting gas, where equation of state is given by,

$$PV = k_i T \left[ 1 - \sum_{l=1}^{\infty} \frac{l}{l+1} \beta_l \left( \frac{V}{N} \right)^{-l} \right],$$

(5.9)

where $\beta_l$ is a so-called irreducible integral, which represents the interactions between objects in the cluster of size $l$. Let us again concentrate on the gas phase and consider
the model of subsection 3.1. The identification to the results in section 3.1 follows, 
\( \beta_i = \gamma_i/l^2 \). The remarkable point is that the sum in eq. (5.9) can be carried out 
extactly. As a result, the equation of state in the model in subsection 3.1 becomes:

\[
P V = S(d, \beta_R) m \left[ -1 + 2 \left( 1 + \frac{1}{A} \right) \ln \frac{1 + \sqrt{1 - 2A}}{2} + \frac{3}{A} \left( 1 - \sqrt{1 - 2A} \right) \right]. 
\]  

(5.10)

The right hand side of eq. (5.10) can be considered as the free energy in the grand 
canonical system. Let us define the right hand side of eq. (5.10) as a function of 
\( A \), represented by \( F(A) \), and consider this function on the complex \( A \)-plane. This 
function has a cut on the real axis for \( ReA > 1/2 \). The existence of this cut in the 
free energy is a typical feature of a first order phase transition \([9,14]\). The model in 
subsection 3.1 has a equilibrium configurations only in the region \( A \leq 0.5 \). When 
\( A \geq 0.5 \), it does not have a equilibrium configurations but will have time-dependent 
solutions. In the standard nucleation theory, the main interest is in fact in these 
time-dependent solutions of a set of non-linear differential equations. One of the 
most important quantity in such solutions is the nucleation rate, which is a constant 
current creating infinitely large clusters (or strings in our case). This nucleation 
rate is obtained as a imaginary part of the free energy which appears when the free 
energy \( F \) is analytically continued to the unstable region of the parameters. In our 
case in eq. (5.10), the analytical continuation is done from the real axis \( A < 0.5 \) 
to the real axis \( A > 0.5 \) in the vicinity of the critical value \( A = 0.5 \). A simple 
calculation gives:

\[
\text{Im} F(A = \frac{1}{2}(1 - \delta e^{\pi i})) \approx 2\delta^{3/2},
\]

where \( \delta \) is a small positive number. It would be an interesting problem to examine 
this nucleation rate by numerical experiment using the full model described in 
section 2.
6 Conclusions

In this paper we have discussed the model of interacting strings based on Nucleation Theory. We did not assume equilibrium, but instead looked for the equilibrium configurations, and investigated their stability, finding that the stationary equilibrium is reached always and that this equilibrium is stable. The main results of the analysis is the presence of a phase transition from a 'gas' phase to a 'liquid' phase. In the gas phase, at low densities, the strings form a gas of massless or nearly-massless particles. The strings obey an energy distribution not too different from the Maxwell-Boltzman one, therefore temperature can be defined and usual thermodynamics appears to be valid. The equilibrium configurations in this gas phase appear to be independent on the number of large scale dimensions of space-time.

At a density of order one (in Planck units) we find the transition to the liquid phase. In this case the energy is carried by a single string or a few strings, with very high excitation number. The equilibrium configurations in this case are very dissimilar from a Maxwell-Boltzman distribution, and traditional thermodynamics does not appear adequate to describe the state of the system. In the liquid phase the equilibrium configurations depend on the number of dimensions in a definite way.

It is worth remembering at this point that a variety of different methods and models concerning strings seem to point in the same direction of a different phase of string theory. First there are the discussion on the statistical mechanics of strings, canonical or heretic, which indicate the presence of a phase in which most of the energy goes into few strings, and this paper is in this same stream. Work on the canonical ensemble [4] discusses a phase with many degrees of freedom less than an ordinary field theory. On the other side there are indications [16] of new phenomena appearing in the scattering of string at super-Planckian kinetic energies. And even the recent matrix models [17] based on a triangulation of surfaces find an infinite genus phase which could again indicate a new phase of strings. The connection among those aspects of string theory however is very far from clear.

Mainly this work uses the input coming from string theory to fix the coefficients of the differential equations and to justify the approximations made, these ingredients are however common to any dual model, including of course hadronic physics.

4We would like to thank R. Hagedorn for a letter calling our attention on this point

38
And in fact the analogy with the hadron transition [18], the deconfinement of quark and gluons, shows that above a certain density the degrees of freedom of have to change from hadrons to quarks and gluons. The added complication (and fascination) in string theory is that the graviton is a string, in the massless state, or the ‘first bin’ to use the language of this paper. Above the phase transition one would have to change the degrees of freedom of the theory, and presumably the graviton would not be an useful state. But what would then happen to the structure of space time itself? Atick and Witten in [4] have suggested the presence of a topological field theory, but the issue is again very very far from being resolved.

The analogy with hadronic physics works both ways, and the model presented here can be easily adapted [19] for a description of the deconfinement phase transition [20], both in the early universe and heavy ion collisions. The model can be further improved both for the fundamental and the hadronic point of view, one could for example consider the effect of fermionic states, or of the red shift. Probably the most important thing to do would be to consider the presence of winding modes, where the strings wrap around the compactified dimensions. For high densities, achieved at the beginning of the universe, winding modes become extremely important in view of the presence of a duality [21] between momentum and winding modes. It would be interesting to see if the presence of winding modes alter sensibly the results of the model presented here.

Acknowledgments

We would like to thank D. Amati, A. Levi and G. Pastore for discussions, A. Nobile for making the ICTP version of Minuit work and K. Heller for help with the computing which went well beyond the call of duty. Some of the calculations were performed using the symbolic manipulation package Maple. We are also grateful to ICTP, IAEA and UNESCO for support.
Appendix A

In this appendix we give the details of the fitting of the equilibrium configurations to the ansatz. We have performed a least square fit of the difference between the numerical equilibrium configurations and the ansatz evaluated at the appropriate values of \( i \). As error on the numerical values we used the one given by the finite precision of our algorithms, which as we said was of \( \approx 10^{-12} \). Obviously with such a small value for the error the \( \chi^2 \) was usually very large, but the numerical equilibrium configurations and the ansatz were usually in agreement for three or four digits. The global minima for the best fit were found using the various searches for a minimum available in the CERN package Minuit.

The main results are in figures 10 and 11 in the main text. Since the discontinuity of \( u_2 \) between the intermediate region and the liquid phase is not clear in these figures, the enlargement of \( u_2 \) is shown in Fig. 18.

The asymptotic form of \( \{c_i\} \) for large \( i \) is difficult to fit in the low \( \Lambda \) region, because of the sharp falling of the \( c_i \)'s. This can be seen for the region of \( \Lambda \): \( 0.3 \leq \Lambda \leq 0.5 \). By using the same program, we performed the fitting of \( \{c_i\} \) for different parts of range of the bins. In figure 19 the result of \( m = 100, d = 3 \) are
Figure 19: Fitting to different regions of the bins range in the gas phase.

shown, the fitting is done for bins $1 \leq i \leq 20$, $21 \leq i \leq 40$ and $41 \leq i$, respectively. In order to compare the result of the fitting with the prediction of subsection 3.1, the value of $\ln 2A$ are also plotted. From this figure, one can see that the value of $u_2$ matches well with the prediction of subsection 3.1. As for $u_1$, it shows a tendency to becomes smaller for larger bins, which means $u_1$ becomes closer to the asymptotic value of the prediction in the subsection 3.1. For completeness, we did the fitting of the exact result in eq.(3.6) using the same program. For the first several bins ($1 \leq i \leq 10$), our fitting program gives a result of $u_1 \sim -2.4$ for any $A$ smaller than 0.5. In order to compare directly, we show in table 1, the values of the first $c_i$'s for the exact result of the idealised model in eq.(3.6) and numerically obtained values in the cases $d = 3$, $d = 25$, $m = 100$. We find that the version of the model in section 3.1 describes the gas phase of the full model quite well. The results of a similar partial fitting in the region $A > 0.5$ is in fig. 20, in the case $d = 3$, $m = 100$ are shown, the partial fitting is done in the three regions of bins $1 \sim 33$, $34 \sim 66$ and $67 \sim 100$. In the intermediate region between the gas phase and the liquid phase, the values of $u_1$ and $u_2$ show a dependence on the region of fitting. In the liquid phase, the fitting of the three regions give nearly the same results. Thus, we can consider those as the proper values of $u_1$ and $u_2$ in the liquid phase.

41
Table 1: The values of the first bins in the numerical and exact solutions.

<table>
<thead>
<tr>
<th>$d = 3, m = 100$</th>
<th>$d = 25, m = 100$</th>
<th>sec. 3.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_1$</td>
<td>0.9500</td>
<td>0.9500</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$0.4508 \times 10^{-1}$</td>
<td>$0.4500 \times 10^{-1}$</td>
</tr>
<tr>
<td>$c_3$</td>
<td>$0.4375 \times 10^{-2}$</td>
<td>$0.4375 \times 10^{-2}$</td>
</tr>
<tr>
<td>$c_4$</td>
<td>$0.4375 \times 10^{-3}$</td>
<td>$0.4375 \times 10^{-3}$</td>
</tr>
<tr>
<td>$c_5$</td>
<td>$0.7437 \times 10^{-4}$</td>
<td>$0.7437 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Figure 20: Fitting to different regions of the bins range in the liquid phase.
REFERENCES


43


Stampato in proprio nella tipografia
del Centro Internazionale di Fisica Teorica