PINCHING CONDITIONS FOR YANG-MILLS INSTABILITY
OF HYPERSURFACES

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A compact Riemannian manifold $M$ is said to be Yang-Mills instable, if for every choice of compact Lie group $G$ and every principle $G$-bundle $P$ over $M$, none of the nonflat Yang-Mills connection in $P$ is weakly stable. This paper gives curvature pinching condition for the Yang-Mills instability of hypersurfaces in space form.
1 Introduction

Let M be a compact Riemannian manifold and P a principal G-bundle over M, where G is a compact Lie group. On the space of connections in P, we consider the Yang-Mills functional $J: \mathcal{C} \rightarrow \mathbb{R}$ defined by

$$J(w) = \frac{1}{2} \int_M \| \Omega \|^2, \quad w \in \mathcal{C},$$

where $\Omega$ is the curvature of the connection and the norm $\| \Omega \|$ is defined by the Riemannian metric of M and a fixed $\text{Ad}_G$-invariant inner product on the Lie algebra $\mathfrak{g}$ of G.

A Yang-Mills connection is a critical point of $J$ and its curvature is a Yang-Mills field. A Yang-Mills connection $w$ is called a weakly stable if for any family of connections $w^t$, $|t|<1$ with $w^t = w$, the second variation of the functional at $w$ is non-negative, i.e.

$$\frac{d^2}{dt^2} J(w^t) \bigg|_{t=0} \geq 0.$$  

M is said to be Yang-Mills instable, if for every choice of compact Lie group G and every principal G-bundle P over M, none of the nonflat Yang-Mills connection in P is weakly stable. A typical example of Yang-Mills instable manifold is the Euclidean sphere $S^n$ with $n > 5$ ([1]). For the case that M is a hypersurface or submanifold in the Euclidean space $E^m$ or $S^m$, some conditions for the Yang-Mills instability of M can be found in [2] and [3]. In this paper, we give an intrinsic condition for the Yang-Mills instability of M where M is a hypersurface in a space form $S^m(c)$ of constant curvature $c$ with $c > 0$. 

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2 Preliminaries

Let $M^n$ be an isometrically immersed compact submanifold in a space form $S^m_p(c)$ ($c > 0$). As known, $S^m_p(c)$ can be isometrically immersed into the Euclidean space $E^m_p$ in a standard manner. We choose a local field of orthonormal base $\{ e_i, e_d, e_c \}$ of $E^m_p$ such that, restricted to $M^n$, $\{ e_i \}$ span the tangent space of $M$, $\{ e_d \}$ span the normal space of $M^n$ in $S^m_p(c)$ and $e_c$ is the unit position vector. Throughout this paper we agree on the following ranges of indices unless otherwise stated:

1 $\leq i, j, k, \ldots \leq n$, $n+1 \leq \alpha, \beta, \gamma, \ldots \leq n+p$, $1 \leq a, b, c, \ldots \leq \dim G$

Now let $w$ be a Yang-Mills connection in a principal $G$-bundle $P$ over $M^n$ with compact Lie group $G$. Let

$$w^t = w + A^t,$$

where $A^t$ is a $G$-valued 1-form on $M^n$ and let

$$B = \frac{dA^t}{dt} |_{t=0}.$$

Evidently, $B$ is a $G$-valued 1-form on $M^n$.

It is known ([1]) that the second variation of the Yang-Mills functional is

$$4 \sum \left< \xi VB + \text{R} - \text{W}(B) , B > \right>,$$

where $d^w$ is the gauge covariant differential operator, $\delta^w$ is the adjoint operator of $d^w$, and $\text{R}^w(B)$ is an operator defined by

$$\text{R}^w(B)(X) = \sum \left[ \left< \Omega(e_i, X), B(e_c) \right>, \forall X \in T_p(M) \right].$$

If $\delta^w B = 0$, (6) can be rewritten ([1]) as

$$\frac{d^2}{dt^2} J(w^t) |_{t=c} = \sum \left< \nabla^w X, \nabla^w B + B \text{R}^w + 2 \text{R}^w(B), B > \right>,$$

where $\nabla^w$ is the trace Laplacian operator.

Set

$$B(e_i) = b_i = \sum \xi b \cdot X_a , \quad \Omega(e_i, e_j) = \sum f_{ij}^a X_a , \quad [X_a, X_b] = \sum C_{ab}^c X_c ,$$

where $\{ X_a \}$ is an orthonormal base of $G$, i.e. $\left< X_a, X_b \right> = \delta_{ab}$, and $C_{ab}^c$ are the structure constants of the Lie group $G$.

Let $V$ be a fixed unit vector in $E^m_p$. $\nabla$ denotes the tangent projection to $M^n$ of $V$. Locally, $\nabla_a$ can be expressed as

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For each \( V \), we can define a \( \mathbb{Q} \)-valued 1-form \( B \) as follows.

\[
B^V = \sum v_i f_{ij} w_j x_a,
\]

where \( \{w_i\} \) is the dual base of \( \{e_i\} \).

It is easy to see that \( \sum w_i B^V = 0 \). So formula (7) applies this case. For each \( V \), denote the correspond second variation of \( w \) by \( Q_w(V) \) and \( Q_w(V) \) can be considered as a quadratic form on \( E^{\mathbb{Q}} \).

We have ([3])

\[
\text{trace } Q_w = \sum c_i (4-n) c_i + \sum c_i (4-n) c_i + 2 \sum c_i (4-n) c_i.
\]

Choose \( \{e_i\} \) such that \( h_{ij} = \lambda_i \lambda_j \), where \( \lambda_1, \ldots, \lambda_m \) are principal curvatures of \( M^n \). From (12), we have, setting \( H = \sum \lambda_i \),

\[
\text{trace } Q_w = \sum (4-n) c_i + 2 \lambda_i + 2 \lambda_j - H \lambda_i \lambda_j.
\]

It is obvious that if trace \( Q \) is negative then \( w \) is weakly stable if and only if \( w \) is flat. So we have the following

**Theorem 1.** If \( M^n \) is a compact hypersurface in \( S^{n+1}(c) \) \((c>0)\) such that its principal curvatures \( \lambda_1, \ldots, \lambda_m \) satisfy

\[
(4-n) c + \lambda_i (2 \lambda_i + 2 \lambda_j - H) < 0, \text{ for any } i \neq j,
\]

at every point of \( M^n \), then \( M^n \) is Yang-Mills instable.
Now we give a pinching condition on the curvatures for a compact hypersurface in $S^{n+1}(c)\ (c>0)$ to be Yang-Mills instable.

Theorem 2. Let $S^{n+1}(c)$ be an $(n+1)$-dimensional simply connected space form with constant sectional curvature $c>0$.
Suppose that $M^n(n+5)$ is a compact hypersurface in $S^{n+1}(c)$ of which the sectional curvatures $\text{Riem}^M$ satisfy the following pinching condition:

$$c + 3a^2/[(n-4)c + (n-1)a] \leq \text{Riem}^M < c + a$$

(15)

for some constant $a>0$. Then $M^n$ is Yang-Mills instable.

From the Gauss equations $R_{ij;kl} = c + \lambda_i \lambda_j$, (15) is equivalent to

$$a^2/[(n-4)c + (n-1)a] \leq \lambda_i \lambda_j < a, \text{ for any } i \neq j.$$ 

(16)

Since $M^n$ is convex, without loss of generality, we may assume

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n.$$ 

(17)

Setting

$$\lambda_k = H - 2\lambda_k,$$

(18)

we have $\lambda_n < \lambda_{n-1} \leq \ldots \leq \lambda_1$ and (14) is equivalent to

$$2\lambda_i^2 - \lambda_i \lambda_j - (n-4)c < 0, \ i \neq j.$$ 

(19)

Due to (17) it is equivalent to

$$\lambda_i < \frac{1}{4} \left[ \lambda_j + \sqrt{\lambda_j^2 + 8(n-4)c} \right], \ i \neq j.$$ 

(20)

Lemma 1. If $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ and $\lambda_n < \frac{1}{4} \left[ \lambda_n + \sqrt{\lambda_n^2 + 8(n-4)c} \right]$, then (14) holds.

Proof Since, for any $k$, we have

$$\lambda_k - \lambda_{k-1} = (H - 2\lambda_k) - (H - 2\lambda_{k-1})^2$$

$$= H^2 - 4\lambda_k H + 4\lambda_k^2 - H^2 + 4\lambda_{k-1} H - 4\lambda_{k-1}^2$$

$$= 4(\lambda_k - \lambda_{k-1})(H - \lambda_k - \lambda_{k-1}) > 0,$$

hence for any $i, k$,

$$\lambda_i < \lambda_k < \frac{1}{4} \left[ \lambda_n + \sqrt{\lambda_n^2 + 8(n-4)c} \right] < \frac{1}{4} \left[ \lambda_k + \sqrt{\lambda_k^2 + 8(n-4)c} \right].$$

The Lemma is proved.
Lemma 2. If
\[ \lambda_n < \frac{1}{8} [ H + \sqrt{H^2 + 16(n-4)c} ], \] (21)
then
\[ \lambda_n < \frac{1}{4} [ A_n + \sqrt{A_n^2 + 8(n-4)c} ]. \] (22)

Proof From (21), we have
\[ 4 \lambda_n^2 - \lambda_n H - (n-4)c < 0 . \]
It follows that
\[ (6 \lambda_n - H)^2 < A_n^2 + 8(n-4)c . \]
Consequently,
\[ 4 \lambda_n < (H - 2 \lambda_n) + \sqrt{A_n^2 + 8(n-4)c} . \]
So
\[ \lambda_n < \frac{1}{4} [ A_n + \sqrt{A_n^2 + 8(n-4)c} . \]

Now let
\[ b^\lambda_1, b^\lambda_j < B^\lambda, i \neq j, \] (23)
where \( b, B \) are positive constants.

Lemma 3. If \( \lambda_j > b \) and \( b, B \) in (23) satisfy
\[ B^\lambda - Lb + (n-1)b^\lambda = 0 , \] (24)
where
\[ L = \frac{1}{b} [ 7(n-1)b + \sqrt{(n-1)b^2 + 12(n-4)c} ] , \] (25)
then (21) holds.

Proof Since \( \lambda_i \lambda_j < B^\lambda \), from (24) we have
\[ \lambda_n < \frac{B^\lambda}{\lambda_i} \leq \frac{B^\lambda}{b} = L - (n-1)b . \] (26)

From (25), we have
\[ 3L^\lambda - 7(n-1)bL + 4(n-1)b^\lambda - (n-4)c = 0 . \] (27)

Hence
\[ L - (n-1)b = \frac{1}{8} [ L + \sqrt{L^2 + 16(n-4)c} ] . \] (28)

Thus
\[ \lambda_n < \frac{1}{8} [ L + \sqrt{L^2 + 16(n-4)c} ] . \] (29)

If \( L < H \), then (29) implies (21). Suppose \( L > H \). Set \( L - H = K \)
We have
\[
\lambda_n = H - \sum_{k \neq n} \lambda_k < H - (n-1)b = L - (n-1)b - K
\]
\[
= \frac{1}{8} \left[ L + \sqrt{L^2 + 16(n-4)c} - 8K \right] = \frac{1}{8} \left[ H + \sqrt{H^2 + 16(n-4)c} - 7K \right]. \tag{30}
\]
On the other hand, it is easy to see
\[
\left( \sqrt{H^2 + 16(n-4)c} + 7K \right)^2 > L^2 + 16(n-4)c. \tag{31}
\]
Hence
\[
\sqrt{L^2 + 16(n-4)c} - 7K < \sqrt{H^2 + 16(n-4)c}. \tag{32}
\]
From (30) and (32), we still have
\[
\lambda_n < \frac{1}{8} \left[ H + \sqrt{H^2 + 16(n-4)c} \right].
\]
The Lemma is proved.

**Lemma 4.** If \( n > 5 \) and (24) holds, then (21) is true.

**Proof** It suffices to prove the Lemma in the case of \( \lambda_i < b \).

Since \( n > 5 \), there exist \( \lambda_i \) and \( \lambda_2 (\lambda_3) \) such that \( \lambda_i \lambda_2 > b^2 \).

Construct
\[
\lambda_i' = \lambda_2' = \frac{1}{2} (\lambda_i + \lambda_2). \tag{33}
\]
Then
\[
(\lambda_i')^2 = \frac{1}{4} (\lambda_i + \lambda_2)^2 > \lambda_i \lambda_2 > b^2,
\]
so that \( \lambda_i' > b \). Obviously we still have \( \lambda_i' - \lambda_2' \leq \lambda_3 \) and
\[
\lambda_i' + \lambda_1' + \lambda_2' = H \text{ unchanged. Thus, applying Lemma 3 to the case}
\]
where \( 0 < \lambda_i' - \lambda_2' \leq \lambda_3 \leq \ldots \leq \lambda_n \), we can prove this lemma.

The proof of Theorem 2. (15) is equivalent to
\[
3a^2 / [(n-4)c + (n-1)a] < \lambda_i \lambda_j < a. \tag{34}
\]
Set
\[
b^2 = 3a^2 / [(n-4)c + (n-1)a] \text{ and } B^2 = a. \tag{35}
\]
we have
\[
3a^2 = (n-4)cb^2 + (n-1)b^2 a. \tag{36}
\]
Thus
\[
a = \frac{b}{c} [(n-1)b + \sqrt{(n-1)b^2 + 12(n-4)c}]
\]
\[
= \frac{b}{c} [7(n-1)b + \sqrt{(n-1)b^2 + 12(n-4)c} - (n-1)b^2]. \tag{37}
\]
It follows from (35) and (37) that
\[ B^2 - bL + (n-1)b^2 = 0, \quad (38) \]
where
\[ L = \frac{1}{6} [7(n-1)b + \sqrt{(n-1)b^2 + 12(n-4)c}]. \quad (39) \]

Now applying Lemma 4 we complete the proof.

**Corollary.** If \( n \geq 5 \) and \( M^n \) is a compact \( n \)-dimensional hypersurface in the Euclidean space \( E^n \) satisfying the condition
\[ \frac{3a}{(n-1)} < \text{Riem}^M < a, \]
for some \( a > 0 \), then \( M^n \) is Yang-Mills unstable.

**Remark.** The constant \( a \) can be replaced by a positive function \( a \in C^\infty(M) \) such that the pinching condition holds at every point \( x \in M \).

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