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ABSTRACT

We study the coupling of an Abelian Chern–Simons field to fermions in space–times of the form $\mathbf{R} \times M^2$, where M^2 is a compact Riemannian manifold. Upon integrating out the non–zero modes of the Chern–Simons field, an effective N–particle Hamiltonian is constructed, which involves a term representing the effects of the zero modes. We also study the transformation to the fractional statistics (anyon) basis. It is shown that unlike the case of the flat Euclidean M^2 the anyon wave equation involves some residual metric dependent interactions, and the wave function is multivalued.

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Fractional Statistics and Chern–Simons Theory

The 2+1–dimensional Chern–Simons theories have been the subject of many recent studies [1]. It is well known that the Abelian gauge field in 2+1 dimensions is without any dynamical degrees of freedom [2]. It can be solved for and eliminated, leaving only a characteristic instantaneous interaction between the charged particles to which it couples [3]. This is true at least when the spacetime is topologically trivial. However, if there are nontrivial homotopies then it is possible for the Chern Simons field to carry zero mode excitations which correspond to true dynamical degrees of freedom: they give rise, in effect, to retarded interactions among the charged particles [4]. In this note we shall consider spacetimes of the form $M^2 \times \mathbf{R}$ where M^2 is a compact Riemannian space and $\mathbf{R}$ is the time. Since the Chern–Simons action functional does not depend on the metric, we can regard M^2 as a Riemann surface and employ the machinery of conformal geometry to analyze the theory. In such cases we can integrate out the Chern–Simons field, apart from the zero mode variables, and express the resulting instantaneous interaction in terms of a scalar Green's function, which is given in terms of well known Riemann surfce quantities such as the prime form. The zero mode couplings, on the other hand, are expressed in terms of the holomorphic 1–forms of the Riemann surface.

One of the motivations for studying Abelian Chern Simons theory in 2+1 dimensions is the intimate association with fractional statistics [5]. In particular, it is possible to construct a (singular) gauge transformation which converts a system of ordinary fermions with Chern–Simons interactions into a system of free particles which obey fractional statistics (anyons) [6]. This is true only if M^2 has the topology of $\mathbf{R}^2$. Otherwise the anyons are not free. There is generally a residual, metric dependent background and, in the case of higher genus, there is a coupling to the zero modes and a residual interaction between the anyons which cannot be gauged away. The purpose of this note is to give a formal description of these residual effects in the case of a simple model in which the Chern–Simons field is coupled to non–relativistic fermions on a compact surface without boundaries.

1. THE MODEL

The action functional describing a system of non–relativistic charged fermions, ψ, to a Chern–Simons field, a_μ, is given by

$$S = \int d^3x \left[\sqrt{g}\, \psi^+ i\nabla_0 \psi - \frac{1}{2m}\sqrt{g}\, g^{ij}\nabla_i\psi^+\nabla_j\psi - \frac{k}{4\pi}\varepsilon^{\lambda\mu\rho}a_\lambda\partial_\mu a_\rho\right] \qquad (1)$$

where g_{ij} is a time independent metric on M^2 and $\varepsilon^{\lambda\mu\rho}$ is the 3–dimensional permutation symbol. We shall not treat the metric as a dynamical variable. The fermions may carry spin in which case the covariant derivative, ∇_i would have to include the spin connection. For simplicity we shall here treat the fermions as scalars. The covariant derivatives therefore contain only the Chern–Simons connetion

$$\nabla_\mu \psi = (\partial_\mu - ia_\mu)\psi, \quad \nabla_\mu \psi^+ = (\partial_\mu + ia_\mu)\psi^+ \tag{2}$$

The action functional (1) is clearly invariant with respect to infinitesimal, or topologically trivial gauge transformations. However, as is well known, it is not invariant with respect to the topologically non–trivial transformations: it is shifted by integer multiples of $2\pi k$. The coupling parameter, k, is required to be an integer in order that $\exp iS$, and hence the quantized theory, shall be invariant.

The equations of motion for the Chern–Simons field take the form

$$\varepsilon^{\lambda\mu\rho}\partial_\mu a_\rho = -\frac{2\pi}{k}J^\lambda \tag{3}$$

which imply that the field strength must vanish in empty space. It follows that the Chern–Simons field can be gauged away from regions where no charged particles are located. The transformation which achieves this is necessarily singular at the particle locations and it leads to the multivalued wave functions characteristic of fractional statistics [5),3)]. One of our aims is to specify this transformation in the case of compact manifolds of finite genus. The strategy is to integrate out the Chern–Simons field to obtain the effective action for the charged fields ψ and ψ^+. From this we can obtain the N–fermion Hamiltonian with coupling effects included. The gauge transformation which eliminates these couplings – up to residual effects – can then be found by inspection.

2. THE EFFECTIVE ACTION

Elimination of the Chern–Simons field is a canonical exercise which can be carried out most conveniently in the temporal gauge, $a_0 = 0$. However, owing to the compact and homotopically non–trivial features of the space, it is necessary to exercise some care, and we therefore sketch the procedure[*)].

Since the Abelian Chern–Simons theory is Gaussian we have

$$\int (da)\delta(a_0) \exp\, i\int d^3x \left[-\frac{k}{4\pi}\varepsilon^{\lambda\mu\rho}a_\lambda\partial_\mu a_\rho - J^\mu a_\mu\right] = \exp\left[-\frac{i}{2}\int d^3x J^\mu a^{cl}_\mu\right] \tag{4}$$

where a^{cl}_μ is a particular solution of the classical equations (3). The result (4) should be independent of the gauge choice if the current, J^μ, is conserved. This will be the case if the effective action is used for the computation of on–shell quantities. In temporal gauge the appropriate classical solution is quite easily expressed,

$$a^{cl}_0 = 0, \qquad a^{cl}_z = -\frac{2\pi i}{k}\int^t dt' J'_z \tag{5}$$

*) Details will be presented elsewhere.

Here, and in the following we use conformal coordinates on M^2. The line element is given by $ds^2 = 2\rho dz d\bar{z}$ where $\rho(z)$ is real and positive. Although the effective interaction (4), bilinear in the current when a^{cl} is given by (5), is independent of the metric, we find it is necessary to use a metric–containing Green's function in order to separate the instantaneous (non–dynamical) effects from the zero–mode (retarded) contributions. To describe this Green's function we must recall some notions from conformal geometry [7].

On a Riemann surface of finite genus, γ, we can define the multivalued Abelian integrals

$$\varphi_i(z) = \int^z dw\, \omega_i(w), \quad i = 1, \ldots, \gamma \tag{6}$$

where the ω_i are holomorphic 1–forms. We also have the prime form, $E(z,w)$, which is holomorphic and multi–valued. It is antisymmetric and, near $z = w$, $E(z,w) = z - w + \ldots$ Out of these we can make the quantity

$$F(z,w) = |E(z,w)|^2 \exp\left[-2\pi(\varphi_2(z) - \varphi_2(w))\tau_2^{-1}(\varphi_2(z) - \varphi_2(w))\right]$$

which is real, symmetric and single–valued. Here $\varphi_{2i}(z)$ denotes the imaginary part of $\varphi_i(z)$ and $(\tau_2)_{ij}$ denotes the imaginary part of the period matrix, τ_{ij}. The quantities φ, E and F are conformal in the sense that they depend on the conformal class of the metric but not on the scale factor, $\rho(z)$. This is in contrast with the Green's function defined by [7]

$$G(z,w) = -\ell n F(z,w) + \int d^2x \frac{\rho(x)}{A}(\ell n F(z,x) + \ell n F(x,w)) - \int d^2x d^2y \frac{\rho(x)\rho(y)}{A^2} \ell n F(x,y) \tag{7}$$

which depends explicitly on ρ (and the area $A = \int d^2 z \rho$) and is therefore not conformal. However, the Green's function is real, symmetric and single–valued, and it transforms as a scalar with respect to analytic reparametrizations. Most important for our purposes, it satisfies the differential equations,

$$\partial_z \partial_{\bar{z}} G(z,w) = -2\pi\delta_2(z,w) + 2\pi\frac{\rho(z)}{A} \tag{8a}$$

$$\partial_z \partial_{\bar{w}} G(z,w) = 2\pi\delta_2(z,w) - \pi\omega(z)\tau_2^{-1}\overline{\omega(w)} \tag{8b}$$

Using these, the effective interaction can be expressed as follows,

$$\begin{aligned}
-\frac{1}{2}\int d^2 z dt (J_{\bar{z}} a_z^{cl} + J_z a_{\bar{z}}^{cl}) &= \\
&= \frac{i\pi}{k}\int d^2 z dt dt' (J_{\bar{z}}\theta(t-t') J'_z - J_z \theta(t-t') J'_{\bar{z}}) \\
&= \frac{i\pi}{k}\int d^2 z d^2 w dt dt' J_{\bar{z}} \varepsilon(t-t')\delta_2(z,w) J'_w \\
&= \frac{i}{2k}\int d^2 z d^2 w dt dt' [\partial_z J_{\bar{z}} \varepsilon(t-t') G(z,w)\partial_{\bar{w}} J'_w + \pi J_{\bar{z}}\omega(z)\tau_2^{-1}\varepsilon(t-t')\overline{\omega(w)} J'_w]
\end{aligned} \tag{9}$$

where $\theta(t-t')$ is the step function, etc. We have used (8b) to introduce $G(z,w)$ into the integrand. With the imposition of current conservation together with suitable assumptions, about the asymptotic current configurations, it is possible to express the right–hand side of (9) in the form

$$\frac{i}{2k}\int d^2z d^2w dt J_0(z,t)G(z,w)(\partial_w J_{\bar{w}} - \partial_{\bar{w}} J_w) + \frac{i\pi}{2k}\int dt dt' \bar{Q}(t)\tau_2^{-1}\varepsilon(t-t')Q(t') \quad (10)$$

where $Q_i(t)$ denotes the source component that couples to the zero modes.

$$Q_i(t) = \int d^2z\, \overline{\omega_i(z)} J_z \quad (11)$$

By introducing a set of dynamical variables $u_i(t), \bar{u}_i(t),\ i = 1\ldots\gamma$ with the 2–point function,

$$\langle T u_i(t)\bar{u}_j(t')\rangle = \frac{\pi}{k}(\tau_2^{-1})_{ij}\varepsilon(t-t') \quad (12)$$

we can simulate the zero mode exchange,

$$\exp\left[-\frac{\pi}{2k}\int dt dt' \bar{Q}(t)\tau_2^{-1}\varepsilon(t-t')Q(t')\right] = \langle \exp\frac{1}{i}\int dt(\bar{u}Q + \bar{Q}u)\rangle \quad (13)$$

The other term in (10) describes an instantaneous interaction between the currents. Taken together these terms define the Chern–Simons induced interactions between the fermions. It can be shown*) that they are governed by the effective action functional

$$S = \int dt\left[\frac{k}{\pi}\bar{u}\tau_2 i\partial_0 u + \int d^2z\left\{\rho\psi^+ i\partial_0\psi - \frac{1}{2m}(\nabla_z\psi^+\nabla_{\bar{z}}\psi + \nabla_{\bar{z}}\psi^+\nabla_z\psi)\right\}\right] \quad (14)$$

where the covariant derivatives are of the form (2) but with the connection

$$a_z = \omega(z)\cdot u + \frac{i}{2k}\int d^2w \partial_z G(z,w) J_0(w) \quad (15)$$

where the charge density is given by

$$J_0 = \rho\psi^+\psi \quad (16)$$

3. THE N–PARTICLE HAMILTONIAN

From the action integral (14) it is straightforward to extract the N–particle Hamiltonian. Define the N–fermion basis vectors $\langle u|\psi(\xi^1)\ldots\psi(\xi^N)$ where $\langle u|\psi^+(\xi) = 0$. The vacua are

*) There is a technical complication arising from the $a^2\psi^+\psi$ interaction in (1). To avoid this it is necessary to introduce auxiliary fermionic variables $\Pi \sim \nabla\psi$ so that the currents do not depend explicitly on a. The auxiliary variables can be eliminated after a itself has been eliminated.

chosen to diagonalize the "holomorphic" zero mode operators u_i, the conjugate operators are given by

$$\langle u|\bar{u}_i = -\frac{i\pi}{k}(\tau_2^{-1})_{ij}\frac{\partial}{\partial u_j}\langle u| \tag{17}$$

The action of the Hamiltonian on the N–fermion states is given by

$$\begin{aligned}
&\langle u|\psi(\xi^1)\ldots\psi(\xi^N)H = \\
&\qquad = \langle u|\psi(\xi^1)\ldots\psi(\xi^N)\int d^2z\frac{1}{2m}(\nabla_z\psi^+\nabla_{\bar z}\psi + \nabla_{\bar z}\psi^+\nabla_z\psi) \\
&\qquad = \sum_{\alpha=1}^{N}\left(-\frac{\rho(\xi^\alpha)^{-1}}{2m}\right)(\nabla_\alpha\nabla_{\bar\alpha} + \nabla_{\bar\alpha}\nabla_\alpha)\langle u|\psi(\xi^1)\ldots\psi(\xi^N)
\end{aligned} \tag{18}$$

where the covariant derivative in N complex dimensions is defined by

$$\nabla_\alpha = \frac{\partial}{\partial\xi^\alpha} - iA_\alpha(\xi), \quad \nabla_{\bar\alpha} = \frac{\partial}{\partial\bar\xi^\alpha} - iA_{\bar\alpha}(\xi)$$

with

$$A_\alpha(\xi) = \omega(\xi^\alpha)\cdot u + \frac{i}{2k}\sum_{\beta\neq\alpha}\partial_\alpha G(\xi^\alpha,\xi^\beta) \tag{19}$$

and likewise for the components $A_{\bar\alpha} = (A_\alpha)^*$. The Hamiltonian is thereby reduced to a kind of N–dimensional Laplace operator. It is covariant with respect to the analytic reparametrizations $\xi^\alpha \to f(\xi^\alpha)$, $\alpha = 1,\ldots N$ and also with respect to N–dimensional gauge transformations, $A_\alpha \to A_\alpha + \partial_\alpha\Lambda$. (It should be pointed out that the result (18), (19) is exact. There is no need to regularize the short distance behaviour of $G(z,w)$ because the sum (19) automatically excludes terms with $\beta = \alpha$.) It remains to consider how much of the interaction can be gauged away.

4. FRACTIONAL STATISTICS

The transition to a new (anyonic) basis in which the N–particle wave functions exhibit fractional statistics is effected by means of a singular gauge transformation. In the original basis the interactions are described by the complex N–vector, A_α, which has singularities on the complex lines, $\xi^1 = \xi^2$, etc., and we can expect the gauge transformation to have these singularities. Away from these singular lines the components of the field strength corresponding to (19) are given by

$$F_{\alpha\beta} = 0; \qquad F_{\alpha\bar\beta} = \begin{cases} \frac{i\pi}{k}\omega(\xi^\alpha)\tau_2^{-1}\overline{\omega(\xi^\beta)}, & \beta\neq\alpha \\ -\frac{i2\pi}{k}(N-1)\frac{\rho(\xi^\alpha)}{A}, & \beta = \alpha \end{cases} \tag{20}$$

where we have used (8). This structure is to be contrasted with the planar case (genus $\gamma = 0$, area $A = \infty$) where all components vanish. Here we shall not be able to gauge away the interaction so as to arrive at a system of free anyons. The best that can be achieved is to write

$$A_\alpha = A_\alpha + \partial_\alpha\Omega \tag{21}$$

where Ω is real, defined so as to remove the singular parts of the connection (19). We choose

$$\Omega = \sum_\alpha \frac{\pi}{k}(N-1)\eta(\xi^\alpha)\tau_1\eta(\xi^\alpha) + \sum_{\alpha<\beta}\left[\frac{1}{2ik}\ell n\left(\frac{E(\xi^\alpha,\xi^\beta)}{E(\xi^\alpha,\xi^\beta)^*}\right) - \frac{2\pi}{k}\eta(\xi^\alpha)\tau_1\eta(\xi^\alpha)\right] \quad (22)$$

where $\tau_1 = Re\tau$ and $\eta = \tau_2^{-1} Im\varphi$. The corresponding residual connection takes the form,

$$\mathcal{A}_\alpha = b(\xi^\alpha) - \frac{i\pi}{k}\sum_{\beta\neq\alpha}\omega(\xi^\alpha)\tau_2^{-1}\bar{\tau}\eta(\xi^\beta) \quad (23)$$

in which b is a metric–dependent background given by

$$\begin{aligned} b(z) =& \frac{i\pi}{2k}(N-1)\omega(z)\tau_2^{-1}\tau_1\eta(z) + \omega(z)\cdot u \\ &+ \frac{i}{2k}(N-1)\int d^2w \frac{\rho(w)}{A}\left[\partial_z \ell n E(z,w) - 2\pi i\omega(z)\eta(w)\right] \end{aligned} \quad (24)$$

The expressions (22)–(24) are somewhat complicated. In fact $\mathcal{A}_\alpha$ is a k–valued vector potential on M^2. For example, on taking the coordinate ξ^β around the cycle $m\cdot a + n\cdot\beta$ the change in $\mathcal{A}_\alpha$ is given by a gauge transformation,

$$\delta^{(\beta)}_{mn}\mathcal{A}_\alpha = \partial_\alpha \Lambda^{(\beta)}_{mn}$$

where

$$\Lambda^{(\beta)}_{mn} = \frac{2\pi}{k}(N-1)\mu(\xi^\beta)\cdot n + \frac{2\pi}{k}\sum_{\alpha\neq\beta}\mu(\xi^\alpha)\cdot n \quad (25)$$

In these formula the real functions, $\mu(z)$ are defined like $\eta(z)$, in terms of the Abelian integrals (6),

$$\varphi = \mu + \tau\eta, \quad \bar{\varphi} = \mu + \bar{\tau}\eta \quad (26)$$

They have simple periods,

$$\delta_{mn}\mu_i = m_i, \quad \delta_{mn}\nu_i = n_i \quad (27)$$

It follows that $\delta^{(\alpha)}_{mn}\Lambda^{(\beta)}_{mn} = \frac{2\pi}{k}\left(N\delta^{\alpha\beta} - 1\right) n\cdot m'$ and hence $\exp ik\Lambda^{(\beta)}_{mn} \in U(1)$. Thus the wave functions are k–valued.

The wave functions in the new basis are obtained from the original ones by multiplication with the phase factor,

$$e^{-i\Omega} = \prod_{\alpha<\beta}\left(\frac{E(\xi^\alpha,\xi^\beta)}{E(\xi^\alpha,\xi^\beta)^*}\right)^{-\frac{1}{2k}} \exp\left[-\frac{ik}{k}(N-1)\sum_\alpha \eta(\xi^\alpha)\tau_1\eta(\xi^\alpha) + \frac{2\pi i}{k}\sum_{\alpha<\beta}\eta(\xi^\alpha)\tau_1\eta(\xi^\beta)\right] \quad (28)$$

which is clearly multivalued with respect to particle interchange. For example, with $\xi^1 - \xi^2 \to e^{i\pi}(\xi^1 - \xi^2)$ we obtain

$$e^{-i\Omega} \to e^{-i\Omega}e^{-i\pi/k} \quad (29)$$

on using the antisymmetry of $E(\xi^1,\xi^2)$. This is the usual anyon phase factor [5),6)] now generalized to the finite genu. cases. The ne v feature here is that the anyon Hamiltonian is no free, but inclu'es interactions in the form of a residual connection, $\mathcal{A}_\alpha$ specified by (23) and (24).

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