SIGMA MODELS AND RENORMALIZATION OF STRING LOOPS

A.A. Tseytlin
SIGMA MODELS AND RENORMALIZATION OF STRING LOOPS

A.A. Tseytlin
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

An extension of the "\(\sigma\)-model \(\beta\)-functions - string equations of motion" correspondence to the string loop level is discussed. Special emphasis is made on how the renormalization group acts in string loops and, in particular, on the renormalizability property of the generating functional \(Z\) for string amplitudes (related to the \(\sigma\) model partition function integrated over moduli). Renormalization of \(Z\) at one and two loop order is analyzed in some detail. We also discuss an approach to renormalization based on operators of insertion of topological fixtures.

MIRAMARE – TRIESTE
May 1989

* Lectures at the 1989 Trieste Spring School on Superstrings.
** Permanent address: Department of Theoretical Physics, P.N. Lebedev Physical Institute, Leninsky prospect 53, Moscow 117924, USSR.
I. INTRODUCTION

At present we are lacking some basic principles of string theory. In particular, in the first quantized approach based on the geometrical path integral over 2-surfaces we do not know a priori the weights with which different string diagrams appear in the perturbation theory expansion in string coupling (or genus).

The remarkable role played by the renormalization group (RG) in determining the string equations of motion and the effective action at the string tree level [11-20] suggest that the RG approach may shed light on (or even be a part of) a more fundamental formulation of string theory. In fact, in order for the RG to operate consistently at string loop level, the relative weights of string diagrams should take some particular values. Also, an understanding of a role of RG and 2-d cutoff is important for an off-shell extension of string theory.

The idea that the equivalence between the string equations of motion and the vanishing of some $\beta$-functions may generalize to string loop level, first suggested in Refs. 1 and 9, was studied, e.g. in Refs. 10-19. If true it would imply that one is able to determine the true "loop corrected" string vacua by classifying some "non-perturbatively conformal invariant" 2-d theories. Below we are going to discuss the basic issues related to the possibility of extension of the "$\sigma$-model approach" to string loop level.

To understand if (and how) the RG acts in string loops we should first determine a basic object of the theory which should be RG invariant, i.e. renormalizable. The object is the (properly defined) generating functional for string amplitudes $\hat{Z}$ [2] which can be expressed in terms of the $\sigma$-model partition function $Z$ (integrated over all moduli parameters). Introducing a universal 2-d short distance cutoff (which regularizes both the $\sigma$-model and integrals over the moduli) we find the "local" and "modular" divergences in $\hat{Z}$. The renormalizability condition implies that we should be able to "absorb" these divergences into the $\sigma$-model couplings $\varphi^{a}$ which can be interpreted as the space-time fields corresponding to the "massless" string models. We would like to stress that it is only if $\hat{Z}$ is renormalizable that we can define the "loop corrected" $\beta$-functions ($\beta \sim \frac{d}{d\varphi^{a}}$) and hence may question their relation to string equations of motion.

As we shall see, it is nontrivial to define $\hat{Z}$ in such a way that it satisfies the renormalizability condition with respect to all ("local" and "modular") infinities **.

The crucial difference between the string theory object $\hat{Z}$ and the $\sigma$-model object $Z$ is that for the on-shell values of its arguments $Z$ should reproduce the usual expressions for the on-shell string amplitudes and hence the definition of $\hat{Z}$ should include a subtraction of the Mobius infinities at the tree level and an integration over the moduli at higher loop orders. It turns out that the procedures of regularization and fixation of the Mobius symmetry do not "commute". To implement the RG in a consistent way it is necessary to treat all the infinities including the Mobius ones on an equal footing. This implies in particular that to ensure the renormalizability property of $\hat{Z}$ one should use an "extended" (e.g. Schottky) set of moduli parameters with which one has formal on-shell projective invariance, i.e. the formal $\Omega^{-1} = (\text{vol} SL(2, C))^{-1}$ Mobius factor for all terms in the expansion in genus. Regularizing the corresponding modular integrals, one is then to replace the $\Omega^{-1} \sim (\Delta - \text{factor})^{a}$ by the RG invariant operator $\partial/\partial e^{a}$ [21,22]. If we first fix the Mobius gauge reducing the number of modular parameters to the usual one (3n-3, n $\geq$ 2) the $\hat{Z}$ we get is not renormalizable with respect to both local and modular infinities. The crucial point is that the Mobius infinities can be interpreted [21] as a subclass of local $\sigma$-model infinities and hence should be regularized on an equal footing with all other local and modular infinities.

We shall start (Sec. 2) with a brief review of the $\sigma$-model approach at the string tree level. In Sec. 2 we shall also discuss the nature of "local" infinities present in the string correlations and give an argument which demonstrates the renormalizability of the string generating functional starting from the assumption that the massless sector of the string $S$-matrix can be reproduced by an effective field theory.

In Sec. 3 we shall first recall the classification of divergences which appear in string loop amplitudes (see Refs. 23-26), emphasizing that the "non-dividing" tachyonic singularities should be avoided by using a kind of analytic continuation prescription [27,28] and hence should be ignored in the RG approach. Then we shall analyze the structure of the "tadpole" logarithmic infinities, which may be interpreted in terms of propagation of the zero momentum dilaton and (scalar part of) graviton. We shall find that, in principle, they may be cancelled out by renormalizing the slope (\alpha') and the string coupling (g) parameters of the string theory in a trivial (flat) vacuum.

In Sec. 4 we introduce the generating functional for string amplitudes $\hat{Z}$ and describe how one may eliminate the tadpole (and external or self energy insertion) infinities by renormalizing the $\sigma$-model couplings (the space-time metric and the dilaton) thus reinterpreting the result of Sec. 3 about the renormalization of $\alpha'$ and $g$. The corresponding "modular" terms in the $\beta$-functions are the "field independent" (tadpole) and the "linear in $\varphi$" (external leg correction). Their complicated dependence on the string coupling, i.e. on the constant part of the dilaton suggests that the $\beta$ functions should also contain other "modular" contributions, which are of higher orders in the fields $\varphi$. These terms correspond, in fact, to the momentum dependent singularities of loop amplitudes which are due to the massless poles in transferred momentum (which, in fact, contribute to the Weyl or BRST anomaly). We stress that the inclusion of these higher order terms is necessary for a correspondence between the $\beta$-functions and the effective action. We also discuss the higher order $\beta e^{a}$ counterterms which presence is dictated by the RG.
The correspondence with the effective action (EA) is analyzed in detail in Sec. 5. We explain how the assumption of an existence of a EA reproducing (by the tree diagrams) the massless loop-corrected string amplitudes imposes constraints on relative weights of string loop corrections and also implies renormalizability of $Z$. We formulate the higher loop generalizations of the tree level relations between the generating functional $Z$, the effective action $S$ and the $\beta$-functions, which already are assumed to contain contributions from all "dividing" (momentum dependent as well as momentum independent) singularities corresponding to the massless poles in the amplitudes. As an application, we discuss a simple solution of the $\beta = 0$ or $\frac{\partial S}{\partial \rho} = 0$ equations in the leading approximation accounting only for the one-loop cosmological term in the EA. While the solution with (anti) de Sitter metric $D_2$ and constant dilaton $\rho$ formally exists if $D \neq D_2$, there is a problem of its interpretation since the one-loop cosmological constant is complex because of the analytic continuation prescription used to regularize the tachyonic loop infinity $^{28}$.

In Secs. 6-9 we are trying to check the condition of renormalizability of the generating functional, analyzing systematically the loop contributions to $Z$. As was already mentioned above, to realize RG in string loops in a consistent way it is necessary to use an "extended" parameterization of the moduli space, in which the Möbius volume factor $\Omega^{-1}$ appears in front of all loop corrections to the amplitudes. A distinguished parameterization of this type is the Schottky parametrization $^{23-26,29}$, where an extensive list of references can be found. In Sec. 6 we present the expression for the generating functional $Z$ in the Schottky parametrization.

The renormalization of $Z$ in the one loop approximation is studied in detail in Sec. 7, where we use the Schottky-type parametrization (for the torus and the disc) to isolate the modular infinities from the local ones. We check the renormalizability of $Z$ with respect to the modular tadpole infinities (to obtain the consistent result in the case of the disc topology one is to use the "corrected" expression for the modular measure $^{15}$). In particular, we discuss the renormalization of the $\alpha R$ term — infinity in the one loop contribution to $Z$ (see also Refs. 19 and 22) and show that one is to use a special definition of $Z$ (special prescription for subtracting of Möbius infinities) in order to satisfy the requirement of renormalizability with respect to the sum of local and modular infinities.

In Sec. 8 we consider the renormalization of the derivative independent part of $Z$, i.e. of the "string partition function" in the two-loop examples (annulus and genus 2 surface). The logarithmic divergence in the genus 2 string partition function is found explicitly in the Schottky parametrization. It is shown how the condition that this divergence should be canceled out by the renormalization of the metric in the torus contribution to $Z$ fixes the overall coefficient in the two-loop expression. We also interpret the results in terms of the tree diagrams of the corresponding effective action.

In the final Sec. 9 we rederive and generalize the results of the previous sections using the approach based on a representation of $Z$ in terms of the operators on insertion of "topological fixtures" like holes, handles, etc. $^{18}$ (see also Ref. 32). For the RG to actually operate in string theory the divergences should "exponentiate" to become counterterms in the string (or-model) action. This suggests a resummation of the standard expansion in genus, which is analogous to a "renormalization group improved" perturbation theory or may be interpreted as a "dilute handle gas" approximation (valid only in some regions of moduli spaces). For the vacuum string partition function we propose an exact expression in terms of the expectation value on the sphere of the exponent of the sum of "subtracted" (or "irreducible") topological fixture operators. We demonstrate how to construct these modified operators on the genus 2 example. This approach seems to be very interesting. A possible point of view is that consistently imposing the requirement that the renormalization group should act in string loops may eventually lead us to a "non-perturbative" formulation of string theory.

2. TREE APPROXIMATION

To motivate the attempt to generalize the "renormalization group" or "sigma model" approach to string loop level it is useful first to recall briefly some basic facts known in the tree approximation (sigma model approach at the string tree level was recently discussed in the review $^7$ where an extensive list of references can be found).

1. Let $V_i$ be the dimension 2 interaction vertices present in the action of a general renormalizable bosonic sigma model and $\psi^i$ be the corresponding dimensionless coupling constants. The partition function for the sigma model $^{33}$ defined on a compact 2-space (of an arbitrary Euler number $\chi$) with some regular 2-metric $g_{ab}$ is

$$Z = \int D\varphi e^{-I}, \quad I = I_0 + I_{tot}, \tag{2.1}$$

$$I_0 = \frac{1}{4\pi\alpha'} \int d^2 z \sqrt{g} \partial_\mu \varphi \partial^\mu \varphi, \quad I_{tot} = \psi^i V_i, \tag{2.2}$$

$$I = \frac{1}{4\pi\alpha'} \int d^2 z \sqrt{g} \partial_\mu \varphi \partial^\mu \varphi G_{\mu\nu}(z) + \frac{1}{4\pi} \int d^2 z \sqrt{g} R^{(2)} \phi(z). \tag{2.3}$$

The measure $D\varphi$ should be defined so that the theory is covariant under the general coordinate transformations of $z$. Naively,

$$D\varphi = \prod_z dz(z) \sqrt{G(z(z))}, \quad G = \det G_{\mu\nu}. \tag{2.4}$$

A particular definition of the product in (2.4) depends on a regularization chosen (see Refs. 7 and 22 for details). One may introduce a short distance 2-d cutoff through the free propagator, e.g. $$D(z, z') = \frac{1}{4\pi} \frac{a^{-2}}{z - z'} \left( 1 + \frac{a^2 e^{-h(z) - h(z')}}{2} \right),$$

2. TREE APPROXIMATION

To motivate the attempt to generalize the 'renormalization group' or 'sigma model' approach to string loop level it is useful first to recall briefly some basic facts known in the tree approximation (sigma model approach at the string tree level was recently discussed in the review $^7$ where an extensive list of references can be found).

1. Let $V_i$ be the dimension 2 interaction vertices present in the action of a general renormalizable bosonic sigma model and $\psi^i$ be the corresponding dimensionless coupling constants. The partition function for the sigma model $^{33}$ defined on a compact 2-space (of an arbitrary Euler number $\chi$) with some regular 2-metric $g_{ab}$ is

$$Z = \int D\varphi e^{-I}, \quad I = I_0 + I_{tot}, \tag{2.1}$$

$$I_0 = \frac{1}{4\pi\alpha'} \int d^2 z \sqrt{g} \partial_\mu \varphi \partial^\mu \varphi, \quad I_{tot} = \psi^i V_i, \tag{2.2}$$

$$I = \frac{1}{4\pi\alpha'} \int d^2 z \sqrt{g} \partial_\mu \varphi \partial^\mu \varphi G_{\mu\nu}(z) + \frac{1}{4\pi} \int d^2 z \sqrt{g} R^{(2)} \phi(z). \tag{2.3}$$

The measure $D\varphi$ should be defined so that the theory is covariant under the general coordinate transformations of $z$. Naively,

$$D\varphi = \prod_z dz(z) \sqrt{G(z(z))}, \quad G = \det G_{\mu\nu}. \tag{2.4}$$

A particular definition of the product in (2.4) depends on a regularization chosen (see Refs. 7 and 22 for details). One may introduce a short distance 2-d cutoff through the free propagator, e.g. $$D(z, z') = \frac{1}{4\pi} \frac{a^{-2}}{z - z'} \left( 1 + \frac{a^2 e^{-h(z) - h(z')}}{2} \right),$$
\[ g_{ab} = e^{2} \delta_{ab}, \quad e^{\phi} = (u + a)^{-2} \to 0, \quad (2.5) \]

or

\[ D(z, z') = \sum_{\nu=0} \epsilon^{-3} \lambda_{\nu}^{-1} \psi_{\nu}(z) \phi_{\nu}(z'), \quad \epsilon = A^{-1} \]

\[ \Delta D = \delta^{(2)}(z, z') - \frac{1}{V}, \quad V = \int d^{2} z \sqrt{g}, \quad \Delta = -\nabla^{2} \quad (2.6) \]

where \( \lambda_{\nu} \) and \( \phi_{\nu} \) are the eigenvalues and eigenfunctions of the Laplacian and \( a \sim V^{1/2} \) is a "scale" of the 2-space. Under a proper choice of the measure all power (quadratic) infinities cancel out and one finds \(21^{*}\)

\[ Z = \int d^{2} y \sqrt{g} e^{-x^{+}w}, \quad x = \frac{1}{4 \pi} \int d^{2} z \sqrt{g} \eta D(2) \quad (2.7) \]

\[ \text{Here } G \text{ and } \phi \text{ are the bare couplings and } W(G, \phi, g_{ab}, \epsilon) \text{ is given by the sum of } 1-\Pi \text{ diagrams constructed with the propagator in which the constant (zero mode) part } \eta^{ab} \text{ of } \sigma^{ab} \text{ is "projected out". } W \text{ is a covariant functional of } G \text{ and } \phi \text{ depending on their derivatives at the point } \epsilon. \text{ In general } \]

\[ W = \gamma_{1} + \gamma_{2} \epsilon \kappa_{1} + \gamma_{3} \epsilon^{2} \kappa_{2} + \ldots. \quad (2.8) \]

\[ \gamma_{1} = -\frac{1}{16 \pi^{2}} \eta^{ab} \int R^{(2)} R^{-1} R^{(2)} + \ldots. \quad (2.9) \]

\( \beta^{a} \beta^{b} \) is the basic Weyl anomaly coefficient related to the \( \beta \)-functions by

\[ \beta^{a} = \beta^{a} - \frac{1}{4} K^{ab} \beta^{b}, \quad K^{ab} = G^{ab} + \ldots, \quad (2.10) \]

\[ \beta^{a} = \beta^{a} + \frac{k}{a} \partial \phi \delta^{a}, \quad \beta^{a} = \beta^{a} + D_{a} \epsilon + D_{a} \epsilon. \quad (2.11) \]

\( \beta^{a} \) are the Weyl anomaly coefficients which appear in the operator expression for the trace of the 2-d energy momentum tensor. Power counting and the proper choice of the measure guarantees the renormalizability of \( Z \) within the loop \((a')\) expansion

\[ Z(\phi(\epsilon), \epsilon) = Z_{R}(\phi, \epsilon), \quad (2.12) \]

\[ \frac{\partial Z}{\partial \lambda} = \beta^{a} \frac{\partial Z}{\partial \phi^{a}} = 0, \quad \beta(\phi) = -\frac{d \phi}{d \lambda}, \quad \lambda \equiv \epsilon \kappa_{1}. \quad (2.13) \]

Direct computation shows that \(22^{*}\)

\[ W = W_{0} - \frac{1}{2} \epsilon \kappa_{1}(R + \chi D^{2} \phi) + \ldots. \quad (2.14) \]

\[ W_{0} = -\frac{1}{6} (D - 26) (\chi \epsilon \kappa_{1} + \frac{1}{16 \pi} \int R^{(2)} R^{-1} R^{(2)} + \text{const} \quad (2.15) \]

\[ \text{Hence (see (2.7))} \]

\[ Z = a_{0} e^{1/2(D-26)} \epsilon \theta a \int d^{2} y \sqrt{g} e^{-x^{+}w} \left[ 1 + \frac{1}{2} \epsilon \kappa_{1}(R + \chi D^{2} \phi) + \ldots \right] \quad (2.15) \]

The coefficients of the \( \epsilon \kappa_{1} \)-terms in (2.14) are consistent with the renormalizability of \( Z \) (Eq.(2.13)) as one can check using the known expressions for the \( \beta \)-functions \(23^{*}, 30, 31^{*}\)

\[ \beta_{\mu}^{G} = a R_{\mu} + \frac{1}{2} a^{2} R_{\mu \nu} R^{\mu \nu} + o(a^{3}), \quad (2.16) \]

\[ \beta_{\phi}^{G} = \frac{1}{6} (D - 26) - \frac{1}{2} a^{2} D^{2} \phi + \frac{1}{16 \pi^{2}} R^{2} + o(a^{3}) \quad (2.17) \]

Substituting the expressions for the bare fields

\[ G_{\mu \nu} = G_{\mu \nu} - \epsilon \kappa_{1} R_{\mu \nu} + \ldots, \quad \phi = \phi_{0} - \frac{1}{6} (D - 26) \epsilon \kappa_{1} + \ldots \quad (2.18) \]

into \( Z (2.7) \) we indeed find the cancellation at the leading \( \epsilon \kappa_{1} \) terms.

It is possible to prove that the \( \beta_{\phi} \)-functions can be expressed in terms of the derivatives of some functional \( S \)

\[ \beta_{\phi} = \kappa_{0} \frac{\partial S}{\partial \phi} \quad (2.19) \]

where the matrix \( \kappa \) is non-degenerate (within the \( a' \)-expansion)

\[ \kappa = \kappa_{0} + o(a'), \quad \kappa_{0}^{G}_{\mu \nu} = kG^{-1/2} G_{\mu \nu} G_{\nu \mu}, \quad \kappa_{0}^{G}_{\phi} = kG^{-1/2} G_{\phi} G_{\phi} + \frac{1}{16} k^{2} G^{-1/2} (D - 2). \quad (2.20) \]

Under a particular choice of the couplings (i.e. in a particular renormalization scheme) \( S \) can be represented in the form \(30^{*}\)

\[ S = a \int d^{2} y \sqrt{g} e^{-x^{+}w} \beta_{\phi}, \quad a = 4 k^{-1}, \quad (2.21) \]

\[ \beta_{\phi} = \frac{1}{6} (D - 26) - \frac{1}{4} a' (R + 4 D^{2} \phi - 4 (\phi^{3}) + \ldots \quad (2.22) \]

The relations (2.19), (2.21) were checked by the explicit perturbative computations up to rather high order in the expansion in \( a' \) (for a discussion and references see Ref. 7). The leading \( a' \)-independent term in \( \kappa_{0} \) is essentially "one": the non-diagonal terms in \( \kappa_{0} \) are due to the "mixing"
between the graviton and the dilaton in the kinetic term in $S$ (2.21). If we redefine the couplings to diagonalize the kinetic term

$$G_{\mu \nu} = G_{\mu \nu} \exp(-\frac{\phi^2}{\sqrt{D-2}}), \quad \phi = \frac{1}{4} \sqrt{D-2} \phi',$$

(2.23)

$$S = \int d^Dy \sqrt{\det G} \left[ \frac{1}{6}(D-26)\epsilon^{\mu \nu \rho \sigma} - \frac{1}{4} \alpha' (R - \frac{1}{4} \partial^2 \phi)^2 + \ldots \right]$$

(2.24)

we find

$$\beta^\prime = \kappa_2^\prime \frac{\partial S}{\partial \phi^2}, \quad \kappa_2^\prime = \kappa_0' + O(\alpha'),$$

(2.25)

$$\kappa_{G^\mu \mu} = kG^{-1/2} [G_{\mu \nu} G_{\rho \sigma} - (D-2) G_{\mu \nu} G_{\rho \sigma}],$$

(2.26)

$$\kappa_{\phi^\mu} = \kappa_{\phi^2} = 0, \quad \kappa_{\phi^\prime} = kG^{-1/2}$$

(2.27)

Expanding near the constant background $(G_{\mu \nu} = \delta_{\mu \nu} + h_{\mu \nu})$

$$S = \frac{1}{2} \phi^2 \Delta^i \phi^i + O(\phi^3),$$

(2.28)

$$\Delta_{ij} = F_{ij}, \quad \Delta = - \square,$$

(2.29)

we find by comparing the leading order terms in (2.19)

$$\kappa_{F_{ik}} = \frac{1}{2} \alpha' \delta_{ik} \Delta, \quad \kappa_0 = \frac{1}{2} \alpha' \Delta^{-1}$$

(2.30)

where we have used that $\beta_{\phi^2} = \frac{1}{2} \alpha' \Delta h_{\mu \nu} + \ldots, \beta^\phi = \frac{1}{2} \alpha' \Delta \phi + \ldots$. Thus $\kappa_{\Delta^{-1}}$ is proportional to the kinetic matrix in $S$.

2. A relation to string theory is established through the observation $3^2$) that $Z (2.1)$ is the generating functional for the correlators of the massless vertex operators (we consider the tree approximation and hence assume the spherical topology, $\chi = 2$)

$$Z = \sum_{N=0}^{\infty} \frac{(-1)^N}{N!} T_N \phi^N, \quad T_N = <V_1 \ldots V_N>.$$ 

(2.31)

The on-shell string tree amplitudes $A_N$ differ from $T_N$ by the infinite factor of the $SL(2, C)$ Möbius group volume, $A_N \sim \Omega^{-1} T_N$. Introducing a short distance cutoff $\epsilon$, one finds that (up to power divergent terms) $\Omega \sim \epsilon^{20 (2.1)}$. Hence naively $A_N \sim (\epsilon \Delta)^{-1} T_N$. However, this is not a correct prescription since it does not treat all the short distance divergences in $T_N$ on an equal footing. $T_N$ contains the following types of logarithmic infinities: (1) Singularities corresponding to the region of integration where $N$ or $N - 1$ integration points $x_i$ are close to each other. These are "momentum independent" (i.e. present for arbitrary on-shell external momenta) Möbius infinities, which are absent in $A_N$ computed with 3 Koba-Nielsen (KN) points fixed. (2) "Momentum dependent" massless pole singularities corresponding to the regions of integration where $M = 2, \ldots, N - 2$ points are close to each other. As discussed in Ref. 21, the correct RG invariant relation between the regularized expressions for $A_N$ and $T_N$ is

$$A_N \sim \frac{\partial}{\partial \epsilon} T_N$$

(2.32)

Hence the generating functional for the massless string tree amplitudes $Z \sim \sum_N A_N \phi^N$ (computed with a 2-d cutoff $\epsilon$ and using the expansion in $\alpha'$) can be represented as

$$\tilde{Z} = \frac{\partial}{\partial \epsilon} Z.$$

(2.33)

Since the partition function $Z$ is renormalizable, i.e. satisfies (2.13), the same is true for $\tilde{Z}$.

$$\frac{\partial Z}{\partial \alpha} = 0, \quad \tilde{Z} (\phi^i, \epsilon) = \tilde{Z}_R (\phi_R),$$

(2.34)

We would like to stress that $Z$ is the basic object which defines the first-quantized string theory. In particular, it is $Z$ and not the amplitudes which satisfies the renormalization property. Since a renormalization of logarithmic divergences in $\tilde{Z}$ corresponds to a subtraction of the massless poles in the amplitudes $1^1, 2^2$, the renormalized value $\tilde{Z}_R$ of $\tilde{Z}$ should coincide $4^0$ with the effective action (EA) $S$ which reproduces the massless sector of the string $S$-matrix. Let $\mu$ be a renormalization parameter, $Z = Z_R (\phi_R (\mu), \mu)$. Then

$$\frac{\partial Z_R}{\partial \mu} = 0, \quad \tilde{Z}_R = \frac{\partial}{\partial \mu} Z_R = -\beta (\phi_R) \frac{\partial}{\partial \phi_R} Z_R (\phi_R).$$

(2.35)

Using (2.7) (for $\chi = 2$) and redefining the fields to absorb the finite part of $W$, we get $Z_R \sim \int d^Dy \sqrt{G} e^{-2 \phi}$. Substituting this into $\tilde{Z}_R (2.35)$ we finish with $2^11$

$$\tilde{Z} (\phi_R) = S (\phi) = a \int d^Dy \sqrt{G} e^{-2 \phi} \tilde{X}_R = \tilde{X}_R (\epsilon)$$

(2.36)

(we used $\phi$ instead of $\phi_R$ to simplify the notation). Comparing (2.21) and (2.36) we conclude that EA which reproduces the massless string amplitudes coincides with the functional appearing in the relation (2.19) for the Weyl anomaly coefficients. Since $\kappa' \neq 0$ (nondegenerate within the $\alpha'$-expansion) the string effective low energy equations of motion $\partial^2 \phi = 0$ are thus equivalent to the conditions of the Weyl invariance of the corresponding renormalizable sigma model.

In view of (2.35) $\tilde{Z}_R$ can be represented in terms of the scale anomaly $2^11$

$$S = \frac{\partial}{\partial \epsilon} Z_R = \int d^Dz \sqrt{g} \partial \phi >.$$

(2.37)
While a straightforward generalization of the scale anomaly action (2.37) may not reproduce the full string equations with all massive modes included (see Ref.43) it certainly gives the correct effective equations of motion in the massless sector. Let us note also that $\frac{\partial S}{\partial \theta} = -\frac{\partial}{\partial \theta} V_i > 0$ is proportional to the massless tadpole computed in a background, i.e.

$$\frac{\partial S}{\partial \theta} = -\frac{\partial}{\partial \theta} < V_i > R = A_{1i} \quad (2.38)$$

Hence the effective equations of motion are indeed equivalent to the vacuum stability condition, i.e. to the vanishing of the massless tadpoles in a background.

Eq. (2.37) gives the explicit representation for the effective action in terms of the partition function of the $\sigma$-model. It is easy to derive for, example, the Einstein term in the EA.

According to (2.7), (2.15) ($\chi = 2$; we set $D = 26$ for simplicity)

$$Z \sim \int d^2 \sqrt{-g} e^{-\frac{\alpha'}{2} R_{\text{ene}} + \ldots} \quad (2.39)$$

Since the order R term corresponds to the (or $k^2$-term in) 3-graviton correlator on the sphere, $T_1 \approx < V_i V_j V_k >$, the $\theta$-divergence in (2.39) can be interpreted as the Möbius divergence present in the regularized expression for $T_1$. Differentiating over $\text{ene}$ we thus obtain the $(k^2 - 1)$-piece of usual 3-graviton (Möbius gauge fixed) amplitude which corresponds to the order $R$ term in the EA

$$S = \frac{\partial Z}{\partial \theta} = \int d^2 \sqrt{-g} e^{-\frac{\alpha'}{2} R_{\text{ene}} + \ldots} \quad (2.40)$$

To summarize, there is a deep connection between the low energy (massless sector) string dynamics and the renormalizable $\sigma$-model at the string tree level which is expressed by the following relations

$$\frac{\partial S}{\partial \theta} = 0 \implies \beta = 0 \implies A_{1i} = 0 \quad (2.41)$$

where $S$ is the effective action reconstructed from the massless string amplitudes, $\beta$ are the Weyl anomaly coefficients ($\beta$-functions) of the $\sigma$-model and $A_{1i}$ are the massless 1-point amplitudes computed in a massless background. Moreover, $S$ can be explicitly represented in terms of the Weyl anomaly coefficients (see (2.36), (2.37)).

A physical explanation of the perturbative renormalizability of the $\sigma$-model is that if the external momenta (or derivatives of $\phi$) are small (compared to $\alpha'^{-1/2}$), we get only the singularities corresponding to massless poles (there is no energy enough to produce massive poles).

It would be very interesting to generalize the above string theory $\sigma$ model correspondence to the string loop level. One of the consequences of the equivalence between the string equations of motion and conditions of conformal invariance of the $\sigma$-model is that it makes possible to avoid the difficult problem of determining the stationary points of the EA (reconstructed order by order from string amplitudes), replacing it by the problem of classification of 2-d conformal field theories. Similar equivalence at the loop level would provide a way of studying exact string vacua solutions by solving some generalized "conformal invariance" conditions.

3. The property of the string generating functional $Z$ which makes possible to establish the connection with $\beta$-functions is its renormalizability with respect to the local world sheet infinities. To be able to extend the string-$\sigma$ model relation to string loop level it is necessary to have the corresponding renormalizability property of the loop-corrected generating functional. Before turning to the analysis of string loop corrections it is instructive to add some remarks about renormalization of $Z$ at the string tree level. Consider the partition function $Z = e^{-\frac{\alpha'}{2} R_{\text{ene}}} > 0$ expanded in powers of $\phi$, see (2.31). The regularized correlators $T_N$ contain divergences proportional (in view of the factorization property) to $T_N$ with $M < N$. In fact, the divergences come from the regions where 2 or more integration points coincide; if we consider only the low momentum massless particles as the external ones, the operator product $V_1 V_2$ closes on dimension 2 massless vertex operators

$$\phi = \phi + \theta e^{-\frac{\alpha'}{2} R_{\text{ene}}} B_1 \phi^2 + B_2 \phi^2 + \ldots + O(\Theta^N) \quad (2.42)$$

The usual argument for perturbative renormalizability of $Z$ is based essentially on dimensional (or power counting) considerations. The coefficients which appear in the counterterms (2.42) are in fact proportional to the string scattering amplitudes (see Refs. 1, 44, 8 and references therein).

To see this one should introduce a background $\delta$ and compute the counterterms extracting the local order $V_i [\delta] e^{-\frac{\alpha'}{2} R_{\text{ene}}}$ term in $W[\delta] = -e^{-\frac{\alpha'}{2} R_{\text{ene}}}$. Expanding in power of $\phi$ it is possible to check that the overall $\phi$ singularity in $< V_i V_j V_k >$ ... $V_N (x + \delta) >$ comes from the region where all $N$ points are close to each other. Factorizing on the pole one finds that the coefficient is the usual $N + 1$-point amplitude $A_{N+1}$ (computed with the Möbius gauge fixed, e.g. $z_1 = 0, z_2 = 1, z_{N+1} = \infty$). Thus the $\beta$-function is $\beta = \sum_{N+1} B_N \phi^N$, $B_N = (A_{N+1}, \text{loop})$ (to obtain the $\beta$-function we should subtract all subleading singularities). This result is obviously consistent with the $\beta \sim \frac{\alpha'}{2} R_{\text{ene}}$ relation (2.19) since the EA is $Z \sim \sum_{N} A_{N+1} \phi^N$.

It would be very interesting to generalize the above string theory $\sigma$ model correspondence to the string loop level. One of the consequences of the equivalence between the string equations of motion and conditions of conformal invariance of the $\sigma$-model is that it makes possible to avoid the difficult problem of determining the stationary points of the EA (reconstructed order by order from string amplitudes), replacing it by the problem of classification of 2-d conformal field theories. Similar equivalence at the loop level would provide a way of studying exact string vacua solutions by solving some generalized "conformal invariance" conditions.
field theory. By this one implies that $\hat{Z}$ should be equal to the generating functional $s$ for the field theory $S$-matrix. The idea is to establish the renormalizability property of $s$ introducing a cutoff through the space-time propagator and assuming the proportionality (2.19) between the $\beta$-functions and the derivative of the EA $S$. Let

$$S(\varphi) = \frac{1}{2} \psi \Delta_U \psi + U(\varphi)$$  \hspace{1cm} (2.43)

Consider the functional

$$S(\varphi) = W(J) - \alpha \Delta_U,$$  \hspace{1cm} (2.44)

where $W$ is the generating functional for connected Green functions. In the tree approximation

$$W(J) = S(\varphi) - \varphi J,$$  \hspace{1cm} (2.45)

The on-shell $S$-matrix generating functional is

$$s(\varphi_m) = s(\varphi_n) - \varphi_m \Delta \varphi_n.$$  \hspace{1cm} (2.46)

It is convenient not to specify $\varphi$ to be equal to $\varphi_m$ and hence $\varphi$ to be equal to $\varphi_n$ at the intermediate stages (see also Ref. 45). Then

$$s(\varphi) = \frac{1}{2} \psi \Delta U\psi + U(\varphi(\varphi)) + \frac{1}{2} (U' \Delta^{-1} U')_m \varphi_m,$$  \hspace{1cm} (2.47)

$$\varphi = \varphi - \Delta^{-1} U(\varphi), \quad U' = \frac{dU}{d\varphi},$$  \hspace{1cm} (2.48)

$$s(\varphi_m) = U(\varphi_m(\varphi_m)) + \frac{1}{2} (U' \Delta^{-1} U')_{m,n} \varphi_m \varphi_n = U(\varphi_m) - \frac{1}{2} (U' \Delta^{-1} U')_{m,n} + \ldots$$

Now let us introduce a cutoff by a formal substitution

$$\Delta^{-1} \rightarrow \Delta^{-1}(\xi, \Lambda) \sim \epsilon \Lambda + 0(\epsilon \Lambda^2).$$  \hspace{1cm} (2.49)

For example, one may put (we use $\alpha'$ to match the dimensions)

$$\left(\alpha' \Lambda\right)^{-1} \rightarrow \int_0^1 \frac{dt}{t^2} e^{\Lambda t} = (\alpha' \Lambda)^{-1} - (\alpha' \Lambda)^{-1} e^{\alpha' \Lambda} =$$

$$-\Lambda \epsilon + \frac{1}{2} \alpha' \Lambda^2 \epsilon + \ldots$$  \hspace{1cm} (2.50)

In general (cf. (2.29))

$$\frac{\partial \Delta^{-1}}{\partial \epsilon} = -\pi(\epsilon, \Lambda), \quad \pi = \alpha' P^{-1} + 0(\epsilon \Lambda^2).$$  \hspace{1cm} (2.51)

Note that $\Delta^{-1}$ appears in $s$ (2.47) explicitly as well as implicitly through $\varphi$. Differentiating $s$ with respect to $\Lambda$ we find

$$\frac{\partial \Delta}{\partial \epsilon} = \frac{1}{2} \psi (U + \Delta^{-1} U') + \frac{1}{2} (U' \Delta^{-1} U')'$$

$$= \frac{\partial}{\partial \epsilon}, \quad \Lambda \equiv \epsilon \Lambda.$$  \hspace{1cm} (2.52)

Using that $\varphi = -(1 + \Lambda^{-1} U') \Delta^{-1} U'$ we finish with

$$\frac{\partial s}{\partial \Lambda} = -\frac{1}{2} (U' \Delta^{-1} U') \varphi = -\frac{1}{2} (U' \Delta^{-1} U') \varphi =$$

(similar relations appeared in Ref. 46).

On the other hand, if $s$ were renormalizable, we would have ($\varphi$ are supposed to be the bare couplings)

$$\frac{\partial s}{\varphi} - \frac{\partial s}{\varphi} = 0,$$  \hspace{1cm} (2.54)

Let us assume that the $\beta$-functions satisfy the relation (2.19) with $S$ given by (2.43). Then (2.54)

implies that

$$\frac{\partial s}{\varphi} = \frac{\partial s}{\varphi} = \frac{\partial s}{\varphi} = 0,$$  \hspace{1cm} (2.55)

If we go on shell ($\varphi = \varphi_m$) and compare (2.55) with (2.53) we get

$$\frac{\partial s}{\varphi_m} = \frac{1}{2} \psi U' \left(\lambda + \frac{1}{2} \Lambda \right) \left(\lambda - \frac{1}{2} \Lambda \right) \left(\lambda + \frac{1}{2} \Lambda \right) \left(\lambda - \frac{1}{2} \Lambda \right)$$

Since $\varphi_m = \varphi_n + 0(\epsilon \Lambda)$ and $\Delta = \frac{1}{2} \alpha' P^{-1} + \ldots + \frac{1}{2} \Lambda + \ldots$ (see (2.30), (2.51)) we conclude that Eqs. (2.53) and (2.55) are in agreement to the leading order. This implies that $\alpha$ is indeed renormalizable at least to the $\epsilon \Lambda^2$-order. In general, (2.53) would be equivalent to (2.55) if $\kappa$ were such that

$$\kappa = -\frac{1}{2} \psi U' \left(\lambda + \frac{1}{2} \Lambda \right) \left(\lambda - \frac{1}{2} \Lambda \right) \left(\lambda + \frac{1}{2} \Lambda \right) \left(\lambda - \frac{1}{2} \Lambda \right)$$

It seems plausible that this relation can be satisfied by properly adjusting $\alpha$ order by order in $\alpha'$.

The argument for the renormalizability of $\hat{Z}$, we have just given should generalize to string loop level. It is natural to assume that the massless sector of the full loop corrected string $S$-matrix can be reproduced by the tree $S$-matrix corresponding to some effective action. This is so if a string field theory exists and should follow in general from the conditions of factorization and unitarity. We shall return to this topic in Sec. 5.

* Note that the "diffeomorphism terms" on which $\hat{P}$ differ from $\hat{P}$ (see (2.11)) drop out from the products if $S$ (and $s$) are covariant.
3. STRING LOOP DIVERGENCES AND THEIR RENORMALIZATION

To understand how the renormalization group can be defined to act in string loops we are first to analyze the divergences which may appear in string loop corrections. We shall limit the discussion to the case of the massless external particles. Let us recall that the tree string amplitudes do not contain momentum independent singularities. Namely, they diverge in some regions of momentum space but are convergent in the others and hence can be defined to be finite (modulo poles) for all momenta using the analytic continuation.

To make contact with the σ-model and RG approach it is necessary, however, not to use the analytic continuation but to introduce a short distance cutoff in the 2-d propagator (and also to expand in external momenta if we are interested in the low-energy RG realized on the massless fields corresponding to the renormalizable couplings of the σ-model).

To regularize the string loop amplitudes it is not sufficient to introduce a cutoff in the 2-d propagator: we should regularize the divergences which may come from the boundaries of the integration regions for both the Koba–Nielsen and the moduli parameters. Such a regularization can be introduced in a more or less systematic way in the parametrizations in which the moduli are represented as coordinates of points on a complex plane (the examples are the branch point and Schottky parametrizations). In these parametrizations all the divergences can be regularized by imposing the restriction that any two points on C must be separated by a distance greater than ε.

Let us now recall the classification of possible singularities in closed string loop amplitudes. There are two basic types of singularities which appear a general n-loop N-point amplitude:

1. Singularities corresponding to shrinking of a "non-dividing" (non-trivial) cycle(s) on the 2-surface. The "momentum-dependent" singularities of this type are the usual unitary singularities (discontinuities) which are due to loops of internal particles. The "momentum-independent" "non-dividing" singularities in the vacuum amplitude (string partition function) and the massless amplitudes may be attributed to the tachyon loop, i.e. interpreted as being due to the tachyon present in the tree level spectrum of the Bose string theory in the standard flat space vacuum. Similar singularities are found in the scattering amplitudes for massive states in the superstring theory.

2. "Non-dividing" singularities in the vacuum amplitude (string partition function) and the massless amplitudes may be attributed to the tachyon loop, i.e. interpreted as being due to the tachyon present in the tree level spectrum of the Bose string theory in the standard flat space vacuum. Similar singularities are found in the scattering amplitudes for massive states in the superstring theory.

Now let us note, however, that such a restriction does not introduce a "built in" cutoff in the theory: there still remains a difference between the regularization of "local" infinities (which can be done by inserting a cutoff in the propagator) and a regularization of "modular" infinities which is done "by hand". For example, nothing prevents us from using different cutoffs for different parameters (say ε for local and λε for modular). This may lead to ambiguities in the coefficients of some divergences.

* Still, from geometrical point of view, it is not at all obvious that one should not regularize the "non-dividing" singularities. In fact, if we introduce a "universal" (e.g. geodesic distance) cutoff on the 2-surface, it automatically regularizes also the "non-dividing" singularities (since the lengths of all cycles then are greater than ε). For a recent attempt to regularize the "non-dividing" singularity of the one-loop partition function see Ref. 50.

** Using, e.g., a kind of analytic continuation prescription or renormalizing the tachyon coupling in the string action (cf. the discussion of "massless" renormalizations below).
The idea is that properly choosing $M_v$ and using (3.1) one can make the total (summed over genera) amplitude free of the external leg divergences. Since $<V_i V_j>$ itself contains self-energy subdivergences

$$<V_i V_j>_n \sim \sum_{\ell \leq n} <V_\ell V_v>_{t \ell} \epsilon v e <V_i V_j>_n + o(\epsilon v^2 e)$$

(3.3)

it is natural to anticipate the exponentiation of the "rudimental" self-energy corrections,

$$z_{i\ell} = \exp(\sum_{n} \mu_n \epsilon v e)$$

(3.4)

where $\mu_n$ is the massless 2-point amplitude on genus $n$ with all self-energy subdivergences subtracted out. This exponentiation was checked on the example of the one loop tachyon leg correction in Ref. 18. Note that (3.4) can be true only if the relative weights of string loop corrections take particular values. As we shall see, the exponentiation of divergences is crucial in order for the RG to act in string loops (see also Ref. 14).

Let us now consider the logarithmic tadpole divergences. A careful analysis of factorization of the closed string amplitudes in the "tadpole" limit gives the following result (see Ref. 15 and refs. therein)*

$$A_N^{n_0} \sim \sum_{n=1}^{n_0} A_n^{n_0} \epsilon v e$$

(3.5)

$$\Lambda_n^{n_0} \sim <V_1 \ldots V_n GHz>_n \delta_n \sim <1 >_n$$

(3.6)

$$\Omega_n = \frac{1}{4 \pi \alpha'} \int d^2 z \sqrt{g} [2 n : \partial_2 x^i \partial^2 x^i : + \alpha'(1 - n) R(2)]$$

(3.7)

where normal ordering is with respect to the propagator on the sphere.

$$: \partial_2 x^i \partial^2 x^i : = \partial_2 x^i \partial^2 x^i + \frac{1}{4 \alpha'} D R(2)$$

(3.8)

We assume that a regular metric is introduced on the world surface, so that the Euler number is

$$\chi_n = \frac{1}{4 \pi} \int d^2 z \sqrt{g} R(2) = 2(1 - n)$$

(3.9)

$\delta_n$ is proportional to a massless tadpole or, equivalently, to the vacuum amplitude. It is possible to rewrite (3.5) in the following way

$$A_N^{n_0} \sim -4 \sum_{n=1}^{n_0} <V_1 \ldots V_n (1/D_v) <V_\ell V_v>_n$$

where

$$V_v = \frac{1}{4 \pi \alpha'} \int d^2 z \partial_2 X^i \partial^2 X^i$$

(3.11)

is the (trace of) soft graviton vertex operator, and (see Refs. 14-16)

$$<V_\ell V_v>_n \sim 1/4 d(D^2 - 2) <1 >_n$$

(3.12)

The $\epsilon v e$ which appears in (3.5) and (3.10) originates from the graviton and dilaton propagators at zero momentum. In fact, one can represent $\Omega_n$ (3.7) as the combination of the vertex operators for the trace of the soft graviton and the soft dilaton

$$\Omega_n = 2 V_v - \chi_n V_v = 2 V_v + 2(n - 1) V_v$$

(3.13)

$$V_v = \frac{1}{4 \pi \alpha'} \int d^2 z \sqrt{g} (\partial_2 x^i \partial^2 x^i : - \frac{1}{2} \alpha' R(2))$$

(3.14)

$$V_v = \frac{1}{4 \pi \alpha'} \int d^2 z \sqrt{g} (\partial_2 x^i \partial^2 x^i + \frac{1}{4 \alpha'} (D - 2) R(2))$$

(3.15)

Note that the dilaton couples to the Euler number (cf. (2.3)). We shall also use another equivalent representation for $\Omega_n$

$$\Omega_n = \frac{1}{4 \pi \alpha'} \int d^2 z \sqrt{g} (\partial_2 x^i \partial^2 x^i : + \alpha'(1 - n))$$

(3.16)

$$f_{n} = 2 n, \quad \Omega_n = 1 + \frac{1}{4 \alpha'} (D - 2)$$

(3.17)

Observing that (3.15) looks similar to the string action (2.3) in a trivial vacuum

$$G_{mn} = \delta_{mn}, \quad \phi + \epsilon v e = const, \quad g = string coupling$$

(3.18)

one may try to cancel the tadpole divergences (3.5) in the total amplitude $A_N = \sum_n A_N^{n_0}$ by adding the counterterms to the vacuum string action 9. To illustrate the idea of renormalization consider, e.g., the genus 3 example. Different tadpole factorizations of 3-loop amplitude give the following divergences

$$(A_N^{(3)} \text{tadpole} \sim <(\Delta_1 \chi) \epsilon v e \Delta_2 \epsilon v e \Delta_3 \epsilon v e>_n + ...) \sim (\Delta_1 \chi) \epsilon v e \Delta_2 \epsilon v e \Delta_3 \epsilon v e>_n$$

(3.19)

where $\dot{\delta}_1 \sim \dot{\delta}_2 \sim \dot{\delta}_3 \sim <1 >_n$ are, in general, divergent, containing tadpole subdivergences

$$\dot{\delta}_1 = \dot{\delta}_2 = e_1 \dot{\delta}_3 \epsilon v e + \dot{\delta}_2$$

(3.19)
\[ d_1 = e_1 d_1 e_1 + e_2 d_2 e_1 e_1 + d_1, \quad d_2 = (d_0)_{\text{finite}}, \quad e_1 = \text{const} \]

Suppose now that the correlators are computed with the "bare" string action \( I = I_R + \delta I \), where the counterterm is a power series in the (renormalized, see below) string coupling and \( \bar{e}n_e \). Expanding in the (renormalized) string coupling we then get insertions of \( \delta I \) in the correlators and hence may hope to cancel the tadpole divergences between different terms in \( A_N \) (for example, \( < \ldots e^{-\delta I} >_0 + < \ldots >_1 + \ldots = \ldots + < \ldots (-\delta I) >_0 + < \ldots d_1 \bar{e}n_e >_0 + \ldots \approx \text{finite} \)). The resulting expression for the counterterm is

\[ \delta I = \sum_{m=1}^{\infty} b_0(e)O_m \bar{e}n_e + 0(\bar{e}^2) \]  
(3.20)

\[ = \sum_{m=1}^{\infty} b_0 O_m \bar{e}n_e + 0(\bar{e}^2), \quad b_0 \sim d_0, \quad b_m \sim d_m. \]  
(3.21)

Though it may seem that the factorization suggests that the \( \bar{e}n^m e, m \geq 2 \) terms are absent in (3.21), the consistency of the RG suggests that they should be present (and have coefficients related to \( (d_m)_{\text{finite}} \)). Using the expression for \( O_m \) (3.15) one can rewrite \( \delta I \) as

\[ \delta I = \frac{1}{4\pi\alpha'} \int d^2 z [\partial_\sigma \sigma^\mu \partial_\sigma z^\mu q_1 + \alpha'(R^{(2)}) q_1], \]  
(3.22)

\[ q_1 = q_1^{(1)} \bar{e}n_e - q_1^{(2)} \bar{e}^2 + \ldots , \]  
(3.23)

where \( q_1^{(m)} \) are power series in the (renormalized) string coupling constant. Comparing (3.22) with the vacuum string action (2.3), (3.17) we conclude that the tadpole divergences may be absorbed into a renormalization of the constant vacuum values of the metric and the dilaton, or, equivalently, into a renormalization of the two basic constants: \( \alpha' \) and the string coupling \( g \)

\[ (\alpha'(e))^{-1} = (\alpha'(q))^{-1} + q_1 (g_R, e), \quad \bar{e}ng(e) = \bar{e}n g + q_2 (g_R, e). \]  
(3.24)

The consistency of this renormalization procedure depends crucially on whether the relative weights of string diagrams are such that the tadpole divergences actually exponentiate. One may consider the exponentiation to be a consequence of the condition of factorization which together with unitarity \(^{30}\) fixes the weights of string amplitudes. Alternatively, one may impose the requirement of renormalizability (which implies exponentiation) and as a consequence, fix the relative weights of string diagrams.

4. GENERATING FUNCTIONAL (OR \( \sigma \) MODEL) APPROACH TO RENORMALIZATION OF STRING LOOPS

The results of the previous section about the elimination of the tadpole and external leg divergences through the renormalization of \( \alpha' \) and \( g \) and the vertex operators can be reinterpreted in a more fundamental way as the renormalizations of the couplings of the \( \sigma \)-model which appears in the generating functional for string amplitudes \( Z \). \( Z \) is the basic object which defines the theory

\[ Z = \Omega^{-1} \sum_{m=0}^{\infty} c_m \int d\mu(g) \int_{\mu} D\sigma \exp(-\Gamma), \]  
(4.1)

\[ I = I_0 + \phi V_\phi, \quad Z \sim \sum_B A_N \phi^N. \]

In general, \( I_0 \) is the string action in a particular vacuum and \( V_\phi \) are the corresponding "massless" vertex operators. We assume that "extended" sets of moduli \( \{m\} \) are used so that the M"obius group volume factor \( \Omega^{-1} \) is present for all genera. \( c_n \) are normalization factors (weights) which we prefer to indicate explicitly. Expanding \( Z \) in powers of \( \phi \) and putting them on shell (\( \phi^0 = 0 \)) we get the on-shell amplitudes as the coefficients.

\( Z \) is formally defined for arbitrary ("off-shell") values of the fields \( \phi \). In (4.1) we do not integrate over the conformal factor of \( 2 \)-metric, fixing a Weyl gauge. The basic consistency requirement is that \( Z \) evaluated for the "true" vacuum values of couplings \( \phi \) (which solve the string equations of motion or generalized Weyl invariance conditions "\( \beta = 0 \)"") should be Weyl gauge independent (i.e. Weyl invariant) \(^*\).

Suppose now that a short distance cutoff is introduced in (4.1), which regularizes, in particular the "local," "tadpole" and "external leg" divergences. We can cancel the local divergences separately for each genus by choosing the bare fields to correspond to the "local" \( \sigma \)-model counterterms (see Sec. 2). To cancel the "modular" divergences we are to combine the contributions of different genera. Since it is the product \( \phi^I V_\phi \) that appears in (4.1) we trade the renormalization of the vertex operators (3.2) for the multiplicative renormalization of the couplings. The counterterm (3.20), (3.22) corresponds directly to the renormalization of the metric and the dilaton in the \( \sigma \)-model action (2.3). The corresponding bare couplings are

\[ \phi^i = \phi_R^i + \delta_{\text{bare}} \phi^i + \delta_{\text{mod}} \phi^i, \]  
(4.2)

\[ \delta_{\text{bare}} \phi^i = q_1^i (e) \phi_R^i + q_2^i (g_R, e), \]  
(4.3)

\* There is, of course, an ambiguity in splitting the metric on a conformal factor and moduli for each particular \( n \). It would be nice to have a universal prescription, which relates the splittings for different \( n \). We anticipate that this ambiguity can be "absorbed" into redefinitions of the couplings \( \phi^i \).
The coefficients which appear here are power series in the string coupling (see also ref. 14). The crucial point is that if \( \hat{Z} \) is renormalizable
\[
\hat{Z}(\varphi(\epsilon), \epsilon) \sim \hat{Z}_0(\varphi_0) + \hat{O}(\epsilon)
\]  

(4.4)
it is possible to introduce the \( \beta \)-functions (see (2.34), (2.35))
\[
\frac{\partial \hat{Z}}{\partial \epsilon e_c} - \beta'(\varphi) \frac{\partial \hat{Z}}{\partial \varphi} = 0, \quad \beta'(\varphi) \equiv - \frac{d \varphi'}{d \epsilon e_c}.
\]  

(4.5)
In general, the relation between the bare and renormalized couplings is
\[
\varphi' = \varphi_R + T_1^2(\varphi_R) \epsilon e_c + T_2^2(\varphi_R) \epsilon^2 e_c^2 + \ldots.
\]  

(4.6)
The basic restriction imposed by the RG is that \( \beta' \equiv - \frac{d \varphi'}{d \epsilon e_c} \) should depend only on \( \varphi' \) but not explicitly on \( \epsilon \). This implies that
\[
\beta'(\varphi) = - T_1^2(\varphi), \quad T_2^2(\varphi) = \frac{1}{2} T_1^2(\varphi) \frac{\partial}{\partial \varphi} T_1^2(\varphi), \text{ etc.}
\]  

(4.7)
We get from (4.2)
\[
\beta' = - a_1 - M_{11}^2 \varphi'.
\]  

(4.8)
Thus the tadpole counterterm corresponds to the "inhomogeneous" term in the \( \beta \)-function while the external leg one- to the term linear in the fields. The coefficients in (4.8) have the following structure (see (3.4), (3.21))
\[
a_1 = \sum_{m=1}^{\infty} b_n^2 \mu_n^2, \quad M_{11}^2 = \sum_{m=1}^{\infty} b_n^2 \mu_n^2,
\]  

(4.9)
where \( b_n^2 \) and \( \mu_n^2 \) are proportional to the finite parts of the massless tadpoles and 2-point functions on genus \( n \); they are both proportional to the genus \( n \); vacuum amplitude.

Recalling the structure of the counterterm (3.22) we can write the metric and the dilaton \( \beta \)-functions in the following symbolic form \((G_{\mu \nu} = \delta_{\mu \nu} + h_{\mu \nu})\)
\[
\beta_{\mu \nu} = A^2 \delta_{\mu \nu} + B_{\mu \nu} \phi + B_{\mu \nu}^a h_{\mu \nu},
\]  

(4.10)
\[
\beta^a = A^a + B_a^a h_{\mu \nu}, \quad A \sim a_1, \quad B \sim M_1.
\]  

(4.11)
If the theory is defined (regularized) in a way consistent with general covariance in \( D \) dimensions, one should be able to rewrite (4.10) as
\[
\beta_{\mu \nu} = A^2 G_{\mu \nu}, \quad \beta^a = A^a + B_a^a \phi
\]  

(4.12)
(Note that A and B should be constants; the terms in (4.11) are the only covariant terms to the linear order in the fields \( h_{\mu \nu} \) and \( \phi \). Eq. (4.10) can reduce to (4.11) only if there are relations between the values of the massless tadpoles and the 2-point amplitudes.

In (4.11) \( G_{\mu \nu} \) and \( \phi \) are the full non-constant fields (constant vacuum values plus fluctuations). An important observation is that \( \beta \)'s are non-trivial functions of the constant parts of \( \phi \) since it is directly related (see (2.3) and Refs. 53, 2, 54) to the string coupling constant which appears in (4.9). Since we renormalize \( \phi \) we also should renormalize \( g \) and it is, in fact, the renormalized value of \( g \) (or \( \phi \)) which appears in the coefficients in (4.3). If we split \( \phi \) on the constant and non-constant parts
\[
\phi = \phi_0 + \phi, \quad \phi_0 = \log = \text{const}
\]  

(4.13)
we get the following expressions for the \( \beta \)-functions
\[
\beta_{\mu \nu} = \sum_{m=1}^{\infty} \lambda_1 e^{2 \gamma m} G_{\mu \nu},
\]  

(4.14)
\[
\beta^a = \sum_{m=1}^{\infty} \nu_a e^{2 \gamma m} + \sum_{m=1}^{\infty} \mu_a e^{2 \gamma m} \tilde{\phi}.
\]  

(4.15)
They look unnatural because of the different dependence on \( \phi \) and \( \tilde{\phi} \) and cry for a generalization.

The complicated dependence on \( \phi \), suggests that the modular counterterms (4.3) and the \( \beta \)-functions (4.8) should, in fact, contain all powers of the fields (in particular, of the dilaton). Then, e.g., the \( \Omega(\tilde{\phi}) \)-term in (4.14) should correspond to the linearization of the exact expression for \( \beta^a \) which is non-linear in \( \phi \). Thus the consistent dependence on \( \phi \) implies that \( \beta^a \) should have the form
\[
\beta_{\mu \nu}^a = F_1(\phi) G_{\mu \nu} + \ldots, \quad \beta^a = F_2(\phi) + \ldots
\]  

(4.16)
\[
F_1 = \sum_{m=1}^{\infty} \lambda_1 e^{2 \gamma m}, \quad F_2 = \sum_{m=1}^{\infty} \nu_a e^{2 \gamma m}.
\]  

(4.17)
where dots in (4.15) stand for other possible terms depending on derivatives of the fields. If we further use the explicit expression for the tadpole counterterm (3.20), (3.15) we get
\[
F_1 = - \sum_{m=1}^{\infty} 2 \nu_0 e^{2 \gamma m},
\]  

(4.18)
\[
F_2 = - \sum_{m=1}^{\infty} (1 + \frac{1}{2} \log D - 2) \nu_0 e^{2 \gamma m}.
\]  

(4.19)
To get a deeper understanding of the structure of the \( \beta \)-functions (4.15), (4.17), (4.18) let us consider the "zero-momentum" part of the generating functional \( \hat{Z} \) (4.1) or the string partition function computed for the constant vacuum values of \( G_{\mu \nu} \) and \( \phi \). Assuming that the string
The path integral is defined in a way preserving $D$-dimensional general covariance (the functional measure is consistent with a regularization, etc., see Refs. 16 and 22) we find (cf. (2.7))

$$\tilde{Z} = \tilde{Z}_0 + \tilde{Z}_1 + \tilde{Z}_2 + \ldots =$$

$$= \tilde{Z}_0 + \int d^Dy \sqrt{g} e^{-2\phi} \left( \alpha' R + \ldots + \alpha' R + \ldots + \phi \right) d^Dy \sqrt{g} (1 + \ldots)$$

$$= \tilde{S}_0 + \int d^Dy \sqrt{g} \omega (\phi) + \ldots,$$

(4.19)

$$\omega = \sum_{n=1}^\infty \delta_n e^{(2n-2)\phi} + \mu_0 \sim (\alpha')^{-1/2},$$

(4.20)

where $\delta_n (d_n)$ are the finite parts of the higher genus contributions to the vacuum amplitude (see (3.19)). In (4.19) we have included the tree level contribution (see (2.36), (2.21); $\tilde{S}_0$ denotes the tree level piece of the effective action). Consider now the divergent parts of $\tilde{Z}$.

$$\tilde{Z}(\phi, \varepsilon) = \tilde{Z}_{\mathrm{R}}(\phi) + \tilde{Z}^{(1)}(\phi) \varepsilon \varepsilon + \tilde{Z}^{(2)}(\phi) \varepsilon^2 \epsilon + \ldots$$

(4.22)

The renormalizability implies that (see (4.5))

$$\tilde{Z}^{(1)} = \beta^R \frac{\partial}{\partial \phi} \tilde{Z}_{\mathrm{R}}, \quad \tilde{Z}^{(2)} = \frac{1}{2} \beta^R \frac{\partial}{\partial \phi} \tilde{Z}^{(1)}.$$

(4.23)

The renormalized part $\tilde{Z}_{\mathrm{R}}$ should coincide with the effective action (since renormalization is simply a subtraction of massless exchanges). The operator $\beta^R \frac{\partial}{\partial \phi} \rightarrow F_1 G_{\mu\nu} - \frac{\partial}{\partial \phi} \varepsilon^2 / 2 F_2 \frac{\partial}{\partial \phi}$ (see (4.15)) represents the insertion of the tadpole counterterm, so that (4.23) is in correspondence with the factorization property of the amplitudes. Let us now show that the expressions for $F_1, F_2$ (4.17), (4.18) which follow from the factorization imply that the $\beta$-functions (4.15) are equivalent to the equations of motion following from the "cosmological term" in loop-corrected effective action (4.20). Namely, let us assume that the tree level relation (2.15) between the $\beta$-functions and the effective action holds also at the loop level

$$\beta^R = \kappa^R \frac{\partial S}{\partial \phi^R}, \quad \beta^R = \sum_{n=1}^\infty \beta_n \epsilon^n,$$

$$S = \sum_{n=0}^\infty \phi_n, \quad \kappa^R = \sum_{n=0}^\infty \kappa_n \epsilon^n.$$  

(4.24)

The loop corrections to $\kappa^R$ are not important to the leading order (recall that $\kappa^R$ (2.20) is necessary in order to account for the mixing between the metric and the dilaton in the kinetic term of the tree level EA). Applying (4.24), (2.20) to the case of $S$ given by (4.20) one finds (cf. (2.16))

$$\beta^R = \beta_0^R + \beta_{10}^R + \ldots, \quad \beta_n^R = \left[ \alpha' (R_{\mu\nu} + 2 \mathcal{D}_\mu \mathcal{D}_\nu \phi) + \ldots \right] + F_1 (\phi) G_{\mu\nu} + \ldots,$$

$$\beta^R = \left[ \frac{1}{6} (D - 26) - \frac{1}{2} \alpha' \mathcal{D}^2 \phi + \alpha' (\mathcal{D} \phi)^2 + \ldots \right] + F_2 (\phi) + \ldots,$$

(4.25)

$$F_1 = -\frac{1}{4} \varepsilon^2 \frac{d\omega}{d\phi}, \quad \omega \equiv \frac{d\mu}{d\phi},$$

(4.26)

$$F_2 = -\frac{1}{16} \varepsilon^4 \frac{d^2 \omega}{d\phi^2} (2 D\omega + (D - 2) \omega).$$

(4.27)

Substituting here the expression (4.21) for the dilaton potential $\omega (\phi)$, one finds that (4.26), (4.27) coincide with (4.17) and (4.18) if

$$b_n = \frac{1}{4} \delta_n d_n^{-1},$$

(4.28)

To find the bare couplings corresponding to the $\beta$-functions (4.15) we are to solve the equations $\beta (\phi) = -\frac{\partial \beta}{\partial \phi}$. If there were no higher order terms in $\beta^R$, i.e. if (4.8) were true, we would get (we take $M$' diagonal for simplicity)

$$\beta (\phi) = -e^{M(\phi)} \beta (\phi_R + M_{\phi_R}^{-1} a_1) - M_{\phi_R}^{-1} a_1,$$

(4.29)

Thus the RG implies the exponentiation of the divergences (cf. (3.4)) and, in particular, the presence of higher order $\varepsilon^2$ counterterms with the coefficients fixed in terms of the coefficients appearing in the $\beta$-function. The bare fields corresponding to (4.15) are

$$G_{\mu\nu} = G_{\mu\nu} - F_1 (\phi) G_{\mu\nu} \varepsilon \varepsilon + \frac{1}{2} F_2 (\phi) F_2 (\phi) \varepsilon^2 \epsilon + 0 (\varepsilon^3 \epsilon)$$

(4.30)

where the $\varepsilon^2$-terms can be found by applying the general relations (4.7), (4.6). It would be a non-trivial check of the consistency of the RG approach to find that the string loops contain the $\varepsilon^2 \epsilon$-divergences which can be cancelled by inserting the counterterms (4.30) (or, equivalently, that the $\varepsilon^2 \epsilon$-part of $\tilde{Z}$ satisfies (4.23)).

Let us now discuss the origin of the higher order $O (\varepsilon^m)$, $m \geq 2$, terms in the $\beta$-functions (4.15). Obviously they correspond to the divergences which appear when an amplitude factorizes into two parts (with the numbers of external particles $N_1$ and $N_2$) and all external
particles on one part (say the second) have zero momentum (the tadpole divergence thus corresponds to the case when \( N_2 = 0 \)). Clearly, this is a particular case of "momentum dependent" singularity due to massless pole in \((k_1 + \ldots + k_N)^2\) which one may try to eliminate by using analytic continuation in external momenta. However, as we have argued above, it is necessary to account for (regularize) such divergences in order to get covariant expressions for the \( \beta \)-functions.

The continuity in external momenta then suggests that we should regularize the massless pole momentum dependent singularity also for non-vanishing external momenta. This is consistent with the fact that the pole singularities contribute to the Weyl (or BRST) anomaly and is obviously necessary for the correspondence between the \( \beta \)-functions and the effective action.

**5. REMARKS ON CORRESPONDENCE WITH EFFECTIVE ACTION**

We have already discussed the role played by the effective action in the RG approach at the tree level (see Sec. 2). Most of this discussion generalizes to loops. To construct the EA for the massless fields we start with the massless string scattering amplitudes (with loop corrections included) and subtract all massless poles (see also Ref. 14). To subtract the massless exchanges we are to consider all possible "dividing" factorizations of a string diagram. At the same time the corrections to the \( \beta \)-functions we discussed in Secs. 3 and 4 come precisely from the massless pole singularities which appear in these factorizations. Hence, qualitatively, we should have

\[
S \sim \sum_N (A_N)_{\text{moder}} \varphi^N, \quad \beta(p) \sim \sum_N (A_N-1)_{\text{moder}} \varphi^{N+1}, \quad \frac{\delta S}{\delta \varphi} \sim \beta.
\]

In general, we expect to find that: (1) the generating functional \( \hat{Z} \), (4.1) is renormalizable with respect to the infinities associated with all "dividing" singularities of massless string amplitudes ("momentum dependent" as well as "momentum independent" ones). As at the tree level, it is necessary to combine particular amplitudes into \( \hat{Z} \) in order to obtain an object which is renormalizable with respect to "momentum - dependent" infinities. (2) The EA \( S(p) \) is a renormalized (finite) value of \( \hat{Z} \) (since all the divergences in \( \hat{Z} \) are due to the massless exchanges, a subtraction of the latter should be equivalent to a renormalization procedure). (3) The resulting effective equations of motion are equivalent to a generalized conformal invariance conditions \( \beta^i = 0 \) and also to generalized vacuum stability conditions \( A_{ii} = 0 \) (the vanishing of tadpoles in a background, see (2.41), (2.38)).

As we have shown in Sec. 2 the renormalizability of \( \hat{Z} \) is closely connected with the existence of an effective action. The important point is that the existence of the EA (and hence renormalizability of \( \hat{Z} \)) depends on the values of relative weights of string diagrams. These weights are usually assumed to be fixed by the condition of factorization (and unitarity) which in turn should guarantee the existence of the EA. Alternatively, it is interesting to observe that the assumption of existence of the EA leads to constraints on the weights. Consider first the ordinary field theory, e.g. \( S_0 = \frac{1}{2} \varphi \Delta \varphi + \frac{1}{4} g \varphi^2 \). It is possible to represent its quantum effective action

\[
S(p) = W(J) - \varphi J, \quad \frac{\delta W}{\delta J} = \varphi, \quad e^{-W(J)} = \int D\varphi e^{-S_0 + \varphi J} (5.2)
\]

in the following way (cf. (4.1))

\[
S(p) = S_0(p) + \sum_n c_n \int dt \mu(t) \int d\tau(t) e^{-1}, \quad I = I_0 + I_{\text{int}},
\]

where \( \gamma_n \) are 1 - \( P_L \) graphs and \( \{ \tau \} \) are the proper time parameters ("moduli" of a 1-d metric \( e(t) \)). To arrive at this expression one should represent \( S \) in terms of the background dependent propagators and to use the path integral representation for the latter. The "weights" \( c_n \) or the relative normalizations of the modular measures are fixed by the functional integral definition of \( S \) (5.2). The generating functional for the quantum \( S \)-matrix is simply the tree generating functional for \( S \) (see (2.41), (2.46)). Thus the relative weights of the terms in the EA automatically determine the weights of various amplitudes in agreement with factorization property.

The situation in string theory is different since if we do not start with a string field theory functional integral we do not know a priori the relative weights in the "first-quantized" path integral (4.1)\(^*\). Suppose now that the massless sector at the string \( S \)-matrix can be reproduced by an effective field theory, i.e. that \( \hat{Z}(\varphi_{\text{mm}}) \) can be represented as the generating functional for the tree level \( S \)-matrix for some action \( S(p) \) (see (2.43), (2.46), (2.48))\(^*\)

\[
\hat{Z}(\varphi_{\text{mm}}) = S(\varphi_{\text{mm}}), \quad S(\varphi_{\text{mm}}) = S(\varphi_{\text{mm}}(\varphi_{\text{mm}})) - \varphi_{\text{mm}} \Delta \varphi_{\text{mm}}(\varphi_{\text{mm}}),
\]

\[
a = U(\varphi_{\text{mm}}) - \frac{1}{2} \left( U^\Delta - U \right) \varphi_{\text{mm}} + \ldots,
\]

\(^*\) It is commonly believed that all weights should be "ones" if one properly defines the measure of the Polyakov integral (to be ultralocal, etc. \(^{50}\)) and integrates over one copy of the moduli space.

\(^*\) A remarkable fact that distinguishes string theory from field theory is that the path integral representation is true not only for loop terms but also for the tree term in \( \hat{Z} \), cf. (4.1), (5.3).
The reason why eq (5.4) imposes constraints on the string weights is that it fixed the relative coefficients of the "one particle reducible" terms in terms of "one particle irreducible" ones. However both the reducible and irreducible graphs originate from one string diagram. Suppose we know the n-loop modular measure (and region of integration) up to one overall constant $c_n$. Expanding in powers of string coupling we have from (5.5)

$$s = \sum_{m=0}^{\infty} s_m, \quad s_n = U_n - \frac{1}{2} \sum_{m=0}^{n} U_m A^{-1} U_{m-n} + \ldots, \quad (5.6)$$

$$U = \sum_{m=0}^{\infty} U_m,$$

As is clear from (5.6), the first non-trivial relation is $c_2 \sim c_1^2$ (we shall discuss it in Sec. 8). It is not possible to fix $c_1$ using (5.6). To fix $c_1$ we should employ the unitarity constraint or compare the coefficient of the one loop term in $S$ with the coefficient of the usual $\log$ term in a field theory EA, (properly fixing the normalization of the background fields, cf. Ref. 54).

Since the massless propagators in $s$ correspond to the infinities in the regularized $\tilde{Z}$, eq (5.4) implies that $S$ coincides with the renormalized (finite) part of $\tilde{Z}$. The renormalizability of $\tilde{Z}$ and the relation (4.24) between $S$ and $\beta$-functions imply

$$\tilde{Z}(\phi, \epsilon) = S(\phi) + \beta(\phi) \frac{\partial S}{\partial \phi} \epsilon \epsilon_n + 0(\epsilon^2),$$

$$\tilde{Z} = S + \frac{\partial S}{\partial \phi^2} \epsilon \epsilon_n + 0(\epsilon^2). \quad (5.7)$$

This gives an "off-shell" analog of (5.5). Expanding in genus we have, to the lowest order (see (4.24))

$$\tilde{Z} = \sum_{n} \tilde{Z}_n, \quad \tilde{Z}_n = s_n + \sum_{m=0}^{\infty} \frac{\partial S_n}{\partial \phi^2} \delta \frac{\partial S}{\partial \phi} \epsilon \epsilon_n + \ldots. \quad (5.8)$$

Hence we can reformulate the above argument about the relation between weights of different string diagrams in the following way: since the divergent part of $\tilde{Z}_n$ should have the same overall coefficient as its finite part $(S_n)$ one can fix this coefficient in terms of the coefficients of the lower genus terms $S_m, m < n$.

Let us finish this section with a remark on possible solutions of the generalized conformal invariance equations $\beta_i + \beta_j + \ldots = 0$ or the effective equations of motion $\frac{\delta S}{\delta \phi} + \frac{\delta S}{\delta \phi^2} = 0$. Let us consider only the leading "cosmological" (dilaton potential) term in the EA (see (4.21)). The corresponding equations of motion are given by (4.25). Let us try to solve them for $\phi = \text{const}$ keeping only the leading term in the curvature, (i.e. assuming $(\alpha'/R) < 1$). We find

$$\alpha'(\phi) = -F_1(\phi) G_{\phi \phi}, \quad \frac{1}{\phi} (D - 26) = -F_2(\phi), \quad \phi = \text{const} \quad (5.9)$$

where $F_i$ are given by (4.26), (4.27) or (4.17), (4.18). Note that no solution exists unless $D \neq 26$.

These equations determine the vacuum values of $G_{\phi \phi}$ and $\phi$ (or $G_{\phi \phi}$ and $D$) (19). The problem with this solution (and, in fact, with all other solutions of loop corrected equations of motion in Bose string theory) is that the coefficients of the loop contributions to $S$ are, in general, complex because of the analytic continuation used to eliminate the tachyonic divergence (see the discussion in Sec. 3). In particular, the coefficients which appear in $F_i$ (4.17), (4.18) have imaginary parts. For example, in the one loop case one has

$$F_1(\phi) = -\frac{1}{2} d_1 d_0^{-1} e^{2\phi}, \quad F_2(\phi) = -\frac{1}{8} d_1 d_0^{-1} D e^{2\phi}$$

where $d_1$ is the coefficient of the one-loop (torus) vacuum amplitude $50-58$.*

As is clear from (5.6), the first non-trivial relation is $c_2 \sim c_1^2$ (we shall discuss it in Sec. 8). It is not possible to fix $c_1$ using (5.6). To fix $c_1$ we should employ the unitarity constraint or compare the coefficient of the one loop term in $S$ with the coefficient of the usual $\log$ term in a field theory EA, (properly fixing the normalization of the background fields, cf. Ref. 56).

Since is clear from (5.6), the first non-trivial relation is $c_2 \sim c_1^2$ (we shall discuss it in Sec. 8). It is not possible to fix $c_1$ using (5.6). To fix $c_1$ we should employ the unitarity constraint or compare the coefficient of the one loop term in $S$ with the coefficient of the usual $\log$ term in a field theory EA, (properly fixing the normalization of the background fields, cf. Ref. 56).

6. GENERATING FUNCTIONAL IN SCHOTTKY PARAMETRIZATION

In discussing renormalization of string loop corrections it is important to use a parametrization of moduli space which is based on the "extended" set of moduli parameters such that the $SL(2, C)$ Möbius group volume $\Omega$ appears as the universal inverse factor of all (tree and loop) contributions to on-shell string amplitudes. In this case one has a clear geometrical picture of factorization of the amplitudes with the moduli and the Koba–Nielsen (KN)

$$d_1 \approx \frac{\pi}{2(D/2)} (\alpha')^{1-D/2} (0.3 \ldots + i) \quad (5.11)$$

$\text{Im} d_1$ can be interpreted as a decay rate of the unstable (tachyonic) classical vacuum. The small (complex) shifts of the massless fields** which we find by solving the effective equations cannot eliminate the tachyon and hence do not, in fact, determine a consistent vacuum. This difficulty is absent in superstring theory.

* This expression is modular invariant only for $D = 26$ but we must keep $D$ arbitrary in order to cancel the total Weyl anomaly.

** It is interesting to note that we may keep $F_i$ real (and hence get real $G_{\phi \phi}$ and $D$) at the expense of making complex the vacuum value of the dilaton or the string coupling constant (see (5.9)).
parameters playing different roles in the degeneration limits. The separation of the universal $\Omega^{-1}$-factor for all genera makes it possible also to study the "exponentiation" of particular types of divergences.

One of such parametrizations is the Schottky parametrization (SP) which is also distinguished by its direct relation to the operator formalism and hence by a large amount of explicitness in the corresponding formulas (e.g. for the 2-d propagator and the string measure). Using SP one represents the $n$-loop Riemann surface (sphere with $n$ handles) in terms of the extended complex plane with $n$ pairs of holes cut out and chooses the $3n$ moduli to be (roughly) the coordinates of the centers of holes, their radii and the "twist" angles which appear in the identification of boundary circles necessary to represent the handles. Since both the KN points and the (relevant subset) of moduli appear as coordinates on $\mathbb{C}$ it is possible to regularize both the "local" and "modular" divergences (corresponding to shrinking of the dividing cycles on $\mathbb{C}$) in a universal way, introducing a short distance cutoff on $\mathbb{C}$. Since the mechanism by which the renormalization groups is implemented in string loops involves cancellations of divergences between contributions of surfaces of different genera it is also important that in SP all surfaces are represented universally in terms of $\mathbb{C}$ with holes (more topological structure, i.e. more holes and hence more moduli parameters for higher genus). These important advantages of the SP in extending the RG approach to string loop level were already appreciated in Refs. 9 and 18.

Let us now briefly represent the solution of the closed Bose string amplitudes in SP (for more information see Refs. 23b, 30, 31). Let $(T_a, \ a = 1, \ldots, n)$ be the elements of $SL(2, \mathbb{C})$ acting on $\mathbb{C}$. The infinite discontinuous Schottky group (SG) is generated by its products: $SG = \{T_a T_b \}$, $a \neq b$, $a, b, \ldots, n$ being rational, under the restriction that the fixed points of all the elements $T_a$ should form a discrete set and that their multipliers should satisfy $|\lambda_a| < 1$. By definition, if $z' = T_a z = z - k \bar{z} + \eta$, $AD - BC = 1$, it is represented as

$$ T_z \eta \eta - \eta = k \eta \eta \eta - \xi \xi \eta - \xi \xi, \quad A = \Delta(\eta - k \bar{\xi}), \quad B = \Delta \xi \eta(k - 1), \quad C = \Delta(1 - k), \quad D = \Delta(k \eta - \xi), 
$$

$$ \Delta^{-1} = \sqrt{\Delta(\xi - \eta)}. \quad (6.1) $$

$k$ is a multiplier of $T$ and $z$ and $\eta$ are the (regulus and attractive) fixed points. We also assume that the isometric circles $I_0$ and $I'_0$ of $T_0$ and $T_0^{-1}$ are exterior to each other. The fundamental domain $D$ is defined as $|\xi - \eta| = 1$. The moduli are thus the parameters of $SG$.

The moduli are thus the parameters of $T_a$, i.e. by the set of $3n$ complex numbers $\{\xi_1, \eta_1, \ldots, \xi_n, \eta_n\}$, $a = 1, \ldots, n$. The freedom of making the "overall" $SL(2, \mathbb{C})$ transformation reduces the number of independent moduli to the usual one $3n - 3$. Note that while $\xi_a$ and $\eta_a$ transform in the standard way under the $SL(2, \mathbb{C})$, $k$ remain invariant. This implies that torus ($n = 1$) is the special case: we cannot fix $SL(2, \mathbb{C})$ completely by fixing the positions of only two points ($\xi$ and $\eta$). The remaining symmetry (related to the M"obius $U(1)$ symmetry of the torus in the usual "parallelogram" representation) can be fixed only by fixing the position of one extra (KN) point. We shall consider the case of the torus in detail in the next section.

$\xi_a$ and $\eta_a$ lie inside the corresponding isometric circles. As is clear from (6.2) there are two singular limits in which the surface looses a handle and which correspond to shrinking of the $a_0$ and $b_0$ cycles on the Riemann surface ($a_0$ go around the isometric circles $I_0$ and $I'_0$ while $b_0$ connects a point on $I_0$ with its image under $T_0$ on $I'_0$). The first is "non-dividing" limit, in which $k_0 \to 0$ and hence the radii of the holes go to zero while their centres remain at a finite distance $|\xi - \eta|$. The second is "dividing" limit in which $k_0 \to 1$. As is clear from (6.1) this implies that at the same time $\xi_0 \to \eta_0$ (and vice versa). In this limit the transformation $T_a$ becomes parabolic (see e.g., the discussion in the paper by Allesandrini and Amati in Ref. 23b).

$\xi = \eta + k, \quad 1 - C \xi, \quad (\xi^2 - C)^{-1} = (\xi - C)^{-1} \xi. \quad (6.3)$

As a result, the invariant circles $I_0$ and $I'_0$ touch each other with their radii and the distance between their centres remaining finite. Let us note that making a global $SL(2, \mathbb{C})$ transformation on points of $\mathbb{C}$ and hence on $\{\xi_a, \eta_a\}$ one, in general, changes the picture of the fundamental domain $D$ and hence may represent the "dividing" limit in some different way.

Let us now give the expressions for the basic quantities in SP. The abelian holomorphic differentials are

$$ \omega_a = \sum_b \left( \frac{1}{x - T_a \eta_a} - \frac{1}{x - T_a \xi_a} \right) dx \equiv \omega_a(x) dx \quad (6.4) $$
where the sum goes over the elements of SG which do not have $T^m_a$ as their right most factor (recall that $\xi$ and $\eta$ are the fixed points). The period matrix is given by

\[ \omega_n = \int \omega_n(z) \, dz = \epsilon_n \langle n | \omega_n \rangle \]

(6.5)

where $T^m_a \neq T^m_b \ldots T^m_m$ and for $a = b$ the sum does not include the unit element. The prime form is

\[ \hat{E}(z, w) = \prod_{r} \left( z - w \right) \hat{E}(z, w), \]

(6.6)

where the product is over all elements of SG except 1 and $T^m_a$ and $T^m_1$ are counted only once.

The above expressions determine the Green function for the scalar Laplacian (with the constant zero mode projected out)

\[ V(z, w) = \sum_{\alpha} \frac{1}{4} \epsilon_{\alpha}^2 | \omega_\alpha(z, w) |^2 \]

(6.7)

\[ \langle \ldots \rangle = \int [dz] \exp \left( -\frac{1}{4} \epsilon(z,w) \right) \phi \phi^* \phi \phi^* \]

< ... >_n = \int [dz] \exp \left( -\frac{1}{4} \epsilon(z,w) \right) \, dz < 1 >_n = 1 ,

(6.11)

where the propagator is given by (6.7) and the modular measure is

\[ d\mu_n = \left( \prod_{\alpha} \left| \omega_\alpha - \omega_n \right|^2 \right) \left| \omega_n \right| \left( \det t_{ab} \right)^{-1/2} \]

(6.12)

where the product $\prod_{\alpha}$ goes over all primitive elements of SG. The factor in power $D$ is the contribution of the integral over $z^a$, while the contribution of the 2-d reparametrization ghosts is

\[ \prod_{\alpha} |1 - k^2_{\alpha}|^{-2} \]

The integration over the moduli $(k, \xi, \eta)$ should be restricted ("by hand") to go over the fundamental region of the modular group (which is not explicitly known in the SP). One may formally not restrict the region of integration assuming that the overall normalization constants can be finally fixed, e.g., from unitarity considerations.

The measure (6.12) is clearly projective invariant. Recalling the remark that the SL(2,C) transformation property of $\hat{E}(z, w)$ is the same as of the propagator on $C$ we conclude that the Möbius volume factor $\langle z^a(z,w) \rangle$ cancels out if the external momenta satisfy the tree level on-shell conditions. One can fix the Möbius symmetry by fixing the positions of the three parameters among $\xi, \eta, u$, and the KN points $z_i$. The usual expressions for the loop amplitudes are obtained by fixing $z_i, \xi, \eta$ and $z_i$ for $n = 1$ and, e.g., $\xi, \eta, \xi_1$ for $n > 2$ (note again that $\eta = 1$ is a special case).

It is, in principle, straightforward to study the factorization of string amplitudes in SP. One draws a cycle $\gamma$ on $C$ which one wants to pinch and considers the limit in which the points $z_1, \xi, \eta$ on $C$ uniformly come close to each other (e.g. $y = \eta + u, z_i = \eta + u, u \to 0$, etc.) A discussion of factorization in SP has already appeared in Ref. 18.

We would like to point out, however, that the general analysis of Ref. 15 implies that so ge! the consistent results, e.g., for the tadpole factorization one should properly account for the additional ghost or 2-d curvature insertions. The example of the factorization on the disc (see

\[ T \in \text{SL}(2,\mathbb{C}) \text{ and } T \in \text{SO}(2) \]

for $n = 1$ is a special case)

It is, in principle, straightforward to study the factorization of string amplitudes in SP. One draws a cycle $\gamma$ on $C$ which one wants to pinch and considers the limit in which the points $z_1, \xi, \eta$ which lie inside $\gamma$ uniformly come close to each other (e.g. $\xi = \eta + u, z_i = \eta + u, u \to 0$, etc.) A discussion of factorization in SP has already appeared in Ref. 18.

We would like to point out, however, that the general analysis of Ref. 15 implies that to get the consistent results, e.g., for the tadpole factorization one should properly account for the additional ghost or 2-d curvature insertions. The example of the factorization on the disc (see

\[ T \in \text{SL}(2,\mathbb{C}) \text{ and } T \in \text{SO}(2) \]

for $n = 1$ is a special case)
There remains a technical problem of finding a special choice of moduli with which $k$ are independent from \{cf. (6.3)\). This problem was ignored in Ref. 18.

Ref. 11, 13, 14, 15, 19 and Sec. 7) suggests that when the string amplitudes diverge one may get different results by using different representations for the world sheet (which are apparently conformally equivalent on shell). A consistent approach is to use a compact curved 2–space representation for the world sheet and to account for the curvature dependence in the modular measure originating from the frame dependence of the coordinates and radii of the holes on the surface. In general, if we start with \( C \) with a topological fixture of size \( a \to 0 \) at point \( w \) we should go to a curved surface representation and define the center of the fixture by \( b \).

\[
\omega = \int_{|\omega|} d^2 z \sqrt{g(z + w)(z + w)} \int_{|\omega|} d^2 z \sqrt{g(z + w)} \simeq \omega + O(a^2). \tag{6.13}
\]

Then

\[
\frac{d\omega}{a^2} = \frac{d\omega}{a^2} \sqrt{g(\omega)} (1 + \frac{1}{a^2} R(\omega) + O(a^4)) \tag{6.14}
\]

We shall return to the discussion of this point on the example of the disc in Sec. 7 below.

Eq. (6.10) suggests the following expression for the generating functional for string amplitudes in SP (cf. (4.1))

\[
\tilde{Z} = \sum_{n=0}^{\infty} Z_n, \quad \tilde{Z}_n = \Omega^{-1} \tilde{Z}_n \tag{6.15}
\]

\[
\tilde{Z}_n = c_n \int d\mu_0 \int_{s_n} [dx] e^{-l}, \quad Z_n = \int [dx] e^{-l}, \tag{6.16}
\]

where \([dx]\) is normalized so that \(\int [dx] \exp(-l_0) = 1\) (i.e. the determinant factors are included into \(d\mu_0\)). \(Z_n\) is the partition function of the \(\sigma\)-model integrated over the moduli. We can introduce a short distance regularization into \(Z_n\) by demanding that any of the points \((\xi, \eta, z)\) cannot come closer than at the distance \(\epsilon \to 0\). While it is possible to regularize the local infinities in a systematic way by using a "built-in" cutoff in the propagator \(D\) (see (2.5)) we are still to insert the cutoff explicitly in the integrals over the moduli. Note that we do not introduce a regularization for the integrals over the multipliers \(k_0\) since, as we have already discussed in Sec. 3, the "non-dividing" limits (corresponding to \(k_0 \to 0\), etc.) do not produce the Weyl or BRST anomaly \(\omega\)-factor in (6.15) (the integrals over \(k_0\) should be defined using an analytic continuation to eliminate the tachyonic divergences). * **

We are assuming that the regularization we use is 2-d covariant, i.e. that \(\epsilon\) is coupled to the conformal factor of the metric so that divergences are directly related to Weyl anomalies.

* Acting by the conformal (or BRST) generator on the integrand of the \(n\)-loop amplitude one finds non–zero contributions coming only from the contour integrals going around \(z_1, \xi, \eta\) (i.e. from the corresponding boundaries of the moduli space). We assume that small discs are cut around these points to regularize the corresponding divergences.

** There remains a technical problem of finding a special choice of moduli \((\xi, \eta, k)\) with which the limits \(\xi_k \to \eta_k\) are independent from \(k_0\) (cf. (6.3)). This problem was ignored in Ref. 18.

Let us note also that in the "generalized" SP in which we use a compact curved 2–space with \(2\) holes to represent the world "surface" the regularized expression for \(\tilde{Z}\) depends on the Weyl factor of the metric chosen. What is expected is that the Weyl anomaly cancels in the total expression for \(\tilde{Z}\) computed at the true vacuum point.

The partition function \(Z_n\) in (6.16) has the following useful representation

\[
Z_n = \{\exp \left( \frac{1}{2} (2 \pi a') D \cdot \frac{\delta^2}{\delta z^2} \right) \exp (-I_{\text{int}}(\omega)) \}_{\omega = 0}, \tag{6.18}
\]

\[
\frac{d}{\delta z^2} = \int d^2 z_1 d^2 z_2 D(z_1, z_2) \frac{\delta^2}{\delta z_1 \delta z_2}. \tag{6.19}
\]

Note that \(Z_n\) depends on moduli explicitly through \(D\) in (6.7) (and also in general, through 2–metric dependence in \(I_{\text{int}}\)) and also implicitly through the region of integration over \(\omega\) (the \(\sigma\) model is defined on a sphere with holes with positions and radii depending on moduli).

Our aim will be to analyze the renormalizability property of \(\tilde{Z}\). Since to find the "non-classical" string vacua we are to move out of the "tree" conformal point we are thus to consider \(\tilde{Z}\) for arbitrary (off–shell) values of the fields \(\omega\). In this case the formal projective invariance of the integrand in \(\tilde{Z}\) (present before regularization and for the classical on–shell values the fields) is absent. Then we are to specify how the \(\Omega^{-1}\)-factor in (6.15) is to be understood. We suggest to use the same prescription as at the tree level, namely, to replace \(\Omega^{-1}\) by the derivative over the cutoff \(\partial/\partial \omega\) (see Sec. 2 and Ref. 22). As we shall see in Sec. 7 on the examples of the torus and the disc this prescription leads to the expression for the generating functional which is renormalizable with respect to both "local" and "modular" infinities. Thus finally we can rewrite (6.15), (6.18) as

\[
\tilde{Z} = \frac{\partial}{\partial \omega} \tilde{Z}_n, \quad \tilde{Z}_n = \sum_{n=0}^{\infty} \tilde{Z}_n, \tag{6.20}
\]

\[
\tilde{Z}_n = c_n \int d\mu_0 \left\{ \exp \left( \frac{1}{2} (2 \pi a') D \cdot \frac{\delta}{\delta z^2} \right) e^{-I_{\text{int}}(\omega)} \right\}_{\omega = 0} \tag{6.18}
\]

7. RENORMALIZATION AT ONE LOOP: TORUS AND DISC TOPOLOGIES

1. In this section we shall consider the renormalization of the generating functional \(\tilde{Z}\) in the one loop approximation. We shall analyze explicitly the divergences present in the torus and disc contributions to \(\tilde{Z}\) and discuss how \(\tilde{Z}\) is to be defined in order to be renormalizable.

Let us start with recalling the expression for the string amplitudes on the torus in the usual "parallelogram" representation (see e.g. Refs. 58, 59, 56)

\[
A^{(1)}_n = c_n \int |d\tau| <V_1 \ldots V_n>. \tag{7.1}
\]

33
where the Koba-Nielsen points $u_i$ belong to the parallelogram $(\tau, 1)$ on $C$ and

\[ \eta^2 = k^2 \prod_{n=1}^{\infty} (1 - k^{2n}) \quad k = e^{i\pi \nu}, \quad D = 26, \]

(7.2)

\[ \eta(\tau) = k^2 \prod_{n=1}^{\infty} (1 - k^{2n}) \quad k = e^{i\pi \nu}, \quad D = 26, \]

(7.3)

\[ \eta(\tau) = k^2 \prod_{n=1}^{\infty} (1 - k^{2n}) \quad k = e^{i\pi \nu}, \quad D = 26, \]

(7.4)

We assume that the corresponding Möbius group $(U(1))$ gauge is fixed by fixing the position of one of the KN points. The tadpole factorization corresponds to the limit in which all the KN points $u_i$ come close to each other. Hence in the parallelogram parametrization there is no clear distinction between "local" and "modular" infinities.

To make the analysis of renormalization more transparent it is useful to go to a Schottky-type parametrization based on the "extended" set of moduli. To this end let us consider the $N > 3$ amplitude on the torus and make the two conformal transformations: first map the parallelogram into the annulus (with coordinates $w = e^{2\pi i u}$ and the radii of the boundary circles $R_1$ and $R_2$) and then map the annulus into the complex plane with the two holes cut out, by the transformation $w \rightarrow z$.

\[ z = (z_1 - z_2) w - u_1 w_2 - w_3 \]

(7.5)

Introducing the two parameters (the corresponding points lie inside the holes)

\[ \xi = z(\omega = 0), \quad \eta = z(\omega = \infty) \]

(6.6)

which can be expressed in terms of $w_1$, $w_2$, $w_3$, $z_1$, $z_2$, $z_3$, we thus finish with the map $(w_1, w_2, w_3) \rightarrow (\xi, \eta, z_1, z_2, z_3)$ (we may assume, e.g., that $u_1$ is fixed as an $U(1)$ Möbius gauge in the parallelogram representation and $z_1, z_2$ and $z_3$ are fixed as an $SL(2, \mathbb{C})$ Möbius gauge). Since the on-shell amplitude is formally invariant under the conformal transformation we can rewrite (7.1) as

\[ A^{(1)} = \frac{d^2 \xi d^2 \eta d^2 k}{(2\pi)^2 [\xi - \eta]^2 |k|^2} \int D\eta \int D\xi \int D^D e^{i\omega V_1 \cdots V_N} \]

(7.6)

where now $V_i$ are defined on the 2-holed plane and instead of (7.2) and (7.4) we have

\[ dV = \frac{d^2 \xi d^2 \eta d^2 k}{(2\pi)^2 [\xi - \eta]^2 |k|^2} \frac{D^D}{(2\pi)^D} \prod_{n=1}^{\infty} |1 - k^{2n}|^{2(D-2)} \]

(7.7)

\[ \mathcal{D}(z_1, z_2) = -\frac{1}{4\pi} \int \frac{d^2 \eta}{|\eta|^2} (1 - |\eta|^2)^2 - \frac{1}{4\pi} \left[ \sum_{m=1}^{\infty} \frac{\int d\eta}{|\eta|^2} \left( \frac{(1 - \lambda k^m)(1 - \lambda^{-1} k^m)}{(1 - k^m)^2} \right) \right] \]

(7.9)

\[ \lambda = \frac{z_1 - \xi}{z_1 - \eta}, \quad k = e^{i\pi \nu}, \quad \nu = \exp(2\pi i u) \]

(7.10)

In (7.7) it is assumed that the $SL(2, \mathbb{C})$ Möbius gauge is fixed on the KN points $u_i$. As in the general case of the Schottky parametrization (see Sec. 6) the integrand in (7.7) is formally $SL(2, \mathbb{C})$ invariant for the on-shell values of momenta (for which the non-invariance of $D_0$ in (7.9) does not matter; note that $\mathcal{D}$ depends on the invariant cross-ratio). This explains why the same expressions (7.7)–(7.9) can be formally obtained by using different conformal maps from the annulus to the 2—holed plane. For example, instead of (7.5) one could use $w = e^{2\pi i u}$ or (see Ref. 32) $z = e^{2\pi i u}$. It is necessary, however, to keep in mind that different conformal maps lead to different integration regions for the parameters in (7.7). Leaving aside the question of integration regions one can rewrite (7.7) in several formally equivalent ways by choosing different $SL(2, \mathbb{C})$ Möbius gauges. For example, fixing $\xi$, $\eta$ and $z$, we return to the expression (7.1) in the parallelogram representation (note that for $\xi = 0, \eta = \infty$ (7.9) reduces to (7.4)).

To compare (7.8), (7.9) with the general expressions (6.12), (6.7) in the Schottky parametrization let us note that for the torus the Schottky group has only one generator $T = (k, \xi, \eta)$ and hence (see (6.6), (6.8))

\[ \mathcal{B} = \sum_{n=1}^{\infty} (z_1 - T^n z_2)(z_2 - T^n z_1), \quad \omega(z) = \frac{1}{z - \eta} \frac{1}{z - \xi} \]

(7.11)

\[ \frac{T^n z - \eta}{T^n z - \xi} = \frac{k^n z - \eta}{k^n z - \xi} \]

It is then straightforward to check (employing e.g. the projective invariance of $\mathcal{D}$) that (7.9) is in agreement with (6.7) and (6.8) (in particular, $\nu = \int d\eta (|\eta|^2) \mathcal{B} = \int \frac{d^2 \eta}{|\eta|^2} (1 - \lambda k^m)(1 - \lambda^{-1} k^m)(1 - k^m)^{-2}$). Using that

\[ \int d\eta (1 - x) = -\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n \]

In the latter case $|k| < |\eta| < 1$ and the radii of the holes are $r_1 = |\eta|, r_2 = k|\eta|$, i.e. $r_1 \neq r_2$ and $\xi \rightarrow \eta$ implies $r_1 \rightarrow 0, r_2 \rightarrow 0$ for arbitrary value of $k$. (so that the boundaries of the holes are not isometric circles).
We have used that the insertion of the operator $O_1$ into a correlator on the sphere can be represented as $\sim \int \mathcal{D} \delta_{0} \mathcal{D} \delta_{0} \cdot \frac{\delta^2}{\delta \theta^2}$. The operator $O_1$ is the same as in (3.7) for $n = 1$. $\Gamma_0$ and $V_0$ are the zero momentum tachyon tadpole and the vertex operator. $\lambda_1$ is proportional to the massless tadpole (or vacuum amplitude) on the torus (in contrast to $\xi_1$, $\lambda_1$ has modular invariant integrand for $D = 26$). One can check (using e.g., the parallelogram representation) that

$$< O_1 > = \mathcal{O}(1)$$

(cf. (3.12); note that $O_1$ is twice the soft graviton operator (3.11)). Eq. (7.15) thus implies that the logarithmically divergent (tadpole) part of the generating functional on the torus is given by

$$Z_0 = c_1 \Omega^{-1} < F(z) >_0 < O_1 >_0 < e^{-i\lambda_1} >_0$$

(7.20)

Thus to cancel this divergence in the total expression $Z = Z_0 + \hat{Z}_0 + \ldots$ we are to add the following counterterm to the string action in $Z_0$

$$Z_0 = \mathcal{O}(\xi) \cdot e^{-i\lambda_1}$$

(7.21)

$$\delta I = \frac{i}{c_1 c_0} \lambda_1 \xi \mathcal{O}(1)$$

(7.22)

This result is consistent with the general expressions (3.21) and (4.28) if

$$b_1 = \frac{i}{c_1 c_0} \lambda_1$$

(7.23)

where $b_0$ are the constants which appear in $Z$ (4.19). If we fix $b_0$ by comparing $Z_0$ with the usual field theory result $\frac{1}{4} \log \det (\alpha'^2) + \ldots$ then $b_1$ is given by (5.10) (see Ref. 56), i.e.

$$d_1 = \frac{1}{4} \mathcal{O}(\xi)$$

(7.24)

$((2 \pi \alpha')^{-D/2}$ comes from the constant in the string action and $(2 \pi \alpha')^{-D/2}$ from the Gaussian integral normalization; the total factor $(4 \pi^2 \alpha')^{-D/2} \sqrt{D}$ in (4.19) is the contribution of the zero mode of $z^a$). The choice of $d_0 \sim (4 \pi^2 \alpha')^{-D/2}$ is a matter of convention, being related to the choice of the string coupling.

2. Let us now analyze how the combined "local" plus "modular" renormalization can be carried out in $Z$. Let us first consider the generating functional for the massless amplitudes in the "parallelogram" representation of the torus $^*$$Z = c_1 \int (d\tau) \omega, \quad Z_1 = \int [dz] e^{\tau}, \quad I = I_0 + I_{\omega}$

(7.25)

$^*$ Note that it is not necessary to fix explicitly the compact $U(1)$ Möbius group.
where \((dr) = |dr|/n\pi\tau\). \(\tilde{Z}_1\) is thus simply the interval over \(\tau\) of the partition function of the \(\sigma\)-model (2.3) defined on a compact torus represented by the parallelogram on \(\mathbb{C}\). As we have already discussed in Sec. 2, the partition function of the \(\sigma\)-model on a compact space of arbitrary topology is given by (see (2.7), (2.15) and Ref. 22).

\[
Z_n = a_n \int d^3y \sqrt{G} e^{2(\tau-1)+1} \left( 1 + \frac{1}{2} \partial \partial e R + \ldots \right) \tag{7.26}
\]

(we put \(c = 2 - 2\pi \) and \(D = 26\) in (2.15)). For all \(n\), \(Z_n\) is renormalizable with respect to "local" infinities, i.e. satisfies (2.12). This leads to the following apparent paradox: if we renormalize all the "local" infinities in \(Z_n\), \(Z_1\) in (7.25) will be finite and hence \(\tilde{Z}_1\) will also be finite (since, as we have already argued, the integral over \(\tau\) does not give any new divergences of the type we are interested in). At the same time, \(\tilde{Z}_1\) should definitely contain the "modular" divergences discussed above. To resolve this paradox one should observe that what appears as a "local" infinity in (7.25) is, in fact, a "modular" infinity from the point of view of the string amplitudes defined on the parallelogram. Consider, for example, the 3-graviton amplitude on the torus in the parallelogram representation. The (momentum)\(^3\)-term in the amplitude corresponds to the \(h_{iuv}^a\)-piece of the \(R\)-term in \(\tilde{Z}_1\) or in \(\tilde{Z}_1\) (7.26). This amplitude contains the infinities corresponding to the limits in which two or all three \(K\)-points come close to each other. These infinities are "local" if considered from the point of view of the \(\sigma\)-model defined on the parallelogram but are modular external leg and tapole infinities from the standard string theory point of view. It is instructive to compare this with what happens on the sphere. The correlator of the three graviton vertex operators on \(\mathbb{C}\) is again divergent in precisely the same limits \((z_1 \rightarrow z_3)\) and \((z_1 \rightarrow z_2 \rightarrow z_3)\) reproducing the \(\text{Re} e\) - term in (7.26) (for \(n = 0\)). From the string theory point of view one interprets these divergences as Möbius divergences (see Sec. 2, and Refs. 20 and 21) and subtracts them (e.g. by fixing the positions of the vertex operators or applying the \(\partial /\partial \text{Re} e\) - prescription (2.33)) obtaining thus the finite expression for the 3-graviton amplitude or the \(R\)-term in the generating functional \(\tilde{Z}_0\) (see (2.39) and (2.40)).

Let us now summarize and generalize the above discussion. Consider the standard "Möbius gauge fixed" approach in which one uses the "restricted" sets of moduli \(\{r\}\) and hence

\[
\tilde{Z}_n = c_n \int d\mu_r(r) Z_n \quad Z_n = \int d\mu_r(r) e^{-1} \quad n \geq 1 \tag{7.27}
\]

Since \(Z_n\) is renormalizable with respect to the local infinities, we can rewrite the \(O(\text{Re} e)\) term in it as \(\beta_0 \partial /\partial \text{Re} e\) (see (2.13)). Hence the part of the \(O(\text{Re} e)\)-term in \(\tilde{Z}_n\) corresponding to the

\[
\beta_0 \partial /\partial \text{Re} e \tag{7.28}
\]

Recalling that \(\beta_0 = \alpha_0 \partial /\partial \text{Re} e\) (see (2.19)) we can absorb the divergence (7.28) into the renormalization of \(\text{Re} e\) in the tree term in \(Z = \sum_{n=0}^{\infty} Z_n\), \(\delta \text{Re} e = \alpha_0 \partial /\partial \text{Re} e + \ldots\), obtaining the leading order modular correction to the \(\beta\)-function

\[
\beta_0 = \alpha_0 \partial /\partial \text{Re} e \quad Z_{\text{ren}} = a_n \int d^3y \sqrt{G} e^{2(\tau-1)+1} \left( 1 + \frac{1}{2} \partial \partial e R + \ldots \right) \tag{7.29}
\]

Using the expression for \(\beta_0\) (2.20) we find that (7.29) is, in fact, in agreement with the previous result for the "momentum independent" contribution to the "modular" \(\beta\)-functions (see (4.25)).

Eq. (7.28) is not, however, what we would expect to find if \(\tilde{Z}\) were renormalizable with respect to the sum of the local and modular infinities

\[
\beta_0 \partial /\partial \text{Re} e + \beta_0 \partial /\partial \text{Re} e + \beta_0 \partial /\partial \text{Re} e + \ldots \tag{7.30}
\]

In particular,

\[
\tilde{Z}_1 = \tilde{Z}_{1\text{R}} + 2 \beta_0 \partial /\partial \text{Re} e + O(\text{Re} e^2) \tag{7.31}
\]

where we have used that to the leading order

\[
\beta_0 \partial /\partial \text{Re} e = \beta_0 \partial /\partial \text{Re} e + \beta_0 \partial /\partial \text{Re} e + \ldots \tag{7.32}
\]

The interpretation of (7.31) and (7.32) is clear: "one half" of the \(\text{Re} e\) - divergence in \(\tilde{Z}_1\) should be "local" and "one-half" - "modular". The "local" divergence is to be cancelled by inserting the local \(\beta_0\) counterterms directly into \(\tilde{Z}_1\) while the "modular" should be cancelled by inserting the modular \(\beta_0\) counterterms into \(\tilde{Z}_0\).

We conclude that the generating functional defined according to the standard Möbius gauge fixed prescription does not satisfy the RG equation (7.30) with the full \(\beta\)-function (which is proportional to the full effective equations of motion). This suggests that we need to modify the definition of \(\tilde{Z}\) in order to ensure its renormalizability. The qualitative reason for the absence of "doubling" of the \(\text{Re} e\) - divergence in \(\tilde{Z}_1\) is that we have used the parameterization in which the Möbius infinities (which, in fact, are a subclass of local infinities, see Sec. 2) are already dropped out. However, the tree level experience suggests that one should first regularize both the Möbius and modular infinities and then use a special prescription of how to "divide"
by the Möbius volume in order to preserve correspondence with the usual results for on-shell amplitudes. Consider once again the 3-graviton correlator on the sphere and the torus, using the Schottky-type (sphere with two holes) representation for the latter. Introducing a short distance cutoff we find (here \( \ell \equiv \infty \)):

\[
\ell > 1 \Rightarrow \ell \in \infty, \quad \ell > 1 \Rightarrow \ell^2 \varepsilon + O(\ell \varepsilon). \tag{7.34}
\]

The \( \ell \varepsilon \)-term in \( \ell > 1 \) has one of the \( \ell \varepsilon \) factors in \( \ell > 1 \) and can be interpreted as Möbius (or local) infinities originating from the limits in which the vertex operators "collide." The second \( \ell \varepsilon \) factor in \( \ell > 1 \) is the modular infinity corresponding to the limit \( (x \to \eta) \) in which the holes collide (the handle disappears). If we use the standard prescription \( (\Omega^{-1} \to \ell \varepsilon) \) for subtracting of the Möbius infinity we get \( A_0^{(1)} \equiv -\Omega^{-1} < \ell \varepsilon > \equiv \ell \varepsilon \in \infty \), finite, where the \( \ell \varepsilon \)-term in \( A_0^{(1)} \) corresponds to the modular divergence in \( \Omega^{-1} \) discussed above. If instead we use the \( \Omega^{-1} \to \partial/\partial \ell \varepsilon \)-prescription \( 21,22 \), we get precisely the desired doubling of the coefficient of the \( \ell \varepsilon \)-term in \( A_0^{(1)} \). What happens is that the derivative \( \partial/\partial \ell \varepsilon \) counts the "local" and "modular" \( \ell \varepsilon \)-factors in the \( \ell \varepsilon \)-term in an independent way, in agreement with (7.31) and (7.32).

Thus we suggest to use the definition of \( \tilde{Z} \) already given in (6.20) and expect to find

\[
\tilde{Z} = \frac{\partial}{\partial \ell \varepsilon} (Z_0 + Z_1 + ...) = \frac{\partial}{\partial \ell \varepsilon} (2 d_0 \int d^2 y \sqrt{G} e^{-2\ell \varepsilon} 
\times (1 + \frac{1}{2} \alpha' \ell \varepsilon R + ...) + d_1 \varepsilon R \int d^2 y \sqrt{G} (1 + \frac{1}{2} \alpha' \ell \varepsilon R + ...)) \tag{7.35}
\]

\[
= d_0 \int d^2 y \sqrt{G} e^{-2\ell \varepsilon} (\alpha' R + ...) + d_1 \int d^2 y \sqrt{G} (1 + \alpha' \ell \varepsilon R + ...) \tag{7.36}
\]

where the overall \( \ell \varepsilon \)-factor in \( Z_1 \) corresponds to the "restored" Möbius group volume factor which should have (to the considered order) the equivalent interpretation of the modular divergence originating from the integral over the "extra" moduli \( \xi \) and \( \eta \). Assuming that \( \tilde{Z}_1 = Z_1 + \ell \varepsilon \frac{\partial}{\partial \ell \varepsilon} Z_1 \) and using the relations \( \beta_0 = \frac{\delta}{\delta \ell \varepsilon} \), \( \beta_1 = \frac{\delta}{\delta \ell \varepsilon} \) (see (7.29) and (7.33)) it is easy to check the renormalizability of the total \( \tilde{Z} \) (7.30)-(7.32).

Let us now explain how the \( \ell \varepsilon \alpha' R \varepsilon \)-term actually appears in the integrated over the moduli \( \sigma \) model partition function

\[
\tilde{Z}_1 = c_1 \int \left( d\Omega \frac{d^2 \ell \varepsilon \eta}{|\xi - \eta|^4} \right) < e^{-\ell \varepsilon} >_1. \tag{7.37}
\]

Recalling the previous result (7.15) and (7.20) for the logarithmic singularity in the \( \xi \to \eta \) limit we get

\[
\tilde{Z}_1 = \frac{1}{2} c_1 \frac{1}{\pi} \lambda \ell \varepsilon < O_1 e^{-\ell \varepsilon} >_0. \tag{7.38}
\]

* Such counting is also necessary in order to have the sum of the local and modular pieces in the total \( \beta \)-function.

Next we note that the insertion of \( O_1 \) (7.18) into the partition function on the sphere can be represented in the following way,

\[
< O_1 e^{-\ell \varepsilon} >_0 = 2 \alpha' \frac{\partial}{\partial \alpha'} < e^{-\ell \varepsilon} >_0 \sim \alpha' \frac{\partial}{\partial \alpha'} Z_0 \sim \int d^2 y \sqrt{G} \frac{1}{2} \alpha' \varepsilon R \varepsilon + ... \tag{7.39}
\]

where in the last line we have used (7.26) (the dilaton factor cancels out since this contribution originates from the torus). Substituting (7.39) into (7.38) we indeed find the \( \ell \varepsilon \) \( R \)-term. The problem which still remains is an apparent absence of the \( \ell \varepsilon \) \( R \)-term in (7.35). This suggests that we are still to improve the above qualitative picture. One subtle point is related to the replacement of \( C \) by a compact 2-sphere for which we can use the expression (7.26) for the partition function. It would be better to start directly with a curved compact 2-sphere with two holes (with identified boundaries). In this case, however, the expression for the modular measure may change.

3. To get a deeper understanding of related issues let us now analyze the case of the disc correction to the generating functional for closed string amplitudes in the theory of open and closed strings. As in the case of the torus, let us first consider the "Möbius gauge fixed" representation of the disc in terms of the interior of the unit circle on the complex plane. It is not necessary to fix the \( SL(2, R) \) Möbius symmetry on the disc explicitly since the corresponding divergences are power-like (not logarithmic), i.e. the \( SL(2, R) \) Möbius group volume is finite if we drop power infinities \( 61,20 \). Then the momentum independent logarithmic modular divergences in the closed string correlators on the disc come from the limits in which all \( N \) or \( N - 1 \) vertex operators collide \( 11 \) (see also Refs. 14 and 19). They are simply the local divergences when considered from the point of view of the \( \sigma \)-model defined on the unit disc. The corresponding generating functional is thus proportional to the partition function (7.26) (formally take \( n = 1/2 \) for the disc since this is in agreement with \( \chi = 1 \)).

\[
\tilde{Z}_{1/2} = d_{1/2} \int d^2 y \sqrt{G} e^{-1}(1 + \frac{1}{2} \alpha' \ell \varepsilon R + ...) \tag{7.40}
\]

and so is renormalizable with respect to the local divergences. Hence just as in the case of the torus in the parallelogram representation (see (7.27)-(7.30)) we find that these local divergences (i) can be interpreted as "modular" from string theory point of view (ii) can be absorbed into a renormalization of the couplings in \( Z_0 \) with the proper "modular" counterterm (corresponding

Note that up to normal ordering \( O_1 \) is twice the free string action. The normal ordering implies that \( \frac{\delta}{\delta \alpha'} \) does not act on the overall zero mode factor \( \alpha' \) \( -D^2 \) in \( Z_0 \) (see Ref. 14). In fact, \( \int d^2 y \frac{\partial}{\partial \alpha'} < e^{-\ell \varepsilon} >_0 \) cancels against the derivative of the \( \alpha' \) \( -D^2 \)-factor.

** Such term is to be expected since the soft graviton amplitudes on the torus are finite only in the standard (Möbius gauge fixed) parallelogram representation.
to (4.15), (4.17) and (4.18) with \( n = \frac{1}{2} \) or (4.26), (4.27) with \( w \sim e^{-\phi} \); (iii) are only "one half" of the divergences that should be present in \( \hat{Z}_{1/2} \) in order for \( \hat{Z} = \hat{Z}_0 + \hat{Z}_{1/2} + \ldots \) to be renormalizable with respect to both local and modular infinities \(^{19}\). In order to resolve the latter problem let us go to the Schottky-type "plane with a hole" representation of the disc \(^{13}\). Conformally transforming (inverting) the unit disc into the plane with a hole one finds the following expression for the generating functional for the closed string amplitudes on the disc \(^{13}\)

\[
Z_{1/2} = c_{1/2} \Omega^{-1} \hat{Z}_{1/2}, \quad \hat{Z}_{1/2} = \int d\mu_{1/2} \left( \exp \left( \frac{1}{2} (2\pi \alpha') D \left( \frac{\delta^2}{\delta z^2} \right) e^{-\phi(w)} \right) e^{\phi(w)} \right)_{\alpha = 0}, \quad (7.41)
\]

\[
d\mu_{1/2} = \frac{da}{\alpha'} d^2 w, \quad (7.42)
\]

\[
D(z_1, z_2) = -\frac{1}{4\pi} \pi \delta(z_1 - z_2)^2 - \frac{1}{4\pi} \pi \delta(1 - w) (z_2 - w)^2 = D_0 + D, \quad (7.43)
\]

where \( a \) and \( w \) are the radius of the hole and the position of its center on \( \mathbb{C} \). \( \Omega \) is the volume of the \( SL(2,\mathbb{C}) \) Möbius group on \( \mathbb{C} \). In the derivation of (7.41) one assumes that the Möbius gauge is fixed by fixing the positions of the KN points (e.g. \( z_1 = 0, z_2 = 1, z_3 = \infty \)). According to (7.41) we first to compute the partition function on the plane with a hole, then average over the "moduli" \( a \) and \( w \) and finally to subtract the Möbius infinities. It is possible to return to the unit disc picture by fixing the formal on-shell \( L(2,\mathbb{C}) \) symmetry by the condition \( a = 1, w = 0 \), \( z_1 = 0 \) and then making the inversion \( u = -\frac{1}{w} \). The result will be the unit disc representation for the amplitudes in the \( SL(2,\mathbb{C}) \) Möbius gauge \( u_1 = 0 \) (the Neumann function (7.43) then reduces to (.4.15), (4.17) and (4.18) with \( n = y \) or (4.26), (4.27) with \( w \sim e^{-\phi} \); (Hi) are only "one half" of the divergences that should be present in \( Z = Z_0(\mathbb{C}) \) and hence (cf. (7.15), (7.16))

\[
Z_{1/2} = c_{1/2} \Omega^{-1} \hat{Z}_{1/2} = -c_{1/2} \Omega^{-1} \pi \delta < \mathcal{O}_{1/2} e^{-\phi} >_0, \quad (7.45)
\]

\[
\mathcal{O}_{1/2} = V_0 = \frac{1}{4 \pi \alpha'} \int d^2 w : \partial_w \phi \partial_w \phi : \quad (7.46)
\]

\[
\mathcal{O}'_{1/2} = V_0 = \frac{1}{4 \pi \alpha'} \int d^2 w : \partial_w \phi \partial_w \phi : \quad (7.47)
\]

The operator \( \mathcal{O}'_{1/2} \) is different however, from the correct counterterm \( \mathcal{O}_{1/2} \) (see (3.7)) needed in order to renormalize the "modular" divergence in the amplitudes on the disc in a way consistent with the effective action. This problem was pointed out by Fischler, Klebanov and Susskind (FKS) \(^{13}\). As we have mentioned above (see also Ref. 19) this problem is absent in the unit disc parametrization if one carefully accounts for all local infinities present in the corresponding partition function, or, equivalently, in the correlators of the vertex operators.

Let us now discuss how the FKS paradox is resolved in the plane with a hole parametrization. The basic point is that to define off-shell or divergent quantities like \( \hat{Z} \) one should use a compact curved 2-space representation for the world surface. This is necessary in order to account for the topology of the world surface in a systematic way and in particular, in order to have a correspondence with the sigma-model approach in which the topology is reflected in the dilaton factor \( e^{-\phi} \). The problem is then to find the measure \( d\mu_{1/2} \) (7.41) in the case of a curved compact sphere with a hole. The approach suggested by Polchinski \(^{15}\) is to consider a general curved surface with a hole with the metric \( g_{ab} = e^{2\phi} \delta_{ab} \) and to compute the measure by expanding in \( a \to 0 \). He pointed out that the 2-curvature dependent terms may appear in the measure due to its frame dependence. In particular, the original parameters \( a \) and \( w \) in (7.49) are frame dependent objects. In the case of a non-trivial metric one should use instead the invariant radius and center \( \tilde{a} \) and \( \tilde{w} \), defined as follows (assuming that the coordinate radius \( a \) is small)

\[
\tilde{a} = e^{\phi(W)} a, \quad \tilde{w} = \int_{|w|=a} d^2 w \sqrt{g(w + \tilde{w})} (w + \tilde{w}) \int_{|w|=a} d^2 w \sqrt{g(w + \tilde{w})}. \quad (7.50)
\]

Then the length of the boundary circle in the coordinate plane is equal to the length of its image and \( \tilde{w} \) is an average position of the center (\( u \) lies inside the disc which is cut out of the coordinate plane). Expanding in \( a \to 0 \) one finds

\[
\tilde{w} = w + \frac{1}{\pi a^2} \int_{|w|=a} d^2 w |w|^2 \delta_\rho(w) + \ldots = w + a^2 \delta_\rho + \ldots, \quad \text{(7.49)}
\]

\[
d^2 w = d^2 \tilde{w} = 2a^2 \delta_\rho d\tilde{\rho} \tilde{w} + \ldots = a^2 d\tilde{w} (1 + \frac{1}{2} \delta_\rho^2 + \ldots) \quad \text{and thus finally} \quad \text{13)} *
\]

\[
d\mu_{1/2} = \frac{da}{\alpha'} d^2 w = \frac{da}{\alpha'} d^2 w \sqrt{g} (w + \tilde{w}) (1 + \frac{1}{4} \delta_\rho^2 \delta_\rho^2 \tilde{w} + \ldots) \quad (7.50)
\]

(to put the measure into the covariant form one is to use that \( g(w) = g(\tilde{w}) - a^2 |\delta_\rho|^2 + \ldots \)). Repeating the analysis of the tadpole \( a \to 0 \) divergence in \( Z_{1/2} \) with the corrected measure (7.50) one obtains (7.46) with the operator \( \mathcal{O}_{1/2} \) replaced by \( \mathcal{O}'_{1/2} \),

\[
Z_{1/2} = -c_{1/2} \Omega^{-1} \pi \delta < \mathcal{O}'_{1/2} e^{-\phi} >_0, \quad (7.51)
\]

* A more complicated definition of the invariant radius \( \tilde{a} = \frac{1}{2\pi} \int d\theta d^2 u \exp \rho(w + u e^{i\theta}) \) suggested in Ref. 63 leads to the same result (7.50) since the \( a^2 \) correction in \( \tilde{a} \) does not matter to the leading order: if \( a = \tilde{a} + c_0 a^3 + \ldots \), \( \tilde{a} = \frac{1}{3} (1 + O(a^4)) \).
which is the correct "modular" counterterm for the disc \( \chi = 1 \) cf. (3.7). Being inserted into the correlators on the sphere this operator is equivalent to \( \alpha' \mathcal{G} + 1 \). Thus the account of the additional curvature dependent \( \alpha^2 \)-term in the modular measure resolves the FKS paradox\(^{15}\).

We would like to stress that the analysis of the renormalization of the leading order "modular" infinity in \( Z_{1/2} \) is independent of how one subtracts the \( SL(2, C) \) M"obius infinities (since the \( \Omega^{-1} \) appears as a common factor in the tree and loop contributions to \( Z \)). However, as was already noted above, one is to use a particular prescription for subtraction of the \( M"obius \) infinities in order to ensure the renormalizability of \( Z \) with respect to all infinities. We suggest again to use the \( \partial/\partial \theta \) prescription (6.20). Using the corrected measure (7.50) (and omitting the waves on \( a \) and \( w \)) we get

\[
Z_{1/2} = \frac{1}{4\pi \alpha'} \int d^2 w \sqrt{\mathcal{G}} : \eta \partial_z \partial_{\bar{z}} \psi^\alpha \frac{1}{2} \alpha' R^{(2)} : + \ldots 
\]

where we have dropped the quadratic divergence and used that the Euler number for the sphere is \( 2 \). Thus the coefficient of the \( \alpha' \mathcal{E} \) term "doubles" in the same way as already suggested in the case of the torus (cf. (7.35), (7.36)) and hence is consistent with the renormalizability of \( Z \) with respect to the sum of the local and modular infinities. We see that in the case of the disc the overall \( \mathcal{E} \)-factor which multiplies the partition function \( Z_{1/2} \), originates from the modular integral and hence may be interpreted as a modular divergence.

Thus we have found that the presence of the additional "topological" term in the modular measure gives the overall \( \mathcal{E} \)-factor in the derivative-independent part of the integrated partition function \( Z_{1/2} \) and hence leads to the correct final result for the generating functional or vacuum amplitude, \( Z_{1/2} = \int d^2 w \sqrt{\mathcal{G}} + \ldots \). In this way we recoup the usual finite expressions for the \( N \)-point soft graviton and dilaton amplitudes (corresponding to \( \sqrt{\mathcal{G}} \)) as well as the divergent parts of the \( N = 2 \) (off shell) and \( N = 3 \) graviton amplitudes (corresponding to \( \sqrt{\mathcal{G}} \)) on the disc.

4. One could expect that a similar curvature dependent \( \partial/\partial \theta \) correction term may be present in the modular measure (7.8) for the torus. It will then produce an overall \( \mathcal{E} \)-factor in the small handle limit and hence lead to the correct final result for the generating functional or vacuum amplitude, \( Z_{1/2} = \int d^2 w \sqrt{\mathcal{G}} + \ldots \). In this way we recoup the usual finite expressions for the \( N \)-point soft graviton and dilaton amplitudes (corresponding to \( \sqrt{\mathcal{G}} \)) as well as the divergent parts of the \( N = 2 \) (off shell) and \( N = 3 \) graviton amplitudes (corresponding to \( \sqrt{\mathcal{G}} \)) on the disc.

\[ \frac{1}{4\pi \alpha'} \int d^2 z \sqrt{\mathcal{G}} : \partial_z \partial_{\bar{z}} \psi^\alpha \psi^\beta : + \ldots + \frac{1}{2} \alpha' R^{(2)} : \]

This problem is absent if we use the "hyperelliptic" or "branch point" parametrization\(^{41}\) for the torus, integrating first over the coordinates of the four branch points and then using \( \partial/\partial \theta \) to subtract the \( M"obius \) volume. A drawback of this parametrization is that it cannot be directly applied for genera higher than 2.

\[ \frac{1}{4\pi \alpha'} \int d^2 z \sqrt{\mathcal{G}} : \partial_z \partial_{\bar{z}} \psi^\alpha \psi^\beta : = \partial_z \partial_{\bar{z}} \psi^\alpha \psi^\beta + \frac{1}{2} \alpha' R^{(2)} : \]

\[ \alpha' \mathcal{G} + 1 \]
Hence finally (cf. (3.12))

\[
< V^{<}_{\mu \nu} > = - \frac{1}{2} \delta^{\mu \nu} - \frac{1}{4 \pi \alpha'} \int d^2 z \sqrt{g} \partial_{\mu} x^\mu \partial_{\nu} x^\nu > = - \frac{1}{2} D
\]  

(7.59)

As was discussed in Ref. 16 (see also Ref. 15) this universal result is consistent with the covariant expression for the "constant" part of the string partition function *

\[
Z_n \sim < e^{-i \omega} > = \int d^2 \gamma \left( 1 - < V^{<}_{\mu \nu} > \right) h_{\mu \nu} + \ldots
\]

\[
= \int d^2 \gamma \left( 1 + \frac{1}{2} h_{\mu \nu} + \ldots \right) \propto \sqrt{g} \left( 1 + \ldots \right), \quad G_{\mu \nu} = \delta_{\mu \nu} + h_{\mu \nu}
\]  

(7.60)

In the case of torus \( \chi = 0 \) and hence the subtlety related to the use of (3.8), (7.58) is irrelevant (we, of course, drop the quadratic divergence \( \propto \partial^2 \partial_0 \)). In the parallelogram representation only the last term in (7.4) contributes to (7.55) and we find again (cf. (7.19))

\[
\frac{1}{\pi \alpha'} \left< \int d^2 u : \partial z^a \partial z^a : > = - \frac{1}{4} \delta^{\mu \nu} \lim_{\eta_1 \to -\eta_2} \int d^2 u \partial_1 \partial_2 (u_{12} - \bar{u}_{12})^2
\]

\[
= - \frac{1}{2} \lambda c \int d^2 u (4 \pi \alpha')^{-1} = - \frac{1}{2} \delta^{\mu \nu}
\]  

(7.61)

(we have used that the area of the parallelogram is \( 4 \pi \alpha' \)). In the case of the Schottky-type parametrization \( D \) (7.9) belongs to the general class (6.7) and hence we get (7.55) with \( u = \frac{\eta - \xi}{(z - \eta)(z - \xi)} \) (see (7.10))

\[
< V^{<}_{\mu \nu} > = - \frac{\delta^{\mu \nu}}{8 \pi \alpha' (4 \pi \alpha')^{-1}} \int d^2 z \left( \frac{(\eta - \xi)}{(z - \eta)(z - \xi)} \right)^2
\]  

(7.62)

If we formally put \( \xi = 0 \), \( \eta = \infty \) and map back to the parallelogram (\( z = e^{i \pi \omega} \)) we get

\[
\int_{\omega \neq 0} d^2 \omega = 4 \pi \int d^2 u = \frac{4 \pi}{4 \pi \alpha'} \left. \right|_{\omega = 0} \text{in agreement with (7.61).}
\]

If instead we substitute (7.62) into the modular integral (7.7), (7.8), we get

\[
A^{(1)} = c_{+} \Omega^{-1} \int d \tau \int d^2 \xi d^2 \eta \left< V^{<}_{\mu \nu} > = - \frac{\delta^{\mu \nu}}{8 \pi \alpha' (4 \pi \alpha')^{-1}} \int d^2 \xi d^2 \eta \right>
\]

\[
= - \frac{\delta^{\mu \nu}}{8 \pi \alpha' c_{+} \lambda_1 \Omega^{-1}} \int \frac{d^2 \xi d^2 \eta d^2 z}{|\xi - \eta|^2 |\xi - z|^2 |z - \eta|^2}
\]  

(7.63)

where \( \lambda_1 \) was defined in (7.17). Thus we find the agreement with the result in the parallelogram representation if

\[
\Omega = \frac{1}{4 \pi^2} \int \frac{d^2 \xi d^2 \eta d^2 z}{|\xi - \eta|^2 |\xi - z|^2 |z - \eta|^2}
\]

(7.64)  

* Note that Eq.(3.8) is true in dimensional regularization in which one can ignore the contribution of the measure \( E^{(2)}(0) = 0 \). In other regularizations one is to account for the contribution of the measure in order to reproduce the covariant result (7.60) \( 16, 22 \). In general, to derive (7.59) and (7.60) one is only to use that \( \Delta D = \delta^{(2)}(1, 2) - \frac{1}{2} \) and to set \( \delta^{(2)}(1, 1) = 0 \).

8. GENUS TWO EXAMPLE OF RENORMALIZATION

Below we are going to explain how the renormalization of the string generating functional can be carried in the two-loop approximation. We shall demonstrate that the condition of renormalizability of \( \tilde{Z} \) (or, equivalently, the condition that the massless sector of the string S-matrix can be reproduced from an effective action) fixes the overall coefficients (or relative weights) of string loop corrections to \( \tilde{Z} \).

Let us first consider the disc and annulus corrections to \( \tilde{Z} \) in the open-closed Bose string theory. Since the annulus factorizes on two discs we expect to find the logarithmic divergence in its contribution to \( \tilde{Z} \) (which we shall denote as \( \tilde{Z}_a \)).

\[
\tilde{Z} = \tilde{Z}_{a} + \tilde{Z}_1 + \ldots = \frac{1}{\pi \alpha'} \int d^2 \gamma \sqrt{g} e^{ik} + \tilde{Z}_1 \int d^2 \gamma \sqrt{g} e^{ik} + \ldots,
\]

(8.1)

\[
\tilde{Z}_1 = \tilde{z}_1 + \tilde{z}_1^{(1)} \phi \text{nc}.
\]

(8.2)

Here \( \phi \) and \( G \) are the bare values of the fields which we know already from the study of the tadpole renormalization on the disc. We shall consider only the renormalization of the zero momentum part of \( \tilde{Z} \) and hence will ignore the "local" counter terms. According to the general expression (4.15)-(4.18), (4.26), (4.27)

\[
G_{\mu \nu} = G_{\text{Rv}} + \frac{1}{4} \delta^{\mu \nu} e^{k} (2 \omega + \omega') \text{nc} G_{\text{Rv}} + \ldots
\]

(8.3)

\[
\phi = \phi_0 + \frac{1}{16} \delta^{\mu \nu} e^{k} (2 \omega + (D - 2) \omega') \text{nc} + \ldots
\]

(8.4)

* For example, \( z \) belongs to the exterior of the two circles and hence the limits \( z \to \xi, z \to \eta \) are excluded from the integration region (unless \( \xi \to \eta \)).
where for the disc \( u = d_{1/2} e^{-\alpha' a} \). Substituting (8.3) into the disc contribution to (8.1) we find that the divergence in the annulus contribution cancels out if

\[
d^4 u = -\frac{1}{16} d^2 z d^2 \theta (D - 2). \tag{8.4}
\]

If we start directly with the well-known expression for the string partition function on the annulus we get

\[
d^4 u = c_1 \int \frac{d^2 z}{\sqrt{g}} \left( \prod_{m=1}^{\infty} (1 - q^{2m}) \right)^{-D/2} \tag{8.5}
\]

\[
= c_1 \left[ \frac{1}{2\pi} - (D - 2) \log \pi + \text{finite} \right].
\]

Dropping the quadratic divergence and comparing with (8.2) and (8.4) we find the agreement if the overall constant \( c_1 \) takes the following value

\[
c_1 = \frac{1}{16} d^2 z d^2 \theta. \tag{8.6}
\]

(note that \( d_0, d_{1/2}, d_f \) and \( c_1 \) are proportional to \( \alpha'^{-D/2} \).) Equivalently, one may say that it is the factorization condition that fixes \( c_1 \).

Let us repeat the same analysis in the case of the torus and genus 2 contributions to \( Z \). Since the genus 2 surface may factorize on the two tori we should find

\[
\mathcal{Z} = \mathcal{Z}_1 + \mathcal{Z}_2 + \ldots = d_1 \int d^2 \sqrt{g} + \mathcal{Z}_2 \int d^2 \sqrt{g} e^{2\phi} + \ldots, \tag{8.7}
\]

\[
\mathcal{Z}_2 = \mathcal{Z}_1 + d_2^{(1)} \mathcal{E} \omega. \tag{8.8}
\]

The renormalizability of \( \mathcal{Z} \) implies that it should be possible to cancel the divergence in \( \mathcal{Z}_2 \) by substituting the expression for the bare metric (8.3) in the torus term in (8.7). Using that for the torus \( \omega = d_1 \), we find the following condition on \( d_2^{(1)} \)

\[
d_2^{(1)} = -\frac{1}{4} d^2 z d^2 \theta D. \tag{8.9}
\]

Let us now check this prediction by directly computing the divergent part of the genus 2 partition function using the Schottky parametrization. According to the general expression for the modular measure in SP (6.12) we have

\[
d_3 = c_2 \Omega^{-1} \int d\omega_2 = c_2 \Omega^{-1} \int d^2 \xi_1 d^2 \xi_2 d^2 \eta_1 d^2 \eta_2 \frac{(1 - \xi_1 \eta_1)(1 - \xi_2 \eta_2)}{[(\xi_1 - \eta_1)(\xi_2 - \eta_2)]^2} \left( \det t_{ab} \right)^{-D/2} \prod_{m=1}^{\infty} (1 - k_m)^{-2D} \tag{8.10}
\]

\[
\times \prod_{m=1}^{\infty} \prod_{m=2}^{\infty} |1 - k_m|^2.
\]

This expression for the genus 2 string partition function was studied in Ref. 66 where the equivalence between the loop measure in SP and in the parameterization in terms of the period matrix \( \mathcal{O} \) (in which the modular invariance is explicit) was checked by expanding the SP measure in powers of the modular parameters \( \xi \). If we formally fix the Möbius group by choosing \( \xi_1 = \infty, \eta_1 = 1, \eta_2 = 0, \xi_2 \equiv \xi \), (8.10) reduces to

\[
d_2 = c_1 \int \frac{d^2 k_1 d^2 k_2 d^2 \xi}{|k_1|^4 |k_2|^4 |\xi|^4} \left| \prod \left( f(k_1, k_2, \xi) \right)^2 \right| \left( \det t \right)^{-D/2} \tag{8.11}
\]

The "dividing" factorization limit corresponds to \( \xi \to 0 \). To find the expansions for \( f \) and \( t = \det t_{ab} \) in powers of \( \xi \) we apply the results of Ref. 66. The generators of the Schottky group are \( T_1 \) and \( T_2 \).

\[
T_1 \xi - 1 \to k_1 \xi - 1, \quad T_1 \xi = k_1 \xi + 1 - k_1, \tag{8.12}
\]

\[
T_2 \xi = 0 \to k_2 \xi = 0, \quad T_2 \xi = -\frac{z k_2 \xi}{z (1 - k_2) - \xi}. \tag{8.13}
\]

Analyzing the \( \xi \)-dependence of the multipliers of the elements of \( \mathcal{G} \) which are products of \( N \) factors \( t_{m_1 \cdots m_N} \), we find that \( k_m \sim \xi^N, N \geq 2 \). Since we are interested in the logarithmic divergence in (8.11) we are to expand to the order \( |\xi|^2 \). We have \( t_{m_1 \cdots m_N} = \prod_{n=1}^{N} t_{m_n} \), \( t_{m_n} = \prod_{n=1}^{N} t_{m_n} \), \( n, m = 1, 2, \ldots \), \( \eta_n = (k_1, k_2), \eta_m = 2 \xi^2 \frac{|k_1|^2 + k_2}{(1 - k_1)(1 - k_2)}, \text{ etc.} \)

\[
f = \prod \left( (1 - k_1)(1 - k_2) \right)^{-2D/2} \left( 1 + O(\xi^2, \xi^3) \right) \tag{8.14}
\]

\[
2 \pi i \eta_1 = \text{en}_1 + O(\xi^2), \quad 2 \pi i \eta_2 = \text{en}_2 + O(\xi^2), \tag{8.15}
\]

\[
2 \pi i \eta_1 = \xi + O(\xi^2), \quad t_{ab} = \det t_{ab}. \tag{8.16}
\]

The non-trivial \( O(1) \) correction factor thus comes only from the expansion of \( \left( \det t \right)^{-D/2} \) factor in (8.10).

\[
\left( \det t \right)^{-D/2} = \left( \det \xi \right)^{-D/2} \left( \det \xi \right)^{-D/2} = \left( \det \xi \right)^{-D/2} \left( 1 + \frac{1}{4} D \left( 2 \pi i \xi \right)^{-1} \left( 2 \pi i \xi \right)^{-1} \left( \xi \right)^2 + O(\xi^3) \right), \tag{8.16}
\]

\[
\left( \det t \right)^{-D/2} = \left( \det \xi \right)^{-D/2} \left( 1 + \frac{1}{4} D \left( 2 \pi i \xi \right)^{-1} \left( 2 \pi i \xi \right)^{-1} \left( \xi \right)^2 + O(\xi^3) \right). \tag{8.16}
\]

Let us note that while the formal agreement of the SP measure with the general Belavin-Knizhnik expression 35 \( d^4 u = d^4 u (f(y))^2 \left( \det t \right)^{-D/2} \) (where \( f \) is the holomorphic (3n - 3, 0) form without zeroes or poles in the interior of the moduli space but with double poles at the parts of the boundary corresponding to surface degenerations) is obvious, the modular invariance of it remains to be proved.
Thus finally (cf. (8.5))
\[ (8.18) \]
where the zero momentum tachyon and massless scalar tadpoles on the torus \( t \) and \( \lambda_1 \) were defined in (7.17)*, and hence (sec (8.8))
\[ (8.19) \]
Comparing this with (8.9) we get
\[ (8.20) \]
This finally determines \( c_1 \) if we use the relation between the coefficients \( d_i \) (the vacuum amplitudes) and \( \lambda_1 \) (the tadpole) for the torus (7.24) (in general \( d_i = c_1 \lambda_1 \)). Let us emphasize that having fixed the overall coefficient in \( d_i \) we automatically determine the normalization of the 2-loop amplitudes generated by \( \lambda_1 \).

The result that the logarithmically divergent part of \( d_i \) is proportional to \( D \) is consistent with the general factorization formula (3.10) (note that \( \chi_1 = 0 \) and \( \langle \Phi \rangle > 1 \sim D \)). To explain why massless propagator contributes effectively let us rederive the restriction (8.9) starting from the assumption that \( \Phi \) can be reproduced by an effective field theory (for similar analysis in the case of the annulus see Refs. 14 and 16). According to the discussion in Sec. 5 (see 5.6) we should have
\[ (8.21) \]
If we use the \( \sigma \)-model parametrization of the fields \( G_{\mu \nu} = \Phi_{\mu \nu} + h_{\mu \nu} \) and \( \phi \), i.e., use the tree action (2.21) (2.22) then \( \Delta_{\phi} \) is non-diagonal and is expressed in terms of \( \epsilon_0 \) (see (2.29), (2.30), (2.20)). We need only the "GG" element of this matrix since in the \( \sigma \)-model parametrization
\[ (8.22) \]
we find \( \exp \left( \frac{\epsilon_0}{2} \Delta_{\phi} \right) \) (where we have used that \( (\alpha' \Delta)^{(0)} \sim -\epsilon \)), see (2.50)). Equivalently, we can use the "S-matrix parametrization" of the fields (2.23)–(2.27) in which the propagator diagonalizes. The sphere plus torus contributions to the EA are then given by (D=26)
\[ (8.23) \]
(we use the harmonic gauge for the graviton). Using (8.21) (or directly integrating over \( h' \) and \( \phi' \)) we get for the \( O(\epsilon^2) \) - terms in \( s_2 \)
\[ (8.24) \]
This result is in agreement with (8.9). The two terms in the brackets are the contributions of the graviton and the dilaton exchanges.

9. APPROACH BASED ON OPERATORS OF INSERTION OF TOPOLOGICAL FIXTURES

In this section we shall rederive some of the results of the previous sections by using the representation of the loop part of the string generating functional in terms of the operators of insertion of holes, handles, etc. Similar approach was originally suggested in Ref. 18 and recently discussed also in Ref. 32 (our formulation is closer to that of Ref. 32).

1. Let \( I_0 = \frac{1}{2} \int d^2 x \sqrt{g} \Delta z, \Delta = -\nabla^2 \) be the free string action on an arbitrary 2-surface (for notational simplicity we shall often set 2 \( \pi \alpha' = 1 \) and do not explicitly indicate the index of \( \alpha' \)). Consider the expectation value
\[ (9.1) \]
* We are assuming, of course, that the integrals in (8.11) are restricted to the fundamental region and hence the integrals over \( k_1 \) and \( k_2 \) in the factorization limit appear restricted to the fundamental region of the modular group of the torus.

* We again replace \( (1/k^2)_{\mu \nu} \) by \( -\epsilon \epsilon' \). This normalization is consistent with the 2-d short distance regularization.
Using the functional Fourier transformation one can prove the validity of the following Hori-type representation (see, e.g. Refs. 68 and 8)

\[ F(z) = \{ \exp\left( \frac{1}{2} D \cdot \frac{\partial^2}{\partial z^2} \right) F(z) \}_{z=0}, \quad D = \Delta^{-1}, \quad (9.2) \]

which we have already used above (see, e.g., (6.18), (6.19)). Suppose now that we split the Green function \( D \) into two parts

\[ D = D_0 + \tilde{D}. \quad (9.3) \]

At this stage the split (9.3) may be arbitrary but actually we will be interested in the case when the Riemann surface is represented in terms of \( C \) with topological fixtures (pairs of holes in the Schottky parametrization); then \( \tilde{D}_0 \) will be the free propagator on the plane and \( \tilde{D} \) will contain all dependence on moduli (see (6.9)). The corresponding split in the Laplace operator is

\[ \Delta = \Delta_0 + \tilde{\Delta}, \quad \Delta_0 = D_0^{-1}, \quad (9.4) \]

\[ \tilde{\Delta} = -(1 + \Delta_0 \cdot \tilde{D})^{-1} \cdot \Delta_0 \cdot \tilde{D} = (1 + \Delta_0 \cdot \tilde{D})^{-1} \cdot \Delta_0 \]

Substituting (9.4) into (9.1) and using (9.2) we get:

\[ \langle F[z] \rangle_0 = \langle F[z] \rangle_{Q[z]} \]

\[ Q[z] = \alpha \exp(-i \int z \cdot \alpha \cdot \tilde{D}) \]

\[ N = \left( \frac{\det \Delta}{\det \Delta_0} \right)^{-D/2} \]

\[ h[z] = \frac{1}{2} \int d^2 z_1 d^2 z_2 (\tilde{D}_0(z_1) \tilde{D}_0(z_2)) \]

(**D** is the number of fields \( x^a \)). The functional \( Q[z] \) has rather complicated form because of the non-local structure of \( \Delta \) (9.4). It is possible to get a simpler representation for \( Q \) using normal ordering with respect to \( D_0 \).

Eq. (9.9) implies that \( F : \langle F[z] \rangle_{\Delta_0 = 0} \) and that there are no pairings of \( x^a \)'s inside \( F[z] \) when \( F \) is inserted into a \( \Delta_0 \)-correlator. Eq. (9.9) and similar relations, e.g.,

\[ F_1[z] F_2[z] = \exp(-\frac{i}{2} D_0 \cdot \frac{\partial^2}{\partial z^2}) F_1[z] : F_2[z] : \quad (9.10) \]

\[ : e^{i \phi} : = e^{i/2(\phi^{n+1})} D_0 (\phi^{n+1}) = e^{i \phi} D_0 \phi ; \quad : e^{i \phi} : = e^{i \phi} \phi \]

Employing (9.9) and (9.11) it is straightforward to prove that if \( A \) is some operator and \( \phi \) is the normal ordering with respect to \( D_0 \), then

\[ e^{-i \phi} \tilde{A} \phi = e^{i \phi} \tilde{A} \phi = C : e^{-i \phi} \tilde{A} \phi : \quad (9.12) \]

\[ B^{-1} = A^{-1} + \Delta_0^{-1} \cdot B \]

\[ C = \left( \frac{\det(1 - D_0 \cdot B)}{\det(1 - D_0 \cdot (\tilde{D} \cdot D_0))} \right)^{D/2} \quad (9.13) \]

Applying (9.12) and (9.13) to the case of the functional \( Q \) (9.7) we find

\[ Q[z] = : \exp h[z] : \quad (9.14) \]

\[ h[z] = \frac{1}{2} \int d^2 z_1 d^2 z_2 (\tilde{D}_0(z_1) \tilde{D}_0(z_2)) \quad (9.15) \]

where we have already specified \( D_0 \) in (9.3) to be the propagator on the plane.

The basic relation (9.5) implies that a gaussian correlator with respect to a complicated Green function can be represented as the correlator with respect to the trivial Green function with the additional insertion of the functional \( Q = : e^{\phi} \) (9.14) with \( \phi \) given by (9.16) where \( \tilde{D} \) is the non-trivial part of the original Green function.

If we apply (9.5) to the case of the string generating functional (4.1) we get \( F = e^{-l_m} \)

\[ Z = \Omega^{-1} \sum_{n=0}^{\infty} Z_n, \quad \tilde{Z}_n = c_n \int d\mu_n Z_n, \quad (9.17) \]

\[ Z_n = e^{-l_m} >_0 < e^{l_m} : e^{-l_m} >_0, \quad < 1 >_n = 1, \quad n = 0, 1, \ldots \]

\[ h_n = \frac{1}{2} \int d^2 z \cdot \tilde{D}_n \cdot \Delta_0 z = 0 \quad (9.19) \]

(the subscript \( n \) indicates that \( D_n = D_n - D_0 \) corresponds to a genus \( n \) surface). The representation (9.17)–(9.19) may be a bit misleading: it may look as if it is possible to integrate first over the moduli and then to compute the expectation value on the plane. However, this is not possible to do in general since the \( \sigma \)-model action \( I_0 + I_{m+1} \) and the expectation values depend implicitly on moduli through the integration region for the 2-d coordinates (for \( n \geq 1 \) the \( \sigma \)-model is, in fact, defined not on a sphere but on a higher genus surface). Still (9.17)–(9.19) may
be useful in the case when we study particular regions of the moduli spaces corresponding to surface degenerations (or when we consider the vacuum partition function, putting $l_{\text{int}} = 0$).

Let us now demonstrate how the representation (9.18) works on the examples of the disc and the torus. Expanding the non-trivial part of the Neumann function on the disc in the "plane with a hole" representation (7.43) in powers of radius of the hole, using $5(z - \tau)$ in order to reduce $5(z - \tau)$ to $5(z - \tau)$ and integrating by parts to get rid of the $\delta$-functions, one finds the following expression for the functional $h_1$ (9.19) (see also Refs. 18 and 32) (2 $\pi \alpha' = 1$)

$$h_{1/2}[z] = 4 \pi \sum_{n=1}^{\infty} \frac{a^{2n}}{n!} \frac{1}{m + (m - 1)} \left( \partial^n z \partial^m z \right)(w). \quad (9.20)$$

If we now consider the limit $a \to 0$ (: $\varepsilon^{n,n} := 1 + 4 \pi a^2 \delta z \delta z + \ldots$) and substitute (9.20) together with the correct modular measure given by (7.50) into (9.17), (9.18) we easily reproduce the consistent result (7.51) and (7.52) for the logarithmically divergent part of the disc contribution to the generating functional $Z_d$. Note that the normal ordering in (9.18) leads to the normal ordering in the operator $O_{1/2}$ in (7.52).

Using the Schottky-type (plane with 2 holes) representation for the torus, substituting the non-trivial part $D$ of the corresponding Green function (7.11) into (9.16) and going through the same steps as in the derivation of (9.20) we get the following expression for the corresponding functional $h_1$ (9.19) $^{18}$

$$h_1 = 4 \pi \left\{ \frac{1}{4 \pi \varepsilon n |k|} \left[ z(\xi) - z(\eta) \right]^2 \right. \
- \sum_{m=1}^{\infty} \frac{1}{m! (m - 1)} \left[ \partial^m \left\{ \left( \xi - \eta \right)^m \partial z \partial x \left( \xi \right) \right\} \right. \left. \left[ \partial^m \left\{ \left( \xi - \eta \right)^m \partial z \partial x \left( \eta \right) \right\} \right] \right. \\
+ c.c. \left\} \equiv h_1(\xi, \eta, k, |z|). \quad (9.21)$$

The first term here originates from the last non-holomorphic term in (7.11). Like the second term in (9.21) it can be represented (by using Taylor expansion) as an infinite series in powers of $\xi - \eta$ and derivatives of $x$ (cf. (9.13))

$$h_1 = 4 \pi \left\{ \frac{1}{2 \pi \varepsilon n |k|} \left[ \xi - \eta \right]^2 \partial x \partial z \partial x(\xi) + O\left( \left( \xi - \eta \right)^2, \left( \xi - \eta \right)^3 \right) \right\}$$

* It is useful to note that

$$\lambda = \frac{(z_1 - \xi)(z_2 - \eta)}{(z_1 - \eta)(z_2 - \xi)} = \frac{1 - \xi - \eta}{z_1 - \eta} \left( 1 - \frac{\xi - \eta}{z_2 - \xi} \right)$$

Considering the tadpole factorization limit $\xi \to \eta$ and integrating over $\xi$ and $\eta$ (see (7.8), (7.15)) we reproduce the result (7.20) for the divergent part of the generating functional $Z_d$.

It is possible also to derive the general expression for $h_d$ and hence for the operator of insertion of a topological fixture $Q_n$ (9.14) for arbitrary genus $\gamma$ representing $E$ and $\omega$ in the Green function (6.7)–(6.9) in terms of the Laurent series in $x$ and $w$ $^{18}$. As we discussed in the previous sections, to have the correct factorization $^{19}$ and hence renormalization properties of $\tilde{Z}$, it is necessary also to go from the flat plane with holes to a curved 2-space. Then the curvature dependent correction terms appear in the normal ordering relations and in the modular measure.

2. In order for the renormalization group to operate at string loop level the tadpole divergences found above should, in fact, exponentiate to become counterterms for the string action (see Secs.3 4, Eq.(3.20)) $\tilde{Z}_d = c_0 \Omega^{-1} \int d\mu < e^{-\xi - 4l} >_a, 1 \equiv \int d\mu c_0 e^{\xi} >_c + \ldots$ Higher powers of $e^{4l}$ appearing from the expansion of $e^{4l}$ in the string coupling should cancel higher order tadpole infinities in higher genus contributions to $\tilde{Z}$.

For this to happen the relative coefficients of the loop contributions to $\tilde{Z}$ should take particular values. Let us assume that the exponentiation does indeed take place, for example, the tadpole divergence on the torus exponentiates. Consider the regions of the moduli space (for each genus $n$) corresponding to surfaces with small and far separated handles. It is then natural to expect that the exponentiation of the 1–loop tadpole divergence implies that the contribution to $\tilde{Z}$ coming from the specified regions of moduli spaces can be effectively represented

$$\tilde{Z} = c_0 g^{-2} \Omega^{-1} < e^{-\xi - 4l} >_d >_a + \ldots \quad (9.23)$$

where (see also Refs. 18 and 32)

$$H_{1/2}[z] = p_1 \int d\mu_1 : e^{h_i} := p_1 \int [d\mu_1 \frac{d^2 \xi d^2 \eta}{|\xi - \eta|^2} : e^{h_i(\xi, \eta, |z|)} :] \quad (9.24)$$

$$\approx \frac{1}{2} p_1 \lambda \pi \varepsilon n \Omega \Omega_{1/2}[z] + \ldots, p_1 = c_1 c_0^{-1} \quad (9.25)$$

(cf. (7.15), (7.20) and (7.38)). Eq. (9.23) corresponds to the "renormalization group improved" perturbation theory or gives a "dilute handle gas" approximation for $\tilde{Z}$. * . The representation (9.23) may be true only for small handles since it is only in this case that we can ignore the dependence of $\Omega$ on the moduli through the integration region (for small handles we may approximately consider the $\sigma$-model as being defined on a sphere).

* Let us emphasize that the possibility to "resum" $\tilde{Z}$ in the form (9.23) depends crucially on the use of the extended parameterization of moduli with the $\Omega^{-1}$ appearing as a universal factor for all contributions to $\tilde{Z}$.
In Eq. (9.23) we have explicitly indicated the dependence on the (bare) string coupling constant. Since $g$ is related to the constant part of the dilaton it should renormalize together with $\phi$ (see Secs. 3 and 4). This observation suggests that there should be higher order terms in (9.23) which should generate the renormalization of $g = e^{\phi}$ in the exponent. A natural guess is

$$\hat{Z} = \Omega^{-1} \sum_{n=0}^{\infty} g^{2(n-1)} \hat{c}_n \int d\mu_n < e^{-i\omega\cdot p\hat{H}_n} >_n + \ldots$$

(9.26)

where the wave in $d\mu_n$ indicates that the corresponding integration regions over moduli do not contain configurations in which all handles are small and separated (it is assumed that we have already summed the contributions of such configurations). The consistency of the RG demands that we should be able to resum (9.26) further, generating higher order terms in the exponent (which produce higher order tadpole renormalizations)

$$e^{-i\omega} \to e^{-i\omega \cdot \hat{p}}, \quad F = g \hat{H}_1 + g^2 \hat{H}_2 + g^3 \hat{H}_3 + \ldots, \quad \hat{H}_1 = H_1,$$

(9.27)

$$\hat{H}_n = p_n \int d\mu_n : e^{\hat{c}_n} : , \quad p_n = \omega_n \hat{c}_0^{-1}$$

(9.28)

The hat in the modular measure indicates that the measure and the integration region are chosen in a way that avoids overcounting equivalent configurations after we expand (9.27) in powers of $g$ and compare with the usual expansion in genus. $\hat{H}_n$ contain the lowest order $O(\hat{c}_n)$ singularities corresponding to the factorization on a finite part of genus $n$ tadpole and a correlator on the sphere. In this way we may reproduce the expression (3.21) for the tadpole counterterm.

As we already noted, a representation based on (9.23), (9.27) can be valid only approximately because of the dependence of $\hat{c}_n$ on moduli through the integration region. However, $l_{int}$ is absent in the case of the vacuum partition function. In this case we may expect that $\hat{Z}$ can be exactly represented as a correlator on the sphere

$$\hat{Z}(0) = \alpha g^{-2} \Omega^{-1} < e^{\phi} >_0,$$

(9.29)

where $F$ is given by (9.27) and $\Omega^{-1}$ should be defined using some regularization prescription (e.g. $\Omega^{-1} \to \partial / \partial \hat{c}_0$). $F$ can be considered as non-local (depending on all powers of derivatives of $\phi$ integrated over moduli) addition to the free string action. Only the leading order $O(\hat{c}_n)$ infinite part of $\hat{F}$ should be local, since it should coincide with the tadpole counterterm (9.21). $F_n = - \sum_{a=1}^{\infty} b_a \hat{c}_a \hat{c}_0 + O(\hat{c}_n^2)$. Eq.(9.29) represents a resummation of the usual perturbation theory.*

3. To illustrate how the functional $F$ in (9.27), (9.29) is constructed let us consider the $O(g^0)$-term in $F$ (9.27). Expanding $e^\phi$ in powers of $g$ we should reproduce the ordinary perturbation expansion in genus $n$. Namely, we should have

$$\hat{Z}_2(0) = \alpha g^{-2} \Omega^{-1} < H_2 + \frac{1}{2} (H_1)^2 >_0$$

(9.30)

where $\hat{Z}_2(0)$ is the usual genus 2 contribution to the string partition function, given e.g. in the Schottky parametrization, by Eq.(8.10). Thus we get the following equation for the modular measure in $\hat{H}_2$ (9.28)*

$$\int d\mu_2 = \int d\mu_1 - \frac{1}{2} q_2 \int d\mu_1^{(1)} \int d\mu_1^{(2)} < e^{\mu_1^{(1)}} : e^{\mu_1^{(2)}} : >_0, \quad q_2 = p_2^2 \hat{c}_2^{-1},$$

(9.31)

where we have noted that $< : e^{\phi} : >_0 = 1$ and used the superscripts $(1,2)$ to denote the two different sets of genus one moduli. $d\mu_1$ is given by (9.24), (7.8) and $h_1$ by (9.21).

The expectation value of the product of the two operators $Q = : e^{\phi} : (9.14)$ can be easily computed in general using (9.7), (9.12), (9.4)

$$< Q^{(1)}(x) Q^{(2)}(x) >_0 = N^{(1)}(x) N^{(2)}(x) < e^{-1/2 (\hat{\phi}^{(1)} + \hat{\phi}^{(2)})} >_0$$

$$= \exp\left( - \frac{D}{2} \text{tr} \hat{c}_n \left( 1 + \hat{A}_0 \cdot \hat{D}^{(1)} \right) \left( 1 + \hat{A}_0 \cdot \hat{D}^{(2)} \right) \right)$$

$$- \frac{D}{2} \text{tr} \hat{c}_n \left( 1 + \hat{A}_0 \cdot \hat{D}^{(1)} \cdot \hat{A}_0 \cdot \hat{D}^{(2)} \right)$$

(9.32)

$$= \exp\left[ - \frac{D}{2} \text{tr} \hat{c}_n \left( 1 + \hat{A}_0 \cdot \hat{D}^{(1)} \cdot \hat{A}_0 \cdot \hat{D}^{(2)} \right) \right] = \hat{M}_2,$$

Substituting here the expression for $\hat{D}$ (9.79) or (7.11) we determine $M_2 (\epsilon^{(1)}, \eta^{(1)}, \xi^{(1)}, \eta^{(2)}, \xi^{(2)}, \eta^{(2)}, \xi^{(2)})$. Note that $M_2$ depends non-trivially on the both sets of moduli (the expectation value $< : e^{\phi} : >_0$ "couples" them). The expression in the exponent can be simplified by using (once) the relation $\hat{D} = - \frac{1}{\alpha} \text{Tr} \hat{c}_n(x) \hat{c}_n(x)$ (see (6.8)). This relation can be used twice in the first term of the expansion of $\hat{M}_2$ in powers of $\hat{D}$ (cf. (7.62))

$$\hat{M}_2 = 1 + \frac{D}{2} \text{tr} \left( \hat{A}_0 \hat{D}^{(1)} \right) \hat{A}_0 \hat{D}^{(2)} + \ldots$$

(9.33)

$$+ \frac{1}{2} \frac{D}{(2\pi)^2} \text{Tr} \hat{c}_n \hat{c}_n \left( 1 + \hat{A}_0 \cdot \hat{D}^{(1)} \hat{A}_0 \hat{D}^{(2)} \right) \chi \left( z_1 - \eta_1 \right) \left( z_2 - \xi_2 \right)$$

$$\times \left( z_1 - \eta_1 \right) \left( z_2 - \xi_2 \right) + \ldots$$

* Note once again the crucial role of the use of a parametrization in which the M"obius volume factorizes.
where we have applied (7.10) and changed the superscripts (1,2) for the subscripts for correspondence with (8.10). Using the formal projective invariance of $M_2$ and the measures $d\mu$, we may fix it in the same way as we did in Sec. 8: $\xi_1 = 0$, $\eta_1 = 1$, $\eta_2 = 0$, $\xi_2 = \xi$. The limit $\xi \to 0$ then corresponds to the factorization of the genus 2 surface on two tori (see (8.11)-(8.18)). According to (7.33) the second term in (9.31) contains the logarithmic singularity which appears to cancel the corresponding singularity (8.18) in $\int d\mu_2$ if $\frac{1}{2} = (2 \pi)^4$, cf. (8.19), (8.20) (the same condition guarantees the cancellation of the quadratic tachyonic infinities originating from the leading order terms in the expansions of the measures) *.

The cancellation of the tadpole divergence in $\int d\mu_2$ against the singularity in the expectation value of the square of the one-handle operator in (9.31) does not imply that all leading multiple divergences in the generating functional $\tilde{Z}$ may be represented by multiple insertions of $\hat{H}_1$. In the case of a non-trivial background ($I_{\text{ref}} \neq 0$) one is to consider factorizations with external "legs" on one part of a diagram. However, if a generalization of (9.29) with $F \to F - I_{\text{ref}}$ is true at least as some approximation (cf. (9.23), (9.26), (9.27)) (which should be sufficient for the analysis of the RG), the singularities which appear in $F$ and hence the corresponding counterterms should be universal. Treating $\hat{H}_n$ as a kind of vertex operators, we expect to find singularities when the integrands of (arbitrary number of) $\hat{H}_n$ and (or) of the ordinary vertex operators in $I_{\text{ref}}$ "collide". This easily explains the exponentiation of the external leg divergences ** and also implies that the loop-corrected $\beta$-functions should contain terms of all powers in the massless fields **. The question of renormalizability of $\hat{Z}$ is thus reduced to that of renormalizability of the theory $I = I - ?$ on the sphere.

There are some subtleties in the renormalization procedure at the $\hat{e}n^k e$, $k \geq 2$, level. Consider, for example, the genus 2 contributions. Splitting $\hat{H}_1$, $\hat{H}_2$ into a finite piece and a logarithmic divergence (proportional to $O_n$), $\hat{H}_n = \hat{H}_n^f + \hat{H}_n^\infty$, we find for the part of $\tilde{Z}_2$ which is divergent to the tadpole singularities

$$\tilde{Z}_{2,\infty} \sim e^{-i\alpha} \hat{H}_{1,\infty} > 0 + e^{-i\alpha} \hat{H}_{1, f} > 0 + e^{-i\alpha} \hat{H}_{1, 0} > 0 + e^{-i\alpha} \hat{H}_{2, 0} > 0 .$$  (9.34)

The first two terms here are $O(\hat{e}n^2 e)$ while the third is $O(\hat{e}n^2 e)$. From the analysis of factorization one, however, expects to find an additional $O(\hat{e}n^2 e)$ term corresponding to the 2-point function on the torus connected with the tadpole on the torus and the sphere with external leg insertions. One possibility could be that the combinatorics of string diagrams is such that this extra term combines together with the term corresponding to the two separate tadpole insertions on the sphere to be in agreement with the last term in (9.34). We expect, however, that there is an additional $O(g^2 \hat{e}n^2 e)$ term present in the exponent which produces an additional $\hat{e}n^2 e$

* Note the presence in (9.33) of the same factor $D$ which we have found in Sec. 8 from the expansion of the determinant of the period matrix factor in $d\mu_2$.

** At this point we disagree with Refs 17 and 32 where no higher order terms in $\beta$ are found (these authors do not account for the contributions of momentum dependent singularities of the amplitudes (see Sec. 3).

...
REFERENCES

23) (a) Lovelace, C., Phys. Lett. B322, 490, 703 (1970);
Olive, D., Nuovo Cim. 3A, 399 (1971);
Kaku, M. and Scherk, J., Phys. Rev. D3, 430 (1971);
(b) Alessandri, V., Nuovo Cim. 2A, 321 (1971);
Pertiwal, V., Mod. Phys. Lett. A4, 33 (1989);
45) Boulware, D.G. and Brown, L.S., Phys. Rev. 112, 1628 (1968);
Aref'eva, I. Ya., Slavnov, A.A. and Faddeev, L.D., Teor. Mat. Fiz. 21, 311 (1974);
Seiberg, N., Phys. Lett., B187, 56 (1987);
64) Fay, J.D., Theta Functions on Riemann Surfaces, Springer Notes in Mathematics 352 (Springer, 1973);
Moore, G., Phys. Lett. B176, 369 (1986);
Stampato in proprio nella tipografia
del Centro Internazionale di Fisica Teorica