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ABSTRACT

A general method for constructing interesting topological gauge theories in arbitrary dimensions is presented. The basic framework upon which these models are built is given by the geometrical data of the 'universal bundle with connection' of Atiyah and Singer. The models considered include theories which represent the moduli spaces of flat connections and solutions to the Yang–Mills equations. The former theories correspond to supersymmetric versions of the recently introduced BF systems. In all cases we show explicitly that the quantization can be carried out through the construction of an off–shell nilpotent BRST operator, thus guaranteeing the metric independence of these models.

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1 Introduction

The construction of arbitrary dimensional topological quantum field theories has recently received much attention [1,2,3]. At present a useful way of classifying these theories is to distinguish between the Schwarz and Witten classes [4]. In the case of a Schwarz type theory [5], one begins with a metric independent classical action $S_{cl}(\Phi)$, where Φ denotes the collective field content. We stress that by Schwarz type we mean that the classical action is non-trivial, excluding for example total derivative actions. Upon gauge fixing the total quantum action takes the form

$$S_q(\Phi, g) = S_{cl}(\Phi) + \{Q, V(\Phi, g)\} \ ,$$

where g is the metric, Q is the (metric independent) BRST operator, and V is some functional of the fields. The metric independence of Q can easily be seen by writing the BRST transformations in terms of differential forms, without reference to the Hodge duality operator. The distinguishing feature of the Witten type theories [6,7] is that the complete quantum action can be written as a BRST commutator, i.e.

$$S_q = \{Q, V(\Phi, g)\} \ .$$

We should note that the Q is generally not the same in the two cases. For the Schwarz case the Q corresponds to the usual gauge symmetry, albeit, in general, of the reducible type. In the Witten case the Q is obtained by combining a 'topological' shift symmetry with conventional gauge symmetry.

The common feature which both classes share is the fact that the stress energy tensor is a BRST commutator,

$$T_{\alpha\beta} = \{Q, \frac{1}{\sqrt{g}}\frac{\delta V}{\delta g^{\alpha\beta}}\} \ .$$

The topological nature of the theory lies in this fact since, assuming that the vacuum is BRST invariant [1], one can easily show that the partition function is metric independent. One can then go on to construct other

[1] In a recent work [30] it is shown that in the four dimensional topological Yang-Mills

metric independent observables, $\delta_g\langle \mathcal{O}\rangle = 0$, from BRST invariant operators $\{Q,\mathcal{O}\} = 0$ which appear in the theory [6].

Given a Witten type model, the metric independence is guaranteed by the above observations. However, the analysis for the case of Schwarz type theories is not completely straightforward. We noted above that, since the ghost and gauge fixing terms are expressible as a BRST commutator, the stress energy tensor can also be written in this way. To prove that this is the case for a general Schwarz type theory is not entirely obvious. It is certainly true for the three dimensional Chern-Simons theory, (the prototype of a Schwarz type theory), as in this case one is simply quantizing the usual Yang-Mills gauge symmetry [8]. For more complicated systems, such as arbitrary dimensional non-abelian BF systems, it has not yet been shown that these models maintain their metric independence upon quantization. By more complicated systems, we simply mean those which suffer from reducibility, and or closure, problems in their gauge algebra. However, there are examples, such as the abelian BF systems in arbitrary dimensions, and the lower dimensional non-abelian models, for which the complete gauge fixing and ghost terms are indeed expressible as a BRST commutator [1]. In general, when quantizing an (on-shell) reducible system, one uses the Batalin-Vilkovisky procedure [9]. The problem is that, within this procedure, there is certainly no guarantee that the classical and quantum actions differ soley by a BRST commutator. Futhermore, one also finds a quantum action which is invariant with respect to only an on-shell nilpotent BRST operator.

There is a class of models, the so-called super BF models, which can be shown to be topological in nature, in any dimension. At first sight one would be tempted to classify these models as being of the Schwarz type, as the classical action is metric independent. For example, in three dimensions [10,3b] the classical takes the form

$$\int BF_A - \chi d_A \psi \ ,$$

theory the Yang-Mills ghost field c acquires a non-zero vacuum expectation value, and that in general the BRST invariance of the vacuum is broken. This symmetry breaking is achieved in such a way as not to effect the usual metric independence arguments. However, it does allow one to establish the non-triviality of the appropriate observables in the theory.

where A, B and χ, ψ are respectively even and odd 1-forms. The extra structure which this system has over a usual BF model is the presence of supersymmetry relating the superpartners A, B and χ, ψ. The theory serves to model the moduli space of flat connections and, as shown by Witten, has a partition function which is the Casson invariant.

It is easy to see that the above classical action can be written as $\{Q, X\}$, for some functional X and symmetry operator Q. Thus at least at the classical level we know that this theory is of Witten class. However, it is not a priori obvious that this property can be maintained upon quantization. In this paper we shall show that indeed this theory is of Witten class. That is, we will explicitly construct an off-shell nilpotent BRST operator Q, and show that the complete quantum action is expressible as $S_q = \{Q, V\}$. This in turn makes explicit the observation by Witten that this theory is equivalent to the one obtained upon dimensional reduction of the four dimensional topological Yang-Mills theory. This dimensionally reduced Witten type theory corresponds to a 3d Yang-Mills Higgs model and was constructed in [11,12]. As such, on a manifold without boundary, its partition function is indeed the Casson invariant, as shown in section 4.

Obviously, since it is so straightforward to write down arbitrary dimensional topological field theories, one needs some way of establishing whether they will possess interesting topological properties. One way of doing this is to construct field theories based upon a certain set of geometric data, for instance that used by Atiyah and Singer in their study of anomalies [13]. In this way we are guaranteed that the resulting theories will encode valuable topological information. We shall adopt this geometrical framework to construct topological field theories which model the moduli spaces of flat connections and solutions to the Yang-Mills equations, in any dimension. All the theories we consider will be shown to be of Witten class.

To do this, as in the 3d case, we show that it is possible to construct an off-shell nilpotent BRST operator with the quantum action expressed as a BRST commutator. The basis for constructing such a Q is to combine the supersymmetry and standard BRST operator. This is already suggested by the Atiyah-Singer construction, where a natural transformation for the gauge potential, (see (8) below),

$$\delta A = \psi - d_A c \ ,$$

arises. The procedure we adopt here is to be contrasted with that of Horne [14], who keeps the 'topological' supersymmetry ($\delta A = \psi$) and BRST symmetry ($\delta A = -d_A c$) split, in his analysis of the 4d topological Yang-Mills theory.

As a final remark we should note that constructing a topological field theory to model the space of Yang-Mills solutions is not without content. We know that the moduli spaces satisfy the inclusive relations

$$\mathcal{M}_{\mathcal{I}} \subset \mathcal{M}_{\mathcal{YM}} \ , \mathcal{M}_{\mathcal{F}} \subset \mathcal{M}_{\mathcal{YM}} \ ,$$

where $\mathcal{F}$, $\mathcal{I}$ and $\mathcal{YM}$ denote flat, instanton and Yang-Mills connections, respectively. One may wonder then why it is not sufficient to simply model the flat and instanton moduli spaces. The question is whether there is any futher structure beyond this which needs to be explored. In fact some recent work has shown that indeed there are new finite action solutions to the Yang-Mills equations which do not correspond to instantons [15]. As such it is useful to have at hand a topological theory which probes this extra structure.

The plan of this paper is as follows. In the next section we begin by setting up the geometrical framework upon which our models are built. Section 3 contains the basic construction of the models. In Section 4 we go on to discuss the Langevin approach to these theories; this establishes, a priori, that they are of Witten class. Section 5 provides explicit details of the quantization while in section 6 we treat the important issue of which metric independent observables are available. We conclude in section 7 with a discussion of the general renormalization properties of topological field theories, treating in particular the four dimensional theory of self-dual connections and the three dimensional super BF system to one loop order.

2 Geometrical Framework

Before diving into the construction of the models in the next section it will be helpful to gain a geometrical understanding of the fields and their transformations used in the construction of topological field theories. As has been noticed independently by the authors of [11,16,17] the appropriate setting is the 'universal bundle with connection' used by Atiyah and Singer in their discussion of anomalies in quantum field theory [13]. We shall here expand slightly on these previous discussions and show how all the BRST transformations of the fundamental fields involved emerge naturally in this setting, and how interesting topological field theories can be constructed by imposing appropriate geometrical constraints.

Let us start off by considering a principal G-bundle P on a compact smooth manifold M. The affine space $\mathcal{A}$ of all connections on P (modelled on $\Omega^1(M, adP)$) is acted upon by the group $\mathcal{G} = Aut_0 P$ of pointed gauge transformations. G acts freely on the base space Q of the principal $\mathcal{G}$-bundle $(P \times \mathcal{A}, \mathcal{G}, Q = (P \times \mathcal{A})/\mathcal{G}))$ and defines the 'universal' G-bundle

$$\mathcal{Q} = ((P \times \mathcal{A})/\mathcal{G}, G, Q/G = M \times \mathcal{A}/\mathcal{G}) \ , \tag{1}$$

over the base space $M \times \mathcal{A}/\mathcal{G}$. Differential forms on $M \times \mathcal{A}/\mathcal{G}$ carry a natural bigrading (for our conventions regarding these graded algebras see the appendix), a (p,q)-form referring to a p-form on M and a q-form on $\mathcal{A}/\mathcal{G}$. Thus locally the universal connection on Q can be written as

$$\mathcal{A} = A + c \ , \tag{2}$$

where c refers to its (0,1)-part. Likewise the exterior derivative splits naturally into $d + \delta$. With this in mind the universal curvature - a horizontal form on Q - is

$$\mathcal{F} = (d+\delta)\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] \ . \tag{3}$$

Upon expanding $\mathcal{F}$ as

$$\begin{aligned} \mathcal{F} &= \mathcal{F}_{(2,0)} + \mathcal{F}_{(1,1)} + \mathcal{F}_{(0,2)} \\ &= F_A + \psi + \phi \ , \end{aligned} \tag{4}$$

one finds that its (2,0), (1,1), and (0,2)-components are respectively given by

$$
\begin{aligned}
F_A &= dA + \frac{1}{2}[A,A] && (5)\\
\psi &= dc + \delta A + [A,c] && (6)\\
\phi &= \delta c + \frac{1}{2}[c,c] \ . && (7)
\end{aligned}
$$

From this we see that we can identify δ precisely with the fully gauge fixed BRST operator used previously [11,17-21] for the construction of Witten type topological field theories in any dimension, once we rewrite (6) and (7) as

$$
\begin{aligned}
\delta A &= \psi - d_A c && (8)\\
\delta c &= \phi - \frac{1}{2}[c,c] \ . && (9)
\end{aligned}
$$

Since by construction $\delta^2 = 0$, (8) and (9) imply the BRST transformations of ψ and ϕ:

$$
\begin{aligned}
\delta\psi &= -[c,\psi] - d_A\phi && (10)\\
\delta\phi &= -[c,\phi] \ . && (11)
\end{aligned}
$$

Since ψ - as a (1,1)-form on $M \times \mathcal{A}/\mathcal{G}$ - should vanish on tangent vectors $X \in \Omega^1(M, adP)$ tangent to the orbits of $\mathcal{G}$, we impose the condition

$$d_A * \psi = 0 \ . \qquad (12)$$

When implemented as a constraint in the path integral in the usual way (12) has the additional virtue of completing the analogy with the universal bundle $\mathcal{Q}$ by realizing the curvature formula [13]

$$\phi(\tau_1,\tau_2) = (d_A^* d_A)^{-1}(*[\tau_1, *\tau_2]) \ , \qquad (13)$$

(with $\tau_i \in \Omega^1(M, adP)$ satisfying $d_A^*\tau_i = 0$) in the form [6]

$$\langle\phi\rangle = \int_M (d_A^* d_A)^{-1}(\psi_\alpha^0, \psi_\alpha^0) \ , \qquad (14)$$

where ψ^0 is the 'zero-mode part' of ψ.

At this point we have pinned our theory down to the space $\mathcal{A}/\mathcal{G}$ (and can in fact recover precisely the BRST structure of ordinary Yang-Mills theory by setting ϕ and ψ to zero). We are however free to impose further conditons on F_A (and hence on ψ) to reduce our configuration space to interesting moduli spaces which are subvarieties of $\mathcal{A}/\mathcal{G}$ for specific choices of P. Precisely how this can be done in practice will be explained in the following section. Let us just note here that the three conditions promising to be most interesting are those characterized by the constraints

$$
\begin{aligned}
F_A &= 0 && (15)\\
F_A^+ &= 0 && (16)\\
d_A * F_A &= 0 \ , && (17)
\end{aligned}
$$

which will lead to topological field theories describing the moduli spaces $\mathcal{M}_\mathcal{F}, \mathcal{M}_I$ and $\mathcal{M}_{\mathcal{YM}}$ of flat, instanton, and Yang-Mills connections respectively. The BRST transformations of these conditions will be the ψ zero modes, which are thus restricted from (co-)tangents to $\mathcal{A}/\mathcal{G}$ to (co-)tangents to $\mathcal{M}$. Since (16) has already been discussed in great detail in the literature (and is only possible in 4 dimensions), we shall mainly restrict our attention to the other two possibilities.

Finally let us mention that the Bianchi identity for $\mathcal{F}$ implies

$$(d+\delta) tr \mathcal{F}^2 = 0 \ , \qquad (18)$$

which - when expanded out in terms of ghost numbers - gives rise [11,16,17] to the hierarchy of observables discussed in [6].

3 Construction of Topological Actions

As discussed in the introduction, we are interested in constructing topological field theories which serve as models for the moduli spaces of flat connections , ($F_A = 0$), and Yang-Mills connections, ($d_A * F_A = 0$). Before presenting the construction of these theories in arbitrary dimensions it is useful to consider first the special case of flat connections in three dimensions. As shown by Witten [10], the partition function of the resulting theory is the Casson invariant. We begin by postulating an $N = 2$ supersymmetry algebra [2] with transformation rules given by

$$\begin{aligned} \delta A = \psi \ , \delta\psi = 0 \ \ , \ \ \delta\chi = B \ , \delta B = 0 \ , \\ \overline{\delta} A = \chi \ , \overline{\delta}\psi = -B \ \ , \ \ \overline{\delta}\chi = 0 \ , \overline{\delta} B = 0 \ . \end{aligned} \tag{19}$$

Here A is the G-valued gauge connection 1-form, ψ , χ , and B being fermionic and bosonic 1-forms in the adjoint representation of G , with ghost numbers $(0, 1, -1, 0)$, respectively. This set of fields can be neatly combined into a single superfield as

$$\mathcal{A} = A + \eta_1\psi + \eta_2\chi - \eta_1\eta_2 B \ . \tag{20}$$

The superfield $\mathcal{A}$ then serves as the superconnection and it is a form with overall degree 1, since η_1 and η_2 have ghost numbers -1 and 1, respectively. We now see that the supersymmetries (19) are simply

$$\delta = \frac{\partial}{\partial\eta_1} \ , \overline{\delta} = \frac{\partial}{\partial\eta_2} \ , \tag{21}$$

with the property that

$$\delta^2 = \overline{\delta}^2 = \delta\overline{\delta} + \overline{\delta}\delta = 0 \ . \tag{22}$$

It is straightforward to compute the supercurvature and we find

$$\begin{aligned} \mathcal{F} &= d\mathcal{A} + \mathcal{A}^2 \\ &= F_A - \eta_1 d_A\psi - \eta_2 d_A\chi - \eta_1\eta_2(d_A B + [\chi, \psi]) \ . \end{aligned} \tag{23}$$

In three dimensions a natural action to consider is the super Chern-Simons action [10]

$$-\frac{1}{2}\int d\eta_1 d\eta_2(\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}^3) = \int (BF_A - \chi d_A\psi) \ , \tag{24}$$

where a trace over group indices and a wedge product between differential forms will always be understood. At this point we see that the role played by the B field is that of a Lagrange multiplier enforcing the constraint $F_A = 0$ while the superpartners ψ and χ ensure a balance of bosonic and fermionic degrees of freedom. In addition, as we have discussed in the introduction, the ψ field is also part of the geometrical data upon which we are building these models. It is easy to check that the super Chern-Simons action is invariant under both the δ and $\overline{\delta}$ symmetries and in fact we have the result

$$\int d\eta_1 d\eta_2(\mathcal{A}d\mathcal{A} + \frac{2}{3}\mathcal{A}^3) = \overline{\delta}\delta\int (AdA + \frac{2}{3}A^3) \ . \tag{25}$$

Given the above symmetries we can ask the question: What other types of topological actions can we write down? Again a natural object to consider is the super Yang-Mills action given by

$$-\frac{1}{2}\int d\eta_1 d\eta_2 \mathcal{F} * \mathcal{F} = \int Bd_A * F_A + \chi(d_A * d_A\psi - [*F_A, \psi]) \ . \tag{26}$$

It is worth noting that the above action is not of the Schwarz class, as the Hodge star operator appears explicitly. In order to provide a basis for a topological field theory we must therefore show that it is of Witten class, and this means that we must explicitly construct the operator Q which allows us to write the complete quantum action as a BRST commutator. The construction of such a Q will be presented in section 5.

In this case we are modelling the moduli space of Yang-Mills connections , i.e., $d_A * F_A = 0$. The action (26) is again invariant under both supersymmetries (19) and can be written as

$$\int d\eta_1 d\eta_2 \mathcal{F} * \mathcal{F} = \overline{\delta}\delta\int F_A * F_A \ . \tag{27}$$

We see that the classical actions (25) and (27) are both of the form $S_{cl} = \{\overline{Q}Q, X\}$. That is, one can classify them as being of the Witten type. However, one must ensure that this assertion is maintained upon quantization, where the Q operator is extended to the full set of fields and the total quantum action action is expressed as a Q-commutator.

An important point to notice here is that the Yang-Mills action, as given in (26), can be written down in an arbitrary dimension. That is, given the δ and $\overline{\delta}$ transformations together with the superconnection $\mathcal{A}$, eqn. (26)

[2] By $N = 2$ supersymmetry we simply mean that we extend our space to include two fermionic coorinates η_1 , η_2 , with corresponing supersymmetry generators δ and $\overline{\delta}$.

makes sense in any dimension. Thus we can use this action to model the Yang-Mills moduli space in any dimension. This is to be contrasted with the super Chern-Simons action which is specific to three dimensions. While one can certainly consider the higher dimensional analogues of the Chern-Simons form, these contain higher derivatives and as such are generally ignored. However, it is still possible to write down an action which will serve as a model for the moduli space of flat connections in any dimension. Before presenting these actions, we pause to make some general remarks.

Let $S(A)$ be a general classical action and consider the following object

$$\delta\bar{\delta}S(A) = B\frac{\delta S}{\delta A} + \chi\frac{\delta^2 S}{\delta A^2}\psi \ . \tag{28}$$

If we now take $\delta\bar{\delta}S$ as our topological action we see that B enforces the classical equation of motion $\frac{\delta S}{\delta A} = 0$, while the ψ and χ fields enforce the linearized equation of motion $\frac{\delta^2 S}{\delta A^2} = 0$. This is precisely the structure we require of a topological action. That is, if we take eqn. (24) as an example, the B equation of motion restricts us to flat connections, while the χ equation of motion ensures that the ψ's are tangents to the solution space.

Together with the δ and $\bar{\delta}$ symmetries, the above actions also possess the usual super gauge symmetries given by

$$\delta\mathcal{A} = -d_{\mathcal{A}}\mathcal{C} \ . \tag{29}$$

In terms of components we can rewrite (29) as

$$\begin{aligned} \delta A &= -d_A c \\ \delta\psi &= -d_A\phi - [c,\psi] \\ \delta\chi &= -d_A\rho_0 - [c,\chi] \\ \delta B &= -d_A\Sigma_0 + [\psi,\rho_0] + [\phi,\chi] - [c,B] \ . \end{aligned} \tag{30}$$

Here the gauge symmetries have already been written in ghost form via the ghost superfield which has degree 1. It is given explicitly by

$$\mathcal{C} = c + \eta_1\phi + \eta_2\rho_0 - \eta_1\eta_2\Sigma_0 \ . \tag{31}$$

At this point we can, if we wish, proceed with the quantization of the 3d super Chern-Simons theory and arbitrary dimensional super Yang- Mills theory. This is done , in the usual way, by simply adding the corresponding gauge-fixing and ghost terms to the classical actions. Since these terms are added via the Lagrange multiplier and ghost superfields it is clear that the resulting quantum action will be invariant under all three symmetries, i.e., Q_{BRST}, δ and $\bar{\delta}$. The important point here, however, is that in this approach the BRST and supersymmetries are split. This is precisely analogous to the treatment of Horne [14] for the four dimensional Witten theory of self-dual connections. In this case there are two symmetry generators, a supersymmetry Q analogous to (19), and a BRST operator for the usual Yang-Mills gauge fixing. The complete quantum action is then expressed as a Q-commutator. However, our aim is to carry out the quantization by introducing a single BRST operator Q and writing the total quantum action as $S_q = \{Q,V\}$. In doing this we can then classify these theories as being of the Witten type; as such metric independence is guaranteed. A futher advantage of this approach is that the field content is reduced.

For the sake of completeness, however, and to allow us to raise an important issue with regard to the gauge fixing of these theories, we present the full quantum action in the usual superfield approach:

$$\begin{aligned} -\frac{1}{2}\int d\eta_1 d\eta_2(\mathcal{A}d\mathcal{A} &+ \frac{2}{3}\mathcal{A}^3 + \mathcal{U}d_{\mathcal{A}} * \mathcal{A} + \bar{\mathcal{C}}d_{\mathcal{A}} * d_{\mathcal{A}}\mathcal{C}) \\ &= \int BF_A + B * d_A u_0 + \cdots \end{aligned} \tag{32}$$

For definiteness we are considering the super Chern-Simons theory which is being used as a model for the moduli space of flat connections. However, we see immediately from (32) that the B equation of motion no longer enforces the desired constraint $F_A = 0$. Rather, due to the necessity of gauge fixing the theory, we find the constraint $F_A = - * d_A u_0$. Thus it is not a priori clear how the presence of the u_0 field will affect the evaluation of the partition function, which in this case is supposed to yield the Casson invariant. We discuss this issue and its resolution in more detail in section 3.

Our discussion so far has concentrated on the construction of topological actions for flat connections in three dimensions and Yang-Mills connections in arbitrary dimensions. In the construction of these actions we made use of the presence of the $N = 2$ supersymmetry (19). In order to construct Witten type topological field theories for flat connections in more than three dimensions we have to be content though with an $N = 1$ supersymmetry. This is already indicated by the fact that the higher dimensional analogues of the $N = 2$ Super Chern-Simons functional do not give rise to the desired equations of motion $F_A = 0$. To see more clearly the need for abandoning $N = 2$ recall that as in three dimensions we want to arrive at actions of the form $S = \int BF_A + \cdots$. For this to make sense however B needs to be (in n dimensions) an $(n-2)$-form, and while this certainly allows us to keep the first set of supersymmetries (19), the second can not possibly be realized any more since ψ is always a one-form. This should be contrasted with the case of Yang-Mills, where the required action (26) of the form $S = \int Bd_A * F_A + \cdots$ permits B to be a 1-form in any dimension.

We can however construct a supersymmetric extension of the BF systems [1,2] mentioned in the introduction in terms of $N = 1$ superfields incorporating the supersymmetry

$$\begin{aligned} \delta A = \psi \quad &, \quad \delta\chi = B, \\ \delta\psi = 0 \quad &, \quad \delta B = 0 \ . \end{aligned} \tag{33}$$

We introduce two superfields $\mathcal{A}$ and $\mathcal{B}$, with degree 1 and $(n-3)$,respectively,

$$\mathcal{A} = A + \theta\psi \ , \mathcal{B} = \chi + \theta B \ , \tag{34}$$

where χ and B are now $(n-2)$-forms. The action is given by

$$\begin{aligned} S = \int d\theta \mathcal{B}\mathcal{F} &= \int (BF_A + (-)^n \chi d_A \psi) \\ &= \delta \int \chi F_A \ , \end{aligned} \tag{35}$$

which is of the desired form and the obvious extension of (24) to higher dimensions. Just as for the $3d$ super Chern-Simons and arbitrary dimensional super Yang-Mills theories, we can now proceed with the superfield quantization of these systems. The important point to notice here is the on-shell reducibility of the theory. That is, since $\mathcal{B}$ is an $(n-2,-1)$-form the super gauge symmetries are given by

$$\delta\mathcal{A} = -d_{\mathcal{A}}\mathcal{C}, \delta\mathcal{B}_{n-2} = d_{\mathcal{A}}\mathcal{L}_{n-3} \ . \tag{36}$$

Letting $\mathcal{L}_{n-3} = d_{\mathcal{A}}\mathcal{L}_{n-4}$, we see immediately that this corresponds to an on-shell symmetry, until the degree of $\mathcal{L}$ reaches zero. To see that the symmetry is on-shell reducible one simple expands out the supergauge transformations (36). As such the quantization of this reducible system requires several stages of gauge fixing. However, there is no obstruction to carrying out this procedure, using for example the Batalin-Vilkovisky algorithm. As mentioned previously, the resulting action is one in which the BRST and supersymmetry are split. In addition, there is no guarantee ,within the Batalin-Vilkovisky procedure, that the complete gauge fixing and ghost terms can be written as a BRST commutator. This is a problem already encountered in the quantization of the usual BF systems in dimensions greater then four [1]. As a result the metric independence of the theory is not guaranteed. In section 5 we will show, however, that indeed a complete off-shell nilpotent BRST operator Q can be constructed such that the total quantum action can be written as $\{Q, V\}$.

At this point it may be worthwile making a remark on the (frequently to be found) statement that topological gauge theories of the Witten type considered here may be obtained by starting with a vanishing classical action and gauge fixing some 'topological symmetry'. To understand the difficulties one faces by adopting this point of view consider the case of gauge theories in four dimensions. As a starting point one demands invariance under arbitrary shifts of the gauge field A. Then the equation $\delta A = \psi$ is to be read as the corresponding BRST transformation, ψ being the ghost for this 'topological' symmetry.

In order to proceed one picks some 'gauge fixing' condition, usually - so as not to interfere with the Yang-Mills gauge symmetry - chosen to be of the form $\mathcal{G}(F_A) = 0$, where $\mathcal{G}$ is some functional. To implement this gauge choice one introduces an antighost χ and a multiplier field B (with $\delta\chi = B$) and postulates the action

$$\begin{aligned} S &= \delta \int \chi \mathcal{G}(F_A) \\ &= \int B\mathcal{G}(F_A) \pm \chi\delta\mathcal{G}(F_A) \ , \end{aligned} \tag{37}$$

where the relative sign depends on the the theory we are dealing with. A look at the actions (24),(26),(35) reveals that they are precisely of this form, where $\mathcal{G}$ is respectively chosen to be

$$\begin{aligned} \mathcal{G}(F_A) &= F_A^+ \ , & (38)\\ &= F_A \ , & (39)\\ &= d_A * F_A \ . & (40) \end{aligned}$$

This however makes the problem apparent: these various conditions are certainly not equally 'strong', since both (38) and (39) imply (40). This is in sharp constrast with the case of ordinary gauge symmetries, where all acceptable gauge choices are required to be equally strong in the sense that they eliminate precisely one (the longitudinal) mode.

Thus even if the topological symmetry is given, the resulting theory depends crucially (again in marked contrast to more familiar theories) on the choice of 'gauge'.

Moreover none of the conditions (38-40) completely fixes the topological symmetry. Indeed the only conditions satisfying this requirement are of the type $A = A_0$ for some choice of A_0, thus leading to trivial theories. Furthermore variations of A orthogonal to gauge directions leaving $\mathcal{G}(F_A) = 0$ invariant, correspond precisely to tangent vectors of the respective moduli spaces and equivalently to zero modes of ψ.

Thus regarding topological field theories as arising from gauge fixing zero may be misleading since

a) the resulting theory depends crucially on the gauge fixing, and

b) the topological symmetry is not fixed completely (and must not be fixed completely in order to arrive at a non-trivial theory)

4 The Langevin Approach

As stated in the previous section, we would like to carry out the quantization of these theories by introducing a single BRST charge and expressing the total quantum action as $S_q = \{Q, V\}$. A first approach in this direction is to make use of the Langevin formulation of topological field theories , as introduced in [18,22]. Let us, for definiteness, consider the case of flat

connections. We begin by introducing an auxiliary, random Gaussian, 2-form field G and choose as our starting classical action the square of the Langevin equation

$$K_2 = G_2 - F_A - *d_A u_{n-3} = 0 \ , \qquad (41)$$

where u is an $(n-3)$-form, whose significance will be explained shortly. The action is given by

$$S = \int K_2 * K_2 \ . \qquad (42)$$

Now in the Langevin approach to $F_A = 0$ one would naively begin with the equation $G_2 - F_A = 0$. One would then hope that the corresponding action had enough symmetry to allow us to set the G field to zero, thereby recovering the flat connection condition. In order to achieve this , however, we must ensure that we have a complete Hodge decomposition for G. It is for this reason that we have introduced the u field. We note that to ensure covariance the decomposition given in (41) involves the covariant derivative d_A rather than simply d. We should also remark here that we are ignoring the harmonic part of the Hodge decomposition. In other words we have enough symmetry to gauge away all of G modulo a finite dimensional harmonic piece. However, it is the very presence of this ungauged piece that allows us to construct nontrivial topological observables for these theories. Another way to see this is from the fact that the Langevin equation defines a complete Nicolai map for the theory [22] over all but a finite dimensional space corresponding to the kernel of the map, this kernel is nothing other than the moduli space of flat connections. Again we should note that the presence of the u field affects this result and we shall now discuss the significance of this field.

By introducing an antighost field χ and a multiplier field B we can rewrite (42) in the form

$$\begin{aligned} (G_2 - F_A - *d_A u_{n-3}) &* (G_2 - F_A - *d_A u_{n-3}) & (43)\\ &= \{Q, \chi_{n-2}(G_2 - F_A - *d_A u_{n-3} - \frac{\alpha}{2} * B_{n-2})\} \\ &= B_{n-2}(G_2 - F_A - *d_A u_{n-3}) - \frac{\alpha}{2} B_{n-2} * B_{n-2} + \cdots \end{aligned}$$

where we are using the symmetry rule $\{Q, \chi\} = B$ and α is a gauge fixing parameter. If we now choose the $\alpha = 1$ gauge and integrate out the multiplier field we recover the square of the Langevin equation. On the other hand we now see that we can interpret the u field as enforcing the gauge fixing condition on B. So apart from the previous requirement of having a complete Hodge decomposition, we have the additional observation that the u field must be present to perform the necessary gauge fixing on the B field. We should also point out that in three dimensions the Langevin equation given above corresponds to the Yang-Mills Higgs model introduced in [11,12]. There the u field, as a 0-form, is nothing other than the Higgs field.

One may wonder if it is possible to choose the gauge $G_2 = *d_A u_{n-3}$, thereby eliminating the dependence on the Higgs field. However, as shown in [11] for the case of supersymmetric quantum mechanics, one must exercise caution when choosing such a gauge, which is in a sense singular. In the latter case, it is still possible to evaluate the index of the theory in this gauge, although the limits of integration must then be treated carefully. The discussion in [11] can be extended to the present case of flat connections.

In more than three dimensions the u-transformations will, of course, be reducible. This will lead to the, now familiar, ghost for ghost phenomenon. This is to be contrasted with the Langevin formulation of Witten's 4d Yang-Mills theory which has the structure [18]

$$S = \int (G_2^+ - F_A^+)^2 \ . \tag{44}$$

Since this is an algebraic equation we do not need to introduce a u field; in other words, the self-duality constraint on the fields is sufficient to guarantee enough symmetry to gauge G to zero. Also, for the case of super Yang-Mills in any dimension, the Langevin equation has the form

$$S = \int (G_1 - d_A * F_A - *d_A u_0)^2 \ . \tag{45}$$

Here u is a zero form in any dimension; this should come as no surprise in view of (26), where we found that the required multiplier field B was always a 1-form, and as such required only one stage of gauge fixing.

Having discussed the formal structure of the Langevin equations for the models of interest we can now proceed with the quantization. Again, as an

example, let us consider the case of flat connections in arbitrary dimensions. From the action (42) we can determine the symmetries (already written in ghost form) as

$$\begin{aligned}
\delta A &= \psi - d_A c \\
\delta u_{n-3} &= -[c, u_{n-3}] + \sigma_{n-3} + d_A \sigma_{n-4} + [F_A, \sigma_{n-5}] \\
&+ [F_A, d_A \sigma_{n-6}] + [F_A, [F_A, \sigma_{n-7}]] + \cdots \\
\delta G_2 &= -[c, G_2] - *d_A \sigma_{n-3} - d_A \psi + *[\psi, u_{n-3}] \\
&- *([F_A, \sigma_{n-4}] + [F_A, d_A \sigma_{n-5}] + [F_A, [F_A, \sigma_{n-6}]] + \cdots) \ .
\end{aligned} \tag{46}$$

At first sight these transformation rules seem rather complicated. This is due to the fact that the reducibility in the u field has been built in from the start. The point to notice here is the presence of zero modes in the gauge algebra. That is, if we let

$$\begin{aligned}
\psi &= d_A c \\
\sigma_{n-3} &= [c, u_{n-3}] - d_A \sigma_{n-4} - [F_A, \sigma_{n-5}] - [F_A, d_A \sigma_{n-6}] + \cdots
\end{aligned} \tag{47}$$

we find that this corresponds to an on-shell zero mode, upon using the G equation of motion (41). For the case of $n > 3$ we also see that we have reducibility problems in the u field, due to the fact that it is an $(n-3)$-form. The ensuing zero modes can easily be read off from (47) by taking $c = 0$ and solving for σ_{n-4} giving

$$\sigma_{n-4} = -d_A \sigma_{n-5} - [F_A, \sigma_{n-6}] + \cdots \tag{48}$$

and so on. A convenient way to carry out the quantization, in these cases, is to use the Batalin-Vilkovisky procedure. However, the resulting quantum action would then necessarily have the property of being BRST invariant with respect to a charge Q_{BRST} which is only nilpotent on-shell, using the quantum G equation of motion.

As we shall show in the following section we can, in actual fact, completely gauge fix the super BF models in any dimension with an *off-shell* nilpotent BRST charge. This situation differs sharply from that encountered in the quantization of the ordinary (in the sense of non-supersymmetric) BF systems. In the latter case the construction of an off-shell nilpotent

BRST charge is, in general, not possible [1]. Futhermore, it is neither to be expected, nor true, that the classical and quantum actions differ solely by a BRST-commutator in more than three dimensions. The crucial difference between the BF and super-BF systems is the greater flexibility one has due to the presence of superpartners for the fields in the theory.

At this point we have to address the question, of whether the gauge fixed action of the Super BF system really describes flat connections (as we originally set out for) despite the fact that the multiplier field B enforces (as we have seen at several places above) the constraint

$$F_A = - * d_A u_{n-3} \quad . \tag{49}$$

This issue is particularly crucial in three dimensions for the following two reasons: On the one hand Witten [10] - using formal path integral arguments with the non-gauge-fixed action - has argued that the partition function of the Super BF system in three dimensions yields the Casson invariant, which is a topological invariant of homology three spheres that can be computed by counting flat connections with appropriate signs.

On the other hand in three dimensions (49) is precisely what is known as the Bogomoln'y equation describing monopoles, which in [11,12] was used as a starting point for constructing topological Yang-Mills Higgs theories.

We might therefore suspect that not only flat connections but also monopole solutions may make significant contributions to the partition function. The answer to the above question depends crucially on the boundary conditions imposed on u. Indeed it is easy to see (as we shall show below) that on a compact closed manifold (49) necessarily implies $F_A = 0$, thus supporting the claim that under those circumstances we do indeed get (in three dimensions) the Casson invariant. It may be instructive however to first see heuristically why there are no non-trivial solutions to (49) on the three-sphere S^3 as opposed to R^3 where monopoles are of course known to exist.

Recall that on R^3 monopoles may be characterized according to the behaviour of the Higgs field u on the sphere S^2 'at infinity'. More precisely they may be classified according to their monopole charge(s) labelling the second homotopy group of a homogeneous space G/H of the gauge group

G. Moreover S^3 may be obtained from R^3 by identifying the sphere at infinity to a point. Therefore the only configurations on R^3 which give rise to configurations on S^3 are those which are constant on the sphere S^2, hence have monopole number zero and correspond to flat connections.

More formally the argument is the following: Let M_n be an orientable closed compact manifold. By the Bianchi identity (49) implies

$$d_A * d_A u_{n-3} = 0 \quad . \tag{50}$$

Multiplying by u and integrating by parts gives

$$\int_M d_A u * d_A u = 0 \quad , \tag{51}$$

which implies $d_A u = 0$ and therefore $F_A = 0$. Note that on a manifold with boundary the boundary term one would have picked up is the analogue of the R^3 (anti)-monopole charge:

$$\int_{\partial M} u * d_A u = - \int_{\partial M} F_A u = - \int_M F_A d_A u \quad . \tag{52}$$

It is worth noting that the anti-BRST symmetry in the Yang-Mills Higgs system of [11] is only present when the Higgs field changes sign. That is, the monopole charge becomes the anti-monopole charge.

5 Quantization

As a warm-up exercise and a first step towards quantization of the super-BF systems in any dimension let us derive the fully gauge-fixed action of the original three-dimensional model (24). The method we use in this section could be called the inductive method, since it is possible (although we shall not proceed in that way here) to arrive at the final BRST-transformations by starting with the transformations for the highest ghost number/zero form degree ghosts and demanding nilpotency at every level.

The BRST-transformations of the (A, ψ, ϕ, c)-system derived in section 1 (8-11) already display the desired feature of combining the 'topological' supersymmetry (19) and the gauge symmetry into a single nilpotent BRST-operator $\delta = \{Q,\ \}$. Our goal will be to extend these transformations to the other fields in the same way. We will then see that this allows us to indeed write the whole quantum action as a BRST-commutator, with all its pleasant consequences.

Writing down the required set of fields is (almost) straightforward. Apart from A, ψ, ϕ and c we need an antighost $\bar{c}$ and a multiplier b for the gauge condition on A, and likewise (due to the reducibility of $\delta A = \psi - d_A c$) a Grassmann even antighost $\overline{\phi}$ and its multiplier η. χ and B as one-forms contribute one ghost/antighost pair each, and these are denoted by $\rho_0, \overline{\rho}_0$ and $\Sigma_0, \overline{\Sigma}_0$, respectively. In addition we need multiplier fields σ_0 and Π_0 to enforce the gauge fixing conditions on χ and B.

In summary we now have arrived at the following set of fields with their form-rank and ghost-numbers as indicated:

A	ψ	ϕ	c	$\bar{c}$	b	$\overline{\phi}$	η
(1,0)	(1,1)	(0,2)	(0,1)	(0,−1)	(0,0)	(0,−2)	(0,−1)

χ	B	ρ_0	Σ_0	$\overline{\rho}_0$	$\overline{\Sigma}_0$	σ_0	Π_0
(1,−1)	(1,0)	(0,0)	(0,1)	(0,0)	(0,−1)	(0,1)	(0,0)

As can be checked immediately the following set of BRST transformations is then *off-shell* nilpotent on all the fields:

$$\begin{aligned}
\delta A &= \psi - d_A c \\
\delta\psi &= -[c,\psi] - d_A\phi \\
\delta\phi &= -[c,\phi] \\
\delta c &= -\frac{1}{2}[c,c] + \phi \\
\delta\bar{c} &= b \\
\delta b &= 0 \\
\delta\overline{\phi} &= \eta \\
\delta\eta &= 0
\end{aligned}$$

$$\begin{aligned}
\delta\chi &= -[c,\chi] - d_A\rho_0 + B \\
\delta B &= -[c,B] - d_A\Sigma_0 + [\phi,\chi] + [\psi,\rho_0] \\
\delta\rho_0 &= -[c,\rho_0] + \Sigma_0 \\
\delta\Sigma_0 &= -[c,\Sigma_0] + [\phi,\rho_0] \\
\delta\overline{\rho}_0 &= -[c,\overline{\rho}_0] + \sigma_0 \\
\delta\overline{\Sigma}_0 &= -[c,\overline{\Sigma}_0] + \Pi_0 \\
\delta\sigma_0 &= -[c,\sigma_0] + [\phi,\overline{\rho}_0] \\
\delta\Pi_0 &= -[c,\Pi_0] + [\phi,\overline{\Sigma}_0]\ .
\end{aligned} \tag{53}$$

Only two kinds of terms in the above transformations may require some explanation: Those of the form $\delta X = [\phi, X] + \cdots$ are required for nilpotency of the $[c,\ ]$-gauge rotations because of the shift by ϕ in δc. And the additional ψ-term in δB is there to compensate the A-variation in $d_A\Sigma_0$.

Choosing the action to be

$$S_q = \{Q, \int \chi F_A + \overline{\rho}_0 d_A * \chi + \overline{\Sigma}_0 d_A * B + \overline{\phi} d_A * \psi + \bar{c} d * A\}\ , \tag{54}$$

we find that indeed all the invariances of the action are completely fixed, the action being (some terms having cancelled)

$$\begin{aligned}
S_q = \int\ & (\ BF_A - \chi d_A\psi) + \\
& (\ \sigma_0 d_A * \chi - \overline{\rho}_0 d_A * B + \Pi_0 d_A * B + bd * A + (\eta - [\overline{\phi}, c]) d_A * \psi) + \\
& (\ \overline{\rho}_0 d_A * d_A\rho_0 - \overline{\Sigma}_0 d_A * d_A\Sigma_0 + \overline{\phi} d_A * d_A\phi - \bar{c} d * d_A c) + \\
& (\ \bar{c} d * \psi + \overline{\rho}_0[\psi, *\chi] + \overline{\phi}[\psi, *\psi] \\
& +\ \overline{\Sigma}_0(-[\psi, *B] + d_A * [\phi,\chi] + d_A * [\psi,\rho_0])\ .
\end{aligned} \tag{55}$$

Here we have indicated the split of the action into its classical part, the gauge fixing terms, the ghost kinetic and interaction terms. Note that again - as expected - integration over B enforces the constraint $F_A = -* d_A(\overline{\rho}_0 - \Pi_0)$ discussed in the previous section. As such $(\overline{\rho}_0 - \Pi_0)$ is the Higgs field we had previously called u or u_0. We have thus arrived at our goal of constructing an off-shell nilpotent BRST-operator Q such that the full quantum action can be written as a BRST-commutator, by combining the super- and gauge-symmetries. Incidentally we have on the way also achieved this for the super Yang-Mills system, since the non-classical part of the above action coincides precisely with the one of the super Yang-Mills system in any dimension, as B and χ will always be one-forms in this case.

Note that the terms $\overline{\phi}(d_A * d_A\phi + [\psi, *\psi])$ originating from the gauge fixing for ψ will - as in [6] - lead to the relation (14), thus completing the identification of ϕ with the $(0,2)$-component of the universal curvature of [13].

For the sake of completeness we note that the second supersymmetry (19) present in the three-dimensional super-BF system can be extended to an off-shell nilpotent anti-BRST operator $\overline{\delta}$ anticommuting with δ and d. In fact it turns out that $\overline{\delta}$ is uniquely determined by these requirements once one postulates $\overline{\delta}A = \chi - d_A\overline{c}$ (expressing the $\psi \leftrightarrow \chi$ symmetry), augmented by $\overline{\delta}\overline{c} = -1/2[\overline{c},\overline{c}]$. The resulting transformations are then

$$
\begin{aligned}
\overline{\delta}A &= \chi - d_A\overline{c} \\
\overline{\delta}\psi &= -[\overline{c},\psi] - B + d_A\rho_0 \\
\overline{\delta}\phi &= -[\overline{c},\phi] \\
\overline{\delta}c &= -[\overline{c},c] - b \\
\overline{\delta}b &= -[\overline{c},b] \\
\overline{\delta}\overline{c} &= -\frac{1}{2}[\overline{c},\overline{c}] \\
\overline{\delta}\,\overline{\phi} &= -[\overline{c},\overline{\phi}] \\
\overline{\delta}\eta &= -[\overline{c},\eta] - [\overline{\phi},b] \\
\overline{\delta}\chi &= -[\overline{c},\chi] \\
\overline{\delta}B &= -[\overline{c},B] + [\chi,\rho_0] \\
\overline{\delta}\rho_0 &= -[\overline{c},\rho_0] \\
\overline{\delta}\Sigma_0 &= -[\overline{c},\Sigma_0] \\
\overline{\delta}\overline{\rho}_0 &= -[\overline{c},\overline{\rho}_0] \\
\overline{\delta}\,\overline{\Sigma}_0 &= -[\overline{c},\overline{\Sigma}_0] \\
\overline{\delta}\sigma_0 &= -[\overline{c},\sigma_0] \\
\overline{\delta}\Pi_0 &= -[\overline{c},\Pi_0] \ .
\end{aligned}
\tag{56}
$$

After having discussed this three-dimensional example the extension to

higher dimensions turns out to be, somewhat surprisingly, fairly straightforward. Normally one would have expected considerable complications arising from the fact that in more than three dimensions the ghosts and ghosts for ghosts for χ and B will have their own (on-shell) gauge invariances. Looking at the B-transformation (53) it is obvious that they will give rise to a term proportional to F_A in addition to those already present involving ψ and ϕ. Mainly as a consequence of the Bianchi identity $(d_{A+c} + \delta)\mathcal{F} = 0$ for the universal curvature $\mathcal{F}$ (3) this is however basically the only modification required to obtain an off-shell nilpotent operator and a quantum action of the form $S_q = \{Q, \ \}$ in any dimension.

While the structure of the transformations in higher dimensions is similar to (53), there is the necessity of introducing extraghosts and their corresponding antighosts and Lagrange multipliers. In other words, one must ensure that the full field content as specified by, for example, the Batalin-Vilkovisky triangles is represented. In order to see this procedure at work, it is instructive to consider a specific example in detail, namely, the five dimensional super BF system.

In five dimensions χ and B are 3-forms with ghost numbers -1 and 0, respectively. Let us denote by ρ_i and Σ_i $(i = 0,1,2,3; \rho_3 = \chi, \Sigma_3 = B)$ collectively χ and B and their hierarchy of ghosts, the subscript 'i' always indicating the form rank of the field. The ghost triangle for the Σ system takes the form

$$
\begin{tikzcd}[column sep=small]
 & & & \Sigma_3^{(0)} \arrow[dl] & & & \\
 & & \overline{\Sigma}_2^{(-1)} \arrow[dl] & & \Sigma_2^{(1)} \arrow[dl] & & \\
 & \Sigma_1^{\prime(0)} \arrow[dl] & & \overline{\Sigma}_1^{(-2)} \arrow[dl] & & \Sigma_1^{(2)} \arrow[dl] & \\
\overline{\Sigma}_0^{\prime\prime(-1)} & & \Sigma_0^{\prime(-1)} & & \overline{\Sigma}_0^{(-3)} & & \Sigma_0^{(3)}
\end{tikzcd}
$$

The structure of this triangle and its field content deserve some explanation. The superscripts in brackets indicate the ghost numbers of the various fields. The horizontal lines contain all the ghosts which first arise at each stage of reducibility of the system. The right hand ledge (i.e. the diagonal connecting Σ_3 to Σ_0) contains the original gauge field Σ_3 together with its ghosts and ghosts for ghosts. The next ledge connecting $\overline{\Sigma}_2$ to $\overline{\Sigma}_0$ contains the antighosts for the ghosts on the right hand ledge. To each arrow connecting the two ledges there corresponds a gauge fixing condition. The fields on the third ledge are called extraghosts, they are simply the antighosts for the second ledge, and so on. Thus in total we need six gauge fixing conditions. To enforce these conditions we must introduce six multiplier fields which we denote by $\Pi_i (i = 0, 1, 2)$, $\Pi'_i (i = 0, 1)$ and Π''_0, with ghost numbers $(-2, -1, 0, 0, 1, 0)$, respectively. Since ρ_3 is also a 3-form the structure of the ρ triangle is identical to the above, except for some obvious modifications with respect to the ghost numbers of the various fields. Having obtained the desired field content we must now specify the BRST transformations. For the Σ system we have

$$\begin{aligned}
\delta\Sigma_i &= -[c, \Sigma_i] - d_A\Sigma_{i-1} + [\phi, \rho_i] + [\psi, \rho_{i-1}] + [F_A, \rho_{i-2}] \\
\delta\overline{\Sigma}_i &= -[c, \overline{\Sigma}_i] + \Pi_i \\
\delta\Pi_i &= -[c, \Pi_i] + [\phi, \overline{\Sigma}_i] \\
\delta\Sigma'_i &= -[c, \Sigma'_i] + \Pi'_i \\
\delta\Pi'_i &= -[c, \Pi'_i] + [\phi, \Sigma'_i] \\
\delta\overline{\Sigma}''_i &= -[c, \overline{\Sigma}''_i] + \Pi''_i \\
\delta\Pi''_i &= -[c, \Pi''_i] + [\phi, \overline{\Sigma}''_i] \ .
\end{aligned} \tag{57}$$

The transformation rules for the ρ system are similar to those in (57), except for the fields on the right hand ledge:

$$\delta\rho_i = -[c, \rho_i] - d_A\rho_{i-1} + \Sigma_i \ . \tag{58}$$

We must now supplement (58) with the transformations of the the remaining fields on the ρ triangle, as well as those of the multipliers. These are obtained by replacing (Σ, Π) by (ρ, σ) in (57). In addition, of course, we have the unaltered transformations of the other fields in (53). It is straightforward to verify that the above rules define an off-shell nilpotent BRST operator. The fully gauge fixed quantum action can be chosen to be

$$S_q = \{Q, \int \chi F_A + \overline{\phi} d_A * \psi + \overline{c} d * A$$

$$\begin{aligned}
&+ \sum_{i=0}^{2} \overline{\Sigma}_i d_A * \Sigma_{i+1} + \sum_{i=0}^{1} \Sigma'_i d_A * \overline{\Sigma}_{i+1} + \overline{\Sigma}''_0 d_A * \Sigma'_1 \\
&+ \sum_{i=0}^{2} \overline{\rho}_i d_A * \rho_{i+1} + \sum_{i=0}^{1} \rho'_i d_A * \overline{\rho}_{i+1} + \overline{\rho}''_0 d_A * \rho'_{i+1}\} \ .
\end{aligned} \tag{59}$$

Here the second and third lines correspond to the relevant gauge fixing terms for the Σ and ρ systems, respectively.

Having presented the five dimensional example in some detail, the generalization to arbitrary dimensional systems is now straightforward. The only extra structure which arises is an expansion of the Σ and ρ ghost triangles. In n dimensions χ and B are $(n-2)$ forms, and we denote by ρ_i and $\Sigma_i, (i = 0, ..., n-2; \rho_{n-2} = \chi, \Sigma_{n-2} = B)$ collectively χ and B and their hierarchy of ghosts. Their corresponding ghost numbers are $\rho_i : (n-3-i)$ and $\Sigma_i : (n-2-i)$. Let us again illustrate the procedure with a study of the Σ ghost triangle. It has the following structure

$$\begin{array}{ccccccccc}
 & & & & & & \Sigma^0_{n-2} & & \\
 & & & & & \swarrow & & & \\
 & & & & \Sigma^1_{n-3} & & \Sigma^0_{n-3} & & \\
 & & & \swarrow & & \swarrow & & & \\
 & & \Sigma^2_{n-4} & & \Sigma^1_{n-4} & & \Sigma^0_{n-4} & & \\
 & \swarrow & & \swarrow & & \swarrow & & & \\
\Sigma^3_{n-5} & & \Sigma^2_{n-6} & & \Sigma^1_{n-6} & & \Sigma^0_{n-5} & & \\
\swarrow & & \swarrow & & \swarrow & & \swarrow & & \\
\cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & & \vdots & & & & \\
\swarrow & & \swarrow & & & & & & \\
\Sigma^{n-2}_0 & & \Sigma^{n-1}_0 & \cdots\cdots & \cdots\cdots & \cdots\cdots & \Sigma^1_0 & & \Sigma^0_0
\end{array}$$

In order to simplify the notation we have introduced the collective label $\Sigma^j_i, (i, j = 0, 1, ..., n-2)$ to denote all the fields in the triangle. The lower index indicates the form rank of the field while the upper index labels the various NW-SE diagonal ledges. Given the jth ledge, the corresponding range of i is given by $i = 0, 1, ..., n-2-j$. Similarly, the ρ triangle can be expressed via the symbol ρ_i^j. Once the ghost numbers of both zeroth ledges are given, the ghost numbers of the remaining ledges and multiplier fields are fixed. The BRST transformations of both triangles can now be written compactly in the following way: For $j = 0$ we have

$$\begin{aligned} \delta\Sigma_i^0 &= -[c, \Sigma_i^0] - d_A\Sigma_{i-1}^0 + [\phi, \rho_i^0] + [\psi, \rho_{i-1}^0] + [F_A, \rho_{i-2}^0] \\ \delta\rho_i^0 &= -[c, \rho_i^0] - d_A\rho_{i-1}^0 + \Sigma_i^0 \ , \end{aligned} \tag{60}$$

while for $j = 1, ..., n-2$ we have

$$\begin{aligned} \delta\Sigma_i^j &= -[c, \Sigma_i^j] + \Pi_i^j \\ \delta\Pi_i^j &= -[c, \Pi_i^j] + [\phi, \Sigma_i^j] \\ \delta\rho_i^j &= -[c, \rho_i^j] + \sigma_i^j \\ \delta\sigma_i^j &= -[c, \sigma_i^j] + [\phi, \rho_i^j] \ . \end{aligned} \tag{61}$$

Note the the multiplier triangles Π_i^j and σ_i^j exist only for $j = 1, ..., n-2$. Again the off-shell nilpotency of the above transformations is easily verified. All that remains is to write down the complete quantum action:

$$\begin{aligned} S_q &= \{Q, \int \chi F_A + \overline{\phi} d_A * \psi + \overline{c} d * A \\ &+ \sum_{j=1}^{n-2}\sum_{i=0}^{n-2-j} (\Sigma_i^j d_A * \Sigma_{i+1}^{j-1} + \rho_i^j d_A * \rho_{i+1}^{j-1})\} \ . \end{aligned} \tag{62}$$

6 Observables

We now turn to a brief discussion of observables in the models we have introduced in the previous sections. Let us start by recalling the consequences of the the main requirement imposed on expectation values of observables (beyond gauge invariance) in topological field theories: their metric independence [6]. The condition

$$\delta_{metric}\langle \mathcal{O} \rangle = 0 \ , \tag{63}$$

is (recalling that the whole quantum action is a BRST commutator in our approach) obviously satisfied if the vacuum is BRST invariant and

$$\{Q, \mathcal{O}\} = 0 \ , \delta_{metric}\mathcal{O} = 0 \ , \tag{64}$$

(the latter condition could be relaxed slightly). Since $\langle\{Q, X\}\rangle$ is zero for any X we learn that we are interested in Q-cohomology classes of metric independent functionals of the fields. A first obvious candidate that comes to mind is the partition function $Z = \langle \mathbf{1} \rangle$ itself, and this brings us directly to yet another requirement a functional has to satisfy if it is to qualify as an observable:

As is well known Z is non-zero only if there are no fermionic (or here: odd ghost number) zero modes. Assuming for the moment that the number of ψ-zero modes (which are (co-)tangents to moduli space) is equal to the dimension m of the relevant moduli space $\mathcal{M}$ this implies in particular that the partition function is zero whenever $m = dim(\mathcal{M}) > 0$. The requirement of 'soaking up the ψ-zero modes' then translates more transparently into the geometric statement that only volume forms on $\mathcal{M}$ can be integrated over $\mathcal{M}$.

More generally there can be obstructions to integrating the infinitesimal deformations of $\mathcal{M}$ described the ψ's, and under favourable circumstances these will be determined by an index theorem, as happens e.g. in Witten's four-dimensional model [6], where the dimension of $\mathcal{M}$ turns out to be determined by the index of the operator acting on ψ.

A little thought shows though that this will not be the case in general. To see this most clearly let us look at our favourite model, the three dimensional super BF system (55). The part of the action relevant for our purposes is

$$I = \int \chi d_A\psi + \eta d_A * \psi + \sigma d_A * \chi \ , \tag{65}$$

and - denoting by

$$T_{BF} : \Omega^1 \oplus \Omega^0 \to \Omega^1 \oplus \Omega^0 \tag{66}$$

the operator mapping (ψ, σ) to $(*d_A\psi + d_A\sigma, *d_A * \psi)$ - we can write I as

$$I = \langle(\chi, \eta), T_{BF}(\psi, \sigma)\rangle \ , \tag{67}$$

where $\langle , \rangle$ denotes the usual scalar product on elements of $\Omega^i = \Omega^i(M, adP)$, this is defined in the appendix. Since T is clearly self-adjoint however, its index is zero. In particular there are always an equal number of ψ and χ (and η and σ) zero modes. Thus the net ghost number violation - which correctly determined $dim(\mathcal{M})$ in [6] - will always be zero, whereas of course the dimension of the moduli space of flat connections may readily be non-zero. But despite the fact that our models differ in that respect from [6] we can nevertheless define observables and topological invariants along those lines by the following procedure:

Assume that there are M ψ and χ zero modes $\psi_1, \ldots, \psi_M$ and $\chi_1, \ldots, \chi_M$, N η and σ zero modes $\eta_1, \ldots, \eta_N$ and $\sigma_1, \ldots, \sigma_N$, and that we have in some way - by counting representations of the fundamental group, say - determined the dimension of moduli space to be $m < M$. The zero mode measure

$$\int d\psi_1 \ldots d\sigma_N$$

has ghost number zero, but in order to reduce it to an integral over $\mathcal{M}$ we have to saturate all the χ, η, σ zero modes and $M - m$ of the ψ_i's by inserting the factor

$$U = \psi_{m+1} \ldots \psi_M \chi_1 \ldots \chi_M \eta_1 \ldots \eta_N \sigma_1 \ldots \sigma_N \quad , \tag{68}$$

into the path integral. U has ghost number $(-m)$, which means that we still have to insert an operator $\mathcal{O}$ with ghost number $m = dim(\mathcal{M})$. We thus recover the prescription given in [6]. This argument also shows why it was just the *net*-ghost number violation which was relevant for the analysis. It is therefore this operator $\mathcal{O}$ which has to satisfy $\delta\mathcal{O} = \delta_{metric}\mathcal{O} = 0$, and not the composite object $\mathcal{O}U$, *since* $\int d\psi_1 \ldots \psi_m$ *is the correct invariant* measure. The fact that the observable $\langle \mathcal{O}U \rangle$ is metric independent, even though U involves the metric follows from the fact that since the zero modes never appear in the action, when one contracts U with its associated measure one gets the result 1, see also [30].

The same reasoning as above applies - mutatis mutandis - to the other models described in this paper. And in particular the super Yang-Mills system (26) (in any dimension) has the feature in common with the three dimensional super BF system, that the relevant (Jacobi) operator

$$T_{YM} : (\psi, \eta) \rightarrow (*d_A * d_A\psi - *[*F_A, \psi] + d_A\eta, *d_A * \psi) \tag{69}$$

is self-adjoint. The dimension of the kernel of T_{YM} is what Atiyah and Bott [23] call the nullity of A. This quantity - being essentially unstable under variations of A - is not directly related to the dimension of moduli space, which will moreover in general consist of several connected components corresponding to self-dual and non-self-dual solutions to the Yang-Mills equations. Indeed, as shown in [24], if A is a self-dual connection and $A + a$ is a small (not necessarily infinitesimal) deformation satisfying the Yang-Mills equations, then $A + a$ is also self-dual. Similar statements hold for flat connections A.

To construct a set of observables $\mathcal{O}$ satisfying $\delta\mathcal{O} = \delta_{metric}\mathcal{O} = 0$ we recall the Bianchi identity $(d_{A+c} + \delta)\mathcal{F} = 0$ from section 1. (That the hierarchy of observables of [6] representing the Donaldson polynomials could be derived in this way has been observed in [11,16,17].) Expanding the equation

$$(d + \delta) tr \mathcal{F}^k = 0 \tag{70}$$

in terms of ghost number and form rank one gets a set of equations

$$\begin{aligned} dW_n &= 0 \\ dW_{n-1} &= \delta W_n \\ dW_{n-2} &= \delta W_{n-1} \\ \vdots \;\; &\;\vdots \;\; \vdots \\ 0 &= \delta W_0 \ , \end{aligned} \tag{71}$$

where $n = dim(M)$ and W_i is a $(i, 2k - i)$-form. Integrating W_i over a non-trivial homology-cycle γ_i of M one then gets a $(2k - i)$-form $I_{2k-i}(\gamma_i)$ on $\mathcal{M}$ by integrating out the non-zero modes in the path integral [6], which is closed and metric independent. By taking products of the W_i for appropriate choices of i and k one may then construct volume forms on $\mathcal{M}$ whose correlation functions lead to topological invariants of M.

Since the issue of 'triviality' of these observables has aroused some interest in the more recent literature on the subject [16,25,26,27] let us pause to make a remark on this point. Since $tr\mathcal{F}^2$ may locally be written as

$$tr\mathcal{F}^2 = (d + \delta) tr(\mathcal{A}\mathcal{F} - \frac{1}{3}\mathcal{A}^3) \quad , \tag{72}$$

this means in particular that

$$W_0 = tr\phi^2 = \delta tr(c\phi - \frac{1}{3}c^3) \tag{73}$$

is δ-exact, implying that it and its descendents W_i should have trivial correlation functions. However we have interpreted c as the (0,1)-part of a connection on the principal bundle Q, which will in general not be trivial over $\mathcal{A}/\mathcal{G}$. This means that although its curvature $\phi = \delta c + 1/2[c,c]$ is globally defined, c will in general not be (this has been confirmed recently by Kanno [28] by explicit calculation of $\langle c \rangle$). Thus exactly like the equation $tr(F_A^2) = d(tr(AF_A - 1/3A^3))$ is only valid locally, (71) by no means implies triviality of W_0 in the de Rham - cohomology of $\mathcal{M}$. Alternatively, as shown in [30], the fact that c acquires a non-zero vacuum expectation value can be used to show that the BRST invariance of the vacuum is broken. This in turn establishes the non-triviality of the observables.

We have attempted to construct other hierarchies of observables as well, but the possibilities are severely limited by the fact that the transformations (53,56,60,61) seem to allow for no non-trivial W_0's (satisfying $\delta W_0 = 0$) apart from $tr\phi^k$. Indeed the δ-cohomology is on general grounds expected to be trivial on the remaining fields [29].

In the Yang-Mills-Higgs system [11,12] or (equivalently) the three-dimensional super BF system on a manifold with boundary however we have nevertheless been able to derive the monopole charge in that way. Indeed on manifolds with boundary (or alternatively, say, in flat space with non-trivial boundary conditions) the whole question of observables has a somewhat different flavour. In that case it may very well be possible to start with a W_0 which is δ-exact but nevertheless arrive at a non-trivial W_n (and conversely).

Starting for instance from $W_0 = tr(\phi\sigma_0) = \delta(tr(\phi u_0))$ one derives

$$\begin{aligned} 0 &= \delta W_0 \\ dW_0 &= -\delta(tr\psi\sigma_0 + \phi d_A u_0) = \delta W_1 \\ dW_1 &= \delta(trF_A\sigma_0 + \psi d_A u_0) = \delta W_2 \\ dW_2 &= -\delta(trF_A d_A u_0) = \delta W_3 \\ dW_3 &= 0 \ , \end{aligned} \tag{74}$$

and ends up with the monopole charge (52) $\int_M F_A d_A u_0 = \int_{\partial M} F_A u_0$ which may not vanish although W_0 is δ-exact. Note that here no use was made of the anti-BRST symmetry (56) which was invoked in [11] to arrive at this

quantity. The decent equations (74) have also been derived in [12].

7 Renormalization

Having constructed several new interesting topological field theories, an important issue to be addressed is one of renormalization. We shall begin by discussing several of the pertinent features of renormalization of topological field theories, in general. Following this we treat more explicitly the four dimensional Witten theory of self-dual connections and the three dimensional super-BF system at 1-loop order.

Let us start by recalling the 1-loop renormalization properties of the four dimensional Witten theory [4]. One finds that the 1-loop effective action is given by

$$\Gamma(A_c) = S(A_c) - \frac{1}{2}log\{\frac{det(D^2\delta_{\alpha\beta} + 2F^-_{\alpha\beta})}{det(D^2\delta_{\alpha\beta} + 2F_{\alpha\beta})}\} \ , \tag{75}$$

where $F^{\pm}$ refer to the self-dual and anti self-dual parts of the curvature. Upon regularization the ratio of determinants in (75) leads to a divergence of the form $\frac{1}{\epsilon}(F^+)^2$. The important point to note here is that it is $(F^+)^2$ which is renormalized, rather than $(F)^2$ as in the case of ordinary Yang-Mills theory. This is essential in preserving the topological nature of the model, as it is the former quantity which appears in the tree level action. Thus the 1-loop effective action remains a BRST commutator, guaranteeing metric independence .

A natural question to ask at this point is: Since a topological quantum field theory is a theory with no local excitations, i.e., its phase space is finite dimensional, why is there a divergence at 1-loop? The answer to this question is in fact quite simple and is as follows: We first note that the infinity mentioned above is an off-shell divergence; on-shell, however, i.e., $F^+ = 0$, we find that the theory is finite. In fact there is no 1-loop correction to the effective action since the ratio of determinants cancel on-shell. This is exactly as we would have expected, since on-shell in this case means we are restricting the theory to the instanton moduli space which is certainly finite dimensional.

A nice way to see this result immediately, without generating the off-shell infinity, is to gauge fix the theory by going to the so-called δ-function gauge. Since the action contains a term of the form B^+F^+, c.f. (43) and (44) with $\alpha = 0$, we can integrate over the B field to enforce a delta function constraint in the path integral. This ensures that only anti self-dual configurations are counted and hence no divergence will appear.

More explicitly, let us examine this theory to 1-loop order within the background field method. For simplicity we assume that the instanton is isolated, so that we do not have to take zero modes into account. The full quantum action is given by

$$S_q = \{Q, \int \chi^+(F_A^+ - \frac{\alpha}{2}B^+) + \overline{\phi} d_{A_{cl}} * \psi + \overline{c}(d_{A_{cl}} * A - \frac{\alpha'}{2}b)\} \ , \qquad (76)$$

where the BRST transformations of the fields are as given by the first eight rows of (53), together with $\delta\chi^+ = B^+, \delta B^+ = 0$. Here χ^+ and B^+ are self-dual 2-forms, i.e. $(1 + *)\chi^+ = \chi^+$, and similarly for B^+, with ghost numbers -1 and 0, respectively. The self-dual and anti self-dual parts of the curvature 2-form are denoted by $F_A^{\pm} = \frac{1}{2}(1 \pm *)F_A$ and α and α' are the gauge fixing parameters.

Expanding (76) and using $\delta F_A^+ = -[c, F_A^+] - \frac{1}{2}(1 + *)d_A\psi$ we find

$$\begin{aligned} S_q \ = \ & \int B^+(F_A^+ - \frac{\alpha}{2}B^+) + \chi^+([c, F_A^+] + \frac{1}{2}(1 + *)d_A\psi) + \eta d_{A_{cl}} * \psi \\ + \ & \overline{\phi} d_{A_{cl}} * ([c, \psi] + d_A\phi) + b(d_{A_{cl}} * A - \frac{\alpha'}{2}b) + \overline{c} d_{A_{cl}} * (\psi - d_A c) \end{aligned} \qquad (77)$$

Since we are interested in working in the delta function gauge we choose the parameters α and α' to be zero. We now decompose the gauge field into a classical plus a quantum piece as

$$A_\alpha = A_\alpha^{cl} + A_\alpha^q \ , \qquad (78)$$

and all other fields are taken to be purely quantum. For the purposes of a 1-loop calculation we insert the decomposition (78) into (77) and retain those terms which are second order in the quantum fields. The resulting action takes the form

$$\begin{aligned} S_q^{(2)} \ = \ & \int B^+ d_{A_{cl}} A_q + b d_{A_{cl}} * A_q - \overline{c} d_{A_{cl}} * d_{A_{cl}} c \\ + \ & \chi^+ d_{A_{cl}} \psi + (\eta + \overline{c}) d_{A_{cl}} * \psi + \overline{\phi} d_{A_{cl}} * d_{A_{cl}} \phi \ . \end{aligned} \qquad (79)$$

One loop corrections to the effective action can be represented as determinants of operators. We can see immediately that the determinants of the $\overline{c}c$ and $\overline{\phi}\phi$ systems cancel against each other, (upon making a simple field redefinition $\eta' = \eta + \overline{c}$). The remaining terms require a little more care. Following [4] we notice that both the $B^+ - b - A_q$ and $\chi^+ - \eta - \psi$ systems define a linear map:

$$T : \Omega^1 \to \Omega^2_+ \oplus \Omega_0 \ , \qquad (80)$$

from the space of 1-forms (Ω^1) to the space of self-dual 2-forms (Ω^2_+) tensored with the space of zero forms (Ω^0). The difficulty here is that the operator T is not a map from a space into itself, so the determinant is not simply defined. However, irrespective of how we choose to define the determinant we see that, since the B^+ and χ^+ systems are of opposite Grassmann character, the corresponding ratio is equal to 1. We thus obtain the result that the entire one-loop correction to the effective action vanishes.

It is important to study for a moment the dependence of this result on the gauge chosen. We have just found that in the Landau gauge (i.e. $\alpha = \alpha' = 0$) there is no one-loop renormalization. This agrees with our arguments concerning the fact that the delta function gauge restricts us immediately to the appropriate finite dimensional moduli space, thereby ensuring the absence of divergences. However, if we choose the Feynman gauge (i.e. $\alpha = \alpha' = 1$) then we obtain the result given in (75). In this case there is a divergence at one-loop, of the form $\frac{1}{\epsilon}(F^+)^2$. If we now go back to (77) and integrate out the B^+ field we obtain a term in the action of the form $\frac{1}{\alpha}(F^+)^2$. Choosing $\alpha = 1$ means that the kinetic term for the A field is the usual Yang-Mills action plus the instanton number. Thus when we use the background field method we will be expanding about a solution of the Yang-Mills equations. It is thus not surprising that in this case we do generate a divergence, as it is only when this background is futher restricted to be an instanton that the divergence cancels. We can interpret the above results as affecting the renormalization of the gauge parameter α as follows. In the Landau gauge, i.e. $\alpha = 0$, there is no renormalization of α. However, in an arbitrary $\alpha \neq 0$ gauge we have that seen that the $\frac{1}{\alpha}(F^+)^2$ term gets renormalized by $\frac{1}{\epsilon}(F^+)^2$; this corresponds to a renormalization of α. This situation is analogous to the one which aries in QCD for the conventional

Yang-Mills gauge fixing parameter, which here we have denoted by α'. In the latter case one finds that only in the Landau gauge does the parameter α' receive no renormalization.

In the case of flat connections we can again apply the same arguments. Here, however, the δ-function constraint takes the form $\delta(F_A + *d_A u_{n-3})$. As shown in section 4, for the case of compact closed manifolds the argument of this delta function can just as well be taken to be F_A. Thus our path integral is constrained to lie on the moduli space of flat connections. As such we are dealing with a finite dimensional system and no divergences will appear. Obviously similar considerations apply for the super Yang-Mills theories.

We can, of course, explicitly check this result by again performing a one-loop computation within the background field method. In this case the analysis is somewhat simplier than for self-dual connections, due to the fact that the operators which appear are linear maps from a space into itself. Proceeding as before, we take the quantum action in the background to be

$$\begin{aligned} S_q &= \{Q, \int \chi(F_A - \frac{\alpha}{2}B) + \overline{\rho}_0 d_{A_{cl}} * \chi + \overline{\Sigma} d_{A_{cl}} * B \\ &+ \overline{\phi} d_{A_{cl}} * \psi + \overline{c}(d_{A_{cl}} * A - \frac{\alpha'}{2} b)\} \ . \end{aligned} \tag{81}$$

Using (78) and the transformations in (53) we find that in the Landau gauge the part of the action which is quadratic in the quantum fields is given by

$$\begin{aligned} S_q^{(2)} &= \int B d_{A_{cl}} A_q - \chi d_{A_{cl}} \psi + \sigma_0 d_{A_{cl}} * \chi - (\overline{\rho}_0 - \Pi_0) d_{A_{cl}} * B + b d_{A_{cl}} * A_q \\ &+ (\eta + \overline{c}) d_{A_{cl}} * \psi + \overline{\rho}_0 d_{A_{cl}} * d_{A_{cl}} \rho - \overline{\Sigma}_0 d_{A_{cl}} * d_{A_{cl}} \Sigma_0 + \overline{\phi} d_{A_{cl}} * d_{A_{cl}} \phi \\ &- \overline{c} d_{A_{cl}} * d_{A_{cl}} c \ . \end{aligned} \tag{82}$$

Looking at (82) we see immediately that the determinants arising from the $(\overline{\rho}_0, \rho_0)$, $(\overline{\Sigma}_0, \Sigma_0)$, $(\overline{\phi}, \phi)$ and $(\overline{c}, c)$ systems cancel against each other. The remaining terms define a linear map

$$T : \Omega^0 \oplus \Omega^1 \to \Omega^0 \oplus \Omega^1 \ , \tag{83}$$

where $\Omega^0 \oplus \Omega^1$ is represented by (b, B), $(\overline{\rho}_0 - \Pi_0, A_q)$, $(\eta + \overline{c}, \chi)$ and (σ_0, ψ). The determinants arising from these two systems again cancel, thus con-

firming our claim that there is no renormalization when one chooses the delta function gauge (i.e. $\alpha = 0$).

To conclude we make some remarks on the structure of BRST Ward identities in topological field theories. The 1-loop renormalization of the Yang-Mills Higgs system (i.e. the 3d super BF system) was treated in detail in [4], in which the BRST Ward identities for all 2-point functions were explicitly verified. For completeness, and to reiterate some of the salient features of topological BRST Ward identities, we present the Ward identity for the model in the form

$$\begin{aligned} 0 &= \int d^3x Tr\{\frac{\delta\Gamma}{\delta A_\alpha}\frac{\delta\Gamma}{\delta S^\alpha} - \frac{\delta\Gamma}{\delta u}\frac{\delta\Gamma}{\delta S} + \frac{\delta\Gamma}{\delta\psi_\alpha}\frac{\delta\Gamma}{\delta T^\alpha} - \frac{\delta\Gamma}{\delta\rho}\frac{\delta\Gamma}{\delta T} \\ &+ \frac{\delta\Gamma}{\delta\chi_\alpha} d_\alpha - \frac{\delta\Gamma}{\delta\phi}\frac{\delta\Gamma}{\delta Y} + \frac{\delta\Gamma}{\delta b} d + \frac{\delta\Gamma}{\delta c}\frac{\delta\Gamma}{\delta V} - \frac{\delta\Gamma}{\delta\lambda}\eta \ . \end{aligned} \tag{84}$$

The point we would like to remark on here is the strength of this identity and the constraints which it puts on the theory. In particular, ones notes that upon differentiation with respect to (A_β, c) the usual transversality constraint on the inverse A propagator is recovered. Futhermore, however, if we differentiate with respect to (A_β, ψ_γ), we obtain the condition that the inverse A propagator in fact vanishes. The reason for the strength of this constraint can be seen as due to the fact that the A transformation rule is $\delta A = \psi - d_A c$ rather than the more familiar Yang-Mills version without the ψ field. Indeed, as shown in [4], this result does hold to 1-loop order. This simple example serves to illustrate the fact that, because we are dealing with a topological symmetry, the structure of the corresponding BRST Ward identities is greatly enhanced.

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Appendix

In this appendix we state our conventions regarding graded forms and present the properties which are necessary for our calculations. We introduce Lie algebra valued differential forms ($\in \Omega^*(M\times \mathcal{A}/\mathcal{G}, adQ)$ on $M\times\mathcal{A}/\mathcal{G}$ which carry a natural bigrading. A (p_1,p_2)-form referring to a p_1-form on M and a p_2-form on $\mathcal{A}/\mathcal{G}$. The p_1 label therefore refers to the usual exterior form degree, while the p_2 label is the ghost number of the form. A (p_1,p_2)-form is then called a graded form of degree p_1+p_2 and is denoted by X_p, where $p=p_1+p_2$. The following are the general properties of graded forms.

$$X_pY_q = (-1)^{pq}Y_qX_p \ . \tag{85}$$

The usual graded commutator is defined as

$$[X_p,Y_q] = X_pY_q - (-1)^{pq}Y_qX_p \ . \tag{86}$$

Thus if p or q is even we obtain the commutator, while if p and q are odd we get the anticommutator. We next note that the usual exterior derivative d and the BRST operator δ are graded derivations, with bigradings $(1,0)$ and $(0,1)$, respectively. The standard result for the exterior derivative acting on a product of forms also holds in this case for both derivations d and δ, e.g.

$$\delta X_pY_q = (\delta X_p)Y_q + (-1)^pX_p\delta Y_q \ . \tag{87}$$

Given a pure ghost form X_p, i.e. where $p=(0,p)$, together with an arbitrary q-form Y_q, we have the following important result

$$*\,(X_pY_q) = X_p * Y_q \ . \tag{88}$$

From this we can derive the properties

$$*\,(Y_qX_p) = (-1)^{np}(*Y_q)X_p \tag{89}$$

and

$$*\,[c,Y_q] = [c,*Y_q] \ , \tag{90}$$

where c is the (0,1)-ghost form, and $*$ is the Hodge star operator. (88) also tells us that the BRST operator commutes with the Hodge star operator:

$$*\,(\delta Y_q) = \delta * Y_q \ , \tag{91}$$

since δ is a pure ghost 1-form. Other properties to note are the trace formulae and Jacobi identity:

$$TrX_pY_q = (-1)^{pq}TrY_qX_p \tag{92}$$

$$TrX_p[Y_q,Z_r] = Tr[X_p,Y_q]Z_r \ , \tag{93}$$

$$[X_p,[Y_q,Z_r]] = [[X_p,Y_q],Z_r] + (-1)^{pq}[Y_q,[X_p,Z_r]] \ , \tag{94}$$

where X,Y,Z are arbitrary degree forms. The integration by parts formula is

$$\int X_pdY_q = (-1)^{(p+1)(q+1)}\int Y_qdX_p \ . \tag{95}$$

If either p or q is odd we get a + sign, while if both p and q are even we get a − sign. Our final result refers to the inner product rule between superforms $X_{(p_1,p_2)}$ and $Y_{(p_1,-p_2)}$ defined by

$$\langle X_p,Y_p\rangle = \int_M tr(X_p * Y_p) \ , \tag{96}$$

which satisfies

$$\langle X_p,Y_p\rangle = (-1)^{p_2}\langle Y_p,X_p\rangle \ . \tag{97}$$

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