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ON HETEROTIC SUPERMANIFOLDS *

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ABSTRACT

The geometry of heterotic supermanifolds is discussed with particular reference to patching conditions and gauge fixing. Superfield formalism is used and the associated torsion constraints are solved explicitly in an arbitrary gauge, that is without imposing gauge conditions. Finite gauge transformations are constructed. The structure group associated with Wess–Zumino type gauges is obtained and is reduced by further refinements of the gauge conditions up to the stage at which the standard description of the super Riemann surface is recovered. It is shown that any invariant functional of the super 3-bein can depend only on a finite number of parameters, i.e. the moduli and super moduli. Chiral superfields and the structure of action functionals are discussed and, finally the integration measure in supermoduli space is derived by an application of the Faddeev–Popov prescription.

1. INTRODUCTION

The geometry of super Riemann surfaces [1] is a topic of crucial importance for the understanding of superstring dynamics. It has been studied intensively in recent years but, in spite of this, there are still some points in need of clarification. The purpose of this paper is to provide a careful examination of the gauge fixing procedure. This procedure amounts to establishing a rule for selecting a representative member from each class of equivalent geometries [2] and defining a measure with which to sum over the classes [3]. Since the discussions of this mechanism so far available in the literature do not, to our knowledge, explicitly address the question of global consistency of the commonly used Wess–Zumino class of gauges, we shall do this here. Our conclusion is that the Wess–Zumino gauges are consistent and that no meaningful information is lost when they are used. In particular, the classification of superconformal geometries by means of moduli, $3(\gamma-1)$ even and $2(\gamma-1)$ odd parameters for surfaces of genus $\gamma > 1$, is complete. There are no hidden moduli in the configurations of the several auxiliary fields that are set to zero in the Wess–Zumino gauges. Although this result only confirms what was believed on heuristic grounds, it is an important matter of principle whose justification requires a study of finite superconformal transformations and patching conditions.

We shall follow the approach of Howe [4] as adapted to the heterotic case by Nelson and Moore [5]. This involves establishing a conformal geometry on a supermanifold whose coordinates (x^m, θ) comprise two commuting and one anti-commuting component. The geometry is intended to be conformal in the sense that it reduces locally to flat superspace. The interesting questions are concerned with the global characterization of the supermanifold and its geometry. The geometry is expressed through frame and connection forms, the superdreibein $dz^M E_M^A$ and spin connection, $dz^M \Omega_M$, respectively. These geometrical elements are globally defined.

The geometrical quantities are subject to a number of restrictions–the torsion constraints. In order to express these elements in terms of independent fields it is necessary to solve the constraints. Although this can be an extremely lengthy process, in the absence of Wess–Zumino gauge [4], [5] simplifications, it is feasible in the case of heterotic geometry, to which our considerations are restricted, where each superfield has only two components, one bosonic and one fermionic. The solution is given in Sec. 3. With this it becomes possible to show explicitly that the most general configuration of the geometrical fields can be reduced to the Wess–Zumino form by a superconformal transformation and that the reduction is globally meaningful. The required transformation is exhibited in Sec. 5.

Although the question of the global characterization of a general configuration is partially solved by the transformation constructed in Sec. 5, this succeeds only in reducing the problem to that of treating Wess–Zumino gauge configurations. There remains the problem of further reducing these to what might be called the "super conformal gauge".

This latter problem is in fact much simpler and can be treated most effectively by means of Ward identities, which are the subject of Sec. 4. There it is shown, by examining the response of invariant functionals to infinitesimal transformations and drawing on the established theorems concerning holomorphic differentials on Riemann surfaces [6], that such functionals can depend only on a finite set of moduli [2], $3(\gamma-1)$ even and $2(\gamma-1)$ odd (anticommuting) moduli for surfaces of genus $\gamma > 1$.

Secs. 3, 4 and 5 comprise the main body of this paper. However, to establish notation and define the problem more precisely, the principles of heterotic geometry are reviewed in Sec. 2. The frame and connection forms are introduced, together with the torsion conditions by which they are restricted. Their response to the gauge group, i.e. reparametrizations, frame rotations and Weyl scalings is defined. It is to be emphasized that the spin–connection transforms canonically with respect to frame rotations but not with respect to Weyl scalings [4]. This peculiar behaviour of the connection, which stems from its origins in solving the torsion constraints, leads to some restrictions in the form of conformal invariant action functionals. These are discussed in Sec. 6. An example of an invariant action functional is the Faddeev–Popov ghost action which is derived in Sec. 7. Some consideration is given here to the role of ghost zero modes in determining the structure of the Faddeev–Popov determinant. This discussion is included for the sake of completeness although equivalent descriptions are available in the literature [7].

2. HETEROTIC GEOMETRY

To describe the geometry of a super Riemann surface it is possible to start from a quasi Riemannian geometry on a supermanifold and proceed by imposing constraints [8]. This procedure was developed by Howe [4] and later adapted to the heterotic case by Nelson and Moore [5]. The underlying supermanifold is parametrized locally by the *bosonic (or even) coordinates* x^m, $m = 1, 2$ and the *fermionic (or odd) coordinates*, θ and $\bar{\theta}$. It is natural to think of θ as a complex odd element of a Grassman algebra, and of $\bar{\theta}$ as its complex conjugate. Local superfields will be polynomials in θ and $\bar{\theta}$. For the discussion of heterotic geometry the coordinate $\bar{\theta}$ is irrelevant and we shall henceforth discard it [*]. All fields of interest will be independent of $\bar{\theta}$.

[*] By this step we are precluding the operation of complex conjugation as applied to Euclidean superfields. In a Lorentzian formulation the Grassman odd coordinates θ and $\bar{\theta}$ would be replaced by independent real variables θ_+ and θ_-. Complex conjugation would there remain a meaningful operation even after dropping θ_-.

To convey the geometry it is necessary to introduce a set of frames

$$\begin{aligned} E^A &= dz^M E_M^A \\ &= dx^m E_m^A + d\theta E_\theta^A, \end{aligned} \tag{2.1}$$

where the index A takes the bosonic values $+, -$ and the fermionic value, s. It is necessary, further, to adopt a group of frame rotations with respect to which the geometry is required to be invariant. One might have considered the supergroup, $OSp(1,2)$, for this role. This would correspond to super Riemannian geometry but it is not conformal. The appropriate choice for the frame group is a combination of the $U(1)$ rotations,

$$\begin{aligned} E^\pm &\to E^\pm e^{\mp iH} \\ E^s &\to E^s e^{-iH/2}, \end{aligned} \tag{2.2}$$

and a somewhat more complicated Weyl scaling to be described below. To implement the invariance with respect to frame rotations it is necessary to introduce a connection form

$$\Omega = E^A \Omega_A \tag{2.3}$$

which transforms according to

$$\Omega \to \Omega + dH \tag{2.4}$$

with the help of this connection it is possible to construct the torsion 2-forms, T^A. They are given by

$$\begin{aligned} T^\pm &= dE^\pm \pm i\Omega E^\pm \\ T^s &= dE^s + \frac{i}{2}\Omega E^s \end{aligned} \tag{2.5}$$

which are invariant with respect to coordinate reparametrizations as well as frame rotations.

To arrive at a conformal geometry it is necessary to impose some restrictions on the components of E and Ω. Such restrictions are given covariant form by requiring that certain parts of the torsion should vanish. The basic idea is that the remaining independent components should be pure gauge degrees of freedom, i.e. removable by gauge transformations. The available gauge transformations comprise three reparametrizations δz^M, the frame rotations, H and the yet to be defined Weyl scalings, W, a total of five superfields. The geometric variables E_M^A and Ω_A comprise twelve superfields. It is therefore necessary to impose seven constraints. They should be chosen so that the supermanifold is locally conformal to flat superspace. The following set was found by Nelson and Moore [5],

$$\begin{aligned} &T^+_{ss} = 2, \quad T^+_{-s} = 0 \\ &T^-_{ss} = 0 \quad T^-_{-s} = 0 \\ &T^s_{ss} = 0 \quad T^s_{-s} = 0 \quad T^s_{+s} = 0 \end{aligned} \quad . \tag{2.6}$$

These constraints are manifestly compatible with coordinate reparametrizations and frame rotations but it remains to be proved that they are also compatible with the Weyl scalings. This problem was solved by Howe [4] who defined the action of Weyl transformations on the frames by

$$\begin{aligned} E^\pm &\to E^\pm e^{-W}, \\ E^s &\to (E^s - E^+ \partial_s W) e^{-W/2}, \end{aligned} \tag{2.7}$$

where the operator ∂_s is defined in terms of the inverse components, $E_A^M \partial_M = \partial_A$. These transform according to

$$\begin{aligned} \partial_+ &\to e^W (\partial_+ + (\partial_s W)\partial_s), \\ \partial_- &\to e^W \partial_-, \\ \partial_s &\to e^{W/2}\partial_s. \end{aligned} \tag{2.8}$$

This means that the action of the Weyl group on the frame components is not linear. However, it is straightforward to prove that (2.7) is a valid non-linear realization of the Weyl group. The action of this group on the connection, Ω, is obtained by requiring that the constraints (2.6) be covariant. One finds that, with respect to the combined frame rotation and Weyl groups,

$$\begin{aligned} \Omega_+ &\to e^{W+iH}(\Omega_+ - i\partial_+(W+iH) + 2i\partial_s^2 W - \partial_s H \partial_s W) \\ \Omega_- &\to e^{W-iH}(\Omega_- - i\partial_-(W+iH)) \\ \Omega_s &\to e^{\frac{1}{2}(W+iH)}(\Omega_s + i\partial_s(W-iH)) \end{aligned} \tag{2.9}$$

or, more succinctly,

$$\Omega \to \Omega + d(H+iW) - 2iE^- \partial_- W. \tag{2.9'}$$

The components Ω_A are of course related to the frame components E_M^A by means of the constraints (2.6). Indeed, all components must be expressed in terms of five independent "superpotentials", V^α, when the constraints are solved [9].

There remains the question as to whether these superpotentials can be gauged away. At the local level this must certainly be possible. It would necessitate integrating the infinitesimal transformations,

$$\delta V^\alpha = R^\alpha_\beta \, \delta\xi^\beta, \quad \alpha = 1, \dots 5 \tag{2.10}$$

where $\delta\xi^\alpha$ represents the collection of infinitesimal variations, δz^M, δH and δW. The coefficients R^α_β are differential operators in superspace. The integration of a set of equations such as (2.10), however, involves global considerations, and the interesting part of the question concerns the possible existence of global obstructions. It is well-known from the classical theory of Riemann surfaces that there is an obstruction represented by the existence of holomorphic quadratic differentials [2]. In this case it is known that an arbitrary deformation of the metric tensor cannot be represented by a combination of coordinate reparametrizations and Weyl scalings,

$$\delta g^{mn} = \nabla^m \xi^n + \nabla^n \xi^m + g^{mn}\xi. \tag{2.11}$$

Rather, these latter variations are subject to the compatibility condition,

$$\int d^2x \; u_{mn}\delta g^{mn} = 0, \tag{2.12}$$

where the integration extends over the entire Riemann surface and the coefficient, u_{mn}, is any symmetric tensor density which satisfies the equations

$$\nabla^m u_{mn} = 0 \quad \text{and} \quad g^{mn}u_{mn} = 0. \tag{2.13}$$

On a compact Riemann surface of genus $\gamma > 1$ (say) there are $6(\gamma-1)$ linearly independent (real) solutions to the system (2.13). This is guaranteed by the Riemann-Roch theorem [6]. It follows that, in the neighbourhood of any given metric, g^{mn}, there is a $6(\gamma-1)$-dimensional family of inequivalent metrics, $g^{mn}+\delta g^{mn}$ with δg^{mn} given by [10]

$$\delta g^{mn} = \Sigma \delta m^i v_i^{mn}, \tag{2.14}$$

where the tensors v_i are chosen to be dual to some basis of solutions to (2.13),

$$\int d^2 z u^i_{mn} v_j^{mn} = \delta^i_j, \quad i,j = 1,2,\ldots,6(\gamma-1). \tag{2.15}$$

It is straightforward in principle to extend this logic to the supermanifold. To discover globally inequivalent geometries on the supermanifold it is necessary to solve the variational problem,

$$\begin{aligned} 0 &= \int d^3 z u_\alpha \delta V^\alpha \\ &= \int d^3 z u_\alpha R^\alpha_\beta \delta\xi^\beta. \end{aligned} \tag{2.16}$$

One expects to find a finite number of linearly independent solutions, some even and some odd. At present, however, the super Riemann-Roch theorem does not have the same status as its classical parent. We shall therefore proceed in a somewhat piecemeal fashion.

There is a technical deficiency in that we do not possess a useful representation of the superpotentials. On the other hand, since there is only a single anticommuting

coordinate each superfield has only two components, one fermionic and one bosonic. It is therefore quite feasible to work with these components directly. The torsion constraints are algebraic in the component fields and can be solved explicitly. All fields can then be expressed in terms of five fermionic and five bosonic fields drawn from among the components of the superdreibein, E_M^A. Their response to an infinitesimal gauge transformation can be evaluated and then used to formulate the variational problem (2.12). This will be undertaken in Sec. 4.

Up to this point the discussion has been general in that no particular choice of gauge has been made. Although it will turn out that the Wess-Zumino gauge is the most useful one, we do not want to prejudice the outcome by eliminating any components which might be capable of harbouring moduli or supermoduli, i.e. which cannot be set to zero in a globally meaningful way. In the next section we shall therefore consider the transformation properties of the component fields and the structure of the torsion constraints, without regard to choice of gauge.

3. COMPONENT FIELDS

The components of the superdreibein, E_M^A, can be expressed in terms of ordinary two-dimensional Fermi and Bose fields as follows,

$$\begin{aligned} E_m^a &= e_m^a + \theta\alpha_m^a, \quad E_m^s = \chi_m + \theta v_m \\ E_\theta^a &= \eta^a + \theta w^a, \quad E_\theta^s = u + \theta\xi, \end{aligned} \tag{3.1}$$

where $m = 1,2$ and $a = +,-$. Bose and Fermi component fields are represented by Latin and Greek letters, respectively. The inverse dreibein, E_A^M, can be expressed in the same fashion,

$$\begin{aligned} E_a^m &= \tilde{e}_a^m + \theta\tilde{\alpha}_a^m, \quad E_a^\theta = \tilde{\eta}_a + \theta\tilde{w}_a \\ E_s^m &= \tilde{\chi}^m + \theta\tilde{v}^m, \quad E_s^\theta = \tilde{u} + \theta\tilde{\xi}. \end{aligned} \tag{3.2}$$

The components entering here are to be expressed in terms of those in (3.1),

$$
\begin{aligned}
\tilde{e}^m_a &= e^m_a + k\chi_a\eta^m \\
\tilde{u} &= (u - \eta\cdot\chi)^{-1} \\
\tilde{\eta}_a &= -k\chi_a \\
\tilde{\chi}_a &= -k\eta_a \\
\tilde{\xi} &= -(u-\eta\cdot\chi)^{-2}\xi + k^2(\eta\cdot\alpha\cdot\chi + w\cdot\chi - \eta\cdot v) \\
\tilde{v}^a &= -(u-\eta\cdot\chi)^{-1}w^a - k\eta^b\alpha^a_b - \tilde{\xi}\eta^a \\
\tilde{w}_a &= -(u-\eta\cdot\chi)^{-1}v_a + k\alpha^b_a\chi_b - \tilde{\xi}\chi_a \\
\tilde{\alpha}^b_a &= -\alpha^b_a - k(\alpha^c_a\chi_c - v_a)\eta^b - k\chi_a(\eta^c\alpha^b_c + w^b) - \chi_a\tilde{\xi}\eta^b,
\end{aligned}
\tag{3.3}
$$

where the factor k is defined by

$$
k = \frac{1}{u} + \frac{1}{u^2}\eta\cdot\chi. \tag{3.4}
$$

In these formulae the two–dimensional frame and world indices are related by means of the zweibein, e^a_m, and its inverse , e^m_a, i.e. $\eta^m = \eta^a e^m_a$, $\eta^a = \eta^m e^a_m$, etc.

The frame components of the torsion 2 forms (2.5) are given by

$$
T^C_{AB} = (-)^{BN}E^N_A E^M_B(\partial_M E^C_N - (-)^{MN}\partial_N E^C_M) - \Omega^C_{AB} + (-)^{AB}\Omega^C_{BA}, \tag{3.5}
$$

where the signature factors are defined in the usual way,

$$
(-)^{BN} = \begin{cases} 1, & \text{for B even (odd) and N even (odd)} \\ \text{-1}, & \text{for B even (odd) and N odd (even)} \end{cases}.
$$

The connection components Ω^C_{AB} are given by

$$
\begin{aligned}
\Omega^C_{AB} &= \Omega_A L^C_B \\
&= \Omega_A \text{ diag } (\mathrm{i}, -\mathrm{i}, \frac{\mathrm{i}}{2})
\end{aligned}
\tag{3.6}
$$

in accordance with (2.5). From (3.5) and (3.6) it is easy to see that among the seven torsion constraints (2.6) there are four which do not involve the connection,

$$
T^+_{ss} = 2 \text{ and } T^-_{ss} = T^+_{s-} = T^s_{ss} - T^-_{-s} = 0. \tag{3.7}
$$

These can be used to eliminate four Bose and four Fermi fields from the set (3.1). The remaining three torsion constraints,

$$
T^s_{+s} = T^s_{-s} = T^s_{ss} = 0 \tag{3.8}
$$

then define the connection components Ω_+, Ω_- and Ω_s, respectively. The constraints (3.7) are solved by strictly algebraic manipulations so that there can be no question of global obstructions at this stage. For example, they can be used to eliminate α^a_m, v_m and w^a, leaving e^a_m, u, η^a, χ_m and ξ as independent variables. A lengthy but straighforward treatment of the constraints (3.7) yields the result,

$$
\begin{aligned}
\alpha^+_+ &= \nabla_+\eta^+ + \frac{2}{u}\xi + 2u\chi_+ - 4\eta\cdot\chi\chi_+ + \frac{2}{u}\eta^a\eta^b\nabla_b\chi_a \\
\alpha^+_- &= \nabla_-\eta^+ + 2u\chi_- - 2\eta\cdot\chi\chi_- \\
\alpha^-_+ &= \nabla_+\eta^- \\
\alpha^-_- &= \nabla_-\eta^- - \frac{2}{u}\xi + 2\eta\cdot\chi\chi_+ - \frac{2}{u}\eta^a\eta^b\nabla_b\chi_a \\
w^+ &= -u^2 - \frac{2}{u}\eta^+\xi + 2u\eta^+\chi_+ + (\eta\cdot\chi)^2 \\
w^- &= \frac{2}{u}\eta^-\xi \\
v_+ &= \nabla_+u + \frac{1}{u}\xi\chi_+ + \frac{1}{2u}\eta^+[\nabla_+,\nabla_-]\eta^- \\
&\quad + \frac{1}{u^2}\eta^+\eta^-[\nabla_+,\nabla_-]u + \eta^-\left(1+\frac{\eta\cdot\chi}{u}\right)(\nabla_-\chi_+ - \nabla_+\chi_-) \\
v_- &= \nabla_-u + \frac{1}{u}\xi\chi_- + u\chi_+\chi_- - \eta^+\left(1+\frac{\eta\cdot\chi}{u}\right)(\nabla_-\chi_+ - \nabla_+\chi_-).
\end{aligned}
\tag{3.9}
$$

In these formulae the covariant derivatives are with respect to the ordinary frame rotations. The component Bose and Fermi fields in E^+_M, E^-_M and E^s_M carry the respective weights $-1, +1$ and $-\frac{1}{2}$ with respect to this group. Hence,

$$
\begin{aligned}
\nabla_\pm\eta^+ &= (\partial_\pm + i\omega_\pm)\eta^+, \\
\nabla_\pm\chi_- &= (\partial_\pm + \frac{3}{2}i\omega_\pm)\chi_-,
\end{aligned}
$$

etc., where *) $\partial_\pm = e^m_\pm\partial_m$ and $\omega_\pm$ is the usual two–dimensional spin connection,

$$
\omega_\pm = ie^n_+e^m_-(\partial_m e^\mp_n - \partial_n e^\mp_m). \tag{3.10}
$$

The existence of the solution (3.9) indicates that the choice of the set

$$
e^a_m, u, \eta^a, \chi_m, \xi \tag{3.11}
$$

as independent variables is an acceptable one. If, alternatively, we had chosen the four fermionic variables α^b_a (say) as independent then the dependent η^a would have to satisfy

*) We use the notation $\partial_\pm$ to represent the two–dimensional frame components when this operator is applied to a component Bose or Fermi field. The same symbol indicates dreibein components when applied to a superfield.

differential equations, and this would have global implications. On the other hand, we could choose to eliminate ξ and χ_- in favour of α_+^+ and α_-^+ without any difficulty. There are several alternatives of this kind. It should be noted that we have made free use of the zweibein and its inverse, as well as u and $1/u$, in these formulae. This implies that det(e) and u should be non–vanishing in any acceptable gauge.

The infinitesimal gauge transformations are comprised in the rules,

$$\begin{aligned}\delta E_M^{\pm} &= -\delta z^N \partial_N E_M^{\pm} - \partial_M \delta z^N E_N^{\pm} - (\delta W \pm i\delta H) E_M^{\pm} \\ \delta E_M^{s} &= -\delta z^N \partial_N E_M^{s} - \partial_M \delta z^N E_N^{s} - \frac{1}{2}(\delta W + i\delta H) E_M^{s} - E_M^{+} \partial_s \delta W. \end{aligned} \tag{3.12}$$

We adopt the following expressions for the component field of $-\delta z^M, \delta W$ and δH,

$$\begin{aligned}-\delta x^m &= p^m + \theta \nu^m \\ -\delta\theta &= \varepsilon + \theta q \\ \delta W &= r + \theta\psi \\ \delta H &= t + \theta\tau. \end{aligned} \tag{3.13}$$

It is straightforward to substitute these expressions into (3.12) and extrac: the component field transformation rules. The result is

1) Reparametrizations

$$\begin{aligned}\delta e_m^a &= p^n \partial_n e_m^a + \partial_m p^n e_n^a + \varepsilon \alpha_m^a + \partial_m \varepsilon \eta^a \\ \delta\eta^a &= p^n \partial_n \eta^a + \varepsilon w^a + \nu^n e_n^a + q\eta^a \\ \delta\chi_m &= p^n \partial_n \chi_m + \partial_m p^n \chi_n + \varepsilon v_m + \partial_m \varepsilon u \\ \delta u &= p^n \partial_n u + \varepsilon\xi + \nu^n \chi_n + qu \\ \delta\alpha_m^a &= p^n \partial_n \alpha_m^a + \partial_m p^n \alpha_n^a - \partial_m \varepsilon w^a + \nu^n \partial_n e_m^a + \partial_m \nu^n e_n^a + q\alpha_m^a + \partial_m q \eta^a \\ \delta w^a &= p^n \partial_n w^a + \nu^n(\partial_n \eta^a - \alpha_n^a) + 2qw^a \\ \delta v_m &= p^n \partial_n v_m + \partial_m p^n v_n - \partial_m \varepsilon\xi + \nu^n \partial_n \chi_m + \partial_m \nu^n \chi_n + qv_m + \partial_m qu \\ \delta\xi &= p^n \partial_n \xi + \nu^n(\partial_n u - v_n) + 2q\xi. \end{aligned} \tag{3.14}$$

2) Frame rotations and Weyl scalings

$$\begin{aligned}\delta e_m^{\pm} &= \mp i t e_m^{\pm} - r e_m^{\pm} \\ \delta\eta^{\pm} &= \mp i t \eta^{\pm} - r\eta^{\pm} \\ \delta\chi_m &= -\frac{i}{2} t\chi_m - \frac{1}{2} r\chi_m - e_m^{+}(\tilde{\chi}^n \partial_n r + \tilde{u}\phi) \\ \delta u &= -\frac{i}{2} tu - \frac{1}{2} ru - \eta^{+}(\tilde{\chi}^n \partial_n r + \tilde{u}\phi) \\ \delta w^{\pm} &= \mp i(t w^{\pm} + \tau\eta^{\pm}) - (r w^{\pm} + \phi\eta^{\pm}) \\ \delta v_m &= -\frac{i}{2}(t v_m + \tau\chi_m) - \frac{1}{2}(r v_m + \phi\chi_m) - \alpha_m^{+}(\tilde{\chi}^n \partial_n r + \tilde{u}\phi) \\ &\quad - e_m^{+}(\tilde{v}^n \partial_n r - \tilde{\chi}^n \partial_n \phi + \tilde{\xi}\phi) \\ \delta\xi &= -\frac{i}{2}(t\xi + \tau u) - \frac{1}{2}(r\xi + \phi u) - w^{+}(\tilde{\chi}^n \partial_n r + \tilde{u}\phi) + \\ &\quad + \eta^{+}(\tilde{v}^n \partial_n r - \tilde{\chi}^n \partial_n \phi + \tilde{\xi}\phi), \end{aligned} \tag{3.15}$$

where $\tilde{\chi}^n, \tilde{u}$, etc. are given by (3.3).

To conclude the section we give the finite transformation formulae. Consider an open superdomain U, and let the finite reparametrizations be expressed in the form

$$\begin{aligned} x^m &= f^m(x') + \theta' \nu^m(x') \\ \theta &= \varepsilon(x') + \theta' g(x'). \end{aligned} \tag{3.16}$$

The component fields e_m^a and η^a then transform according to

$$\begin{aligned} e_m'^{\pm} &= e^{-(r\pm it)}[\partial_m' f^n(e_n^{\pm} + \varepsilon\alpha_n^{\pm}) + \partial_m' \varepsilon(\eta^{\pm} + \varepsilon w^{\pm})] \\ \eta'^{\pm} &= e^{-(r\pm it)}[g(\eta^{\pm} + \varepsilon w^{\pm}) + \nu^n(e_n^{\pm} + \varepsilon\alpha_n^{\pm})], \end{aligned} \tag{3.17}$$

where e' and η' are functions of x' as are also the parameters $f^n, g, \nu^n, \varepsilon, r$ and t. The unprimed components e, η, α, etc., are functions of the variable

$$\tilde{x}^m = f^m(x') \tag{3.18}$$

i.e. the θ'–independent part of x^m. Explicit formulae like (3.17) for the components χ_m' and u' would be very complicated and we therefore give only an implicit version,

$$\begin{aligned} \chi_+' &= e^{-\frac{1}{2}(r+it)}[\partial_+' f^n(\chi_n + \varepsilon v_n) + \partial_+' \varepsilon(u + \varepsilon\xi)] + \tilde{u}'(\eta'^k \partial_k' r - \phi) \\ \chi_-' &= e^{-\frac{1}{2}(r+it)}[\partial_-' f^n(\chi_n + \varepsilon v_n) + \partial_-' \varepsilon(u + \varepsilon\xi)] \end{aligned} \tag{3.19}$$

$$u' = e^{-(r+it)} \tilde{u}'(\eta'^k \partial_k' r + \phi)[\nu^n(e_n^{+} + \varepsilon\alpha_n^{+}) + g(\eta^{+} + \varepsilon w^{+})], \tag{3.20}$$

where $\partial_\pm' = e_\pm'^m \partial_m'$ and $\tilde{u} = (u' - \eta' \cdot \chi')^{-1}$. Finally, we have an equation for ξ'

$$
\begin{aligned}
\xi' = & - e^{-(r+it)}[\tilde{u}'(\eta'^k\partial'_k r + \phi)(\phi + i\tau) + \tilde{v}'^k\partial'_k r - \\
& - \tilde{u}'\eta'^k\partial'_k\phi + \tilde{\xi}'\phi][\nu^n(e_n^+ + \varepsilon\alpha_n^+) + g(\eta^+ + \varepsilon w^+)] \\
& + e^{-(r+it)}\tilde{u}'(\eta'^k\partial'_k r + \phi)[\nu^n(\nu^k\tilde{\partial}_k e_n^+ + g\alpha_n^+ - \varepsilon\nu^k\tilde{\partial}_k\alpha_n^+) \\
& - g(\nu^k\tilde{\partial}_k\eta^+ + gw^+ - \varepsilon\nu^k\tilde{\partial}_k w^+)] \\
& e^{-\frac{1}{2}(r+it)}[-\frac{1}{2}(\phi + i\tau)(\nu^k(\chi_k + \varepsilon v_k) + g(u + \varepsilon\xi)) - \\
& \quad - \nu^n(\nu^k\tilde{\partial}_k\chi_n + gv_n - \varepsilon\nu^k\tilde{\partial}_k v_n) \\
& \quad + g(\nu^k\tilde{\partial}_k u + g\xi - \varepsilon\nu^k\tilde{\partial}_k\xi)],
\end{aligned} \tag{3.21}
$$

where

$$
\tilde{v}'^k = -\tilde{u}'w'^k - k'\eta'^b\alpha_b'^k - \tilde{\xi}'\eta'^k.
$$

The formulae (3.17)–(3.21) although not fully explicit, can be used to deduce the form of gauge transformation needed to reduce a general configuration to the Wess–Zumino form. This will be the subject of Sec. 5. These formulae can also be read as transition elements on the overlap $U \cap U'$ of two open superdomains U and U'. In this case the parameter functions r, t, ε, etc. are defined up to gauge equivalences.

4. WARD IDENTITIES

The world sheet, seen as a supermanifold with heterotic geometry imposed, is not globally equivalent to flat superspace. It is now well-known that these geometries fall into equivalence classes characterized by a finite set of moduli, the number of which depends on the topology. To gain some understanding of these global distinctions it is helpful to examine the structure of a functional, $\Gamma(E, \Omega)$, which depends on the geometry but which is invariant with respect to the superconformal transformations, i.e. the coordinate super-reparametrizations, frame rotations and Weyl scalings. Such a functional, which might, for example, be the vacuum amplitude of a matter supermultiplet "propagating" on the world sheet, can depend only on the conformal class of the world sheet geometry. It must take the same value for any two geometries related by a superconformal transformation. In other words, it can depend only on the moduli. The natural place to begin the investigation of such functionals is with their Ward identities.

The Ward identities are obtained by considering the response of $\Gamma(E, \Omega)$ to an infinitesimal change in the supergeometry. The most convenient way to achieve this seems to be by choosing a set of independent Bose and Fermi component fields to represent the supergeometry. When the torsion constraints are solved there remain five each of Bose and

Fermi components from among the fields appearing in (3.1). For the reasons indicated in Sec. 3 we choose the components

$$
e_m^a, \eta^a, \chi_m, u \quad \text{and} \quad \xi \tag{4.1}
$$

as independent variables. The other components, α_m^a, v_m and w^a, as well as all components of Ω are given in terms of the set (4.1) by the formulae (3.9). An infinitesimal variation of Γ therefore takes the form

$$
\delta\Gamma = \int d^2x(\delta e_m^a T_a^m + \delta\eta^a\gamma_a + \delta\chi_m\varsigma^m + \delta u h + \delta\xi\beta), \tag{4.2}
$$

where the coefficients T_a^m, etc. are variational derivatives of Γ. If Γ is invariant then (4.2) must vanish when the variations δe_m^a, etc, are induced by infinitesimal super conformal transformations, i.e. when they take the form (3.14) or (3.15). On substituting from (3.14) and (3.15) into the general variation (4.2) and requiring $\delta\Gamma$ to vanish, one obtains the various Ward identities. Here it is advisable to proceed in steps.

Observe firstly that the infinitesimal parameter, τ, appears only in $\delta\xi$. The invariance of Γ with respect to this particular transformation therefore implies

$$
\begin{aligned}
0 &= u\beta \\
&= u\frac{\delta\Gamma}{\delta\xi}.
\end{aligned} \tag{4.3}
$$

If we require that $u(x)$ shall not vanish in any admissible gauge, it follows that Γ must be independent of $\xi(x)$.

We can therefore discard the contribution , $\delta\xi\beta$, to the remaining identities. Next consider the contributions of r, t and ϕ to $\delta\Gamma$. These give

$$
\begin{aligned}
0 = \int d^2x[& -(r+it)(T_+^+ + \eta^+\gamma_+ + \frac{1}{2}\chi_m\varsigma^m + \frac{1}{2}hu - \frac{1}{2}\partial_m(\tilde{\eta}^m\varsigma^+ - h\tilde{\eta}^m\eta^+)) \\
& -(r-it)(T_-^- + \eta^-\gamma_- - \frac{1}{2}\partial_m(\tilde{\eta}^m\varsigma^+ - h\tilde{\eta}^m\eta^+)) \\
& -\phi(\varsigma^+\tilde{u} - h\eta^+\tilde{u})\ ]
\end{aligned}
$$

The requirement $u \neq 0$ implies also that $\tilde{u}$ is non–vanishing. Hence

$$
\begin{aligned}
0 &= \varsigma^+ - h\eta^+ \\
0 &= T_-^- + \eta^-\gamma_- \\
0 &= T_+^+ + \eta^+\gamma_+ + \frac{1}{2}\chi_m\varsigma^m + \frac{1}{2}hu.
\end{aligned} \tag{4.4}
$$

Two more identities result from a consideration of the contributions of the super–reparametrization components ν^n and g. These are,

$$
\begin{aligned}
0 &= \gamma_n + \chi_n h \\
0 &= \eta^n\gamma_n + hu.
\end{aligned} \tag{4.5}
$$

Taken together, the identities (4.5) imply $h(u + \eta^n \gamma_n) = 0$, which can be satisfied only if $h = 0$.

The identities (4.3), (4.4) and (4.5) lead directly to the conclusion that Γ must be independent of ξ, u and η^a. The most general conformal invariant functional can depend only on e^a_m and χ_m, and this dependence is constrained to satisfy

$$\begin{aligned} T^+_+ &= \frac{1}{2}\varsigma^- \chi_- \\ T^-_- &= 0 \\ \varsigma^+ &= 0. \end{aligned} \tag{4.6}$$

Finally, the contributions of p^n and ε to $\delta\Gamma$ give rise to the respective identities,

$$0 = \partial_m(e^a_n e^m_b T^b_a) - \partial_n e^a_m e^m_b T^b_a + \partial_m(\chi_n \varsigma^m) - \partial_n \chi_m \varsigma^m \tag{4.7}$$

$$0 = \partial_m(u\varsigma^m) - v_m \varsigma^m + \partial_m(\eta^a T^m_a) - \alpha^a_m T^m_a \tag{4.8}$$

The latter identity involves the dependent variables, v_m and α^a_m, as well as the independent variables, η^a and u. But we now know that η^a, u, χ_+ and ξ are absent from the functional Γ. These absent variables must of course disappear from (4.8) when the dependent variables are eliminated by substituting from (3.9). It will therefore be sufficient to evaluate (4.8) on the subspace,

$$u = 1, \quad \xi = \eta^a = \chi_+ = 0, \tag{4.9}$$

where the relevant dependent variables given in general by (3.9), take the simple form

$$\begin{aligned} v_- &= \frac{i}{2}\Omega_- \\ \alpha^+_+ &= \alpha^-_+ = 0 \\ \alpha^+_- &= 2\chi_-. \end{aligned} \tag{4.10}$$

When the values (4.9) and (4.10) are substituted into (4.8) it reduces to

$$0 = \partial_m(\varsigma^- e^m_-) - \frac{i}{2}\omega_- \varsigma^- - 2\chi_- T^-_+. \tag{4.11}$$

The total content of the requirement that Γ be invariant with respect to infinitesimal superconformal transformations is now expressed in the Ward identities (4.6), (4.7) and (4.11). The generality of this conclusion is limited only by our particular choice of independent variables (4.1) and the requirements that $u(x)$ and $\det e(x)$ should have no zeros in any admissible gauge.

The structure of the Ward identities can be simplified by expressing them in a conformal frame. By this we mean choosing complex coordinates z and $\bar{z}$ such that two of the zweibein components vanish,

$$e^-_z = 0 = e^+_{\bar{z}}. \tag{4.12}$$

These coordinate conditions are compatible with holomorphic coordinate reparametrizations as well as frame rotations and Weyl scalings,

$$\begin{aligned} e^+_z &\to \frac{dz}{dz'} e^+_z \exp(-r - it) \\ e^-_{\bar{z}} &\to \frac{d\bar{z}}{d\bar{z}'} e^-_{\bar{z}} \exp(-r + it). \end{aligned}$$

In the frames (4.12) it is usual to redefine the independent variables so as to make them neutral with respect to Weyl scalings and frame rotations, viz.

$$\begin{aligned} T_{zz} &= e^+_z (e^-_{\bar{z}})^{-1} T^-_+ \\ T_{\bar{z}\,\bar{z}} &= (e^+_z)^{-1} e^-_{\bar{z}} T^+_- \\ \chi &= (e^+_z)^{-1/2} e^-_{\bar{z}} \chi_- \\ \varsigma &= (e^+_z)^{1/2} (e^-_{\bar{z}})^{-1} \varsigma^-. \end{aligned} \tag{4.13}$$

With respect to the holomorphic reparametrizations the new quantities T_{zz}, $T_{\bar{z}\,\bar{z}}$, χ and ς transform as (2,0), (0,2), $(-\frac{1}{2},1)$ and $(\frac{3}{2},0)$ forms, respectively. In terms of the new variables the general infinitesimal variation (4.2) takes the form

$$\delta\Gamma = \int d^2z \big(-\delta e^z_- e^-_{\bar{z}} T_{zz} - \delta e^{\bar{z}}_+ e^+_z T_{\bar{z}\,\bar{z}} + \delta\chi\varsigma\big). \tag{4.14}$$

The only surviving variational derivatives, $T_{zz}, T_{\bar{z}\,\bar{z}}$ and ς are subject to the Ward identities,

$$0 = \partial_{\bar{z}} T_{zz} - \partial_z \chi\varsigma - \frac{1}{2}\partial_z(\chi\varsigma) \tag{4.15a}$$

$$0 = \partial_z T_{\bar{z}\,\bar{z}} + \chi \partial_{\bar{z}} \varsigma \tag{4.15b}$$

$$0 = \partial_{\bar{z}} \varsigma - 2\chi T_{zz} \tag{4.15c}$$

to which the identities (4.7) and (4.11) reduce in the conformal frames. These identities reflect the invariance of $\Gamma(e, \chi)$ with respect to the infinitesimal transformations

$$\begin{aligned} \delta e^z_- &= -e^{\bar{z}}_- (\partial_{\bar{z}} p^z + 2\varepsilon\chi) \\ \delta e^{\bar{z}}_+ &= -e^z_+ \partial_z p^{\bar{z}} \\ \delta\chi &= p^z \partial_z \chi - \frac{1}{2}\partial_z p^z \chi + \partial_{\bar{z}}(p^{\bar{z}}\chi) + \partial_{\bar{z}}\varepsilon \end{aligned} \tag{4.16}$$

and it is possible to make a further simplification by exploiting this invariance. In analogy with the selection of a conformal frame wherein $e^-_z = e^+_{\bar{z}} = 0$, it is possible to choose $\chi = 0$ as well. This can be effected in every open set of the covering of the Riemann surface *). In consequence, the Ward identities (4.15) reduce to the statement that T_{zz} is

*) The global validity of this remark is justified by the discussion in Sec. 5.

a holomorphic (2,0) form, $T_{\bar{z}\bar{z}}$ is an antiholomorphic (0,2) form and ς is an holomorphic $(\frac{3}{2},0)$ form. As such, these variational derivatives of the invariant functional Γ can be regarded as belonging to finite dimensional vector spaces

$$\begin{aligned} T_{zz} &= \Sigma T_i u^i(z) \\ T_{\bar{z}\bar{z}} &= \Sigma T_{\bar{i}} \overline{u^i(z)} \\ \varsigma &= \Sigma \varsigma_\alpha v^\alpha(z). \end{aligned} \tag{4.17}$$

For a Riemann surface of genus $\gamma > 1$, the space of holomorphic (2,0) forms has complex dimension $3(\gamma-1)$ and is spanned by the basis $u^i(z)$. The space of antiholomorphic (0,2) forms is spanned by their complex conjugates, $\overline{u^i(z)}$, and the space of holomorphic $(\frac{3}{2},0)$ forms is spanned by the basis $v^\alpha(z)$, $\alpha = 1, \ldots 2(\gamma-1)$. The coefficients $T_i, T_{\bar{i}}$ and ς_α are related to derivates of Γ with respect to the moduli. To make the relation precise it is necessary to define Beltrami differentials, which we now consider briefly.

The expressions (4.17) for the variational derivatives indicate that $\Gamma(e, \chi)$ depends on a finite set of parameters, the moduli,

$$m^A = (m^i, m^{\bar{i}}, \mu^\alpha),$$

where $i, \bar{i} = 1, \ldots, 3(\gamma-1)$ and $\alpha = 1, \ldots, 2(\gamma-1)$. These parameters must distinguish the inequivalent superconformal geometries. If we suppose the classes are represented by the zweibein, $e_n^a(m)$ and gravitino $\chi(m)$, specified functions of the moduli, then we can define the Beltrami differentials,

$$\begin{aligned} \mathrm{v}_A(z) &= -e_z^+ \partial_A e_+^{\bar{z}}, \\ \overline{\mathrm{v}}_A(z) &= -e_{\bar{z}}^- \partial_A e_-^z \\ \nu_A(z) &= \partial_A \chi. \end{aligned} \tag{4.18}$$

where $\partial_A = \partial/\partial m^A$, The variation $\delta\Gamma$ due to an infinitesimal change in the moduli is then given by

$$\begin{aligned} \delta\Gamma &= \delta m^A \int d^2z (\mathrm{v}_A T_{zz} + \overline{\mathrm{v}}_A T_{\bar{z}\bar{z}} + \nu_A \varsigma) \\ &= \delta m^A [(\mathrm{v}_A, u^i) T_i + (\overline{\mathrm{v}}_A, \overline{u}^i) T_{\bar{i}} + (\nu_A, v^\alpha) \varsigma_\alpha], \end{aligned} \tag{4.19}$$

where

$$(\mathrm{v}_A, u^i) = \int d^2z \mathrm{v}_A(z) u^i(z), \tag{4.20}$$

etc. The linear form (4.19) thus serves to define the modular derivatives of Γ in terms of the components $T_i, T_{\bar{i}}$ and ς_α. A more refined discussion would yield more detailed properties of the differentials (4.18). For instance, one should be alble to prove [11] that $e_+^{\bar{z}}$ and χ depend holomorphically, on m^i and μ^α while e_-^z depends only on $\overline{m}^i$, i.e.

$$\mathrm{v}_{\bar{j}} = \overline{\mathrm{v}}_j = \nu_{\bar{j}} = \overline{\mathrm{v}}_\alpha = 0 \tag{4.21}$$

The analysis of Ward identities given here has shown that the invariant functional $\Gamma(E, \Omega)$ can depend only on the moduli. The same result could have been obtained by finding a finite gauge transformation, globally defined, which reduces an arbitrary configuration, e_n^a, χ_n, η^a, u and ξ to the specific form,

$$e_n^a = e_n^a(m), \quad \chi_n = \chi_n(m), \quad \eta^a = 0, \quad u = 1, \quad \xi = 0. \tag{4.22}$$

The transformation which effects part of this reduction, viz.

$$\chi_+ = 0, \quad \eta^a = 0, \quad u = 1, \quad \xi = 0 \tag{4.23}$$

is given in Sec. 5. This transformation can be used quite generally to "gauge away" the variables (4.23) in functionals which involve supermultiplets of matter fields as well as the geometrical fields. For example, the action functional which governs the propagation of a neutral scalar superfield,

$$X = y(x) + \theta\psi(x)$$

can be shown to be a functional of e_m^a, χ_-, y and ψ', where ψ' is given by

$$\psi' = \frac{\psi - \eta \cdot \partial y}{u - \eta \cdot \chi}. \tag{4.24}$$

In effect, the auxiliary geometrical fields χ_+, η and u are elinimated by a redefinition of ψ.

5. WESS–ZUMINO GAUGES

In Sec. 4 it was argued on the basis of Ward identities that an invariant functional of the supergeometry, $\Gamma(E, \Omega)$, must be independent of the components, χ_+, η^a, u and ξ. It can depend only on the zweibein, e_m^a, and gravitino, χ_-. This observation can be generalized to include invariant functionals which contain matter fields as well. That is, by means of a gauge transformation it must be possible to explicitly eliminate these redundant components. The appropriate transformation will be given in the following.

The Wess–Zumino class of gauges is defined by the conditions

$$\chi_+ = \eta^a = \xi = 0 \quad \text{and} \quad u = 1. \tag{5.1}$$

If these conditions are imposed in every neighbourhood of a coordinate covering of the supermanifold then a number of restrictions on the transition elements are indicated. These transition elements, which are maps from the overlaps of coordinate neighbourhoods into the gauge group, define the global structure of the geometry. Our first problem is to show that the restrictions implied by (5.1) can be satisfied, i.e. that it is indeed possible to impose the gauge conditions (5.1) in overlapping neighbourhoods. We shall verify that the transition elements which are reduced versions of the transformations (3.17)–(3.21) are well defined. They take their values in a subgroup of the full structure group. This subgroup, given by (5.14), can be identified as the stability group of the WZ section defined by (5.1). The choice of a WZ gauge implies a partial fixing of the original five superfield degrees of freedom. Still allowed are the two–dimensional reparametrization and tangent space transformations and a local supersymmetry. These transformations constitute what we call the stability group of the WZ configurations.

There is a technical peculiarity concerning the non–canonical response of the connection form to super Weyl transformations that is worth remarking. The connection is determined by the torsion conditions (2.6) and its transformation rule is given by (2.9) or

$$\Omega \to \Omega + d(H + iW) - 2iE^{-}\partial_{-}W. \tag{5.2}$$

where H and W are superfield parameters of the frame rotations and Weyl scalings, respectively. It is clear that Ω does not transform like a connection unless $\partial_{-}W = 0$. Because of this, it is possible to construct invariant action functionals only for a restricted class of matter superfields, those which are neutral with respect to one or other (or both) of the tangent space subgroups generated by $H + iW$ and $H - iW$. This effect will be discussed in Sec. 6. Here we wish only to remark on the global structure of the Weyl group. To this end, let the collection $\{E_{\alpha}^{A}\}$ represent the superdreibein in a covering of the manifold by neighbourhoods, $\{U_{\alpha}\}$. In the overlap $U_{\alpha} \cap U_{\beta}$ we have the transition rules

$$E_{\alpha}^{+} = e^{-\Lambda_{+\alpha\beta}} E_{\beta}^{+}, \tag{5.3}$$

etc., where $\Lambda_{+\alpha\beta}$ is a superfield defined over $U_{\alpha} \cap U_{\beta}$. In each U_{α} the frames can be modified by a tangent space operation.

$$E_{\alpha}^{+} \to E_{\alpha}^{'+} = e^{-\Lambda_{+\alpha}} E_{\alpha}^{+}, \tag{5.4}$$

etc. where $\Lambda_{+\alpha}$ is defined in U_{α}. The transition elements are modified accordingly,

$$\Lambda_{+\alpha\beta} \to \Lambda'_{+\alpha\beta} = \Lambda_{+\alpha\beta} - \Lambda_{+\alpha} + \Lambda_{+\beta}, \tag{5.5}$$

and similarly for $\Lambda_{-\alpha\beta}$. The Grassman even part of the transition element, $\exp(-\Lambda_{+\alpha\beta})$, takes its values in $\mathbb{C}^*$, the set of non–zero complex numbers, which is not simply connected. The implied *topological non–triviality of these elements*, however, can be confined to the imaginary part $2iH_{\alpha\beta} = \Lambda_{+\alpha\beta} - \Lambda_{-\alpha\beta}$. In other words, it should always be possible to choose the frames such that the transition elements are mere phase factors,

$$e^{-\Lambda_{\pm\alpha\beta}} = e^{\mp i t_{\alpha\beta}}. \tag{5.6}$$

This would amount to nothing more than a particular choice of gauge. The WZ gauges, however, are not of this type. Their associated transition elements are rather more complicated, as we shall see. Something of their nature can be seen by considering the analogous

but simpler choice of gauge in the case of ordinary Riemann surfaces. Here the frames are represented by collections of 1–forms $\{e_{\alpha}^{\pm}\}$ relative to the coordinate covering, $\{U_{\alpha}\}$. In the overlap, $U_{\alpha} \cap U_{\beta}$ we have

$$e_{\alpha}^{+} = \phi_{\alpha\beta}\, e_{\beta}^{+} \quad \text{and} \quad e_{\alpha}^{-} = \overline{\phi}_{\alpha\beta} e_{\beta}^{-} \tag{5.7}$$

where the transition element $\phi_{\alpha\beta}$ takes its values in $\mathbb{C}^*$. By means of conformal transformations in each U_{α} it is possible to bring the frame representatives into the simple form *)

$$e_{\alpha}^{+} = dz_{\alpha} \quad \text{and} \quad e_{\alpha}^{-} = d\overline{z}_{\alpha} \tag{5.8}$$

in which case the transition elements must be given by

$$\phi_{\alpha\beta} = \frac{dz_{\alpha}}{dz_{\beta}} \tag{5.9}$$

As a result, the tangent space transition elements are determined by the coordinate transition functions and they are holomorphic.

Consider now the heterotic supergeometry. The problem is to find a gauge transformation which, in a given neighbourhood, reduces a general configuration to the WZ form. The resulting frames and connection should be

$$\begin{aligned} E^{+} &= dx^{m}(e_{m}^{+} + 2\theta e_{m}^{-}\chi_{-}) - d\theta\, \theta \\ E^{-} &= dx^{m} e_{m}^{-} \\ E^{*} &= dx^{m}(e_{m}^{-}\chi_{-} + \frac{i}{2}\omega_{m}\theta) + d\theta \\ \Omega &= E^{+}\omega_{+} + E^{-}(\omega_{-} - 2i\theta(\partial_{+}\chi + i\omega_{+}\chi)) + E^{*}\theta\omega_{+} \end{aligned} \tag{5.10}$$

corresponding, in our choice of independent variables to (5.1). The available gauge transformations are, as described in Sec. 2,

$$\begin{aligned} E'^{\pm}(z') &= e^{-\Lambda_{\pm}(z')} E^{\pm}(z) \\ E'^{*}(z') &= e^{-\frac{1}{2}\Lambda_{+}(z')} E^{*}(z) - e^{-\Lambda_{+}(z')} E^{+}(z)\partial'_{*}W(z'), \end{aligned} \tag{5.11}$$

*) In this gauge the torsionless connection (3.10) must vanish along with its associated curvature form. However, it does not follow that the Euler characteristic must vanish as well. This is because the torsionless connection fails to transform canonically with respect to the Weyl group and therefore cannot be used to compute the Euler characteristic in gauges such as (5.8), where the transition elements (5.9) involve Weyl transformations. Instead one should use a connection which responds canonically to the structure group which is $\mathbb{C}^*$.

where, in compact notation,

$$x^m = f^m(x') + \theta'\nu^m(x')$$
$$\theta = \varepsilon(x') + \theta' g(x')$$
$$\Lambda_\pm = r \pm it + \theta'(\phi \pm i\tau). \tag{5.12}$$

The components transform according to the rules given implicitly in the formulae (3.16)-(3.21). Using these, a lengthy but straightforward analysis gives the result,

$$\nu^n = -(u - \eta\cdot\chi)^{-1}\eta^n$$
$$g = (u - \eta\cdot\chi)^{-1}$$
$$\phi = \chi_+$$
$$i\tau = \chi_+ + 2(u - \eta\cdot\chi)^{-2}(\xi + \eta\cdot v - \eta\cdot\partial u - \eta^n\eta^m\partial_m\chi_n) \tag{5.13}$$

together with $r = t = \varepsilon = 0$ and $f^m(x') = x'^m$. This transformation will serve to reduce a general configuration to the form (5.1). It is not unique, of course. The most general such transformation is obtained by combining (5.13) with an element of the stability group of the WZ configuration. The elements of the latter group are given by

$$\nu^n(x') = e^{\frac{1}{2}(r+it)}\varepsilon(x')e^n_+(\tilde{x})$$
$$g(x') = e^{\frac{1}{2}(r+it)}$$
$$\phi(x') = e^{\frac{1}{2}(r+it)}(\tilde{\partial}_+ + \frac{i}{2}\omega_+(\tilde{x}))\varepsilon(x')$$
$$i\tau(x') = e^{\frac{1}{2}(r+it)}(\tilde{\partial}_+ - \frac{i}{2}\omega_+(\tilde{x}))\varepsilon(x'), \tag{5.14}$$

where $\tilde{x}^m = f^m(x')$, $r = r(x'), t = t(x')$ and $\varepsilon(x')$ are unrestricted. In effect, the elements of the stability group are defined by the θ–independent parts of the various superfield parameters.

If the WZ gauge conditions (5.1) are imposed in all neighbourhoods of the covering then, the structure group must coincide with the stability group. All transition elements must satisfy (5.14), where (x^m, θ) and (x'^m, θ') are now to be interpreted as alternative coordinates in the overlap of two neighbourhoods, U and U'. With such transition elements the collection of WZ gauge representatives $\{e^\pm_m, \chi_-; e'^\pm_m, \chi'_-; \ldots\}$ will patch together consistently to provide a global description of the supergeometry. In the overlap $U \cap U'$ patchings are given by

$$e'^+_m = e^{-(r+it)}[\partial'_m f^n(e^+_n + 2\varepsilon e^-_n\chi_-) + \varepsilon\partial'_m\varepsilon]$$
$$e'^-_m = e^{-(r-it)}\partial'_m f^n e^-_n$$
$$\chi'_- = e^{\frac{1}{2}(r-3it)}[\chi_- + (\tilde{\partial}_- + \frac{i}{2}\omega_-)\varepsilon + 2\varepsilon\tilde{\partial}_+\varepsilon\chi_- + \varepsilon\tilde{\partial}_+\varepsilon\tilde{\partial}_-\varepsilon], \tag{5.15}$$

where the primed fields together with the group parameters are evaluated at x' and the unprimed fields are evaluated at $\tilde{x} = f(x')$.

One can proceed further with the gauge fixing to arrive at what might be called the "superisothermal" gauge,

$$e^+ = dz e^+_z$$
$$e^- = d\bar{z} e^-_{\bar{z}}$$
$$\chi_- = 0. \tag{5.16}$$

In a given neighbourhood, this configuration is obtained by means of a gauge transformation in the stability group (5.15). It must satisfy the conditions

$$0 = \partial'_{\bar{z}} f^n(e^+_n + 2\varepsilon e^-_n\chi_-) + \varepsilon\partial'_{\bar{z}}\varepsilon$$
$$0 = \partial'_z f^n e^-_n$$
$$0 = \chi_- + (\tilde{\partial}_- + \frac{i}{2}\omega_-)\varepsilon + 2\varepsilon\tilde{\partial}_+\varepsilon\chi_- + \varepsilon\tilde{\partial}_+\varepsilon\tilde{\partial}_-\varepsilon. \tag{5.17}$$

These equations are to be solved for $f^n(z', \bar{z}')$ and $\varepsilon(z', \bar{z}')$. The stability group of the superisothermal gauge is easily discovered by imposing the conditions (5.16) on both sides of (5.15). One finds

$$\partial'_{\bar{z}} f^z = \partial'_z f^{\bar{z}} = 0 \quad \text{and} \quad \partial'_{\bar{z}}((e^+_z)^{-1/2}\varepsilon) = 0. \tag{5.18}$$

The tangent space elements r and t are as yet unrestricted. On going one step further and imposing the gauge conditions

$$e^+_z = e^-_{\bar{z}} = 1 \tag{5.19}$$

the stability group will be determined entirely in terms of $f^z, f^{\bar{z}}$ and ε. The tangent space elements are now given by

$$e^{r+it} = \partial'_z f^z + \varepsilon\partial_{z'}\varepsilon$$
$$e^{r-it} = \partial'_{\bar{z}} f^{\bar{z}}. \tag{5.20}$$

In this "superconformal" gauge the frames are represented in each neighbourhood, U_α, by the simple forms

$$E^+_\alpha = dz_\alpha - d\theta_\alpha\theta_\alpha$$
$$E^-_\alpha = d\bar{z}_\alpha$$
$$E^*_\alpha = d\theta_\alpha \tag{5.21}$$

and the connection vanishes everywhere. The transition elements appropriate to this gauge are obtained by substituting from (5.16), (5.19) and (5.20) into (5.14),

$$\nu^z = (\partial_{z'} f^z + \varepsilon\partial_{z'}\varepsilon)^{1/2}\varepsilon = (\partial_{z'} f^z)^{1/2}\varepsilon$$
$$\nu^{\bar{z}} = 0$$
$$g = (\partial_{z'} f^z + \varepsilon\partial_{z'}\varepsilon)^{1/2}$$
$$e^{\Lambda_+} = (\partial_{z'} f^z + \varepsilon\partial_{z'}\varepsilon)(1 + 2\theta\partial_z\varepsilon) = \partial_{z'} f^z + \varepsilon\partial_{z'}\varepsilon + 2\theta\partial_{z'}\varepsilon$$
$$e^{\Lambda_-} = \partial'_{\bar{z}} f^{\bar{z}}. \tag{5.22}$$

where $\partial_{\bar{z}} f^z = \partial_z f^{\bar{z}} = \partial_{\bar{z}} \epsilon = 0$. Notice, in particular, that the reparametrization part of the transition element takes the form

$$\begin{aligned} z &= f^z(z') + \theta'(\partial_{z'} f^z)^{1/2} \epsilon(z') \\ \theta &= \epsilon(z') + \theta'(\partial_{z'} f^z + \epsilon \partial_{z'} \epsilon)^{1/2} \end{aligned} \tag{5.23}$$

in the overlap $U \cap U'$. This agrees with the standard definitions of the super Riemann surface [1].

6. LAGRANGIANS

To describe the behaviour of matter fields on a supermanifold with heterotic geometry one can use the superdreibein and connection defined in Sec. 2 to set up invariant action functionals. Since a density factor is available in the superdeterminant of E_M^A, we can assume without loss of generality that all matter fields transform as scalars with respect to the super–reparametrizations. This leaves only their classification with respect to the tangent space group to be considered. The representations of this group are characterized by a pair of weights (λ_+, λ_-) which may be integers or half–integers. In general then, we have

$$\Phi \to e^{\lambda_+ \Lambda_+ + \lambda_- \Lambda_-} \Phi \tag{6.1}$$

for a superfield of type (λ_+, λ_-). However, most of these cannot be used in the construction of conformal field theories. This is because the tangent space group is not fully gauged. Heterotic geometry as described in Sec. 2 involves only a single connection form Ω, defined by the torsion constraints (2.6). Its transformations under the action of the tangent space group are given by (2.9) and these are clearly not canonical. With respect to pure frame rotations, $\Lambda_\pm = \pm iH$, it does indeed behave like a $U(1)$ connection. With respect to the Weyl scalings, however, it is not a connection. The frame components Ω_- and Ω_s do transform in a fashion very like that of connection components, viz.

$$\begin{aligned} \Omega_- &\to e^{\Lambda_-}(\Omega_- - i\partial_- \Lambda_+) \\ \Omega_s &\to e^{\frac{1}{2}\Lambda_+}(\Omega_s + i\partial_s \Lambda_-) \end{aligned} \tag{6.2}$$

but Ω_+ does not. Because of the transformations (6.2) it follows that we can define a covariant derivative ∇_- acting on fields of type $(\lambda_+, 0)$, and ∇_s acting on fields of type $(0, \lambda_-)$. These are

$$\nabla_- \Phi_+ = (\partial_- - i\lambda_+ \Omega_-)\Phi_+$$

and

$$\nabla_s \Phi_- = (\partial_s + i\lambda_- \Omega_s)\Phi_- \tag{6.3}$$

where Φ_+ and Φ_- are fields of type $(\lambda_+, 0)$ and $(0, \lambda_-)$, respectively. It is also clear from (6.2) that these covariant derivatives transform as fields of type $(\lambda_+, 1)$ and $(\frac{1}{2}, \lambda_-)$ respectively.

In the case of a field X of type $(0,0)$, the two components, $\partial_- X$ and $\partial_s X$, are covariant. They transform as fields of type (0,1) and $(\frac{1}{2}, 0)$, respectively.

There are no other possibilities. The most general heterotic conformal field theory must be constructed out of fields which are either neutral, type (0,0), or chiral, types $(\lambda, 0)$ or $(0, \lambda)$. In addition to these fields, which comprise the dynamical variables of the theory, there is of course the superdreibein E_M^A which characterizes the geometrical background in which the other fields propagate. It is involved implicitly in the covariant derivatives of these fields and also in the density factor, sdet$E$, which must appear as a factor in every term in the Lagrangian. This factor, which transforms as a reparametrization density, also carries weights with respect to the tangent space group,

$$\text{sdet}E \to e^{-\frac{1}{2}\Lambda_+ - \Lambda_-} \text{sdet}E. \tag{6.4}$$

This is obtained from the transformation rules for the frames,

$$E^+ \to e^{-\Lambda_+} E^+, \; E^- \to e^{-\Lambda_- E^-}, \;\; E^s \to e^{-\frac{1}{2}\Lambda_+}(E^s - E^+ \partial_s W). \tag{6.5}$$

It follows that the action functional for a heterotic conformal field theory takes the form

$$S = \int d^3 z \text{sdet} E \mathcal{L}(X, \partial_- X, \partial_s X, \Phi_+, \nabla_- \Phi_+, \Phi_-; \nabla_s \Phi_-), \tag{6.6}$$

where $\mathcal{L}$ is a scalar of type $(\frac{1}{2}, 1)$. At the classical level the Lagrangian is otherwise unconstrained. When quantum effects are taken into account there will arise, of course, the usual considerations about anomaly cancellation and modular invariance.

The archetypal theory is the original heterotic superstring model [12] for which the Lagrangian is given by

$$\mathcal{L} = \partial_- X^\mu \partial_s X^\mu + \Psi^\alpha \nabla_s \Psi^\alpha + B_+ \nabla_- C_+ + B_- \nabla_s C_-. \tag{6.7}$$

where $\mu = 1, \ldots, 10$ and $\alpha = 1, \ldots 32$. The fields in (6.7) are assigned the following weights,

supercoordinates	$X^\mu \sim (0,0)$
"internal" coordinates	$\Psi^\alpha \sim (0, \frac{1}{2})$
Faddeev – Popov ghosts	$B_+ \sim (\frac{3}{2}, 0)$
	$C_+ \sim (-1, 0)$
	$B_- \sim (0, 2)$
	$C_- \sim (0, -1)$.

The Lagrangians (6.6) and (6.7) can be reduced to the component notation discussed in Sec. 3. Generally it is necessary to have expressions for the components of the connection as well as the dreibein in terms of a set of independent Bose and Fermi components, e.g. e_m^a, u, χ_m, η^a and ξ. However, these are all gauge fields and they can be largely eliminated by means of gauge transformations. It was shown in Sec. 4 that any invariant functional must in fact be independent of u, χ_+, η^a and ξ. In the case of the invariant action functional (6.6) which involves a set of matter fields as well as the dreibein components, this argument can be generalized to show that the latter fields can be absorbed in redefinitions of the former. It is therefore possible to express a functional such as (6.6) in terms of e_m^a, χ_- and the redefined (i.e. gauge transformed) matter field components (such as (4.24)). The results are obtained most directly by employing the Wess–Zumino gauge description of the dreibein and connection *),

$$E_M^A = \begin{pmatrix} e_m^+ + 2\theta e_m^- \chi_- & e_m^- & e_m^- \chi_- + \frac{i}{2}\theta\omega_m \\ -\theta & 0 & 1 \end{pmatrix} \tag{6.8}$$

$$E_M^A = \begin{pmatrix} e_+^m & -\frac{i}{2}\theta\omega_+ \\ e_-^m - \theta e_+^m \chi_- & -\chi_- - \frac{i}{2}\theta\omega_- \\ \theta e_+^m & 1 \end{pmatrix} \tag{6.9}$$

$$\Omega_A = \begin{pmatrix} \omega_+ \\ \omega_- - 2i\theta(\partial_+ + i\omega_+)\chi_- \\ \theta\omega_+ \end{pmatrix} \tag{6.10}$$

The superdeterminant of (6.8) is given by

$$\mathrm{sdet} E = \det e_m^a = e \tag{6.11}$$

In the expressions (6.8)–(6.10) the notation is two-dimensional: frame components are defined with respect to the zweibein, e_m^a and its inverse e_a^m; the spin connection is defined by the vanishing of two-dimensional torsion, i.e.

$$\omega_\pm = i e_+^n e_-^m (\partial_m e^\mp - \partial_n e_m^\pm).$$

The notation $\partial_\pm$ is intended to indicate zweibein components of ∂_m when applied to two-dimensional Bose or Fermi fields. There should be no confusion with the operators $\partial_\pm = E_\pm^M \partial_M$ which are applied to superfields.

The generic fields and their covariant derivatives in WZ gauge are given by the following list, with the corresponding weights at the left,

*) Our notation differs from that of Nelson and Moore [5] but we should point out that their equation $\Omega_s = 0$ seems to be in error.

$$\begin{aligned}
&(0,0) & \mathrm{X} &= y + \theta\psi \\
&(0,1) & \partial_- X &= \partial_- y - \chi_-\psi + \theta(\nabla_-\psi - \chi_-\partial_+ y) \\
&(\tfrac{1}{2},0) & \partial_s X &= \psi + \theta\partial_+ y \\
&(\lambda,0) & C_+ &= c + \theta\gamma \\
&(\lambda,1) & \nabla_- C_+ &= \nabla_- c - \chi_-\gamma + \theta(\nabla_-\gamma - \chi_-\nabla_+ c - 2\lambda\nabla_+\chi_- c) \\
&(0,\lambda) & C_- &= \bar{c} + \theta\bar{F} \\
&(\tfrac{1}{2},\lambda) & \nabla_s C_- &= \bar{F} + \theta\nabla_+\bar{c}
\end{aligned} \tag{6.12}$$

The covariant derivatives which appear among the components on the right–hand sides of these equations are defined by

$$\begin{aligned}
\nabla_-\psi &= (\partial_- - \tfrac{i}{2}\omega_-)\psi \\
\nabla_\pm c &= (\partial_\pm - i\lambda\omega_\pm)c \\
\nabla_-\gamma &= [\partial_- - i(\lambda + \tfrac{1}{2})\omega_-]\gamma \\
\nabla_+\chi_- &= (\partial_+ + i\tfrac{3}{2}\omega_+)\chi_- \\
\nabla_-\bar{c} &= (\partial_+ + i\lambda\omega_+)\bar{c}
\end{aligned} \tag{6.13}$$

With the exception of $\nabla_+ c$ and $\nabla_+\chi_-$ they are covariant with respect to the two-dimensional reparametrizations, frame rotations and Weyl scalings. Although $\nabla_+ c$ and $\nabla_+\chi_-$ are not separately covariant with respect to Weyl scalings, the combination $\chi_-\nabla_+ c + 2\lambda\nabla_+\chi_- c$ which appears in $\nabla_- C_+$, is covariant. The action of the two-dimensional gauge transformations on the various WZ gauge component fields is derived from the θ–independent parts of the transformations discussed in Sec.3 together with a supplementary rule, $\theta \to \theta e^{\frac{1}{2}(r+it)}$, needed to maintain the WZ gauge. The WZ gauge also admits a local supersymmetry associated with the infinitesimal parameter $\varepsilon(x)$ as in Sec.5. The derivatives (6.13) are of course not covariant with respect to these transormations although the action functional will be invariant. In the case of the heterotic string (6.7), the action reduces in component notation to the form

$$S = \int d^2xe\big[\partial_- y^\mu \partial_+ y^\mu + \nabla_- \psi^\mu \psi^\mu - 2\chi_- \psi^\mu \partial_+ y^\mu$$
$$+ \lambda^a \nabla_+ \lambda^a + F^a F^a$$
$$+ b\nabla_- c + \beta\nabla_- \gamma + \chi_- b\gamma - \beta(\chi_- \nabla_+ c - 2\nabla_+ \chi_- c)$$
$$+ \bar{b}\nabla_+ \bar{c} + \overline{G}\,\overline{F}\big], \tag{6.14}$$

where the components of X^μ, C_+ and C_- are as indicated in (6.12) and, in addition, we write

$$\Psi^a = \lambda^a + \theta F^a$$
$$B_+ = \beta + \theta b$$
$$B_- = \bar{b} + \theta\overline{G}. \tag{6.15}$$

The conformal gauge version of the action (6.14) is obtained by setting $e_z^+ = e_{\bar{z}}^- = 1$, $e_z^- = e_{\bar{z}}^+ = 0$, $\chi_- = 0$, and $\omega_\pm = 0$, so that $\partial_+ = \partial_z$ and $\partial_- = \partial_{\bar{z}}$. The ghost field terms in this action can be obtained by the Faddeev–Popov construction which will be examined in Sec. 7.

7. FADDEEV–POPOV FACTOR

Although the construction of the Fadeev–Popov determinant is well known, we give here, for the sake of completeness, a brief treatment of its main features as they apply in the case of heterotic geometry. The problem is to define a sensible prescription for integrating over the space of equivalence classes of geometries on the world sheet. The integrand generally includes an invariant functional, $e^{-\Gamma(e,\chi)}$, which stands for some amplitude of the underlying superconformal field theroy. It must also include a factor, $\Delta(e,\chi)$, the Faddeev–Popov determinant, which can be thought of as the "volume" of a class of equivalent geometries. The argument goes as follows.

Within each class it should be possible to single out a representative element $(e(m), \chi(m))$, where $m = (m^A)$ stands for a set of moduli. A general configuration in the class can be represented in the form

$$(e,\chi) = (e(m)^\Omega, \chi(m)^\Omega),$$

where the superstript, Ω, denotes the action of an element of the supergroup. The general transformation involves the entire collection of independent gauge fields, e, χ, η, u and ξ, but we have seen that Γ depends only on e and χ, so we can ignore the others. This means that, without loss of generality, we can restrict Ω to the subgroup of gauge transformations

that act only on e and χ (Eq. (5.15)). As we saw in Sec. 5, it is the transition elements belonging to this subgroup which completely determine the global structure of the world sheet. In the neighbourhood of the identity the transformations are parametrized by p^n, r, t and ε,

$$\delta e_m^+ = p^n\partial_n e_m^+ + \partial_m p^n e_n^+ - (r+it)e_m^+ + 2\varepsilon\chi_- e_m^-$$
$$\delta e_m^- = p^n\partial_n e_m^- + \partial_m p^n e_n^- - (r-it)e_m^-$$
$$\delta\chi_- = p^n\partial_n\chi_- + \frac{1}{2}(r-3it)\chi_- + (\partial_- + \frac{1}{2}\omega_-)\varepsilon. \tag{7.1}$$

The central feature of the Faddeev–Popov construction is the change of integration variables from e and χ to Ω and m. This is effected by the formal insertion,

$$1 = \int (d\Omega) \int dm \Delta(e,\chi)\delta(e - e(m)^\Omega)\delta(\chi - \chi(m)^\Omega), \tag{7.2}$$

where the pair (e,χ) represents a general configuration and the invariant functional $\Delta(e,\chi)$ is supposed to be determined by (7.2). By means of a formal interchange of the orders of integration one writes,

$$\int (ded\chi)\exp(-\Gamma(e,\chi)) = \int (d\Omega) \int dm \Delta(e(m),\chi(m))\exp(\Gamma(e(m)^\Omega,\chi(m)^\Omega)).$$

If Γ is invariant then the integrand is independent of Ω and it becomes meaningful to discard the volume factor, $\int(d\Omega)$. The quantity of interest is the remaining (modular) integral,

$$\int dm\Delta(e(m),\chi(m))\exp(-\Gamma(e(m),\chi(m))).$$

Although the modular integration is not well understood for world sheets of genus > 2, the first problem is to compute the Faddeev Popov factor,

$$\Delta(m) = \Delta(e(m),\chi(m))$$

which is given, according to (7.2) by

$$\frac{1}{\Delta(e,\chi)} = \int (d\Omega) \int dm\delta(e - e(m)^\Omega)\delta(\chi - \chi(m)^\Omega). \tag{7.3}$$

This representation is meaningful to the extent that for any configuration (e,χ) there must exist an Ω and m in the integration domain for which the delta functionals are non-vanishing. Indeed, it is only the infinitesimal neighbourhood of this point which needs to be considered. In effect the integral (7.3) can be replaced by an integral over the tangent space at the points in question,

$$\int (d\Omega) \int dm \to \int (d^2pdrdtd\varepsilon) \int d\hat{m}.$$

To facilitate the evaluation one can choose $e = e(m_0)$, $\chi = \chi(m_0)$ and expand the arguments, $e(m)^\Omega - e(m_0)$ and $\chi(m)^\Omega - \chi(m_0)$ to first order in $\hat{m}, p^n$, etc.

$$
\begin{aligned}
(e(m)^\Omega - e(m_0))^+_m &= p^n\partial_n e^+_m + \partial_m p^n e^+_n - (r+it)e^+_m \\
&\quad + 2\varepsilon e^-_m \chi_- + \hat{m}^A\partial_A e^+_m \\
(e(m)^\Omega - e(x_0))^-_m &= p^n\partial_n e^-_m + \partial_m p^n e^-_n - (r-it)e^-_m + \hat{m}^A\partial_A e^-_m \\
\chi_-(m)^\Omega - \chi_-(m_0) &= p^n\partial_n\chi_- + \frac{1}{2}(r-3it)\chi_- + (\partial_- + \frac{i}{2}\omega_-)\varepsilon + \hat{m}^A\partial_A\chi_-,
\end{aligned}
\tag{7.4}
$$

where $\hat{m} = m - m_0$ and the fields on the right–hand sides are evaluated at m_0.

The integrand of (7.3) comprises four even delta functionals corresponding to the four components, e^a_m, and one odd one corresponding to χ_-. Their arguments can be simplified by choosing a conformal frame,

$$
e^-_z(m_0) = e^+_{\bar z}(m_0) = 0. \tag{7.5}
$$

However, it is necessary to retain the modular derivatives, $\partial_A e^-_z$, $\partial_A e^+_{\bar z}$ which need not vanish in the frame (7.5). Derivatives of the diagonal components, $\partial_A e^+_z$ and $\partial_A e^-_{\bar z}$, on the other hand, can be absorbed by Weyl scalings and frame rotations. These can therefore be set equal to zero. The integral representation for $\Delta(m_0)$ then takes the form,

$$
\begin{aligned}
\frac{1}{\Delta(m_0)} = \int (d^2p\,dr\,dt\,d\varepsilon) \int d\hat{m}\,&\delta[\partial_z(p^z e^+_z) + p^{\bar z}\partial_{\bar z}e^+_z - (r+it)e^+_z] \\
&\delta[\partial_{\bar z}(p^{\bar z}e^-_{\bar z}) + p^z\partial_z e^-_{\bar z} - (r-it)e^-_{\bar z}]\,\delta[\partial_z p^{\bar z}e^-_{\bar z} + \hat{m}^A\partial_A e^-_z] \\
&\delta[\partial_{\bar z}p^z e^+_z + 2\varepsilon e^-_{\bar z}\chi_- + \hat{m}^A\partial_A e^+_{\bar z}] \\
&\delta[p^z\partial_z\chi_- + p^{\bar z}\partial_{\bar z}\chi_- + \frac{1}{2}(r-3it)\chi_- + (\partial_- + \frac{i}{2}\omega_-)\varepsilon + \hat{m}^A\partial_A\chi_-].
\end{aligned}
\tag{7.6}
$$

The values of r and t, whose derivatives make no appearance here, are fixed by the first two delta functionals. These give, in particular, the combination

$$
\frac{1}{2}(r-3it) = -\frac{1}{2}e^z_+[\partial_z(p^z e^+_z) + p^{\bar z}\partial_{\bar z}e^+_z] + e^{\bar z}_-[\partial_{\bar z}(p^{\bar z}e^-_{\bar z}) + p^z\partial_z e^-_z]
$$

which is to be substituted in the argument of the last delta functional. Having eliminated two of the delta functionals by integrating over r and t, one can represent the remaining three by Fourier integrals. The result is

$$
\begin{aligned}
\frac{1}{\Delta(m_0)} = \int (d^2p\,d^2q\,d\varepsilon\,d\xi) \int d\hat{m}& \\
\exp\int d^2z\,\big[&q(\partial_{\bar z}p^{\bar z}e^+_z + 2\varepsilon\chi_- e^-_{\bar z} + \hat{m}^A\partial_A e^+_z) + \overline{q}(\partial_z p^{\bar z}e^-_{\bar z} + \hat{m}^A\partial_A e^-_z) + \\
&+ \xi\{p^z\partial_z\chi_- + p^{\bar z}\partial_{\bar z}\chi_- + (\partial_- + \frac{i}{2}\omega_-)\varepsilon + \hat{m}^A\partial_A\chi_- - \\
&- \frac{1}{2}e^z_+(\partial_z(p^z e^+_z) + p^{\bar z}\partial_{\bar z}e^+_z)\chi_- + e^{\bar z}_-(\partial_{\bar z}(p^{\bar z}e^-_{\bar z}) + p^z\partial_z e^-_{\bar z})\chi_-\}\big]
\end{aligned}
\tag{7.7}
$$

when $q, \overline{q}$ and ξ are Lagrange multipliers. The zweibein components can be largely eliminated from this expression by converting to field variables which are neutral with respect to r and t transformations. These are

$$
\begin{aligned}
b &= e^+_z q, & \overline{b} &= e^-_{\bar z}\overline{q}, & \beta &= (e^+_z)^{\frac{1}{2}}(e^-_{\bar z})^{-1}\xi \\
c &= p^z, & \overline{c} &= p^{\bar z}, & \gamma &= (e^+_z)^{-1/2}\varepsilon
\end{aligned}
\tag{7.8}
$$

and

$$
\chi = (e^+_z)^{-1/2}e^-_{\bar z}\chi_-
$$

Finally, in order to have an integral expression for $\Delta(m)$ rather than $1/\Delta(m)$, it is necessary to reverse the Grassman type of all integration variables including the quantities $\hat{m}^A$. This gives

$$
\begin{aligned}
\Delta(m) = \int (d^2b\,d^2c\,d\beta\,d\gamma) \int d\hat{m}& \\
\exp\int d^2z\,\big[&b(\partial_{\bar z}c + \hat{m}^A e^z_+\partial_A e^+_{\bar z}) + \overline{b}(\partial_z\overline{c} + \hat{m}^A e^{\bar z}_-\partial_A e^-_z) \\
&+ \beta(\partial_{\bar z}\gamma + \hat{m}^A\partial_A\chi) \\
&+ 2b\gamma\chi + \beta\left(-\frac{1}{2}\partial_z c\chi + c\partial_z\chi + \partial_{\bar z}(\overline{c}\chi)\right)\big],
\end{aligned}
\tag{7.9}
$$

where $b, \overline{b}, c$ and $\overline{c}$ are odd and β, γ are even. If the components $\hat{m}^A$, tangent to the space of moduli, are integrated one obtains the final result *),

$$
\begin{aligned}
\Delta(m) = \int (d^2b\,d^2c\,d\beta\,d\gamma) \prod_A \delta((b, \mathrm{v}_A) + (\overline{b}, \overline{\mathrm{v}}_A) + (\beta, \nu_A))& \\
\exp\int d^2z\,\big[&b\partial_{\bar z}c + \overline{b}\partial_z\overline{c} + \beta\partial_{\bar z}\gamma \\
&+ 2b\gamma\chi + \beta\left(-\frac{1}{2}\partial_z c\chi + c\partial_z\chi + \partial_{\bar z}(\overline{c}\chi)\right)\big],
\end{aligned}
\tag{7.10}
$$

where $\mathrm{v}_A, \overline{\mathrm{v}}_A$ and ν_A are the Beltrami differentials defined in Sec. 4. The inner products appearing in the arguments of the delta functions are defined by

$$
(b, \mathrm{v}_A) = \int d^2z\,b(z)\mathrm{v}_A(z), \tag{7.11}
$$

etc., as in (4.20).

We do not have expressions for the representatives $e(m), \chi(m)$ or their derivatives, the Beltrami differentials. Since the moduli space of the classical Riemann surface

*) The ghost action (7.10) agrees with the conformal gauge version of the ghost part of the invariant action (6.14) if we make the redefinitions $\beta \to -2\beta$, $\gamma \to -\gamma/2$ and discard the irrelevant term $\beta\partial_{\bar z}(\overline{c}\chi)$.

has a complex structure, one expects that something analogous is true for the heterotic geometries discussed here [11]. More specifically, one would like to have a natural partition of the modular coordinates,

$$(m^A) = (m^j, m^{\bar{j}}, \mu^\alpha), \tag{7.12}$$

where $m^{\bar{j}} = \overline{m^j}$, $j, \bar{j} = 1, \ldots, 3(\gamma - 1)$ and $\alpha = 1, \ldots, 2(\gamma - 1)$ such that, in a local system of coordinates on the world sheet with $e_z^- = e_{\bar{z}}^+ = \chi = 0$, the Beltrami differentials are restricted by

$$\bar{\mathrm{v}}_j = \bar{\mathrm{v}}_\alpha = \mathrm{v}_{\bar{j}} = \nu_{\bar{j}} = 0. \tag{7.13}$$

The product of delta functions in the integrands of (7.10) would then assume the form

$$\prod_j ((b, \mathrm{v}_j) + (\beta, \nu_j)) \prod_{\bar{j}} (\bar{b}, \bar{\mathrm{v}}_{\bar{j}}) \prod_\alpha \delta((b, \mathrm{v}_\alpha) + (\beta, \nu_\alpha)). \tag{7.14}$$

It is then possible to effect a formal evaluation in terms of the determinants $\det \overline{\partial}_2$ and $\det \overline{\partial}_{3/2}$ defined by

$$\int (dbdc) b(z_1) \ldots b(z_{3\gamma-3}) e^{\int d^2 z b \partial_{\bar{z}} c} = \det \overline{\partial}_2 \det u^i(z_j)$$

$$\int (d\beta d\gamma) \beta(z_1) \ldots \beta(z_{2\gamma-2}) e^{\int d^2 z \beta \partial_{\bar{z}} \gamma} = (\det \overline{\partial}_{3/2})^{-1} \det v^\alpha(z_\beta), \tag{7.15}$$

where $u^i(z)$ and $v^\alpha(z)$ denote bases for the holomorphic 2 forms and 3/2 forms, respectively. The quantities $\det \overline{\partial}_2$ and $\det \overline{\partial}_{3/2}$ are sensitive to the choice of basis but the products which appear on the right–hand sides of (7.15) must be independent of this choice. The integral (7.10) with $\chi = 0$ and with the assumptions (7.13) then reduces to

$$\Delta(m) = \int (d^2 b d^2 c d\beta d\gamma) \prod_j [(b, \mathrm{v}_j) + \beta, \nu_j)] \prod_{\bar{j}} (\bar{b}, \bar{\mathrm{v}}_{\bar{j}}) \prod_\alpha [\delta((b, \mathrm{v}_\alpha) + (\beta, \nu_\alpha)]$$

$$\exp \int d^2 z (b \partial_{\bar{z}} c + \beta \partial_z \bar{c} + \beta \partial_{\bar{z}} \gamma)$$

$$= \frac{|\det \overline{\partial}_2|^2}{\det \overline{\partial}_{3/2}} \det(\bar{u}^i, \bar{\mathrm{v}}_j) \mathrm{sdet} \begin{vmatrix} (u^i, \mathrm{v}_j) & (v^\alpha, \nu_j) \\ (u^i, \mathrm{v}_\beta) & (v^\alpha, \nu_\beta) \end{vmatrix}. \tag{7.16}$$

In this formula the 2 forms, $u^i, \overline{u^i}$ and the 3/2 form v^α are all even in the Grassman sense. So also are the Beltrami differentials $\bar{\mathrm{v}}_{\bar{j}}, \mathrm{v}_j$ and ν_β. The remaining Beltrami differentials, v_β and ν_j, are odd.

In conclusion it should be remarked that the Faddeev–Popov fields β and γ have a wider role in superstring theory. They, or rather one of their bosonized components, is involved also in the stucture of emission vertices [13]. More complicated multiple correlators of the $\beta\gamma$ system than indicated in (7.15) are therefore needed. According to the Verlindes [14] such correlations can develop "unphysical" singularities on the world sheet. We have nothing to add to that discussion [15].

8. SUMMARY AND CONCLUSIONS

We have analyzed the question of gauge fixing in heterotic super conformal field theories and shown that the various auxiliary components, describing the geometry, can be chosen in a globally well defined way. To do this we realize the geometry in terms of a set of independent Bose and Fermi fields defined relative to a covering of the supermanifold by coordinate neighbourhoods. The transition functions which define the patching are given as elements of a structure group which is generally a subgroup of the super–reparametrizations, frame rotations and Weyl scalings. The subgroup is defined relative to a choice of gauge since it can be interpreted as the stability group of a gauge slice – a space of configurations restricted by gauge conditions. We give explicit formulae for the action of this group on the Wess–Zumino class of configurations and also for the more restricted "superconformal" configurations. In the latter case we recover the transition elements employed in the standard definition of super Riemann surfaces.

We find that superfield notation is useful to the extend that it makes clear the group property of the various finite transformations. It is also useful in the treatment of superconformal matter multiplets, particularly in the construction of action functionals. However, because of the torsion constraints, which take the form of differential equations in superspace, but which can be solved algebraically in component notation, it is advisable to adopt the latter in order to be able to deal with independent variables. We have solved these constraints in component notation. This has enabled us to express the transformations, at least implicitly, through their action on sets of independent variables, and in particular to give explicit form to the finite transformations of the WZ stability group. We have further been able to show that the more stringent gauge conditions which define the superconformal gauge (wherein the geometrical elements are reduced to flat superspace form) define a "holomorphic" stability group. Transition elements drawn from this group define the standard super Riemann surface.

In general the various geometrical fields can be absorbed by redefining the matter fields. Such redefinitions are effected by means of finite gauge transofrmations. For the particularly simple case in which no matter fields are present, we have used the Ward identities to argue, on the basis of infinitesimal gauge transformations, that the dependence on geometrical fields is expressed entirely through a finite set of moduli.

In addition to the generalized discussion of heterotic conformal field theories involving chiral (and neutral) matter superfields, in which we give the superspace rules together with the WZ gauge expressions for the various covariant derivatives, we have included a treatment of the Faddeev–Popov procedure. This leads to an expression for the integration measure in super moduli space.

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