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THE INDUCTION ON A CONTINUOUS VARIABLE *

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ABSTRACT

Mathematical induction is a useful tool. But it could be used to prove only the proposition with form $P(n)$ for the natural number n. Could the natural number n be replaced by a continuous variable x? Yes, and then we have the continuous induction. The continuous induction is very easy to grasp by the students who have learned mathematical induction. And it can be used to prove many basical propositions in the elementary calculus.

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In the elementary calculus teaching, the theory concerning the real number system and its continuity is very important. But, unfortunately, it is a difficult point for most students in colleges or universities, especially for some students who do not want to become mathematicians but have to learn more mathematics for other reasons. To overcome this difficult point, an interesting proposition is proposed, the so-called continuous induction.

To explain what the continuous induction is let us make a comparison between continuous induction and the well-known mathematical induction:

The continuous induction:	The mathematical induction:
Assume $P(x)$ be a proposition on the real number x	Assume $P(n)$ be a proposition on the natural number n
If:	If:
(1) There exists a real number x_0 such that $P(x)$ is true for any $x < x_0$.	(1) There exists a natural number n_0 such that $P(n)$ is true for any $n < n_0$.
(2) If $P(x)$ is true for any $x < y$, then there exists $\delta_y > 0$ such that $P(x)$ is true for any $x < y + \delta_y$.	(2) If $P(n)$ is true for any $n < m$, then $P(n)$ is true for any $n < m + 1$.
Then $P(x)$ is true for any real number x.	Then $P(n)$ is true for any natural number n.

We see that the two are very much alike!

Some questions could arise immediately. Is the continous induction really true? How to prove it? Where is its position in the theory of the real number? What is its use?

The answers to the first three questions will be given by the following theorem:

Theorem 1 The continuous induction (CI) is equivalent to the Didekind axiom (DA).

Let us recall the Didekind axiom first.

DA If the real number set R is divided into two non–empty sets, say L and H, such that $x < y$ for any $z \in L$ and $y \in H$, then either L has a maximal element or H has a minimum element.

Proof of Theorem 1 We use the reduction to absurdity. Suppose CI is true but DA false, and assume that R has been divided into L and H as in DA but neither L has a maximum not H has a minimum. Make a proposition $P(x) =$ "$x \in L$", then check the two conditions in CI:

(1) Since L is non–empty, there exists $x_0 \in L$. So $x \in L$ for any $x < x_0$ by the definition of L and H.

(2) If $x \in L$ for any $x < y$, then $y \notin H$ because there is no minimum in H. So $y \in L$ and there must exist $y_1 \in L$ such that $y_1 > y$ because L has no maximum. Then $x \in L$ for any $x < y_1 = y + \delta y$. So $x \in L$ for every real number x by CI. This is a contradiction since H is non–empty.

Now we are going to prove $DA \Rightarrow CI$.

Use the reduction to absurdity again. Suppose DA is true and CI false, and assume that there is a proposition $P(x)$ such that the conditions (1) and (2) in CI are satisfied but $P(x)$ is false for some real number x. Let

$$L = \{t : P(x) \text{ is true for any } x < t\}$$
$$H = R \setminus L$$

Obviously, both L and H are non–empty and $x < y$ for any $x \in L$ and $y \in H$, by our assumption. Since DA is true, there exists y such that $(-\infty, y) \subset L$ and $(y, +\infty) \subset H$, i.e. $P(x)$ is true for any $x < y$ but false for some $x \in (y, y + \delta)$ for any $\delta > 0$. It is contrary to the condition (2) which has been assumed by us.

□

Now we learned that CI is true, and that it could be set as an axiom in the theory of the real number to replace the well–known axiom DA.

As applications of CI, we give below a series of examples.

Example 1 If M is a real number set which is non–empty and bounded from above, then M has the least upper bound.

Proof Use the reduction to absurdity. Suppose M has no least upper bound and let U be the set of all the upper bound of M. Make a proposition $P(x)$ = "$x \notin U$". Then

(1) Since M is non–empty, there exists $x_0 \in M$ and so $P(x)$ is true for any $x < x_0$.

(2) If $P(x)$ is true for any $x < y$, then y must not be an upper bound of M (otherwise, y will be the least upper bound.) i.e., there is $y_1 \in M$ such $y_1 > y$ so $P(x)$ is true for any $x < y_1 = y + \delta$.

We have that $P(x)$ is true for every real number x by CI. It is contrary to that M is bounded from above.

□

Example 2 Suppose the number sequence $\{a_n\}$ is increasing and bounded. Then there exists a real number a such that $\lim_{n\to\infty} a_n = a$.

Proof Use the reduction to absurdity. Suppose there does not exist the limit of $\{a_n\}$ and make a proposition $P(x)$ = "$\{a_n\} \cap (x, +\infty) \neq \phi$", then

(1) $P(x)$ true for $x < x_0 = a_1$.

(2) If $P(x)$ true for any $x < y$, then $\{a_n\} \cap (y, +\infty) \neq \phi$, (otherwise, we have $\lim_{n\to\infty} a_n = y$) i.e., there exists some term $a_m > y$, such that $P(x)$ true for any $x < y + \delta_y$, here $\delta_y = a_m - y > 0$.

By CI, $P(x)$ is true for every real number x. It is contrary to that $\{a_n\}$ are bounded.

□

Example 3 If there is a nest of closed intervals $\Delta_1 \supseteq \Delta_2 \supseteq \ldots \supseteq \Delta_n \supseteq \Delta_{n+1} \supseteq \ldots$ Then there must exist at least one real number which belongs to every interval Δ_n.

Proof Use the reduction to absurdity. Suppose that there is no such real number which belongs to every interval $\Delta_n = [a_n, b_n]$. Make a proposition $P(x) =$ "$(-\infty, x) \cap \{b_n\} = \phi$", then

(1) $P(x)$ is true for any $x < x_0 = a_1$.

(2) If $P(x)$ is true for any $x < y$, then $(y, +\infty) \cap \{a_n\} \neq \phi$, (otherwise, $a_n \leq y$ and $y \leq b_n$ for every $\Delta_n = [a_n, b_n]$) and so $P(x)$ is true for any $x < y + \delta_y = a_m$, here a_m is a term belonging to $(y, +\infty)$.

By CI, $P(x)$ is true for every real number x. This is impossible.

□

Example 4 If a closed interval $[a, b]$ is covered by a set of open intervals $U = \{\Delta_t\}$, then there is a finite set of intervals $\Delta_k \in U, k = 1, 2, \ldots, n$, which still cover the interval $[a, b]$.

Proof Make a proposition. $P(x) =$ "$(-\infty, x) \cap [a, b]$ could be covered by a finite set of intervals in U". We have

(1) $P(x)$ is true for any $x < x_0 = a$.

(2) If $P(x)$ is true for any $x < y$ and $y \in [a, b]$. (If $y > b$, the conclusion we want is obvious) then there is an interval $\Delta = (\alpha, \beta) \in U$ such that $y \in (\alpha, \beta)$. Take $\delta_y = \frac{1}{2}(\beta - y)$, it is easy to know $P(x)$ is true for any $x < y + \delta_y$ by the inductive assumption. By CI, $P(x)$ is true for any real number x. Especially, take $x = b + 1$, our proposition is proved.

□

Example 5 If M is an infinite point set which is contained by $[a, b]$, there exists at least one point in $[a, b]$, which is a limit point of M.

Proof Use the reduction to absurdity. Suppose that there is no limit point of M in $[a, b]$. Make a proposition $P(x) =$ "$(-\infty, x) \cap M$ is a finite set", we have

(1) $P(x)$ is true for any $x < x_0 = a$.

(2) If $P(x)$ is true for any $x < y$, then there is $\delta > 0$ such that $(y - \delta, y + \delta) \cap M$ is a finite set because y is not a limit point of M. So $P(x)$ is true for any $x < y + \frac{1}{2}\delta$.

By CI, $P(x)$ is true for every real number x. Take $x > b$, we have a contradiction.

□

Example 6 Suppose $f(x)$ is a continuous function on $[a, b]$. If $f(a) < 0$ and $f(b) > 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

Proof Use the reduction to absurdity. Suppose $f(x) \neq 0$ for any $x \in [a, b]$. Let $f(x) = f(a)$ when $x < a$ and $f(x) = f(b)$ when $x > b$. Make a proposition. $P(x)$ = "$f(t) < 0$ for any $t < x$". We have

(1) $P(x)$ is true for any $x < x_0 = a$.

(2) If $P(x)$ is true for any $x < y$, then $f(y) \leq 0$ since $f(x)$ is continuous. By the supposition of the reduction to absurdity, $f(y) < 0$. So there is $\delta > 0$ such that $f(x) < 0$ for any $x \in (y - \delta, y + \delta)$ by continuity. Then $P(x)$ is true for any $x < y + \delta$.

By CI, $P(x)$ is true for every real number x. It is contrary to the assumption $f(b) > 0$ when $x > b$.

□

Example 7 Suppose $f(x)$ is a continuous function on $[a, b]$. Then there exists $M > 0$ such that $|f(x)| \leq M$ for any $x \in [a, b]$.

Proof Let $f(x) = f(a)$ when $x < a$ and $f(x) = f(b)$ when $x > b$. Make a proposition $P(x)$ = "f is bound on $(-\infty, x)$". We have

(1) $P(x)$ is true for any $x < x_0 = a$.

(2) Assume that $P(x)$ is true for any $x < y$. There exists $\delta > 0$ such that $|f(x)| < |f(y)| + 1$ for any $x \in (y - \delta, y + \delta)$ by continuity. So $P(x)$ is true for any $x < y + \delta$.

By CI, $P(x)$ is true for any real number x. This is we want.

□

Example 8 Suppose $f(x)$ is a continous function on $[a, b]$. Then there exists $x^* \in [a, b]$ such that $f(x) \leq f(x^*)$ for any $x \in [a, b]$.

Proof Use the reduction to absurdity. Suppose f has no maximum on $[a, b]$, i.e., for any given $x \in [a, b]$ there exists some $x_1 \in [a, b]$ such that $f(x) < f(x_1)$. Let $f(x) = f(a)$ when $x < a$ and $f(x) = f(b)$ when $x > b$. Make a proposition $P(x)$ = "There exists $c \in [a, b]$ such that $f(t) < f(c)$ for any $t < x$". We have

(1) $P(x)$ is true for any $x < x_0 = a$.

(2) Assume that $P(x)$ is true for any $x < y$. By the supposition of the reduction to

absurdity, there exists $y_1 \in [a, b]$ such that $f(y) < f(y_1)$. So there exists $\delta > 0$ such that $f(x) < f(y_1)$ for any $x \in (y - \delta, y + \delta)$ since f is continuous. By our inductive assumption, there exists $c_1 \in [a, b]$ such that $f(x) < f(c_1)$ for any $x < y - \frac{\delta}{2}$. Let c is c_1 or y_1 such that $f(c) = \max\{f(c_1), f(y_1)\}$ then $f(x) < f(c)$ for any $x < y + \delta$, i.e., $P(x)$ is true for any $x < y + \delta$.

By CI, $P(x)$ is true for any real number x. But this is impossible when $x > b$.

□

Example 9 If $f(x)$ is a continuous function on $[a, b]$, then $f(x)$ is uniformly continuous on $[a, b]$.

Proof Let $f(x) = f(a)$ when $x < a$ and $f(x) = f(b)$ when $x > b$. Given any $\varepsilon > 0$, we are going to prove that there exists $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ when $|x_1 - x_2| < \delta$. Make a proposition $P(x)$ = "There exists $\delta_x > 0$ such that $|f(t_1) - f(t_2)| < \varepsilon$ when $|t_1 - t_2| < \delta_x$, for any $t_1 < x$ and $t_2 < x$". We have

(1) $P(x)$ is true for any $x < x_0 = a$.

(2) Assume that $P(x)$ is true for any $x < y$. Since $f(x)$ is continuous at point y, there exists $\delta_1 > 0$ such that $|f(t) - f(y)| < \frac{\varepsilon}{2}$ when $t \in (y - \delta_1, y + \delta_1)$, and so $|f(t_1) - f(t_2)| < \varepsilon$ for any $t_1, t_2 \in (y - \delta_1, y + \delta_1)$. By our inductive assumption, $P\left(y - \frac{\delta_1}{2}\right)$ is true, i.e., there exists $\delta_2 > 0$ such that $|f(t_1) - f(t_2)| < \varepsilon$ when $|t_1 - t_2| < \delta_2$, for any $t_1, t_2 \in \left(-\infty, y - \frac{\delta_1}{2}\right)$. Let $\delta = \min\left\{\frac{\delta_1}{2}, \delta_2\right\}$. It is easy to know $|f(t_1) - f(t_2)| < \varepsilon$ when $|t_1 - t_2| < \delta$ for any $t_1, t_2 \in (-\infty, y + \delta)$. Then $P(x)$ is true for any $x < y + \delta$.

By CI, $P(x)$ is true for any real number x. We have the conclusion we want when $x > b$.

□

Example 10 Suppose $f(x)$ is a continuous function on $[a, b]$. If f is differentiable in (a, b) and $f'(x) = 0$ for any $x \in (a, b)$, then $f(a) = f(b)$.

Proof It is enough to prove that $f(x)$ is a constant in (a, b) by continuity.

Take any two points $x_1, x_2, a < x_1 < x_2 < b$, and let

$$\bar{f}(x) = \begin{cases} f(x) & (x \in [x_1, x_2]) \\ f(x_1) & (x < x_1) \\ f(x_2) & (x > x_2) \end{cases}$$

then $\bar{f}'(x) = 0$ for every $x \in (-\infty, +\infty)$.

Use the reduction to absurdity. Suppose

$$|f(x_1) - f(x_2)| = M > 0.$$

Make a proposition

$$P(x) = \text{“}|\bar{f}(x) - \bar{f}(x_1)| \leq \frac{M|x - x_1|}{2|x_1 - x_2|}\text{”}.$$

We have

(1) $P(x)$ is true for any $x < x_1$.

(2) Assume that $P(x)$ is true for any $x < y$. Since $\bar{f}'(y) = 0$, there exists $\delta > 0$ such that

$$|\bar{f}(x) - \bar{f}(y)| \leq \frac{M}{2} \cdot \frac{|x-y|}{|x_1 - x_2|} \quad \text{for any} \quad x \in (y-\delta, y+\delta).$$

By our inductive assumption, we have

$$|\bar{f}(x') - \bar{f}(x_1)| \leq \frac{M}{2} \cdot \frac{|x' - x_1|}{|x_1 - x_2|} \quad \text{for any} \quad x' < y$$

and so

$$|\bar{f}(y) - \bar{f}(x_1)| \leq \frac{M}{2} \cdot \frac{|y - x_1|}{|x_1 - x_2|}.$$

Then for $x \in [y, y+\delta)$ we have

$$|\bar{f}(x) - \bar{f}(x_1)| \leq \frac{M}{2} \cdot \frac{|x-y| + |y - x_1|}{|x_1 - x_2|} = \frac{M}{2} \cdot \frac{|x - x_1|}{|x_1 - x_2|}$$

so $P(x)$ is true for any $x < y + \delta$.

By CI, $P(x)$ is true for any real number x. It is contrary to the supposition of the reduction to absurdity when $x = x_2$.

□

We have now seen that one can prove a series of important propositions using the continuous induction in a similar way. This is interesting and easy to grasp for the students who have learned the mathematical induction.

We can build a more general induction which includes the mathematical induction, the continuous induction and the transfinite induction.

Definition Suppose $H = \{M_\xi\}$ is a family of sets, and $D = \{\xi\}$ is the set of its index. If

$$\Big(\bigcup_{\xi \in D_1} M_\xi\Big) \in H \qquad (\text{for any } D_1 \subset D)$$

then we call H an inductive family of sets.

We give some examples of inductive family of sets as follows:

(1) $H_1 = \{M_n : M_n = \{1, 2, \ldots, n\},\ n = 1, 2, \ldots\}$.

(2) $H_2 = \{M_y : M_y = (-\infty, y),\ y \in (-\infty, +\infty)\}$.

(3) Let M be a normal set. We have the inductive family of sets

$$H_3 = \{M_\alpha : M_\alpha = \{\beta : \beta < \alpha\},\ \alpha \in M\}.$$

(4) Let S be a simply ordered set. A subset $P \subset S$ will be called a passage of S, if for any three elements $x < y < z$ in S, we have $y \in P$ when $x, z \in P$. Then for a given fixed $c \in S$,

$$H_4 = \{P : P \text{ is a passage of } S \text{ and } c \in P\}$$

is also an inductive family of sets.

The following theorem is obvious.

Theorem 2 Suppose $H = \{M_\xi\}$ is an inductive family of sets, and M is the maximal set in H. If $P(\alpha)$ is a proposition on $\alpha \in M$, such that

(1) There exists $M_{\xi_0} \in H$ such that $P(\alpha)$ is true for any $\alpha \in M_{\xi_0}$.

(2) If $P(\alpha)$ is true for any $\alpha \in M_\xi \neq M$, then there exists $M_{\xi_1} \in H$, $M_{\xi_1} \supset M_\xi$ and $M_{\xi_1} \neq M_\xi$, such that $P(\alpha)$ is true for any $\alpha \in M_{\xi_1}$.

Then $P(\alpha)$ is true for any $\alpha \in M$.

In the above theorem, we will have the mathematical induction when $H = H_1$, the continuous induction when $H = H_2$, and the transfinite induction when $H = H_3$. Therefore, three inductions are integrated in the same form.

By the way, we do not know of any use of the general inductive principle other than the three.

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