IC/89/283

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

CHERN-SIMONS SUPERCONDUCTIVITY AT FINITE TEMPERATURE

S. Randjbar-Daemi

Abdus Salam

and

J. Strathdee

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1989 MIRAMARE - TRIESTE

IC/89/283

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization
INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

CHERN–SIMONS SUPERCONDUCTIVITY AT FINITE TEMPERATURE *

S. Randjbar–Daemi, Abdus Salam and J. Strathdee
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

A simple gauge theory discussed recently in the literature as a model of high temperature superconductors is examined. The model contains a Maxwell field and a Chern–Simons field coupled to fermions in 2+1–dimensional spacetime. This model has been shown to exhibit a kind of Meissner effect at zero temperature which originates in the 1–loop mixing between the two gauge fields. We use a Euclidean effective action formulation to show that the effect persists at all finite temperatures. Although a long range magnetic type interation arises at non–zero temperatures, in competition with the finite range forces which dominate the zero temperature interaction, the effect varies smoothly with temperature. In our perturbation treatment, we find no indication of a critical transition at which the Meissner effect is extinguished.

MIRAMARE – TRIESTE
September 1989

* To be submitted for publication.

1. INTRODUCTION

Gauge symmetries in 3+1–dimensional spacetime are normally associated with long range forces such as electromagnetism. In the appropriate circumstances, however, the gauge symmetry can be spontaneously broken, in which cases the associated force has a finite range. This mechanism is invoked in the BCS theory of superconductivity where a weakly attractive force between electrons gives rise to a condensate of Cooper pairs. The BCS ground state is not invariant with respect to the electromagnetic gauge transformations and the photon, in effect, acquires a mass. Magnetic fields are consequently expelled from the superconductor. This mechanism is non–perturbative in the sense that the formation of quasi–bound states, the Cooper pairs, is an essential feature. On the other hand, it has been known for some years that gauge symmetries in 2+1–dimensional spacetime are not necessarily associated with long range forces [1]. Here it is consistent with unbroken gauge symmetry to have massive gauge quanta. In the Lagrangian formulation the mass term is represented by a Chern–Simons density,

$$\mathcal{L} = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \frac{\nu}{2\pi}\varepsilon^{\lambda\mu\rho}A_\lambda\partial_\mu A_\rho + j^\mu A_\mu \tag{1.1}$$

This density is invariant, up to a total derivative with respect to the $U(1)$ gauge transformations, $A_\mu \rightarrow A_\mu + \partial_\mu\Lambda$, provided the current is conserved, $\partial_\mu j^\mu = 0$. The coupling parameter, e^2, has the dimensions of mass (in natural units) while the second coupling parameter, ν, is dimensionless. It is an elementary exercise to show that the effective current–current interaction, due to the exchange of a gauge quantum, is given by

$$\frac{1}{k^2 + (\frac{\nu}{\pi}e^2)^2}\frac{1}{2}\left[e^2 j_\mu(-k)j^\mu(k) - \frac{i\pi}{\nu}\varepsilon^{\mu\lambda\rho}j_\mu(-k)k_\lambda j_\rho(k)\right]$$

$$+\frac{1}{k^2}\frac{1}{2}\frac{i\pi}{\nu}\varepsilon^{\mu\lambda\rho}j_\mu(-k)k_\lambda j_\rho(k) \tag{1.2}$$

This indicates that the force has both finite and long range components. Both of these forces have a parity violating piece with strength π/ν. (The long range component in fact describes an instantaneous interaction of the currents. See Eq.(2.24).)

The parameter ν is arbitrary if the 2–dimensional space has Euclidean topology. However, if the space is compactified then it becomes possible to consider topologically non–trivial gauge transformations under which the Chern–Simons term in (1.1) is not invariant. Then, in order to maintain invariance of the phase, $\exp(i\int\mathcal{L})$, it becomes necessary to restrict ν to be an integer [2].

The relevance of 2+1–dimensional gauge theory to the description of high temperature superconductors [3] is suggested by the possibillty that the layered structure of Copper oxide crystals causes the electron motion to be effectively 2–dimensional. One could then consider (1.1) as an effective Lagrangian in which the coupling parameter, e^2, is obtained by averaging over the thickness of the layers,

$$e^2 \sim \alpha/\delta$$

where $\alpha \sim 1/137$ is the usual fine structure constant and $\delta \sim 10^{-7}$ cm is a measure of thickness. The parameter, ν, is not so easy to interpret – it would have to be generated by some parity violating effect in the underlying crystal structure. In this paper we shall examine a simple model which has been considered recently by a number of authors [4),8)]. It contains a fundamental Chern–Simons field, a_μ, which couples to the "electrons" exactly like the electomagnetic A_μ. This Chern Simons field should be viewed as a kind of vector excitation of the underlying microscopic dynamics. It can be shown that the introduction of a fundamental Chern–Simons gauge field coupled to 2–diemnsional fermions is mathematically equivalent to modifying the statistics of these fermions [5)] – changing them into what are called "anyons" [6)]. It is known that such particles, although otherwise free, can exhibit an effective interaction which is caused by their fractional statistics [7)]. This is demonstrated by eliminating the Chern–Simons field as an independent dynamical variable [4)]. Here we shall adopt the point of view that it is more practical to keep the Chern–Simons field among the independent variables and treat it as a gauge field on the same footing as the electromagnetic field. The electrons will therefore obey normal Fermi statistics. It is then possible to carry out perturbative calculations by expanding in powers of π/ν, which is to be regarded as a small parameter. Using this approach, the authors of Refs.4 and 8 were able to show that, at the level of 1–loop quantum corrections, an effective Chern–Simons type of term involving the electromagnetic potential, A_μ, is induced along with the usual ground state polarization effects. In other words, the photon develops a mass in the lowest order of perturbation theory. Specifically, the effective Lagrangian includes the term

$$\frac{\nu}{2\pi}\varepsilon^{\lambda\mu\rho}(A_\lambda + a_\lambda)\partial_\mu(A_\rho + a_\rho) \tag{1.3}$$

along with other bilinear terms involving the electric and magnetic field strengths. This phenomenon can be interpreted as a kind of 2–dimensional superconductivity with a Meisener effect [4),8)] (finite penetration depth) and a supercurrent which is proportional to the Chern–Simons electric type field strength.

In Sec. 2 we review the model and describe the 1–loop computation in terms of an effective action formulation. Although this simple model may be somewhat artificial, it is at least arguable that it may be typical of a class of microscopic theories which incorporate P and T violation and which lead to an induced electromagnetic Chern–Simons term. We have studied the effective interaction and shown that there are two independent short range forces, corresponding to the exchange of massive quanta. In the static limit one of these ranges is identical to the penetration depth found in Refs.4 and 8. The other is generally shorter.

To test this theory further, we have made a finite temperature computation which is described in Sec.3 and Appendix C. The results are used in Sec.4 to obtain the effective current–current interaction at finite temperature. We find that, generally there are three independent components to the magnetic interaction, two of finite range, corresponding to the exchange of massive quanta and, in addition a long range component which vanishes in the zero temperature limit. The relative strengths of these interactions vary smoothly with temperature. In the low temperature regime we obtain a formula for the temperature dependence of the parameter, λ, identified as "pen-

etration depth" in Refs.4 and 8,

$$\frac{1}{\lambda^2} = \frac{e^2 n_e}{m}(1 - 4y\, e^{-y}) + \frac{4\pi n_e}{\nu^2}\, y\, e^{-y}$$

where m and n_e denote the electron mass and number density, respectively, and $y = \pi n_e/2\nu mT \sim 10^4 K/\nu T \gg 1$. Notice that $1/\lambda$ *grows* with T. Further, we find that the long range component dominates at high temperatures. The two finite penetration depths appear to vary smoothly with temperature and, at least in the low and high regimes studied in Sec.4, cannot diverge to infinity. There is no indication of critical behaviour.

The failure of the present model in perturbation theory to provide a realistic description of high temperature superconductivity, notwithstanding, we believe that it may still be possible to develop a more successful version based on the Chern Simons idea. We have therefore set out our calcualtions in some detail in the Appendices.

2. EFFECTIVE ACTION

The model to be considered in 2+1 dimensions involves a gas of non–relativistic electrons coupled to two independent gauge fields. The latter are represented by the electromagnetic potential, A_μ, and the Chern–Simons potential, a_μ. The Lagrangian for this system is

$$\begin{aligned}\mathcal{L} = &-\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \frac{\nu}{2\pi}\varepsilon^{\lambda\mu\rho}a_\lambda\partial_\mu a_\rho \\ &+ \psi^+ i\nabla_0\psi - \frac{1}{2m}\nabla_k\psi^+\nabla_k\psi \end{aligned} \tag{2.1}$$

where the electron field, ψ, is a two–component spinor whose covariant derivatives are given by

$$\nabla_\mu\psi = (\partial_\mu - iA_\mu - ia_\mu)\psi$$

$$\nabla_\mu\psi^+ = (\partial_\mu + iA_\mu + ia_\mu)\psi^+$$

The Lagrangian of Hosotani and Chakravarty [8] includes a magnetic moment term $\sim F_{12}\psi^+\sigma_3\psi$ but we shall ignore this for the present. The system described by (2.1) can be quantized in the standard way and Feynman rules given for the perturbative development in powers of the coupling parameters e^2 and π/ν. In principle one could construct an effective action functional to any given order by evaluating irreducible graphs. However, since our main interest lies in finding effective field equations for the two gauge fields which are accurate to 1–loop order, we shall simply integrate out the electron field to obtain

$$\Gamma(A,a) = -\frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \frac{\nu}{2\pi}\varepsilon^{\lambda\mu\rho}a_\lambda\partial_\mu a_\rho + \frac{1}{i}\ell n\det G(A+a) \tag{2.2}$$

where $G(A+a)$ denotes the electron Green's function in a specified background. The functional determinant in (2.2) gives the complete 1–loop contribution to Γ. Higher loop contributions would

require the evaluation of graphs involving internal A– and a–lines and so would be proportional to positive powers of the small parameters e^2 and π/ν. We shall therefore ignore them. Because of the assumed minimal couplings, the 1–loop term is a functional of the sum, $A_\mu + a_\mu$. The introduction of non–minimal terms in (2.1) would destroy this simplicity *)

The next simplification that we shall make is to assume that the fields A_μ and $a'_\mu = a_\mu - a^0_\mu$ are small and we shall evaluate Γ up to second order in these small quantities. Here, the "background" a^0_μ represents the ground state expectation value and corresponds to a uniform Chern–Simons magnetic field. We shall take

$$a^0_0 = a^0_2 = 0, \quad a^0_1 = -\frac{x_2}{\ell^2} \tag{2.3}$$

where ℓ^2 is a constant to be determined. This background enters both the electron propagator and the vertices. It causes the electron energy levels to be quantized in units of $1/m\ell^2$ (Landau levels) and the resulting electron propagator $G(a^0)$ is neither translation nor rotation invariant. However, it should be recognized that the breaking of these continuous symmetries is only a gauge artifact since the background field strength, $f^0_{\mu\nu} = \partial_\mu a^0_\nu - \partial_\nu a^0_\mu$, is invariant. Covariance with respect to combined rotations, translations and suitable background gauge transformations will be maintained (see Appendix A). Gauge invariant quantities such as the ground state polarization tensors will turn out to have the correct spacetime covariance. On the other hand, with respect to discrete symmetries, such as the reflection $x_1 \to -x_1$, $x_2 \to x_2$, the background magnetic field is not invariant and these symmetries are broken.

The leading terms in the development of the 1–loop effective action can be expressed in terms of ground state expectation values of the time ordered products of electron current operators,

$$\begin{aligned}\Gamma_{(1)} = &\int d^3x \Gamma^\mu(x)(A_\mu(x) + a'_\mu(x)) \\ &+ \frac{1}{2}\int d^3x d^3x' \Gamma^{\mu\nu}(x,x')(A_\mu(x) + a'_\mu(x))(A_\nu(x') + a'_\nu(x')) \\ &+ \dots\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}\Gamma^\mu(x) &= - < j^\mu(x) > \\ \Gamma^{\mu\nu}(x,x') &= i < T j^\mu(x) j^\nu(x') > - < j^{\mu\nu}(x,x') >\end{aligned} \tag{2.5}$$

with the current operators given by

$$\begin{aligned}j_0(x) &= \psi^+\psi \\ j_k(x) &= \frac{i}{2m}(\psi^+\nabla_k\psi - \nabla_k\psi^+\psi) \\ j_{0\mu}(x,x') &= 0 \\ j_{k\ell}(x,x') &= \frac{1}{m}\delta_{k\ell}\delta_3(x-x')\psi^+\psi\end{aligned} \tag{2.6}$$

*) At the end of this section we include the contribution due to an electron magnetic moment, see Eq. (2.21).

in which ∇_k indicates the covariant derivative with the background connection a^0_μ. (We are using the Minkowskian metric for which $j_0 = -j^0$, $j_k = j^k$).

The electron Green's function,

$$G_{\sigma\sigma'}(x,x') = i < T\psi_\sigma(x)\psi^+_{\sigma'}(x') > \tag{2.7}$$

is obtained by solving the equation

$$(i\partial_0 + \frac{1}{2m}\nabla_k^2)G(x,x') = -\delta_3(x-x') \tag{2.8}$$

subject to appropriate boundary conditions. It can be expressed by the integral

$$G(x,x') = -\int_{C_N}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{dk_1}{2\pi}e^{ik_1(x_1-x_1')-i\omega(t-t')}$$

$$\cdot\frac{1}{\ell}\sum_n \frac{v_n(\frac{x_2}{\ell}+k_1\ell)v_n^*(\frac{x_2'}{\ell}+k_1\ell)}{\omega-\varepsilon_n} \tag{2.9}$$

where the functions v_n, $n = 0,1,\ldots$ are orthonormalized harmonic oscillator wave functions. They satisfy the eigenvalue equations

$$\frac{1}{2m}\left[\left(k_1+\frac{x_2}{\ell^2}\right)^2 - \partial_2^2\right]v_n = \varepsilon_n v_n,$$

$$\varepsilon_n = (n+\frac{1}{2})\frac{1}{m\ell^2} \tag{2.10}$$

The choice of contour C_N for the integration over frequencies is dictated by the boundary conditions which define the ground state, viz.

$$G(x,x') \sim \begin{cases} \sum_{n<N} e^{-i\varepsilon_n(t-t')}, & t<t' \\ \sum_{n\geq N} e^{-i\varepsilon_n(t-t')}, & t>t' \end{cases}$$

where N is an integer which counts the number of filled Landau levels. Hence the contour C_N must pass below the poles at $\omega = \varepsilon_n$ for $n = 0,1,\ldots N-1$ and above all the rest, $n \geq N$. Using this propagator it is straightforward to compute the quantities (2.5),

$$\begin{aligned}
\Gamma^0(x) &= iTrG(x,x')|_{\underline{x}'=\underline{x},t'=t+0} \\
\Gamma^k(x) &= \frac{1}{2m}(\nabla_k - \nabla_k')TrG(x,x')|_{\underline{x}'=\underline{x},t'=t+0} \qquad (2.11)\\
\Gamma^{00} &= iTrG(x,x')G(x',x) \\
\Gamma^{k0}(x,x') &= \frac{1}{2m}Tr(\nabla_k G(x,x')G(x',x) - G(x,x')\nabla_k G(x',x)) \\
\Gamma^{k\ell}(x,x') &= -\frac{i}{4m^2}Tr(\nabla_k G(x,x')\nabla_\ell' G(x',x) - G(x,x')\nabla_k\nabla_\ell' G(x',x)) \\
&\quad + \nabla_\ell' G(x,x')\nabla_k G(x',x) - \nabla_k\nabla_\ell' G(x,x')G(x'x)) \\
&\quad - \frac{i}{m}\delta_{k\ell}\delta_3(x-x')TrG(x,x')|_{\underline{x}'=\underline{x},t'=t+0} \qquad (2.12)
\end{aligned}$$

The linear components (2.11) are given by

$$\Gamma^0(x) = \frac{N}{\pi\ell^2}$$
$$\Gamma^k(x) = 0 \qquad (2.13)$$

The polarization components (2.12), being gauge independent, must have translation and rotation symmetry. Further, because of current conservation their Fourier components can be expressed in terms of three independent invariants *),

$$\tilde{\Gamma}_{00} = \underline{k}^2 \Pi_0$$
$$\tilde{\Gamma}_{j0} = -\omega k_j \Pi_0 - i\varepsilon_{j\ell} k_\ell \Pi_1$$
$$\tilde{\Gamma}_{j\ell} = \omega^2 \delta_{j\ell} \Pi_0 + i\varepsilon_{j\ell}\omega\Pi_1 + (k^2\delta_{j\ell} - k_j k_\ell)\Pi_2 \qquad (2.14)$$

where $\omega = k^0 = -k_0$. The general structure of the invariants Π_0, Π_1 and Π_2 is considered in Appendix B. They are functions of ω^2 and k^2 but here we need only their threshold values,

$$\Pi_0 = \frac{N}{\pi} m\ell^2, \quad \Pi_1 = \frac{N}{\pi}, \quad \Pi_2 = -\frac{N^2}{\pi m} \qquad (2.15)$$

These are zeroth order terms in expansions in powers of $(k\ell)^2$ and $(\omega m\ell^2)^2$.

The results (2.13) and (2.15) define the behaviour of the 1–loop effective action for small fluctuations of the gauge potentials around the ground state values $A_\mu = 0$, $a_\mu = a^0_\mu$. For a consistent determination of the ground state it is necessary to satisfy the effective equations of motion,

$$\frac{\delta\Gamma}{\delta A_\mu} = n_e \delta^\mu_0$$
$$\frac{\delta\Gamma}{\delta a_\mu} = 0 \qquad (2.16)$$

where n_e denotes an external uniform charge density representing an average density of compensating positive charge. (It is assumed that only the electromagnetic potential is coupled to this charge.) The functional derivates in (2.16) are to be evaluated at $A = 0$, $a = a^0$ where the non–vanishing components are,

$$\frac{N}{\pi\ell^2} = n_e$$
$$-\frac{\nu}{\pi}\frac{1}{\ell^2} + \frac{N}{\pi\ell^2} = 0 \qquad (2.17)$$

*) Where $\varepsilon_{12} = -\varepsilon_{21} = 1$. The three-dimensional permutation symbol is defined such that $\varepsilon^{012} = 1 = -\varepsilon_{012}$, etc.

respectively. The first term in the latter equation represents the classical contribution $\delta S/\delta a_0$ evaluated at the point (2.3). It follows that the ground state parameters, N and ℓ^2 are fixed by the consistency requirement,

$$N = \nu \quad \text{and} \quad \ell^2 = \frac{\nu}{\pi n_e} \tag{2.18}$$

This means, in particular, that the coupling parameter, ν, which was introduced in the classical Lagrangian (2.1) must be an integer, positive or negative (In the above treatment we have assumed that ν is positive. Consistency for the case of negative ν requires merely a change of sign in (2.3) and subsequent formulae.)

Having dealt with the linear term in the effective action and its role in determining the ground state parameters we can now combine the polarization terms (2.15) with the classical terms to define an effective Lagrangian which is bilinear in the gauge potentials A_μ and a'_μ.

$$\begin{aligned}\mathcal{L}_{eff} = & \frac{1}{2e^2}(E_j^2 - B^2) - \frac{\nu}{2\pi}\varepsilon^{\lambda\mu\rho}a'_\lambda\partial_\mu a'_\rho \\ & + \frac{1}{2}(\frac{\nu}{\pi})^2\frac{m}{n_e}(E_j + e_j)^2 - \frac{1}{2}(\frac{\nu}{\pi})^2\frac{\pi}{m}(B + b)^2 \\ & + \frac{\nu}{2\pi}\varepsilon^{\lambda\mu\rho}(A_\lambda + a'_\lambda)\partial_\mu(A_\rho + a'_\rho)\end{aligned} \tag{2.19}$$

where the electric and magnetic field strengths are defined by

$$E_j = \partial_0 A_j - \partial_j A_0\,, \quad B = \partial_1 A_2 - \partial_2 A_1$$

$$e_j = \partial_0 a'_j - \partial_j a'_0\,, \quad b = \partial_1 a'_2 - \partial_2 a'_1$$

In obtaining (2.19) we have discarded higher derivative terms. Hence this Lagrangian should be used for the description of fields which are slowly varying in the sense

$$\ell\partial_j E << E, \quad m\ell^2\partial_0 E << E\,, \tag{2.20}$$

etc. At this point it may be remarked that the inclusion of an electron magnetic moment term $-\mu_e B\psi^+\sigma_3\psi$ in the original Lagrangian would give rise to a higher derivative term in (2.19),

$$\frac{1}{2}(\frac{\nu}{\pi})^2\frac{m}{n_e}\mu_e^2(\partial_j B)^2 \tag{2.21}$$

With $\mu_e = \frac{1}{2m}$ this term would become significant at longer wavelengths than those which are excluded by (2.20) but we shall nevertheless discard it along with all the other higher derivative terms.

From the effective Lagrangian (2.19) we can obtain expressions for the gauge field Green's functions. The procedure for doing this is straightforward in principle but the computations are somewhat lengthy because of the mixing and they are not very illuminating. Some of the details

are given in Appendix D. The main result is that the energy spectrum has two branches given by the zeroes of the function

$$\Delta(\omega^2,\underline{k}^2) = \left(\frac{\pi}{\nu e^2}\right)^2 \left[\left\{(\omega^2-\underline{k}^2)\Pi_0 - \frac{\nu}{\pi}e^2\Pi_1\right\}^2 - (\omega^2-\underline{k}^2)\left(\frac{\nu}{\pi}\right)^2\left(1-\frac{\pi}{\nu}\Pi_1+e^2\Pi_0\right)^2 + \right.$$
$$\left. + \underline{k}^2(\Pi_0+\Pi_2)\left\{(\omega^2-\underline{k}^2)\Pi_0 - \left(\frac{\nu}{\pi}\right)^2 e^2(1+e^2\Pi_0)\right\}\right] \qquad (2.22)$$

which is quadratic in ω^2. The roots are real and positive if $e^2\Pi_0 > 0$. In the limit $\underline{k} \to 0$ they define the rest energies,

$$\mu_\pm^2 = \left(\frac{\nu}{2\pi}\right)^2 \frac{1}{\Pi_0^2}\left[\left(1-\frac{\pi}{\nu}\Pi_1+e^2\Pi_0\right) \pm \sqrt{\left(1-\frac{\pi}{\nu}\Pi_1-e^2\Pi_0\right)^2+4e^2\Pi_0}\ \right]^2 \qquad (2.23)$$

(It may be remarked that in the special situation, $\Pi_2 = -\Pi_0$, the Lagrangian (2.19) becomes Lorentz invariant as does the function (2.22). In this case the rest energies (2.23) can be interpreted as rest masses. In practice, as we shall see, $\Pi_2 \ll \Pi_0$, and the system is far from relativistic.)

The static interaction, mediated by the exchange of zero frequency gauge quanta, is governed by the zeroes of the function $\Delta(0,\underline{k}^2)$ which is quadratic in $\underline{k}^2$. There are two roots, $k_\pm^2$, which are both negative, corresponding to forces of finite range. In the relativistic case these ranges are given by the compton wavelengths $\mu_\pm^{-1}$, but in general they are given by quite complicated expressions. The finite ranges of the static interaction are a manifestation of the Meissner effect in this system. They define two independent "penetration depths". Since our main concern in this paper is to study the temperature dependence of this effect, we postpone further discussion to Sec.4, where the finite temperature version of the function $\Delta(0,\underline{k}^2)$ is obtained.

3. THERMODYNAMIC POTENTIAL

To obtain a description of the system at finite temperature we shall apply the methods of Euclidean field theory and the grand canonical ensemble [9]. That is, we shall make the substitution $t \to -i\tau$ and regard the real parameter τ as a coordinate on a circle of circumference $\beta = 1/T$. Bosonic fields are required to be single–valued (periodic) on this circle while fermions are double–valued (antiperiodic). The time components of vectors such as A_μ and J_μ are likewise redefined,

$$A_0 \to iA_\tau, \quad J_0 \to iJ_\tau$$

and the conjugate electron field ψ^+ is replaced by $\bar{\psi}$. We obtain the Euclidean action,

$$S^E = \int_0^\beta d\tau \int d^2x \left[\frac{1}{4e^2}F_{\mu\nu}F_{\mu\nu} + \frac{i\nu}{2\pi}\varepsilon_{\lambda\mu\rho}a_\lambda\partial_\mu a_\rho \right.$$
$$\left. + \bar{\psi}\nabla_\tau\psi + \frac{1}{2m}\nabla_k\bar{\psi}\nabla_k\psi - \mu\bar{\psi}\psi\right] \qquad (3.1)$$

where $\varepsilon_{\tau 12} = 1$, and the covariant derivatives are defined in the natural way,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \mu, \nu = \tau, 1, 2$$
$$\nabla_\mu \psi = (\partial_\mu - iA_\mu - ia_\mu)\psi$$
$$\nabla_\mu \bar{\psi} = (\partial_\mu + iA_\mu + ia_\mu)\bar{\psi}$$

It is important to keep in mind that the components A_τ and a_τ are imaginary and that $\bar{\psi}$ is not the hermitian conjugate of ψ. Note that the coefficient of the Chern–Simons term is now pure imaginary.

The Euclidean action (3.1) contains a new parameter, μ, the chemical potential. It will be determined as a function of temperature and the other parameters in (3.1) by the conditions of thermal equilibrium to be described.

A Euclidean effective action can be defined in close analogy to the Minkowskian case discussed in Sec. 2. One proceeds by introducing external sources I_μ and J_μ for the gauge potentials A_μ and a_μ respectively. Define the functional $W(I, J)$ by path integration over periodic boson and antiperiodic fermion fields,

$$e^{-W(I,J)} = \int (dAda d\psi d\bar{\psi})\, e^{-S^E - \int d^3x (I_\mu A_\mu + J_\mu a_\mu)}$$

The averaged fields are defined by functional differentiation,

$$A_\mu = \frac{\delta W}{\delta I_\mu}, \quad a_\mu = \frac{\delta W}{\delta J_\mu}$$

and are to be treated as independent variables. The Euclidean effective action, Γ^E, is then given by the Legendre transform of W,

$$\Gamma^E(A, a) = W(I, J) - \int d^3x (I_\mu A_\mu + J_\mu a_\mu)$$

It can be expressed as a loop expansion in which the zeroth term is the classical Euclidean action,

$$\Gamma^E(A, a) = S^E(A, a) + \Gamma^E_{(1)}(A, a) + \ldots \tag{3.2}$$

The successive terms are represented by irreducible graphs. In the following we shall obtain the leading 1–loop terms in a weak field expansion around the constant Chern–Simons background (2.3),

$$\Gamma^E_{(1)} = \int d^3x \Gamma^E_\mu(x)(A_\mu(x) + a'_\mu(x)) + \frac{1}{2}\int d^3x d^3x' \Gamma^E_{\mu\nu}(x, x')(A_\mu(x) + a'_\mu(x))(A_\nu(x) + a'_\nu(x')) + \ldots \tag{3.3}$$

The coefficient functions in this case are expressed in terms of τ–ordered correlation functions,

$$\begin{aligned}\Gamma^E_\mu(x) &= - < j_\mu(x) > \\ \Gamma^E_{\mu\nu}(x,x') &= - < T_\tau j_\mu(x) j_\nu(x') > - < j_{\mu\nu}(x,x') >,\end{aligned} \tag{3.4}$$

where the Euclidean current components are given by

$$\begin{aligned} j_\tau(x) &= i\bar\psi\psi \\ j_k(x) &= -\frac{i}{2m}(\bar\psi\nabla_k\psi - \nabla_k\bar\psi\psi) \\ j_{\tau\mu}(x,x') &= 0 \\ j_{k\ell}(x,x') &= -\frac{1}{m}\delta_{k\ell}\delta_3(x-x')\bar\psi\psi \end{aligned} \tag{3.5}$$

To evaluate the correlators (3.4) we need the electron thermal Green's function [9],

$$\mathcal{G}(x,x') = - < T_\tau \psi(x)\bar\psi(x') > \tag{3.6}$$

This Green's function is obtained by solving the equation

$$\left(\partial_\tau - \frac{1}{2m}\nabla^2_k - \mu\right)\mathcal{G}(x,x') = -\delta_3(x-x') \tag{3.7}$$

subject to the requirement of antiperiodicity under the translation $\tau \longrightarrow \tau + \beta$. It can be expanded in normal modes,

$$\mathcal{G}(x,x') = \frac{1}{\beta}\sum_s \int \frac{dk_1}{2\pi} e^{ik_1(x_1-x_1') - i\zeta_s(\tau-\tau')} \frac{1}{\ell}\sum_n \frac{v_n(\frac{x_2}{\ell} + k_1\ell)v_n^*(\frac{x_2'}{\ell} + k_1\ell)}{i\zeta_s - \varepsilon_n + \mu} \tag{3.8}$$

where the functions v_n are the same harmonic oscillator eigenfunctions that appeared in Sec. 2. The discrete frequencies, ζ_s, are half–integer multiples of $2\pi/\beta$,

$$\zeta_s = \left(s + \frac{1}{2}\right)\frac{2\pi}{\beta}, \quad s \in \mathbb{Z} \tag{3.9}$$

In terms of the thermal Green's function (3.8) the correlators (3.4) are given by

$$\begin{aligned} \Gamma^E_\tau(x) &= -iTr\mathcal{G}(x,x')|_{\underline{x}'=\underline{x},\tau'=\tau+0} \\ \Gamma^E_k(x) &= \frac{i}{2m}(\nabla_k - \nabla'_k)Tr\mathcal{G}(x,x')|_{\underline{x}'=\underline{x},\tau'=\tau+0} \\ \Gamma^E_{\tau\tau}(x,x') &= -Tr(\mathcal{G}(x,x')\mathcal{G}(x',x)) \\ \Gamma^E_{k\tau}(x,x') &= \frac{1}{2m}Tr(\nabla_k\mathcal{G}(x,x')\mathcal{G}(x',x) - \mathcal{G}(x,x')\nabla_k\mathcal{G}(x',x)) \end{aligned} \tag{3.10}$$

$$\begin{aligned} \Gamma^E_{k\ell}(x,x') = &-\frac{1}{4m^2}Tr(\nabla_k\mathcal{G}(x,x')\nabla'_\ell\mathcal{G}(x',x) - \nabla_k\nabla'_\ell\mathcal{G}(x,x')\mathcal{G}(x',x) \\ &+ \nabla'_\ell\mathcal{G}(x,x')\nabla_k\mathcal{G}(x'x) - \mathcal{G}(x,x')\nabla_k\nabla'_\ell\mathcal{G}(x',x)) \\ &+ \frac{1}{m}\delta_{k\ell}\delta_3(x-x')Tr\mathcal{G}(x,x')|_{\underline{x}'=\underline{x},\tau'=\tau+0} \end{aligned} \tag{3.11}$$

where all covariant derivatives refer to the background Chern–Simons field (2.3).

Some details of the computation are discussed in Appendix C. The result for the components (3.10) is given by

$$\Gamma^E_\tau(x) = \frac{-i}{\pi\ell^2}\sum_n (e^{\beta(\varepsilon_n-\mu)}+1)^{-1}$$
$$\Gamma^E_k(x) = 0 \tag{3.12}$$

The polarization components (3.11) can be represented by Fourier expansions,

$$\Gamma^E_{\mu\nu}(x,x') = \frac{1}{\beta}\sum_s \int \frac{d^2k}{(2\pi)^2} e^{i\underline{k}\cdot(\underline{x}-\underline{x}')-i\omega_s(\tau-\tau')}\tilde{\Gamma}^E_{\mu\nu}(\omega_s,\underline{k}) \tag{3.13}$$

where, since the currents are periodic under $\tau \to \tau+\beta$, the frequencies are integer multiples of $2\pi/\beta$,

$$\omega_s = \frac{2\pi s}{\beta}, \quad s\in\mathbb{Z} \tag{3.14}$$

Again, because of current conservation and rotation symmetry, there are three independent invariants,

$$\tilde{\Gamma}^E_{\tau\tau} = \underline{k}^2\Pi_0^E$$
$$\tilde{\Gamma}^E_{j\tau} = \omega_s k_j\Pi_0^E + \varepsilon_{j\ell}k_\ell\Pi_1^E$$
$$\tilde{\Gamma}^E_{j\ell} = \omega_s^2\delta_{j\ell}\Pi_0^E + \varepsilon_{j\ell}\omega_s\Pi_1^E + (k^2\delta_{j\ell}-k_jk_\ell)\Pi_2^E \tag{3.15}$$

The invariants Π_0^E, Π_1^E and Π_2^E are functions of ω_s^2 and $\underline{k}^2$ and we shall be interested in their behaviour near $\omega_s = k = 0$. It turns out that Π_1^E, Π_2^E are regular there but Π_0^E has a simple pole at $\underline{k}^2 = 0$ in the amplitude with $\omega_s = 0$. The results, for $\omega_s = 0$, are

$$\Pi_0^E = \frac{1}{\underline{k}^2}\frac{\beta}{4\pi\ell^2}\sum_n sech^2\frac{\beta}{2}(\varepsilon_n-\mu)$$
$$+\frac{m\ell^2}{\pi}\sum_n (e^{\beta(\varepsilon_n-\mu)}+1)^{-1} - \frac{\beta}{8\pi}\sum_n (2n+1)sech^2\frac{\beta}{2}(\varepsilon_n-\mu)$$
$$\Pi_1^E = \frac{1}{\pi}\sum_n (e^{\beta(\varepsilon_n-\mu)}+1)^{-1} - \frac{1}{8\pi}\frac{\beta}{m\ell^2}\sum_n (2n+1)sech^2\frac{\beta}{2}(\varepsilon_n-\mu)$$
$$\Pi_2^E = \frac{1}{\pi m}\sum_n (2n+1)(e^{\beta(\varepsilon_n-\mu)}+1)^{-1}$$
$$-\frac{1}{16\pi}\frac{\beta}{m^2\ell^2}\sum_n (2n+1)^2 sech^2\frac{\beta}{2}(\varepsilon_n-\mu) \tag{3.16}$$

The results (3.12) and (3.16) define the behaviour of the 1–loop contribution to the Euclidean effective action for weak long wave fluctuations of the gauge potentials around the equilibrium values $A_\mu = 0$, $a_\mu = a_\mu^0$. For a consistent determination of the equilibrium configuration

it is necessary to satisfy the Euclidean equations of motion,

$$\frac{\delta\Gamma^E}{\delta A_\mu} = -in_e\delta_{\mu\tau}$$
$$\frac{\delta\Gamma^E}{\delta a_\mu} = 0 \qquad (3.17)$$

where, as before, n_e represents an external uniform charge density. The functional derivatives are to be evaluated at $A_\mu = 0\,, a_\mu = a_\mu^0$. According to (3.15) the non–vanishing components are

$$\frac{1}{\pi\ell^2}\sum_n (e^{\beta(\varepsilon_n-\mu)}+1)^{-1} = n_e$$
$$\frac{\nu}{\pi}\frac{1}{\ell^2} - \frac{1}{\pi\ell^2}\sum_n (e^{\beta(\varepsilon_n-\mu)}+1)^{-1} = 0 \qquad (3.18)$$

where the first term in the latter equation represents the 0–loop contribution. The equilibrium value of the magnetic length, ℓ, is therefore given by

$$\ell^2 = \frac{\nu}{\pi n_e} \qquad (3.19)$$

independently of the temperature. On the other hand, the chemical potential, μ, is determined by the equation

$$\nu = \sum_n (e^{\beta(\varepsilon_n-\mu)}+1)^{-1} \qquad (3.20)$$

which indicates that the Chern–Simons parameter ν is to be identified with the thermal average of the electron occupation numbers. (At zero temperature the right hand side of (3.20) reduces to the number of occupied Landau levels in the ground state.)

The coefficients (3.16) determine the bilinear part of the Euclidean effective action for slowly varying fields,

$$|k| << \frac{1}{\ell}, \quad 2\pi|s| << \frac{\beta}{m\ell^2} \qquad (3.21)$$

If we restrict our considerations to fields which are independent of τ then the Euclidean action can be interpreted as a thermodynamic potenital,

$$\Gamma^E = \beta\Omega \qquad (3.22)$$

with

$$\Omega = \int d^2x\,[\frac{1}{2e^2}(E_j^2+B^2) + \frac{i\nu}{\pi}a_\tau b$$
$$+ \frac{1}{2}(E_j+e_j)\Pi_0^E(E_j+e_j) + \frac{1}{2}\Pi_2^E(B+b)^2$$
$$- i\Pi_1^E(A_\tau+a_\tau)(B+b)] \qquad (3.23)$$

where, because of its pole at $\underline{k}^2 = 0$, Π_0^E is to be understood as an integral operator. The functional (3.23) can be used to evaluate the thermal averages of interaction energies associated with static current distributions. For example, the thermal averaged fields A_μ and a'_μ associated with an external electromagnetic current distribution, $I_\mu(x)$, are obtained by solving the equations,

$$\frac{\delta\Omega}{\delta A_\mu(x)} = -I_\mu(x), \quad \frac{\delta\Omega}{\delta a_\mu(x)} = 0$$

The value of the thermodynamic potential is then given by the integral,

$$\Omega_I = \frac{1}{2}\int d^2x I_\mu A_\mu \tag{3.24}$$

which is a bilinear functional of the currents. Some aspects of this computation will be examined in Sec. 4.

Next we give the leading terms in the low temperature expansions of the expressions (3.12) and (3.16). The characterization "low" is taken here to mean that the tempeature is small in comparison with the level spacing, or

$$\beta >> m\ell^2 = \frac{\nu}{\pi}\frac{m}{n_e} \tag{3.25}$$

The first step is to obtain an approximate expression for the chemical potential by solving (3.20). To this end, write

$$e^{\beta(\varepsilon_n-\mu)} = w^{\rho-\frac{1}{2}-n} \tag{3.26}$$

- where

$$w = e^{-\beta/m\ell^2}, \quad \rho = \mu m\ell^2 \tag{3.27}$$

For small w it is possible to expand the right hand side of (3.21) in fractional powers of w,

$$\sum_n (1 + w^{\rho-\frac{1}{2}-n})^{-1} = [\rho + \frac{1}{2}] + \sum_{r\geq 1}(-)^r \frac{w^{r\delta} - w^{r(1-\delta)} - w^{r(\rho+\frac{1}{2})}}{1 - w^r} \tag{3.28}$$

where $[\rho + \frac{1}{2}]$ denotes the positive integer defined by

$$\rho + \frac{1}{2} = [\rho + \frac{1}{2}] + \delta, \quad 0 < \delta < 1 \tag{3.29}$$

If w is sufficiently small then (3.20) takes the form

$$\nu = [\rho + \frac{1}{2}] - w^\delta + w^{1-\delta} + w^{[\rho+\frac{1}{2}]+\delta} + \ldots$$

and is solved in leading order by

$$[\rho + \frac{1}{2}] = \nu$$

$$\delta = \frac{1}{2} + \frac{w^{\nu}}{2\ell n w}$$

The chemical potential is therefore given by

$$\mu = \frac{\nu}{m\ell^2} - \frac{1}{2\beta} e^{-\frac{\beta\nu}{m\ell^2}} + \ldots \tag{3.30}$$

On substituting this result into (3.26) we obtain

$$e^{\beta(\epsilon_n - \mu)} = \left(\exp\left[\frac{\beta}{m\ell^2}\left(n + \frac{1}{2} - \nu\right)\right]\right)\left(1 + \frac{1}{2} e^{-\frac{\beta(\nu+\frac{1}{2})}{m\ell^2}} + \ldots\right) \tag{3.31}$$

It is now a simple exercise to find the leading terms in the expressions (3.16),

$$\begin{aligned}
\Pi_0^E &= \frac{1}{\underline{k}^2} \frac{2\beta}{\pi\ell^2} e^{-\frac{\beta}{2m\ell^2}} + m\ell^2 \frac{\nu}{\pi}\left(1 - \frac{2\beta}{m\ell^2} e^{-\frac{\beta}{2m\ell^2}}\right) \\
\Pi_1^E &= \frac{\nu}{\pi}\left(1 - \frac{2\beta}{m\ell^2} e^{-\frac{\beta}{2m\ell^2}}\right) \\
\Pi_2^E &= \frac{\nu^2}{\pi m}\left[1 + \frac{2}{\nu^2} e^{-\frac{\beta}{2m\ell^2}} - \frac{2\beta}{m\ell^2}\left(1 + \frac{1}{4\nu^2}\right) e^{-\frac{\beta}{2m\ell^2}}\right].
\end{aligned} \tag{3.32}$$

These results will be applied in Sec. 4 to find the tempeature dependence of the penetration depth.

For completeness we now give the high temperature approximations for these amplitudes. When the temperature is large in comparison with the level splitting, $\beta << m\ell^2$, it is possible to approximate the sums (3.16) by integrals and so to evaluate the leading terms in expansions in powers of $\beta/m\ell^2$. The details of this computation are given in Appendix C. The results are,

$$\begin{aligned}
\Pi_0^E &= \frac{m}{\pi\underline{k}^2}\left(\frac{1}{1+e^{-\beta\mu}} + \frac{1}{96}\left(\frac{\beta}{m\ell^2}\right)^2 \sinh\frac{\beta\mu}{2}\operatorname{sech}^3\frac{\beta\mu}{2} + \ldots\right) \\
&\quad - \frac{\beta}{48\pi}\left(\operatorname{sech}^2\frac{\beta\mu}{2} - \frac{7}{120}\left(\frac{\beta}{m\ell^2}\right)^2\left\{\operatorname{sech}^2\frac{\beta\mu}{2} - \frac{3}{2}\operatorname{sech}^4\frac{\beta\mu}{2}\right\} + \ldots\right) \\
\Pi_1^E &= -\frac{1}{48\pi}\frac{\beta}{m\ell^2}\left(\operatorname{sech}^2\frac{\beta\mu}{2} - \frac{7}{120}\left(\frac{\beta}{m\ell^2}\right)^2\left\{\operatorname{sech}^2\frac{\beta\mu}{2} - \frac{3}{2}\operatorname{sech}^4\frac{\beta\mu}{2}\right\} + \ldots\right) \\
\Pi_2^E &= \frac{1}{12\pi m}\left(\frac{1}{1+e^{-\beta\mu}} + \frac{7}{160}\left(\frac{\beta}{m\ell^2}\right)^2 \sinh\frac{\beta\mu}{2}\operatorname{sech}^3\frac{\beta\mu}{2} + \ldots\right)
\end{aligned} \tag{3.33}$$

where the chemical potential μ is determined by the formula

$$\nu = m\ell^2\mu + \frac{m\ell^2}{\beta}\ell n(1 + e^{-\beta\mu}) - \frac{1}{96}\left(\frac{\beta}{m\ell^2}\right)\operatorname{sech}^2\frac{\beta\mu}{2} + 0\left(\frac{\beta}{m\ell^2}\right)^3 \tag{3.34}$$

The approximations are valid if $\beta/m\ell^2 << \beta\mu$. If we impose the stronger condition that $\beta\mu >> 1$ then equation (3.34) can be solved by iteration to give

$$\mu = \frac{\nu}{m\ell^2} - \frac{1}{\beta} e^{-\beta\nu/m\ell^2} + \ldots \tag{3.35}$$

This expression can be used to eliminate μ from (3.33), leading to series expansions in powers of the small quantity $\exp(-\beta\nu/m\ell^2)$. The steps are justified if $\nu >> 1$, in which case the high temperature approximations (3.33) can be looked upon as expansions in powers of $1/\nu$.

4. STATIC COUPLINGS OF ELECTROMAGNETIC CURRENTS

The Euclidean action functional obtained in the last section can be used to evaluate the thermal averages of interaction energies associated with static current distributions. Here we shall consider only the simplest case, computation of the functional $W(I,J)$ with $J_\mu = 0$ and I_μ small. With the understanding that the background equations which determine the equilibrium are satisfied, we can treat I_μ as a weak perturbation and determine the response by solving the linear equations,

$$\frac{\delta\Gamma^E}{\delta A_\mu} = -I_\mu \quad \text{and} \quad \frac{\delta\Gamma^E}{\delta a'_\mu} = 0 \tag{4.1}$$

where Γ^E comprises the bilinear parts of the classical and 1–loop terms,

$$\Gamma^E = \int d^3x \left(\frac{1}{4e^2} F_{\mu\nu}F_{\mu\nu} + \frac{i\nu}{2\pi}\varepsilon_{\mu\lambda\rho}(a'_\mu \partial_\lambda a'_\rho)\right.$$

$$\left. + \frac{1}{2}\int d^3x d^3x' \Gamma^E_{\mu\nu}(x-x')(A_\mu(x) + a'_\mu(x))(A_\nu(x') + a'_\nu(x'))\right. \tag{4.2}$$

On substituting the solution of the inhomogeneous equations (4.1) into (4.2) and taking the Legendre transform as discussed in Sec. 3 we obtain the functional

$$W(I,0) = \Gamma^E(A,a) + \int d^3x I_\mu A_\mu$$

$$= \frac{1}{2}\int d^3x I_\mu A_\mu \ ,$$

an expression which is bilinear in the external current distribution, I_μ,

$$W(I,0) = \frac{1}{2}\int d^3x d^3x' I_\mu(x) D_{\mu\nu}(x-x') I_\nu(x')$$

$$= \frac{1}{2\beta}\sum_s \int \frac{d^2k}{(2\pi)^2} \tilde{I}_\mu(-\omega_s, -\underline{k})\, \tilde{D}_{\mu\nu}(\omega_s, k)\, \tilde{I}_\nu(\omega_s, \underline{k}) \tag{4.3}$$

Since we are interested here only in static current distributions we shall take

$$\tilde{I}_\mu(\omega_s, \underline{k}) = \delta_{s0}\beta \tilde{I}_\mu(\underline{k})$$

in which case (4.3) reduces to the form

$$W(I,0) = \frac{\beta}{2}\int \frac{d^2k}{(2\pi)^2} \tilde{I}_\mu(-\underline{k})\, \tilde{D}_{\mu\nu}(0,\underline{k})\, \tilde{I}_\nu(\underline{k})$$

$$= \beta F(I) \tag{4.4}$$

where $F(I)$ represents the free energy of the given current distribution. The matrix $D_{\mu\nu}(x-x')$ can of course be interpreted as the thermal Green's function for the electromagnetic field, and the functional $F(I)$ represents the interaction energy due to 1–photon exchange, including the effects

due to mixing between electromagnetic and Chern–Simons fields. We shall show that this mixture simulates the exchange of two distinct massive states. One of these states may be interpreted as a massive "photon" and its presence is indicative of a kind of Meissner effect. The other state, which is generally much heavier, probably plays the role of the Higgs state, familiar in Landau Gizburg theory, although here it is a vector rather than a scalar. In addition, the functional $F(I)$ will be found to contain a long range static magnetic interaction $\sim -\tilde{I}_j\tilde{I}_j/\underline{k}^2$ which vanishes at zero temperature.

The equations of motion (4.1) take the form,

$$\frac{1}{e^2}(k^2\tilde{A}_\mu - k_\mu k_\nu \tilde{A}_\nu) + \tilde{\Gamma}^E_{\mu\nu}(k)(\tilde{A}_\nu + \tilde{a}'_\nu) = -\tilde{I}_\mu$$

$$-\frac{\nu}{\pi}\varepsilon_{\mu\lambda\rho}k_\lambda \tilde{a}'_\rho + \tilde{\Gamma}^E_{\mu\nu}(k)(\tilde{A}_\nu + \tilde{a}'_\nu) = 0 \qquad (4.5)$$

where the components of $\tilde{\Gamma}^E_{\mu\nu}$ are given by (3.15) and (3.16). In addition to the static condition, $k_\tau = 0$, we shall impose the gauge conditions $k_\mu \tilde{A}_\mu = k_\mu \tilde{a}'_\mu = 0$. It is then a simple algebraic exercise to construct the solution,

$$\tilde{A}_\mu = \tilde{D}_{\mu\nu}\tilde{I}_\nu \ .$$

The components of the propagator can be expressed in terms of three independent invariants,

$$\begin{aligned} D_{\tau\tau} &= D_0 \\ D_{\tau j} &= -D_{j\tau} = \varepsilon_{j\ell}k_\ell D_1 \\ D_{j\ell} &= (\delta_{j\ell} - \frac{k_j k_\ell}{\underline{k}^2})D_2 \end{aligned} \qquad (4.6)$$

The expressions for D_0, D_1 and D_2 are not very transparent and we shall simplify their appearance by introducing a number of parameters to characterize the low momentum behaviour of the tensor $\tilde{\Gamma}^E_{\mu\nu}$ defined in (3.15),

$$\begin{aligned} \Pi_0^E &= \frac{a}{\underline{k}^2} + b \\ \Pi_1^E &= c \\ \Pi_2^E &= d \end{aligned} \qquad (4.7)$$

These temperature dependent parameters are defined by the sums (3.16). A lengthy but straightforward computation gives

$$\begin{aligned} D_0 &= -\frac{1}{\Delta}\left[\frac{1}{e^2}\left(1-\frac{\pi}{\nu}c\right)^2 + d\left(1+\frac{1}{e^2}\left(\frac{\pi}{\nu}\right)^2(a+k^2b)\right)\right] \\ D_1 &= -\frac{1}{2k^2\Delta}\left[c\left(1-\frac{\pi}{\nu}c\right) - \frac{\pi}{\nu}(a+k^2b)d\right] \\ D_2 &= -\frac{1}{\Delta}\left[\frac{1}{e^2}\left(1-\frac{\pi}{\nu}c\right)^2 + \left(\frac{a}{k^2}+b\right)\left(1+\frac{1}{e^2}\left(\frac{\pi}{\nu}\right)^2 k^2 d\right)\right] \end{aligned} \qquad (4.8)$$

where Δ is quadratic in k^2,

$$\Delta = \frac{a}{e^2} + c^2 + ad + \frac{k^2}{e^4}[(1 - \frac{\pi}{\nu}c)^2 + e^2 b + e^2 d + e^4 bd + (\frac{\pi}{\nu})^2 ad]$$

$$+(\frac{k^2}{e^4})^2(\frac{\pi}{\nu})^2 bd \qquad (4.9)$$

To gain some understanding of these quantities it is essential to make approximations. We shall therefore assume firstly, that we are in the low temperature regime, $\beta >> m\ell^2$, and, secondly, that the Chern–Simons parameter is large, $\nu >> 1$. The formulae (3.32) then give

$$\begin{aligned} a &= \frac{4m}{\pi} y e^{-y} \\ b &= \frac{m}{n_e}\left(\frac{\nu}{\pi}\right)^2 (1 - 4ye^{-y}) \\ c &= \frac{\nu}{\pi}(1 - 4ye^{-y}) \\ d &= \frac{\nu^2}{\pi m}(1 - 4ye^{-y}) \end{aligned} \qquad (4.10)$$

where $y = \beta/2m\ell^2$. Higher powers of e^{-y} and π/ν are neglected in (4.10) *). Among the dimensionful parameters e^2, n_e and m there are two dimensionless ratios for which we shall adopt the estimates used by Hosotani and Chakravarty [8]. There authors regard the Maxwell Lagrangian as a 3–dimensional remnant of the standard 4–dimensional theory, obtained by integrating over the thickness, $\delta \sim 10^{-7}$ cm, of some crystal layer. This suggests the value

$$e^2 \sim \frac{e_4^2}{\delta} \sim (10^{-5} cm)^{-1}$$

They also assume that the density of charge carriers is similar to the density of ions in the lattice,

$$n_e \sim (10^{-7} cm)^{-2}$$

Finally, they assume that the "electron" mass is the usual one,

$$m \sim 10^{10} cm^{-1}$$

with these estimates we have

$$\frac{e^2}{m} \sim 10^{-5} \quad \text{and} \quad \frac{n_e}{m^2} \sim 10^{-6} \qquad (4.11)$$

and it becomes feasible to make significant simplifications.

*) At zero temperature, $y \to \infty$, we have $a = 0$ while the coefficients b, c and d reduce to Π_0, Π_1 and $-\Pi_2$, respectively, given by (2.15). Also in this limit the expression (4.9) reduces to (2.22) evaluated at $\omega = 0$.

Firstly, in order to justify the low temperature expansions, we should require $y \geq 5$ (say). This implies

$$\frac{1}{\beta} = \frac{1}{2m\ell^2 y} = \frac{n_e}{2m}\frac{\pi}{\nu}\frac{1}{y} \lesssim \frac{\pi}{\nu} \times 10^3 K \tag{4.12}$$

Secondly, the dominant parts of Δ are given by

$$\begin{aligned}\Delta &\simeq \frac{a}{e^2} + c^2 + k^2 e^2 b + (\frac{\pi}{\nu}\frac{k^2}{e^2})^2 bd \\ &= (\frac{\pi}{\nu e^2})^2 bd(k^2 + M_+^2)(k^2 + M_-^2)\end{aligned} \tag{4.13}$$

where the masses are given approximately by

$$\begin{aligned}M_+^2 &= \frac{e^2}{b}(\frac{a}{e^2} + c^2) \\ &= \frac{e^2 n_e}{m}(1 - 4ye^{-y}) + \frac{4\pi n_e}{\nu^2} ye^{-y} \\ M_-^2 &= (\frac{\nu}{\pi})^2 \frac{e^2}{d} \\ &= \frac{e^2 m}{\pi}(1 + 4ye^{-y})\end{aligned} \tag{4.14}$$

Here we have discarded terms of order

$$(\frac{\pi}{\nu})^2 \frac{d}{b}(\frac{a}{e^2} + c^2) \sim \frac{n_e}{m^2} \sim 10^{-6}$$

is comparison with unity. The value of M_+^2 at zero temperature coincides with the formulae for penetration depth given by others [4),8)]. Our result gives the low temperature corrections and we see that M_+^2 *increases* with temperature.

Although, regrettably, this model does not appear to undergo a phase transition at finite temperature*), we can define a pseudo–critical temperature at which the thermal contribution to the Meissner effect overtakes the zero temperature contribution. The two contibutions to M_+^2, according to the approximate formula (4.14) become comparable at a temperature dependent coupling, $\nu_c(y)$ given by

$$\nu_c^2 = \frac{4\pi m}{e^2} ye^{-y}(1 + 4ye^{-y}) \tag{4.15}$$

The expression for M_+^2 then takes the form

$$M_+^2 = \frac{e^2 n_e}{m}\left(1 + \frac{\nu_c^2}{\nu^2}\right)(1 - 4ye^{-y}) \tag{4.16}$$

*) One should bear in mind that this is a perturbative result. It is conceivable that non–perturbative effects might lead to a qualitatively different conclusion particularly in the case of small ν where our methods cannot be applied.

and we see that the thermal contribution dominates for $\nu < \nu_c$. The pseudocritical coupling ν_c is plotted in Fig.2 for a range of temeperatures.

The invariant components (4.8) reduce, in the low temperature regime and with the neglect of higher powers of the ratios (4.11), to the following form.

$$
\begin{aligned}
-D_0 &= \frac{e^2}{\underline{k}^2 + M_-^2} + \frac{e^2}{\underline{k}^2 + M_+^2} \frac{8\pi n_e}{m^2} y e^{-\nu} \\
-D_1 &= -\frac{1}{\underline{k}^2 + M_+^2} \frac{\nu e^4 n_e}{2 M_+^2 m^2} \left[1 - 4 y e^{-\nu} \left(2 - \frac{\pi m}{\nu^2 e^2}\right)\right] \\
&\quad + \frac{1}{\underline{k}^2 + M_-^2} \frac{\nu e^4}{2\pi M_-^2} \left[1 + 4 y e^{-\nu} \left(\frac{\pi}{\nu}\right)^2 \frac{n_e}{m e^2}\right] \\
-D_2 &= \frac{e^2}{\underline{k}^2 + M_+^2} \left[1 - \frac{4\pi m}{\nu^2 e^2} y e^{-\nu}(1 + 8 y e^{-\nu}) \left(1 + \frac{4\pi m}{\nu^2 e^2} y e^{-\nu}\right)^{-1}\right] \\
&\quad + \frac{m}{\underline{k}^2} \left(\frac{4\pi}{\nu^2}\right) y e^{-\nu}(1 + 8 y e^{-\nu}) \left(1 + \frac{4\pi m}{\nu^2 e^2} y e^{-\nu}\right)^{-1}
\end{aligned}
\tag{4.17}
$$

In order to have a rough idea of the relative strengths of the various forces involved here, we give the energy functional at zero temperature with the Chern–Simons parameter assigned to an intermediate range such that

$$
\frac{n_e}{m e^2} << (\frac{\nu}{\pi})^2 << \frac{m}{e^2} \tag{4.18}
$$

which means $1 << \nu << 10^3$. At the same time we restore the real charge density $I_0 = i I_\tau$. The result is

$$
\begin{aligned}
E(I) = \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \Big\{ & \tilde{I}_0(-\underline{k}) \frac{e^2}{\underline{k}^2 + M_-^2} \tilde{I}_0(\underline{k}) \\
& + \tilde{I}_0(-\underline{k}) \left[\frac{1}{\underline{k}^2 + M_-^2} - \frac{1}{\underline{k}^2 + M_+^2}\right] \frac{\nu e^2}{m} \varepsilon_{j\ell}\, i k_\ell\, \tilde{I}_j(\underline{k}) \\
& - \tilde{I}_j(-\underline{k}) \frac{e^2}{\underline{k}^2 + M_+^2} - \tilde{I}_j(\underline{k}) \Big\}
\end{aligned}
\tag{4.19}
$$

This indicates that the static force between currents is of intermediate range and attractive. There is a two component force, one of which is longer ranged, between charge and current.

To complete the discussion, we now consider the high temperature regime, $\beta << m\ell^2$ If $\nu >> 1$ then we have, according to (3.35)

$$
\mu \simeq \frac{\nu}{m\ell^2} = \pi \frac{n_e}{m} \tag{4.20}
$$

and the limiting forms of the expressions (3.33) give

$$
\begin{aligned}
a &\simeq \frac{m}{\pi} \\
b &\simeq 0 \simeq c \\
d &\simeq \frac{1}{12\pi m}.
\end{aligned}
\tag{4.21}
$$

which can be substituted into (4.8) and (4.9). We obtain

$$D_0 \simeq -\frac{e^2}{\underline{k}^2 + me^2/\pi}$$
$$D_1 \simeq \frac{1}{24}\frac{e^2}{\nu m}\left(\frac{1}{\underline{k}^2} - \frac{1}{\underline{k}^2 + me^2/\pi}\right)$$
$$D_2 \simeq -\frac{e^2}{\underline{k}^2} \qquad (4.22)$$

The pole at $\underline{k}^2 = 0$ in D_2 indicates the presence of a long range magnetic interaction. This does not imply a phase transition, however. Indeed, the long range effect, which derives from the pole in Π_0^E, is present at any finite temperature. Its coupling strength depends on the temperature and, as can be seen from the low temperature expression for D_2 given by (4.17), this coupling goes smoothly to zero at zero temperature. The pure Meissner effect which is observed at zero temperature, for example in the formula (4.19) is certainly compromised by the apppearance of this pole in the finite temperature amplitude. We must conclude that the magnetic field does in fact develop a long range component along with the finite range ($\sim M_+^{-1}$) component which dominates at low temperature. What we have found is that the relative strengths of these components varies smoothly with temperature and that the long range part comes to dominate at very high temperatures. We have also found that one of the ranges drops to zero ($M_+ \to \infty$) as the temperature goes to infinity.

5. CONCLUSIONS

In this paper we have applied standard field theoretic perturbation theory to calculate thermal corrections to the Messner effect recently demonstrated in a simple model of Chern–Simons superconductivity [4),8)]. The calculation is expressed in the language of effective action functionals, Minkowskian for the zero temperature case and Euclidean for the finite temperature case. Our main approximations are the restriction to weak and slowly varying fields, and the neglect of 2–loop contributions. The model involves two coupling parameters, e^2, with the dimensions of mass, and π/ν, which is dimensionless. Both of these parameters must be small in order to justify the 1–loop approximation. It may eventually become possible to improve on this, extending the range to small values of the Chern–Simons parameter, ν, by applying the methods of fractional statistics, but we have not attempted such a calculation. In this paper the (2+1)–dimensional "electrons" are treated as ordinary fermions.

The mechanism which gives rise to a kind of Meissner effect in this model is the induction, at 1–loop level, of a Chern–Simons term for the electromagnetic potential. As we have pointed out in Sec. 1, the presence of such a term along with the standard Maxwell kinetic term leads to a massive photon. Since there are two independent gauge fields in this model it is necessary

to take account of their mixing, which is substantial. There are in fact two massive states representing the independent mixtures. Because of the assumed values of the parameters, n_e, m and e^2, one of the states is considerably lighter than the other and it is reasonable to consider it as the massive "photon". However, it should be kept in mind that there are two independent states which mediate the interaction between currents. The Meissner effect in this model is kinematically more complicated than in the classic Ginzburg–Landau description. At finite temperatures, the situation is further complicated by the appearance of a long range component in the static interactions. We have shown that the relative strengths of the long and finite range components varies continuously with temperature and that, in the limit of very high temperatures, only the long–range component persists.

It is a great virtue of this model that the electon energies are quantized. The level splitting is inversely proportional to the Chern–Simons parameter, ν. Relative to this splitting it is possible to define low and high temperature regimes where the sums which define the thermal averages can be evaluated approximately. For the values assumed in this paper, the level splitting is given by $(m\ell^2)^{-1} = \pi n_e/m\nu \sim 10^4\, K/\nu$ and this means that the physically interesting temperatures, $\sim 10^2\, K$, may lie in either the low or high regime, depending on the value of ν. If ν is very large then we can view the high temperature approximation as an expansion in powers of $1/\nu$.

It was of course a disappointment to find no indication of a phase transition in the temperature dependence of the penetration depth. However, this conclusion seems to be inescapable. The general criterion for the existence of a critical temperature, manifested as a divergence in the penetration depth, would be the vanishing at $k^2 = 0$ of the determinant, $\Delta(k^2)$, given by (4.9). But the value,

$$\Delta(0) = \frac{a}{e^2} + c^2 + ad$$

where a, c and d are defined by (4.7) and (3.16), seems to be inevitably positive. The coefficient a is given as a sum of positive terms and $e^2 d$ is relatively small. In the high and low temperature approximations we have shown that $e^2 d$ is in fact positive. Although we have not been able to show that this is always the case it is hard too see how it could become sufficiently large and negative to make $\Delta(0)$ vanish. However, our conclusion is based on the 1–loop approximation which makes sense only if π/ν is small.A more sophisticated approach would be necessary if ν is small, and the result could well be different*).

The Chern Simons parameter must be an integer. This restriction on the allowed values of ν is based on two independent considerations. Firstly, as discussed in Sec. 2, the consistency of the assumed ground state configuration, with a uniform Chern–Simons magnetic background, requires that ν should equal the number of filled Landau levels and therefore be a positive integer [4),8)]. Secondly, the Lagrangian density (2.1) is invariant with respect to gauge transformation only

*) For example, such a difference might come about because of higher order effects. These could be estimated by a renormalization group approach which would make the coupling parameters, e^2 and ν into functions of the temperature.

up to a total derivative. If the theory is formulated on a compact space then it becomes possible to consider gauge transformations which are topologically non–trivial and for which the integral of the total derivative does not vanish. Invariance of the phase, e^{iS}, with respect to such transformations is ensured only if ν is an integer [8),2)]. On the other hand, we should point out that the temperture dependent radiative correction to ν, represented by the coefficient c, are not integral. This seems to indicate a breakdown of the topologically non–trivial parts of the gauge symmetry.

It would be interesting to see if any modifications of the present simple model are able to exhibit critical behaviour at finite temperature. Even more interesting would be the discovery of some explanation, in terms of a more fundamental microscopic theory, for the appearance of a Chern–Simons field and its associated parameter, ν.

Acknowledgments

We are indebted to E. Witten for a very useful discussion which stimulated our interest in this subject. We are also grateful to Y. Hosotani for pointing out an error in an early version of this paper.

APPENDIX A

Kinematics of Landau states

Because of the presence of a uniform magnetic field, the 1–electron Hamiltonian is not invariant with respect to rotations and translations. However, because the field is uniform it should be possible to compensate for the action of these space transformations by means of gauge transformations. In this appendix we shall construct the appropriate gauge action and show that the Hamiltonian is indeed invariant with respect to the combined group.

The background Chern–Simons field, used in Sec. 2, is given by

$$a_1(x) = -x_2, \quad a_2(x) = 0 \tag{A.1}$$

Here, in order to streamline the notation we shall adopt a system of units in which the magnetic length, ℓ, is equal to unity. The action on the Chern–Simons vector of an arbitrary space transformation,

$$x_1 \rightarrow x_1' = x_1 \cos\theta + x_2 \sin\theta + \varepsilon_1$$

$$x_2 \rightarrow x_2' = -x_1 \sin\theta + x_2 \cos\theta + \varepsilon_2$$

followed by a gauge transformation, is given by

$$a_1(x) \rightarrow a_1'(x') = a_1(x)\cos\theta + a_2(x)\sin\theta + \partial_1\Lambda$$

$$a_2(x) \rightarrow a_2'(x') = -a_1(x)\sin\theta + a_2(x)\cos\theta + \partial_2\Lambda \tag{A.2}$$

In order to have $a_i'(x) = a_i(x)$ or, more specifically,

$$a_1'(x') = -x_2', \quad a_2'(x') = 0$$

for arbitrary θ and ε_j, it is necessary to choose

$$\Lambda(x) = \frac{1}{2}(x_1^2 - x_2^2)\sin\theta - x_1\varepsilon_2 \tag{A.3}$$

This is the compensating gauge transformation. The action of the combined transformation on the electron field is given by *)

$$\psi(x) \rightarrow \psi'(x') = e^{i\Lambda(x)}\psi(x) \tag{A.4}$$

We can define the generators of infinitesimal transformations in the 1–electron Hilbert space by expanding (A.4),

$$\begin{aligned}\delta\psi(x) &= \psi'(x) - \psi(x) \\ &= -i\varepsilon_j P_j\psi(x) + i\theta J\psi(x)\end{aligned}$$

*) We treat the electron field as a space scalar.

This gives

$$
\begin{aligned}
P_1\psi &= -i\partial_1\psi \\
P_2\psi &= (-i\partial_2 + x_1)\psi \\
J\psi &= -i(x_1\partial_2 - x_2\partial_1)\psi + \frac{1}{2}(x_1^2 - x_2^2)\psi
\end{aligned}
\qquad (A.5)
$$

It is of course true that the space group does not commute with the gauge group and, in deriving (A.5) we have required the compensating gauge transformation to follow the space transformation. The non commutativity of these groups gives rise to a modification of the Lie algebra of the effective invariance group in comparison with that of the pure space group. Indeed, from (A.5) it follows that

$$
\begin{aligned}
[P_1, J] &= -iP_2 \\
[P_2, J] &= iP_1 \\
[P_1, P_2] &= -i
\end{aligned}
\qquad (A.6)
$$

Notice that the "translation" generators do not commute. Instead, they behave like canonically conjugate variables.

It is instructive and useful to replace the original canonical variables x_j and $\pi_j = -i\partial_j$ by the linear combinations,

$$
\begin{aligned}
Q &= \pi_1 + x_2, \quad & q &= \pi_2 + x_1 \\
P &= \pi_2, & p &= \pi_1
\end{aligned}
\qquad (A.7)
$$

which define two independent canonical pairs (q,p) and (Q,P). It is a simple exercise to express the generators (A.5) in terms of the new variables,

$$
\begin{aligned}
P_1 &= p \\
P_2 &= q \\
J &= \frac{1}{2}(q^2 + p^2) - \frac{1}{2}(Q^2 + P^2)
\end{aligned}
\qquad (A.8)
$$

Moreover, the 1–electron Hamiltonian becomes

$$
H = \frac{1}{2m}(Q^2 + P^2) \qquad (A.9)
$$

which clearly commutes with the generators (A.8).

Since the angular momentum is now expressed as the difference of two independent oscillator Hamiltonians (one the true Hamiltonian and the other a kind of shadow Hamiltonian that makes no contribution to the energy) it will be relatively easy to perform rotation invariant computations by using a basis of oscillator eigenstates,

$$
|n, r> = \frac{1}{\sqrt{n!}}\left(\frac{Q - iP}{\sqrt{2}}\right)^n \frac{1}{\sqrt{r!}}\left(\frac{q - ip}{\sqrt{2}}\right)^r |0> \qquad (A.10)
$$

where the ground state is defined by

$$(Q + iP)|0 >= (q + ip)|0 >= 0 \qquad (A.11)$$

With respect to the coordiante basis $|Q, q >$ the wave functions factorize

$$< Q, q|n, r >= v_n(Q)v_r(q) \qquad (A.12)$$

where the $v's$ are expressed in the familiar way in terms of Hermite polynomials,

$$< Q|n >= v_n(Q) = (\sqrt{\pi}2^n n!)^{-1/2} H_n(Q)e^{-Q^2/2} \qquad (A.13)$$

We shall have occasion to employ a momentum basis $|P >$ as well. These states are normalized according to the convention,

$$< Q|Q' >= \delta(Q - Q')$$

$$< P|P' >= 2\pi\delta(P - P') \qquad (A.14)$$

and the transition amplitudes are given by

$$< Q|P >= e^{iQP} \qquad (A.15)$$

APPENDIX B

The zero temperature loop integral

The 1–electron Green's function of Sec. 2 satisfies the differential equation,

$$(i\partial_0 - H)G(x,x') = -\delta_3(x-x')$$

and it can be expressed in the form of an integral over frequencies,

$$G(x,x') = \int_{C_N} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} < \underline{x}|G(\omega)|\underline{x}' > \qquad (B.1)$$

where the operator $G(\omega)$ is defined by

$$G(\omega) = (H-\omega)^{-1} \qquad (B.2)$$

provided ω does not coincide with any of the eigenvalues of H. The arrangement of the integration contour C_N is described in Sec. 2. Here we shall be concerned mainly with the matrix elements of $G(\omega)$. We shall also need expressions for the derivatives $\nabla_i G(x,x')$, $\nabla_i' G(x,x')$, etc. In the notation of Appendix A we have

$$\nabla_1 = \partial_1 + ix_2 = i(\pi_1 + x_2) = iQ$$

$$\nabla_2 = \partial_2 = i\pi_2 = iP \qquad (B.3)$$

and it is useful to define the 2–vector Π_i according to

$$\Pi_1 = Q, \quad \Pi_2 = P \qquad (B.4)$$

In terms of this operator the covariant derivatives of ψ and ψ^+ are expressed by

$$\nabla_i\psi = i\Pi_i\psi \quad \text{and} \quad \nabla_i\psi^+ = -i\psi^+\Pi_i \qquad (B.5)$$

It follows that the covariant derivatives $\nabla_i G(x,x')$ and $\nabla_i' G(x,x')$ can be represented by contour integrals like (B.1) with the operator $G(\omega)$ replaced by $i\Pi_i G(\omega)$ and $-iG(\omega)\Pi_i$, respectively. Higher order covariant derivatives of $G(x,x')$ are given by obvious generalizations of this rule.

Let us now consider the structure of the 1–loop ground state polarization tensor. In the typical 1–loop computation we arrive at the integral

$$\int d^2x d^2x' e^{-i\underline{k}\cdot\underline{x}+i\underline{k}'\cdot\underline{x}'} < \underline{x}|G(\omega_1)|\underline{x}' >< \underline{x}'|G(\omega_2)|\underline{x} >$$

$$= Tr(e^{-i\underline{k}\cdot\underline{x}}G(\omega_1)e^{-i\underline{k}'\cdot\underline{x}}G(\omega_2)) \qquad (B.6)$$

However, because the operator $G(\omega)$ involves only the canonical pair (Q,P) and not the pair (q,p) we can affect an immediate simplification. Substituting from (A.7),

$$\begin{aligned} \underline{k}\cdot\underline{x} &= k_1(q-P)+k_2(Q-p) \\ &= k_1 q - k_2 p - k_1 P + k_2 Q \\ &= k_1 q - k_2 p + \underline{k}\cdot\underline{\tilde{\Pi}} \end{aligned} \tag{B.7}$$

where $\tilde{\Pi}_i$ denotes the dual of Π_i defined above,

$$\tilde{\Pi}_1 = -\Pi_2, \quad \tilde{\Pi}_2 = \Pi_1 \tag{B.8}$$

The trace (B.6) factorizes,

$$\begin{aligned} &Tr(e^{-i\underline{k}\cdot\underline{x}}G(\omega_1)e^{i\underline{k}'\cdot\underline{x}}G(\omega_2)) = \\ &= Tr(e^{-ik_1q+ik_2p}e^{ik_1'q-ik_2'p})Tr(e^{-i\underline{k}\cdot\underline{\tilde{\Pi}}}G(\omega_1)e^{i\underline{k}'\cdot\underline{\tilde{\Pi}}}G(\omega_2)) \end{aligned}$$

The first factor can be evaluated explicitly,

$$\begin{aligned} &Tr(e^{-ik_1q+ik_2p}e^{ik_1'q-ik_2'p}) = \\ &\quad = Tr\left(e^{-i(k_1-k_1')q}e^{i(k_2-k_2')p}\right)e^{i(k_1k_2-k_1'k_2')/2} \\ &\quad = \int dq e^{-i(k_1-k_1')q} < q|e^{i(k_2-k_2')p}|q > e^{i(k_1k_2-k_1'k_2')/2} \\ &\quad = 2\pi\delta_2(\underline{k}-\underline{k}') \end{aligned} \tag{B.9}$$

Thus, the integral (B.6) reduces to the form,

$$\begin{aligned} &Tr(e^{-i\underline{k}\cdot\underline{x}}G(\omega_1)e^{i\underline{k}'\cdot\underline{x}}G(\omega_2)) = \\ &\quad = 2\pi\delta_2(\underline{k}-\underline{k}')Tr(e^{-i\underline{k}\cdot\underline{\tilde{\Pi}}}G(\omega_1)e^{i\underline{k}\cdot\underline{\tilde{\Pi}}}G(\omega_2)) \\ &\quad = 2\pi\delta_2(\underline{k}-\underline{k}')\sum_{n,m}\frac{<n|e^{-i\underline{k}\cdot\underline{\tilde{\Pi}}}|m><m|e^{i\underline{k}\cdot\underline{\tilde{\Pi}}}|n>}{(\varepsilon_m-\omega_1)(\varepsilon_n-\omega_2)} \end{aligned} \tag{B.10}$$

Generalization to integrals which involve covariant derivatives of the Green's function is immediate. One has only to make the appropriate modifications, $G(\omega_1) \rightarrow i\Pi_i G(\omega_1)$, etc. The rotation and translation invariance of the polarization tensor is manifest in these expressions.

The oscillator matrix elements in (B.10) can be evaluated to any desired accuracy by employing the wave functions (A.13). In this paper we are interested only in the leading terms in an expansion in powers of k. For this it is sufficient to use algebraic methods. To illustrate, define the irreducible components,

$$k_\pm = \frac{1}{\sqrt{2}}(k_1 \pm ik_2) \tag{B.11}$$

and write

$$\begin{aligned} i\underline{k}\cdot\underline{\tilde{\Pi}} &= -ik_1P + ik_2Q \\ &= k_+a^+ - k_-a \end{aligned} \tag{B.12}$$

where a^+ and a are the raising and lowering operators,

$$a = \frac{1}{\sqrt{2}}(Q + iP), \quad a^+ = \frac{1}{\sqrt{2}}(Q - iP) \tag{B.13}$$

The oscillator eigenstates are defined by

$$|n>= \frac{1}{\sqrt{n!}}(a^+)^n|0> \tag{B.14}$$

and it is straightforward to evaluate the first few terms of the expansion

$$\begin{aligned} e^{i\underline{k}\cdot\underline{\tilde{\Pi}}}|n> &= \sum_r \frac{1}{r!}(k_+a^+ - k_-a)^r|n> \\ &= |n> + k_+\sqrt{n+1}\,|n+1> - k_-\sqrt{n}|n-1> + \\ &+ \frac{1}{2}k_+^2\sqrt{(n+1)(n+2)}\,|n+2> - k_+k_-(n+\frac{1}{2})|n> + \frac{1}{2}k_-^2\sqrt{n(n-1)}\,|n-2> + \ldots \end{aligned} \tag{B.15}$$

etc. It is clear that the matrix element $< m|e^{ik\Pi}|n >$ has the threshold behaviour $k^{|m-n|}$ and it follows that, in evaluating the sum (B.10) to order k^{2s} we need to keep only the terms with $|m-n| \leq s$.

It remains to consider the integration over frequencies. The most direct way to deal with these integrals is to displace the eigenvalues of H by infinitesimal imaginary parts and then take the contour C_N along the real axis. Specifically, we make the displacements

$$\begin{aligned} \varepsilon_n &\to \varepsilon_n + i\delta, \quad 0 \leq n < N \\ \varepsilon_n &\to \varepsilon_n - i\delta, \quad n \geq N \end{aligned} \tag{B.16}$$

where δ is real and positive. The computation then proceeds in a straightforward way. To illustrate, we consider the polarization component Γ_{00} defined in (2.12). The 3–dimensional Fourier transform is given by

$$\begin{aligned} \tilde{\Gamma}_{00}(k,k') &= \int d^3x d^3x' e^{-ikx+ik'x'} iTrG(x,x')G(x',x) \\ &= 2i\int dtdt' e^{i\omega t - i\omega' t'} \int \frac{d\omega_1}{2\pi} e^{-i\omega_1(t-t')} \\ &\quad \cdot \int \frac{d\omega_2}{2\pi} e^{+i\omega_2(t-t')} Tr(e^{-i\underline{k}\cdot\underline{x}}G(\omega_1)e^{i\underline{k}'\cdot\underline{x}}G(\omega_2)) \\ &= 2i\cdot 2\pi\delta(\omega-\omega')\int \frac{d\omega_1}{2\pi} Tr(e^{-i\underline{k}\cdot\underline{x}}G(\omega_1)e^{i\underline{k}'\cdot\underline{x}}G(\omega_1-\omega)) \\ &= 2\pi\delta(\omega-\omega')(2\pi)^2\delta(\underline{k}-\underline{k}')\tilde{\Gamma}_{00}(\omega,\underline{k}) \end{aligned}$$

In the first line of this development there is a trace over the spin degree of freedom which yields the factor 2 in the succeeding lines. Substitute the expression (B.10) and extract the momentum

conservation delta functions to obtain,

$$\begin{aligned}\tilde{\Gamma}_{00}(\omega,k) &= \frac{i}{\pi}\int\frac{d\omega_1}{2\pi}Tr(e^{-i\underline{k}\cdot\underline{\tilde{\Pi}}}G(\omega_1)e^{i\underline{k}\cdot\underline{\tilde{\Pi}}}G(\omega_1-\omega)) \\ &= \frac{i}{\pi}\sum_{m,n}<n|e^{-i\underline{k}\cdot\underline{\tilde{\Pi}}}|m><m|e^{i\underline{k}\cdot\underline{\tilde{\Pi}}}|n> \\ &\quad\cdot\int\frac{d\omega_1}{2\pi}(\varepsilon_m-\omega_1)^{-1}(\varepsilon_n-\omega_1+\omega)^{-1} \qquad (B.17)\end{aligned}$$

With the energies displaced off the real axis according to the rule (B.16) the integral over ω_1 is well defined and easy to evaluate. It is non–vanishing only if the two poles are situated on opposite sides of the real axis. Using Cauchy's theorem,

$$\int\frac{d\omega_1}{2\pi i}(\varepsilon_m-\omega_1)^{-1}(\varepsilon_n-\omega_1+\omega)^{-1}=\begin{cases}(\varepsilon_n-\varepsilon_m+\omega)^{-1}, & m\geq N,\ n<N\\ -(\varepsilon_n-\varepsilon_m+\omega)^{-1}, & m<N,\ n\geq N\\ 0,\ otherwise\end{cases} \qquad (B.18)$$

Hence the result,

$$\tilde{\Gamma}_{00}(\omega,\underline{k})=\frac{1}{\pi}\left[-\sum_{m\geq N}\sum_{n<N}+\sum_{m<N}\sum_{n\geq N}\right]\frac{<n|e^{-i\underline{k}\cdot\underline{\tilde{\Pi}}}|m><m|e^{i\underline{k}\cdot\underline{\tilde{\Pi}}}|n>}{\varepsilon_n-\varepsilon_m+\omega} \qquad (B.19)$$

The other components of the polarization tensor are obtained in the same way

$$\tilde{\Gamma}_{j0}(\omega,k)=-\frac{1}{2m\pi}\left[-\sum_{m\geq N}\sum_{n<N}+\sum_{m<N}\sum_{n\geq N}\right]\frac{<n|\{e^{-ik\cdot\tilde{\Pi}},\Pi_j\}|m><m|e^{ik\cdot\tilde{\Pi}}|n>}{\varepsilon_n-\varepsilon_m+\omega} \qquad (B.20)$$

$$\tilde{\Gamma}_{j\ell}(\omega,k)=\frac{1}{4\pi m^2}\left[-\sum_{m\geq N}\sum_{n<N}+\sum_{m<N}\sum_{n\geq N}\right]\frac{<n|\{e^{-ik\cdot\tilde{\Pi}},\Pi_j\}|m><m|\{e^{ik\cdot\tilde{\Pi}},\Pi_\ell\}|n>}{\varepsilon_n-\varepsilon_m+\omega}-\frac{N}{\pi m}\delta_{j\ell} \qquad (B.21)$$

where the symbol { , } denotes the anticommutator. On expanding in powers of k_j and ω, using the formula (B.15) for the oscillator matrix elements, the results (2.15) are obtained.

Finite temperature computations

The 1–electron thermal Green function of Sec. 3 satisfies the differential equation

$$(\partial_\tau + H - \mu)\mathcal{G}(x,x') = -\delta_3(x-x')$$

and it can be expanded in the form of a sum over discrete frequencies,

$$\mathcal{G}(x,x') = -\frac{1}{\beta}\sum_s e^{-i\zeta_s(\tau-\tau')} < \underline{x}|G(i\zeta_s+\mu)|\underline{x}' > \qquad (C.1)$$

where $G(\omega)$ is the operator (B.2) defined in Appendix B and

$$\zeta_s = (s+\frac{1}{2})\frac{2\pi}{\beta}, \quad s \in \mathbb{Z}$$

The chemical potential, μ, is determined by the equilibrium conditions (3.17) as discussed in the main text. This determination depends on the formula (3.12) for the tadpole amplitude $\Gamma^E_\mu(x)$ which we now derive.

The component $\Gamma^E_\tau(x)$ is defined by the first of equations (3.10).

$$\begin{aligned}\Gamma^E_\tau(x) &= iTr\mathcal{G}(x,x')|_{\underline{x}'=\underline{x},\tau'=\tau+\delta} \\ &= -\frac{2i}{\beta}\sum_s e^{i\zeta_s\delta} < \underline{x}|G(i\zeta_s+\mu)|\underline{x}> \end{aligned} \qquad (C.2)$$

where δ is an infinitesimal positive quantity. For fixed $\underline{x} = \hat{\underline{x}}$, the diagonal matrix element in (C.2) can be expressed as a trace in the 1–electron Hilbert space,

$$\begin{aligned}< \hat{x}|G(i\zeta_s+\mu)|\hat{x} > &= Tr(\delta(\underline{x}-\hat{\underline{x}})G(i\zeta_s+\mu)) = \\ &= \int \frac{d^2k}{(2\pi)^2} e^{-i\underline{k}\cdot\hat{\underline{x}}} Tr(e^{i\underline{k}\cdot\underline{x}}G(i\zeta_s+\mu)) \\ &= \int \frac{d^2k}{(2\pi)^2} e^{i\underline{k}\cdot\hat{\underline{x}}} Tr(e^{ik_1q-ik_2p}) Tr(e^{-i\underline{k}\cdot\underline{\Pi}}G(i\zeta_s+\mu))\end{aligned}$$

where we have used (B.7) to separate the trace into indpendent factors. The first factor is purely kinematical,

$$Tr(e^{ik_1q-ik_2p}) = 2\pi\delta_2(\underline{k})$$

according to (B.9). Hence we have

$$\begin{aligned}< \hat{\underline{x}}|G(i\zeta_s+\mu)|\hat{\underline{x}} > &= \frac{1}{2\pi}Tr(G(i\zeta_s+\mu)) \\ &= \frac{1}{2\pi}\sum_n (\varepsilon_n - i\zeta_s - \mu)^{-1}\end{aligned} \qquad (C.3)$$

and, on substituting into (C.2),

$$\Gamma_\tau^E(x) = \frac{i}{\pi}\sum_n \frac{1}{\beta}\sum_s (i\zeta_s - \varepsilon_n + \mu)^{-1} e^{i\zeta_s\delta} \qquad (C.4)$$

The sum over s does not converge very well but it can be defined in a plausible way. Thus, by taking a derivative with respect to μ the convergence is improved and we can set $\delta = 0$. The resulting sum can be evaluated by contour methods,

$$-\frac{1}{\beta}\sum_s (i\zeta_s - \varepsilon_n + \mu)^{-2} = -\frac{i}{2\beta}\int_C \frac{dz}{\cot \pi z}\left(\frac{2\pi i}{\beta}z - \varepsilon_n + \mu\right)^{-2}$$

where the contour C encircles all the zeroes of $\cot \pi z$, i.e. $z = s + \frac{1}{2}$, $s \in \mathbb{Z}$, in the positive sense. This contour can be replaced by a pair of straight lines, one running to the right just below the real axis and one running to the left just above the real axis. These two segments can be closed by means of large semicircles in the lower and upper half planes, respectively (see Fig.3). Their evaluation is then trivial since one or other of these contours will contain the dipole at $z = (\beta/2\pi i)(\varepsilon_n - \mu)$, depending on whether $\varepsilon_n - \mu$ is positive or negative. The result is

$$-\frac{1}{\beta}\sum_s (i\zeta_s - \varepsilon_n + \mu)^{-2} = \frac{\beta}{4}sech^2\frac{\beta}{2}(\varepsilon_n - \mu) \qquad (C.5)$$

This can be integrated to give, up to a constant,

$$\begin{aligned}\frac{1}{\beta}\sum_s (i\zeta_s - \varepsilon_n + \mu)^{-1} &= \frac{1}{2}\left(1 - th\frac{\beta}{2}(\varepsilon_n - \mu)\right) \\ &= (e^{\beta(\varepsilon_n - \mu)} + 1)^{-1} \qquad (C.6)\end{aligned}$$

The constant of integration is fixed by requiring appropriate behaviour in the zero temperature limit, $\beta \to \infty$. Thus, we obtain the formula,

$$\Gamma_\tau^E(x) = \frac{i}{\pi}\sum_n (e^{\beta(\varepsilon_n - \mu)} + 1)^{-1} \qquad (C.7)$$

It is easy to see that the space components Γ_j^E must vanish since, according to (3.10), they must contain the factor $Tr(\Pi_j G)$ which clearly vanishes.

Now consider the components of the polarization tensor,

$$\begin{aligned}\Gamma_{\tau\tau}^E(x, x') &= -Tr\mathcal{G}(x, x')\mathcal{G}(x', x) \\ &= -\frac{2}{\beta^2}\sum_{s_1, s_2} e^{-i(\zeta_{s_1} - \zeta_{s_2})(\tau - \tau')} \\ &\quad \cdot < \underline{x}|G(i\zeta_{s_1} + \mu)|\underline{x}' >< \underline{x}'|G(i\zeta_{s_2} + \mu)|\underline{x} >\end{aligned}$$

where we have substituted the expansion (C.1) and taken the trace over the spin degree of freedom. As in Appendix B we take the Fourier transform and remove the delta functions associated with energy and momentum conservation to obtain

$$\tilde{\Gamma}^{E}_{\tau\tau}(\omega_s, \underline{k}) = -\frac{1}{\pi\beta}\sum_{s_1} Tr[e^{-ik\cdot\tilde{\Pi}} G(i\zeta_{s_1} + \mu) e^{ik\cdot\tilde{\Pi}} G(i\zeta_{s_1} - i\omega_s + \mu)] \qquad (C.8)$$

where $\omega_s = 2\pi s/\beta$, $s \in \mathbb{Z}$. The trace refers to the Hilbert space of the oscillator variables P and Q. (This formula could have been obtained directly from the zero temperature expression (B.17) by making the replacements $\omega \to i\omega_s, \omega_1 \to i\zeta_1 + \mu$ and

$$\int \frac{d\omega_1}{2\pi} \to \frac{i}{\beta}\sum_{s_1}\Big)$$

On expressing the trace in (C.8) in terms of oscillator matrix elements we arrive at the sum

$$-\frac{1}{\beta}\sum_{s_1}(\varepsilon_m - i\zeta_{s_1} - \mu)^{-1}(\varepsilon_n - i\zeta_{s_1} - \mu + i\omega_s)^{-1} \qquad (C.9)$$

which can be evaluated by the contour method discussed above. For the case $m = n$ and $s = 0$ the result is given by (C.5). Otherwise, the sum (C.9) reduces to

$$(\varepsilon_n - \varepsilon_m + i\omega_s)^{-1}\left[(e^{\beta(\varepsilon_m-\mu)} + 1)^{-1} - (e^{\beta(\varepsilon_n-\mu)} + 1)^{-1}\right] \qquad (C.10)$$

On substituting into (C.8) we obtain firstly for $s \neq 0$,

$$\tilde{\Gamma}^{E}_{\tau\tau}(\omega_s, \underline{k}) = \frac{1}{\pi}\sum_{n,m}\frac{<n|e^{-ik\cdot\tilde{\Pi}}|m><m|e^{ik\cdot\tilde{\Pi}}|n>}{\varepsilon_n - \varepsilon_m + i\omega_s}\left[\frac{1}{e^{\beta(\varepsilon_m-\mu)}+1} - \frac{1}{e^{\beta(\varepsilon_n-\mu)}+1}\right]$$

and, for $s = 0$,

$$\tilde{\Gamma}^{E}_{\tau\tau}(0, \underline{k}) = \frac{1}{\pi}\sum_{n\neq m}\frac{<n|e^{-ik\tilde{\Pi}}|m><m|e^{ik\tilde{\Pi}}|n>}{\varepsilon_n - \varepsilon_m}\left[\frac{1}{e^{\beta(\varepsilon_m-\mu)}+1} - \frac{1}{e^{\beta(\varepsilon_n-\mu)}+1}\right]$$

$$+\frac{1}{\pi}\sum_{n}<n|e^{-ik\tilde{\Pi}}|n><n|e^{ik\tilde{\Pi}}|n>\frac{\beta}{4}sech^2\frac{\beta}{2}(\varepsilon_n - \mu) \qquad (C.11)$$

Expressions for the other components, $\tilde{\Gamma}^{E}_{j\tau}$ and $\tilde{\Gamma}^{E}_{j\ell}$ are obtained by making the replacements

$$e^{-ik\cdot\tilde{\Pi}} \to \frac{i}{2m}\{e^{-ik\cdot\tilde{\Pi}}, \Pi_j\}$$

$$e^{ik\cdot\tilde{\Pi}} \to \frac{i}{2m}\{e^{ik\cdot\tilde{\Pi}}, \Pi_\ell\} \qquad (C.12)$$

as in the zero temperature case.

The leading terms in the expansion around $\underline{k} = 0$ are obtained by restricting the sum over m to the values $m = n, n \pm 1$, etc. and using the formula (B.15). The results are expressed through the formulae (3.16).

The low temperature approximation is discussed in Sec.3. Here we give some of the background for computations in the high temperature regime $\beta << m\ell^2$. When the temperature is large in comparison with the level spacing it becomes feasible to approximate the sums over discrete energy levels by integrals. For the sums appearing in (3.15) it is possible to evaluate the resulting integrals in terms of elementary functions and so obtain the leading terms in expansions in powers of the small quantity $\beta/m\ell^2$. The coefficients in these expansions will turn out to be functions of $\beta\mu$, where μ is the chemical potential, and we need not assume that this quantity is small.

To begin with the general problem, suppose we are to approximate the sum, $\sum_n f_n$, where the quantities f_n are interpolated by a smooth function,

$$\begin{aligned} f_n &= f(\beta\varepsilon_n) \\ &= \phi\left(\left(n+\frac{1}{2}\right)\alpha\right) , \end{aligned} \tag{C.13}$$

$$\alpha = \frac{\beta}{m\ell^2} . \tag{C.14}$$

Let us further suppose that the function $\phi(z)$ is slowly varing on the scale of α ,

$$\alpha^2\phi''(z) << \phi(z) ,$$

etc. In order to convert the sum into an integral, it is necessary to construct a function $F(z,\alpha)$ such that

$$\phi\left(\left(n+\frac{1}{2}\right)\alpha\right) = \int_{n\alpha}^{(n+1)\alpha} dz F(z,\alpha) \tag{C.15}$$

With such a function we can write

$$\sum_n f_n = \int_0^\infty dz F(z,\alpha) \tag{C.16}$$

It is a simple exercise to find the first few terms in the expansion of $F(z,\alpha)$ in powers of α,

$$F(z,\alpha) = \frac{1}{\alpha}\phi(z) - \frac{\alpha}{24}\phi''(z) + \frac{7}{5760}\alpha^3\phi''''(z) + \ldots \tag{C.17}$$

One can easily verify that this function satisfies (C.15) by expanding in powers of $z - (n+\frac{1}{2})\alpha$. On substituting (C.17) into (C.16) we obtain the formula

$$\sum_n f_n = \frac{1}{\alpha}\int_0^\infty dz\phi(z) + \frac{\alpha}{24}\phi'(0) - \frac{7}{5760}\alpha^3\phi'''(0) + \ldots \tag{C.18}$$

which is valid under the assumptions that the integral converges and that the interpolating function vanishes with sufficient rapidity at $z \to \infty$. These assumptions are justified in the cases to be considered.

Now, to treat the sum (3.20) which defines the chemical potential,

$$\nu = \sum_n (e^{\beta(\varepsilon_n - \mu)} + 1)^{-1}$$

we must choose the interpolating function

$$\phi(z) = (e^{z - \beta\mu} + 1)^{-1}$$

and evaluate the quantities

$$\int_0^\infty dz\phi(z) = \ell n(1 + e^{\beta\mu})$$
$$\phi'(0) = -\frac{1}{4} sech^2 \frac{\beta\mu}{2}$$
$$\phi'''(0) = -\frac{1}{4} sech^2 \frac{\beta\mu}{2} + \frac{3}{8} sech^4 \frac{\beta\mu}{2}$$

which are to be substituted into (C.18). This gives

$$\nu = \frac{1}{\alpha}\left(\beta\mu + \ell n(1 + e^{-\beta\mu})\right) - \frac{\alpha}{96} sech^2 \frac{\beta\mu}{2} + \frac{7\alpha^2}{46080}\left(2\, sech^2 \frac{\beta\mu}{2} - 3\, sech^4 \frac{\beta\mu}{2}\right) + \ldots \qquad (C.19)$$

where $\alpha = \beta/m\ell^2$. If we make the further assumption that $\beta\mu$ is *large*, then (C.19) can be solved by iteration to give

$$\mu = \frac{\nu}{m\ell^2} - \frac{1}{\beta} e^{-\beta\nu/m\ell^2}\left(1 - \frac{1}{24}\left(\frac{\beta}{m\ell^2}\right)^2 + \frac{7}{5760}\left(\frac{\beta}{m\ell^2}\right)^4 + \ldots\right) +$$
$$+ O(e^{-2\beta\nu/m\ell^2})$$

The assumption that $\beta\mu$ is large while $\beta/m\ell^2$ is small requires that ν is large *). We are, in effect, obtaining an expansion in powers of $1/\nu$. On substituting for ℓ^2 from the formula (3.19) we obtain

$$\mu = \pi\frac{n_e}{m} - \frac{1}{\beta} e^{-\beta\pi n_e/m}\left(1 - \frac{1}{24}\left(\frac{\beta\pi n_e}{\nu m}\right)^2 + \ldots\right) + \ldots \qquad (C.20)$$

*) The convergence of the expansion (C.17) necessitates that $\phi(z)$ should vary slowly on the scale of α. This implies $\beta/m\ell^2 << \beta\mu$ which is consistent with $\nu >> 1$.

Turning next to the sums which define the low momentum behaviour of the polarization tensor, consider firstly the quantity

$$a = \frac{\beta}{4\pi\ell^2}\sum_n sech^2\frac{\beta}{2}(\varepsilon_n - \mu) \qquad (C.21)$$

Here we choose

$$\phi(z) = sech^2\frac{z-\beta\mu}{2}$$
$$= -4\frac{d}{dz}(e^{z-\beta\mu}+1)^{-1}$$

To obtain the first correction term we need

$$\phi'(0) = sinh\frac{\beta\mu}{2}sech^3\frac{\beta\mu}{2}$$

On substituting into (C.18) we obtain

$$a = \frac{m}{\pi}\left[\frac{1}{1+e^{-\beta\mu}} + \frac{1}{96}\left(\frac{\beta}{m\ell^2}\right)^2 sinh\frac{\beta\mu}{2}sech^3\frac{\beta\mu}{2} + \ldots\right] \qquad (C.22)$$

Next, for the quantity

$$c = \frac{\nu}{\pi} - \frac{1}{8\pi}\frac{\beta}{m\ell^2}\sum_n (2n+1)\ sech^2\frac{\beta}{2}(\varepsilon_n - \mu)$$
$$= \frac{1}{\pi}\sum_n\left[(e^{\beta(\varepsilon_n-\mu)}+1)^{-1} - \frac{1}{4}\beta\varepsilon_n\ sech^2\frac{\beta}{2}(\varepsilon_n-\mu)\right] \qquad (C.23)$$

choose

$$\phi(z) = \frac{1}{\pi}\left[(e^{z-\beta\mu}+1)^{-1} - \frac{1}{4}z\ sech^2\frac{1}{2}(z-\beta\mu)\right]$$
$$= \frac{1}{\pi}\frac{d}{dz}\left[z(e^{z-\beta\mu}+1)^{-1}\right]$$
$$\phi'(0) = \frac{-1}{2\pi}sech^2\frac{\beta\mu}{2}, \quad \phi'''(0) = \frac{-1}{\pi}sech^2\frac{\beta\mu}{2} + \frac{3}{2\pi}sech^4\frac{\beta\mu}{2}$$

In this case the integral of $\phi(z)$ vanishes and we are left with the correction term,

$$c = \frac{-1}{48\pi}\frac{\beta}{m\ell^2}sech^2\frac{\beta\mu}{2} + \frac{7}{5760\pi}\left(\frac{\beta}{m\ell^2}\right)^3\left(sech^2\frac{\beta\mu}{2} - \frac{3}{2}sech^4\frac{\beta\mu}{2}\right) + \ldots \qquad (C.24)$$

Since, according to the definitions (3.16) and (4.7) we have $b = -m\ell^2 c$, there remains only the quantity

$$d = \frac{1}{\pi m}\sum_n\left[(2n+1)(e^{\beta(\varepsilon_n-\mu)}+1)^{-1} - \frac{1}{16}\frac{\beta}{m\ell^2}(2n+1)^2 sech^2\frac{\beta}{2}(\varepsilon_n-\mu)\right]$$
$$= \frac{2}{\pi m}\frac{m\ell^2}{\beta}\sum_n\left[\beta\varepsilon_n(e^{\beta(\varepsilon_n-\mu)}+1)^{-1} - \frac{1}{8}(\beta\varepsilon_n)^2 sech^2\frac{\beta}{2}(\varepsilon_n-\mu)\right] \qquad (C.25)$$

Here we choose

$$\phi(z) = \frac{2}{\pi m}\frac{m\ell^2}{\beta}\left[z\left(e^{z-\beta\mu}+1\right)^{-1} - \frac{z^2}{8}sech^2\frac{z-\beta\mu}{2}\right]$$
$$= \frac{1}{\pi m}\frac{m\ell^2}{\beta}\frac{d}{dz}\left[z^2(e^{z-\beta\mu}+1)^{-1}\right]$$
$$\phi'(0) = \frac{2}{\pi m}\frac{m\ell^2}{\beta}(1+e^{-\beta\mu})^{-1}, \quad \phi'''(0) = -\frac{3}{\pi m}\frac{m\ell^2}{\beta}sinh\frac{\beta\mu}{2}sech^3\frac{\beta\mu}{2}$$

Again the integral of $\phi(z)$ vanishes and we therefore obtain

$$d = \frac{1}{12\pi m}(1+e^{-\beta\mu})^{-1} + \frac{21}{5760\pi m}\left(\frac{\beta}{m\ell^2}\right)^2 sinh\frac{\beta\mu}{2}sech^3\frac{\beta\mu}{2} + \ldots \qquad (C.26)$$

APPENDIX D

The effective interaction at zero temperature

To gain a better understanding of the dynamics of the two gauge fields, A_μ and a_μ, with their various mixing terms, it is instructive to derive the effective current–current interaction. Part of the result of this computation is exhibited in Sec.2. Here we provide some of the details.

The bilinear part of the effective action is given by

$$\Gamma = \int d^3x \Big[-\frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} - \frac{\nu}{2\pi} \varepsilon^{\mu\lambda\rho} a_\mu \partial_\lambda a_\rho \Big] + \frac{1}{2} \int d^3x\, d^3x'\, \Gamma^{\mu\nu}(x-x')\,(A_\mu(x) + a_\mu(x))(A_\nu(x') + a_\nu(x')) \qquad (D.1)$$

where the Fourier components of $\Gamma_{\mu\nu}$ are expressed in terms of the three invariants, Π_0, Π_1 and Π_2, which are constants in the long wavelength, low frequency approximation. See Eqs.(2.14) and (2.15).

To obtain the effective interaction between a pair of conserved electromagnetic currents we must solve the field equations,

$$\delta\Gamma/\delta A_\mu(x) = -I^\mu(x), \quad \delta\Gamma/\delta a_\mu(x) = 0 \qquad (D.2)$$

and evaluate the amplitudes

$$\begin{aligned} W(I) &= \Gamma(A,a) + \int d^3x\, I^\mu(x)\, A_\mu(x) \\ &= \frac{1}{2} \int d^3x\, I^\mu(x)\, A_\mu(x) \\ &= \frac{1}{2} \int d^3x\, d^3x'\, I^\mu(x)\, D_{\mu\nu}(x-x')\, I^\nu(x') \end{aligned} \qquad (D.3)$$

In Fourier space the field Eqs.(D.2) are

$$\begin{aligned} -\frac{1}{e^2}\, k_\nu (k^\nu \tilde{A}^\mu - k^\mu \tilde{A}^\nu) + \tilde{\Gamma}^{\mu\nu}(k)(\tilde{A}_\nu + \tilde{a}_\nu) &= -\tilde{I}^\mu \\ -\frac{i\nu}{\pi}\, \varepsilon^{\mu\lambda\rho} k_\lambda\, \tilde{a}_\rho + \tilde{\Gamma}^{\mu\nu}(k)(\tilde{A}_\nu + \tilde{a}_\nu) &= 0 \end{aligned} \qquad (D.4)$$

where $k^\mu \tilde{I}_\mu = \omega\, \tilde{I}_0 + k_j \tilde{I}_j = 0$. Since we do not generally have Lorentz invariance (which obtains only for $\Pi_2 = -\Pi_0$) we cannot choose any particularly advantageous frame. However, it is helpful to employ a complex basis, writing

$$\tilde{A}_\pm = \frac{1}{\sqrt{2}} (\tilde{A}_1 \pm i\tilde{A}_2)\,,$$

etc. Since Eq.(D.5) are gauge invariant we can look for the solution with

$$\tilde{A}_0 = \tilde{a}_0 = 0 \ . \qquad (D.5)$$

With this restriction the equations reduce to the form

$$\frac{1}{e^2}\left[(\omega^2 - k_+k_-)\tilde{A}_\pm + k_\pm^2\tilde{A}_\mp\right] + \Gamma_{\pm\mp}(\tilde{A}_\pm + \tilde{a}_\pm) + \Gamma_{\pm\pm}(\tilde{A}_\mp + \tilde{a}_\mp) = -I_\pm$$
$$\mp\frac{\nu}{\pi}\,\omega\, a_\pm + \Gamma_{\pm\mp}(\tilde{A}_\pm + a_\pm) + \Gamma_{\pm\pm}(\tilde{A}_\mp + \tilde{a}_\mp) = 0 \ . \qquad (D.6)$$

On eliminating $a_\pm$ they reduce to

$$\alpha_+\tilde{A}_+ + \beta_+\tilde{A}_- = -j_+$$
$$\beta_-\tilde{A}_+ + \alpha_-\tilde{A}_- = -j_- \qquad (D.7)$$

where

$$j_+ = \left(1 - \frac{\pi}{\nu}\,\Pi_1 - \frac{\pi}{\nu}\,\omega\,\Pi_0\right)\tilde{I}_+ + \frac{\pi}{\nu}\,\Pi_2\,k_+\,\tilde{I}_0$$
$$j_- = \left(1 - \frac{\pi}{\nu}\,\Pi_1 + \frac{\pi}{\nu}\,\omega\,\Pi_0\right)\tilde{I}_- - \frac{\pi}{\nu}\,\Pi_2\,k_-\,\tilde{I}_0 \qquad (D.8)$$

and the coefficients $\alpha_\pm, \beta_\pm$ are given by

$$\alpha_\pm = \mp\frac{\pi}{\nu e^2}\,\omega\left(\omega^2\Pi_0 - \frac{\nu}{\pi}\,e^2\Pi_1 + k_+k_-(\Pi_2 - \Pi_0)\right) +$$
$$+ \frac{\omega^2}{e^2}\left(1 - \frac{\pi}{\nu}\,\Pi_1 + e^2\Pi_0\right) - \frac{k_+k_-}{e^2}\left(1 - \frac{\pi}{\nu}\,\Pi_1 - e^2\Pi_2\right)$$
$$\beta_\pm = \mp\frac{\pi}{\nu e^2}\,k_\pm^2\,\omega(\Pi_0 + \Pi_2) + \frac{k_\pm^2}{e^2}\left(1 - \frac{\pi}{\nu}\,\Pi_1 - e^2\Pi_2\right) . \qquad (D.9)$$

The solution is

$$\tilde{A}_+ = \frac{1}{\omega^2\Delta}\,(\alpha_-j_+ - \beta_+j_-)$$
$$\tilde{A}_- = \frac{1}{\omega^2\Delta}\,(\alpha_+j_- - \beta_-j_+) \qquad (D.10)$$

where the denominator is given by

$$\Delta = -\frac{1}{\omega^2}\,(\alpha_+\alpha_- - \beta_+\beta_-)$$
$$= \left(\frac{\pi}{\nu e^2}\right)^2\left[\left\{(\omega^2 - 2k_+k_-)\Pi_0 - \frac{\nu}{\pi}\,e^2\Pi_1\right\}^2 - \left(\frac{\nu}{\pi}\right)^2(\omega^2 - 2k_+k_-)\left(1 - \frac{\pi}{\nu}\,\Pi_1 + e^2\Pi_0\right)^2\right.$$
$$\left. + 2k_+k_-(\Pi_0 + \Pi_2)\left\{(\omega^2 - 2k_+k_-)\Pi_0 - \left(\frac{\nu}{\pi}\right)^2 e^2(1 + e^2\Pi_0)\right\}\right] . \qquad (D.11)$$

This expression is quadratic in ω^2 which implies that there are two branches to the dispersion relation. Their values at $\underline{k} = 0$ define the "masses" $\mu_\pm^2$, referred to in the text. They are real and positive if $e^2\Pi_0$ is positive,

$$\mu_\pm^2 = \frac{1}{4}\left(\frac{\nu}{\pi}\right)^2\frac{1}{\Pi_0^2}\left[\left(1 - \frac{\pi}{\nu}\,\Pi_1 + e^2\Pi_0\right) \pm \sqrt{\left(1 - \frac{\pi}{\nu}\Pi_1 - e^2\Pi_0\right)^2 + 4e^2\Pi_0}\ \right]^2 \qquad (D.12)$$

The solutions (D.10) are singular at $\omega = 0$ and one might suspect that this is merely a gauge artifact. However, this may not be the case, at least in higher orders. A careful evaluation of the functional (D.3) shows that the singularity may persist and give rise to a long range instantaneous interaction between the currents. The final result is

$$
\begin{aligned}
W(I) = & \\
& = \frac{1}{2}\int \frac{d\omega\, d^2k}{(2\pi)^3}\,\frac{1}{e^2\Delta}\,\Big[\Big\{\Big(1-\frac{\pi}{\nu}\Pi_1\Big)^2 + e^2\Pi_0 - \Big(\frac{\pi}{\nu}\Big)^2\Pi_0^2(\omega^2-\underline{k}^2) \\
& \quad - \frac{1}{2}\Big(\frac{\pi}{\nu}\Big)^2\Pi_0(\Pi_0+\Pi_2)\underline{k}^2\Big\} I_i^* I_i \\
& \quad - \Big\{\Big(1-\frac{\pi}{\nu}\Pi_1\Big)^2 - e^2\Pi_2 + \Big(\frac{\pi}{\nu}\Big)^2\Pi_0\Pi_2(\omega^2-\underline{k}^2)\Big\} I_0^* I_0 \\
& \quad - \Big\{\frac{\pi}{\nu}\Pi_1\Big(1-\frac{\pi}{\nu}\Pi_1\Big) + \Big(\frac{\pi}{\nu}\Big)^2\Pi_0^2(\omega^2-\underline{k}^2) + \\
& \qquad + \Big(\frac{\pi}{\nu}\Big)^2\Pi_0(\Pi_0+\Pi_2)\underline{k}^2\Big\}\frac{\nu e^2}{\pi}\,\frac{i}{\omega}\,\varepsilon_{ij}\, I_i^* I_j\Big] \,. \qquad (D.13)
\end{aligned}
$$

The combination of currents that appears in the last term is in fact Lorentz invariant. On using current conservation it can be expressed in the Chern–Simons form,

$$
\begin{aligned}
-\frac{1}{\omega}\,\varepsilon_{ij}\, I_i^* I_j &= \frac{1}{\underline{k}^2-\omega^2}\,\varepsilon^{\mu\lambda\rho} I_\mu^*\, k_\lambda\, I_\rho \\
&= \frac{1}{\underline{k}^2}\,\varepsilon_{ij}(I_0^* k_i I_j - I_j^* k_i I_0) \,. \qquad (D.14)
\end{aligned}
$$

Indeed, the entire functional (D.13) becomes Lorentz invariant, as it should, when $\Pi_2 = -\Pi_0$. If we substitute the 1–loop value for Π_1 as given in Eq.(2.15) the residue of the pole at $\omega = 0$ vanishes. However, this may not be the case in higher orders.

REFERENCES

1) W. Siegal, Nucl. Phys. **B156** (1979) 135;
J. Schonfeld, Nucl. Phys. **B185** (1981) 157;
S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. **48** (1983) 975; Ann. Phys. **140** (1982) 372.

2) Y. Hosotani, "*On the gauge invariance in the Chern–Simons theory on a torus*", IAS preprint IASSNS–HEP–89/8 (1989);
S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, "*Remarks on the canonical quantization of the Chern-Simons-Witten theory*", IAS preprint, IASSNS–HEP–89/20 (1989).

3) V. Kalmeyer and R.B. Laughlin, Phys. Rev. lett. **59** (1987) 2095;
R.B. Laughlin, *ibid.* **60** (1988) 1057;
I. Dzyaloshinskii, A.M. Polyakov and P. Wiegmann, Phys. Lett. **127A** (1988) 112.

4) Yi-Hong Chen, F. Wilczek, E. Witten and B.I. Halperin, "*On Anyon Superconductivity*", IAS preprint, IASSNS–HEP–89/27 (1989).

5) F. Wilczek and A. Zee, Phys. Rev. Lett. **51** (1983) 2250;
Yong–Shi Wu and Z. Zee, Phys. Lett. **147B** (1984) 325;
D.P. Arovas, R. Schrieffer, F. Wilczek and A. Zee, Nucl. Phys. **B251** (1985) 117.

6) F. Wilczek, Phys. Rev. Lett. **49** (1982) 957.

7) J.M. Leinaas and J. Myrheim, Nuovo Cim. **37B** (1977) 1;
F. Wilczek, Phys. Rev. Lett. **48** (1982) 1144.

8) Y. Hosotani and S. Chakravarty, "*Superconductivity with no Cooper pairs*", IAS preprint, IASSNS–HEP–89/31 (1989).

9) E.M. Lifshitz and L.P. Pitaevskii, *Statistical Physics* (Part 2) (Pergamon Press, Oxford, 1980).

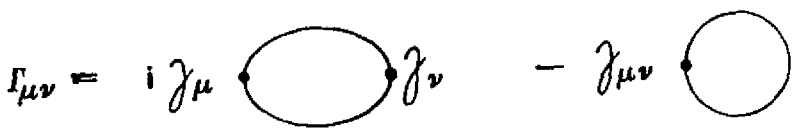

Fig.1 One loop ground state polarization tensor.

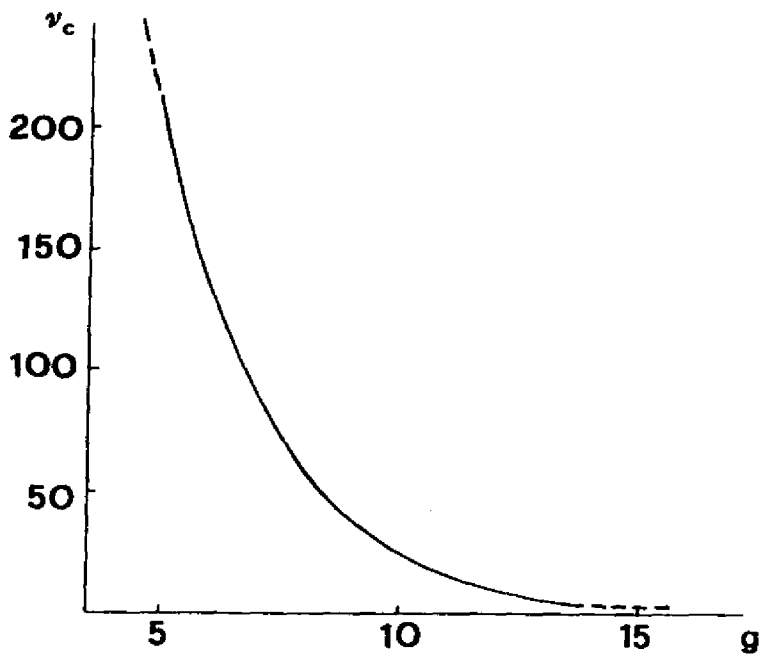

Fig.2 Plot of pseudocritical coupling, $\nu_c(y)$. For $\nu < \nu_c$ the penetration depth is dominated by the thermal contribution. The curve is reliable provided ν and $y = 1/2\, m\ell^2 T$ are not small.

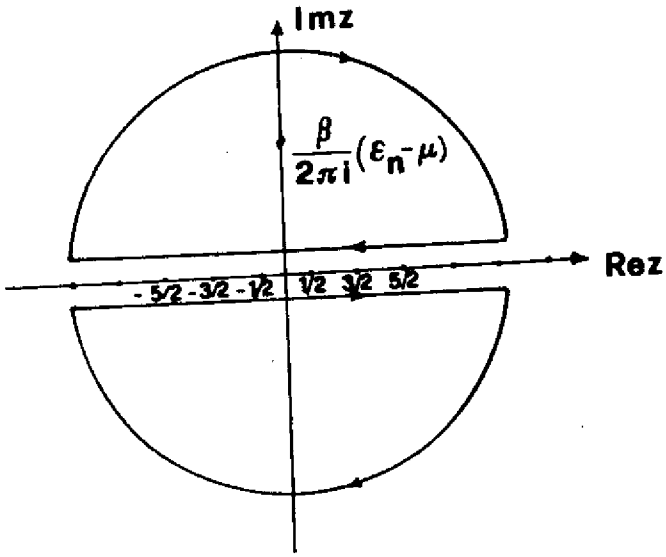

Fig.3 The contour used in evaluating thermal sums. Poles on the real axis are given by the zeroes of cot πz (Appendix C).

Stampato in proprio nella tipografia
del Centro Internazionale di Fisica Teorica