GEOMETRY OF CONVEX SPACES

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In this article we first give the geometry of Moishezon spaces systematically namely the relative place at the category of Moishezon spaces among others, the characterization of Moishezon spaces in terms of various notions of positivity on coherent sheaves. Then the relation between the existence of an almost positive coherent sheaf of rank 1 over a compact complex space X and its being Moishezon is given and the solution to the Grauert–Riemenschneider conjecture is outlined. Moreover the Grauert–Riemenschneider conjecture in strong version is also discussed and its relation with the eigenvalue conjecture is also given. Having cleared the Grauert–Riemenschneider conjecture out of our way the complete picture of the characterization of Moishezon spaces is stated.

Then the notion of $q$-convex spaces $(q \geq 1)$ is introduced. In particular the geometry of 1-convex spaces is studied and their embedding problem also. There is a close relation between Moishezon spaces (respectively...
projective algebraic spaces) and 1-convex spaces
(respectively embeddable 1-convex spaces). Now our
understanding of 1-convex spaces is neat. In fact, the
notion of strongly pseudo convex spaces (due to Grauert)
finitely-point blow ups of stein spaces due to Narasimhan-
fornezi and 1-convexity in the sense of Vo Van Tan are
all essentially the same, though they are introduced in
different contexts. Analogous to the Grauert's understand-
ing of strongly pseudo convex spaces as those complex
spaces X admitting a unique maximal compact analytic subset
A, the existence of such an analytic subset for holomorphic
convex spaces is also discussed.

The geometry of q-convex (respectively q-concave
spaces (for q \geq 1) (more generally, of mixed (p , q) -
convex - concave spaces) is rather more involved and
difficult and in particular immersion and embedding type
theorems for them. There are a few scattered results on
these by Andreotti - Tomassini, Andreotti-Siu, A.Silva,
Forness-Narasimhan. However, in the case of 1-convex
spaces, the geometry is nice because 1-convex spaces occur
as good fibrations over Stein spaces that enjoy rich analytic
geometry which can be transferred to 1-convex spaces.

On the other hand for q > 1, a q-convex space need
not be holomorphically convex. For (p,q) - type complex
spaces cohomology vanishing theorems in general are
difficult and so their embedding or immersion theorems
are even more difficult. In some sense the (p,q) - type
complex spaces lie intermediate between the compact case and
the Stein case. The Kodaira-Grauert type embedding theorems
(as a consequence of cohomology vanishing theorems) are
valid for compact case only under extra conditions such as
the existence of positive line bundles, whereas the
cohomology vanishing and hence embedding theorems hold with-
out any extra conditions for Stein spaces in view of
Cartan - Serre Theorem B. This explains the complexity
involved in the geometric studies of (p,q) - type complex
spaces. Note that holomorphic convexity is free for
1-convex spaces and for holomorphic convex spaces X
algebraic properties such as cohomological vanishing or
finitude can be understood in terms of geometric properties
like the existence of a maximal compact analytic subset
(namely the so called exceptional set A). For q-convex
spaces for $q > 1$ we do not have the holomorphic convexity without extra conditions. Also we have an intimate relationship between $1$-convex manifolds and Stein spaces with isolated singularities which can be exploited to advantage in either direction. The geometry of $1$-convex spaces (more generally $q$-convex spaces) will be dealt with in a subsequent article from the viewpoint of singularity theory. Here our attempt is to emphasize more on the conceptual aspects and it is informative and narrative in style rather than giving rigorous proofs. Nevertheless, a detailed outline is given on the recent solution for Grauert-Riemenschneider conjecture thereby completing the characterizational understanding of Moishezon spaces.

The main plan is to cover the following topics in order:

I. Geometry of Moishezon Spaces

1. Moishezon Spaces
2. Kodaira and Grauert's embedding theorems
3. Categorical place of Moishezon Spaces
4. Relation between Categories $K$ and $M$
5. Positivity notion on a coherent sheaf
6. Primary positivity on a coherent sheaf
7. Finsler structures
8. Cohomological positivity and Moishezon Spaces

II. Geometry of $(1-)$ Convex Spaces

10. Stein Spaces
11. Generalities on Convexity
12. Complex spaces of type $(p,q)$
13. Geometry of Convex Spaces
In the following a summary of 5 lectures given at ICTP is presented in a rough form without proofs, more in an informative style of the results and problems.

I. Geometry of Moishezon Spaces.

1. Let $X$ be a compact complex space of complex dimension $n$. $\mathcal{H}(X)$ denote the meromorphic function field of $X$ and $d$ the transcendence degree of $\mathcal{H}(X)$ over $\mathbb{C}$, that is, the maximum number of algebraically independent meromorphic functions over $\mathbb{C}$. Then we have

$$\text{Th 1 (Siegel [43]) } \quad d \leq n.$$ 

Defn. 2: A compact complex space (manifold) of complex dimension $n$ with $d = n$, that is, having $n$ algebraically independent meromorphic functions is called a Moishezon space (manifold).

Defn. 3: A proper subjective holomorphic map $\pi : \tilde{X} \to X$ from a complex manifold $\tilde{X}$ to a compact complex manifold $X$ is called a desingularization of $X$ if there exists a closed complex subspace $E$ of $X$ such that $\mathcal{H} | X - E : X - E \to X - \pi(E)$ is a biholomorphism. $E$ is called the exceptional set of $\mathcal{H}$.

Theorem 4 (Hironaka [20]): Every compact complex space $X$ admits a desingularization $\tilde{X}$.

Theorem 5 (Moishezon [29]): (a) Every compact irreducible complex space is Moishezon if and only if $X$ admits a projective algebraic desingularization i.e. there exists a projective algebraic manifold $\tilde{X}$ with $\pi : \tilde{X} \to X$ desingularization.

(b) Let $p : X \to Y$ be a surjective holomorphic map with $X$ moishezon. Then $Y$ is also Moishezon.

2. Let $X$ be a compact complex manifold. Let $L$ be a holomorphic line bundle over $X$. We say $L$ is positive if $c_1(L) > 0$ i.e. there exists a hermitian metric $h$ on $L$ such that its curvature form $\omega_h = i \partial \bar{\partial} \log \|h\|$ is a positive $(1,1)$ form. That is, for each holomorphic tangent vector $v \in T^+_X$, $\langle \omega_h(v, \bar{v}) \rangle > 0$.

Defn. 6: $X$ is called projective algebraic if $X$ can be holomorphically embedded in some projective space $\mathbb{P}^N$ as a closed subvariety. Then we have the celebrated

Theorem 7 (Kodaira [25]): Let $M$ be a compact complex manifold then $M$ is projective algebraic iff $M$ admits a positive holomorphic line bundle.
More generally

**Defn. 8:** Let \( X \) be a complex space and let \( \{ U_i \} \) be an open cover of \( X \). We say \( X \) admits a **Kaehler metric** if there exist strongly pluri subharmonic functions \( \{ \varphi_i \} \) such that on \( U_i \cap U_j \), \( \varphi_i - \varphi_j \) is pluriharmonic, i.e., locally, it is the real part of some holomorphic function.

We say \( X \) is Kaehler if \( X \) admits a Kaehler metric.

If \( X \) is non-singular, this agrees with the usual definition on manifold.

**Theorem 9** (Grauert [15]) Let \( X \) be a compact complex space. Then \( X \) is projective algebraic iff \( X \) admits a positive holomorphic line bundle.

**Problem 10:** Since theorems 7 and 9 characterize "algebraicity" of complex spaces (manifolds) which occur as desingularizations of Moishezon spaces, it is a very natural problem to characterize "Moishezonness" of a compact complex space in terms of "some geometric objects" and "certain notion of positivity on them".

3. Categoricial place of Moishezon Spaces:

Let \( \mathcal{A} \) denote the category of projective algebraic spaces;

Let \( \mathcal{M} \) denote the category of Moishezon spaces

Let \( \mathcal{C} \) denote the category of compact complex spaces with Kaehler metric

Let \( \mathcal{C} \) denote the category of compact complex spaces.

We want to discuss the relations between these categories.

We have the following results:

(a) Every projective algebraic space is a Moishezon space;

(b) Every Moishezon space is in \( \mathcal{C} \);

(c) \( \mathcal{C} \) strictly contains \( \mathcal{M} \) because there exist complex Tori \( T^n \) with arbitrary transcendence degree \( d \) over \( \mathbb{C} \).

In fact, there exists \( T^n \) with \( d = 0 \) (Siegel).

(d) There exist Moishezon spaces which are not projective algebraic (Moishezon).

Thus we have the following.

**Theorem 10:** \( \mathcal{A} \subset \mathcal{C} \subset \mathcal{M} \subset \mathcal{C} \)

However, in three special cases the categories \( \mathcal{A} \) and \( \mathcal{M} \) coincide, namely

(i) Let \( M \) be a compact complex manifold of complex dimension \( 2 \).

Then \( M \) is Moishezon iff \( M \) is projective algebraic.

(Kodaira - Chow [8])

In fact, Moishezon-Hironaka constructed a 3-dimensional compact complex manifold which is Moishezon but not algebraic.

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However, the situation is different for the case of complex manifolds. For any $n \geq 2$, we can construct a compact complex space of complex dimension $n$ which is Moishezon but not projective algebraic (Grauert - Riemenschneider for $n=2$, Riemenschneider [43] for $n > 2$).

In fact, Grauert's construction is a "one point blow up" procedure.

1. Let $X = \mathbb{C} \times \Gamma$ be a $n$-dimensional complex torus where
   \[ \Gamma = \{ \sum_{i=1}^{2n} \lambda_i w_i \mid \lambda_i \in \mathbb{Z} \} \]
   is a lattice and $\{ w_i \}$ are $2n$ linearly independent vectors of $\mathbb{C}^n$. Then $\Omega = (\omega_{ij})_{n \times 2n}$ is the period matrix of $\Gamma$ over $\mathbb{C}$. $\Omega$ is called a Riemann matrix if there exists a $2n \times 2n$ non-singular integral skew symmetric matrix $\Omega$ such that (a) $\Omega^T \Omega = 0$
   (b) $-\Omega^T \Omega$ is positive definite where $A = \mathbb{C}^2$.

Theorem 11 (Lefschetz): Let $M = \mathbb{C}^n \times \Gamma$ be $n$-dimensional complex torus with period matrix $\Omega$. Suppose $M$ is Moishezon with $\Omega$ a Riemann matrix. Then $M$ is projective algebraic.

Remark: We have a stronger result, namely, any complex torus $\mathbb{T}^n$ with a Riemann period matrix $\Omega$ is Hodge and hence projective algebraic and conversely.

Example 12: Consider the complex forms $T^2 = \mathbb{C}^2 | \Gamma$ where
   \[ \Omega = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \]

You can check that $\Omega$ is not a Riemann matrix and hence by above remarks, $T^2$ is not projective algebraic and hence not Moishezon, by Kodaira-Chow.

Nevertheless, note that each complex torus, $T^n = \mathbb{C}^n | \Gamma$ admits a Kaehler metric descending form $\mathbb{C}^n$. Thus above $T^2$ is a Kaehler manifold which is not Moishezon; not algebraic.

In fact, $T^2$ admits no meromorphic functions at all (Siegel).

For $(T^n, \Omega)$ with $\Omega$ a Riemann matrix the categories $A$ and $M$ coincide.

Recall every projective algebraic space is Kaehler as well as Moishezon. On the other hand, we have the following deep result:

Theorem 13 (Moishezon [29]): A Moishezon manifold $M$ is projective algebraic iff $M$ admits a Kaehler metric.

Finally, thus, in the presence of a Kaehler metric the categories $A$ and $M$ coincide.

Note that we have the inclusion relation:
For every Kaehler space need not be algebraic as Example 12 above shows; more over, the Hopf surface \( \mathbb{C}^2 \times \mathbb{C}^2 \) which is diffeomorphic with \( S^1 \times S^3 \), is a compact complex manifold which cannot admit any Kaehler metric (The Hopf surface can be generalized to a larger class of manifolds called Eckmann - Calabi manifolds which we discuss later).

In summary, we have the following situation:

**Theorem 14:**

- **(a)** \( X \) which is both Kaehler and Moishezon : projective algebraic ones.
- **(b)** \( X \) is Kaehler but not Moishezon : Siegel torus of Example 12; \( K \) - surface is Kaehler but not Moishezon (Siu [53], Todorov [55]) for \( d = 0 \); \( q^d \) with \( d < n \); elliptic surface (\( d = 1 \)) and Atiyah - Wirth example.
- **(c)** \( X \) not Kaehler and not Moishezon : The Hopf surface \( X \) is not Kaehler and hence not projective algebraic and hence not Moishezon by Kodaira - Chow.
- **(d)** \( X \) Moishezon but not Kaehler : There are standard techniques known for constructing non-Kaehler manifolds by violating the necessary topological (or otherwise) conditions for the existence of Kaehler metric and then we make these Moishezon by some sort of vanishing theorem technique. This is the general idea of this construction.

**Theorem 15** (Siu [49]) Let \( M \) be a compact complex manifold and let \( L \) be a hermitian holomorphic line bundle over \( M \) whose curvature form is semipositive everywhere and is strictly positive outside a set of measure zero (*). Then

\[
H^q(M, L_K) = 0 \quad \text{for} \quad q \geq 1 \quad \text{where} \quad K_M \quad \text{is the canonical line bundle of} \quad M.
\]
As a consequence of Siu's and Hennihaus's vanishing theorems we get

Theorem 16: Let \( M \) be a compact complex manifold which admits a line bundle \( L \) as in Siu's theorem (called the Siu bundle). Then \( M \) is Moishezon.

(+) strictly positive almost at one point is enough.

(\( \text{cf. appendix to the GA Conjecture at the end} \)).

Remark 17: It should be pointed out that a more general vanishing theorem for a line bundle \( L \) satisfying certain data on eigenvalues to conclude \( H^1(M, L^s \otimes G) = 0 \) for any line bundle \( G \) vanishes for \( s \gg 0 \), can be proved (cf. Siu [49]), and then use a sufficient condition namely: Let \( X \) be any compact complex space, \( L \) be any holomorphic bundle over \( X \) satisfying there exists \( x \in X \) such that \( H^1(X, L \otimes \mathcal{O}_x) = 0 \) for \( r = 1,2 \), where \( \mathcal{O}_x \) is the sheaf of holomorphic functions vanishing at \( x \). Then \( X \) is Moishezon (cf. R & S Cat, place [38]).

There are standard techniques of constructing non-Kaehler manifolds as follows:

(a) for a Kaehler manifold \( M \), necessarily its second Betti number \( b_2(M) \) is positive. Generalizing the Hopf surface we can construct a family of manifolds \( S^{p,q} \otimes \mathbb{T}_{1,2} \) called the Eckmann-Caalbi manifolds which are diffeomorphic with \( S^{p+1} \times S^{q+1} \) and having their second betti number zero and hence these are non-Kaehlerian.

(b) Following a method due to Atiyah [5] we can construct a parametrized family of complex tori \( M_n \) where \( M_n = J_n \times T_j \) where the fiber torus varies with each complex structure \( J \neq J_n \). Then by a necessary and sufficient condition for a fibration to be Kaehler we can prove \( M_n \) cannot admit any Kaehler metric (cf. Blanchard [6]).

(c) Hironaka [21] constructed a holomorphic family of compact complex manifolds \((M,T,p)\) parametrized by a space \( T = \{\} \) such that (i) for all \( t \neq 0 \) of \( T \), \( w_t \) is projective algebraic and hence Kaehler and (ii) \( W_0 \) is non-Kaehlerian. In fact, one can show that on \( W_0 \) there exists an algebraic 1-cycle \( z \) which is algebraically equivalent to zero say \( z = \sum \lambda_i C_i \) where \( C_i \) is an irreducible curve and \( \lambda_i \in \mathbb{Z} \).

On this \( W_0 \), we cannot have any Kaehler metric with non-degenerate fundamental form.

(d) Thurston constructed a 4-dimensional manifold \( X \) which is symplectic but \( b_1(X) = 3 \). However, for this \( X \), \( \pi_1(X) \) is trivial. Since for a Kaehler \( X \), its odd betti numbers are even, Thurston manifold cannot admit any Kaehler metric. Now we can apply a blow up technique to Thurston
manifolds to obtain the so called "Generalized Thurston Manifolds".

X having (i) X has a symplectic structure
(ii) $\Sigma_1(X)$ is nontrivial.
(iii) $\beta_1(X)$ is odd.

These $\tilde{X}$ are examples of symplectic manifolds which are non-Kaehlerian.

Then by taking generalized Thurston manifold $\tilde{X}$ and requiring that the almost complex structure $J$ defined by the symplectic structure is indeed a complex structure on $\tilde{X}$ and then putting a Siu line bundle on $\tilde{X}$, we get $\tilde{X}$ to be a non-Kaehler Moishezon manifold. (cf. [48]).

(e) A recent result of Lichnerowicz states

**Theorem:** Let $(W,g)$ be an arbitrary spin manifold of dimension $(n \geq 2)$ admitting a non-trivial killing spiner Then any non-trivial parallel $k$-forms ($k \neq 0, n$). In particular such $W$ is irreducible and non-Kaehlerian. We can put a Siu line bundle on above $W$ to get an example of a non-Kaehler Moishezon manifold. (cf. [26]).

In summary we have

**Theorem 10** (Rama and Sitaramayya [38]): Let $M$ be a complex manifold belonging to one of the classes described in a), b), c), d), e) above. Let $L$ be a Siu line bundle over $M$. Then $M$ is a Moishezon non-Kaehlerian manifold.
Now we proceed to the characterization of Moishezon spaces.

5. Positivity notion on a coherent sheaf:

Let \( X \) be a connected complex space and \( S \) be a coherent sheaf on \( X \). Then we have \( \mathcal{O}_X^n \rightarrow \mathcal{O}_X \rightarrow S \rightarrow 0 \). Let \( L(S) \) be the linear fiber space associated to \( S \). That is, \( L(S) \) is given by an \((n \times m)\)-holomorphic matrix on \( X \) whose transpose defines a homomorphism \( \lambda : X \times \mathbb{C}^m \rightarrow X \times \mathbb{C}^n \) and \( L(S) \) is the kernel of \( \lambda \).

**Definition 19:** By a hermitian form on \( L(S) \) we mean a collection of hermitian forms \( h = (h_x) \) on each fiber \( L_x \), \( x \in X \) satisfying:

- If for each \( x \in X \), there exists a neighborhood \( U_x \) and an isomorphism \( \psi : L_x | U_x \rightarrow U \times \mathbb{C}^m \), and there exists a hermitian form \( h \) on \( U \times \mathbb{C}^m \) such that \( h_{U_x} = h_x \) for all \( x \in U \).

We denote by \( R(X, S) \) the set \{ \( x \in X \mid x \) is a regular point of \( X \) and \( S \) is locally free at \( x \) \}.

**Definition 20:** A coherent sheaf \( S \) over \( X \) is called quasipositive if there exists a hermitian form \( h \) on \( L(S) \) and a dense open subset \( R^0 \subseteq R(X, S) \) such that the vector bundle \( L \mid R^0 \) is negative definite at each point of \( R^0 \).

In this connection we recall the following facts:

1. \( X \) a reduced compact complex space, \( A \subseteq X \) an analytic subset of \( X \). By a proper modification of \((X, A)\) we mean a pair \((Y, \phi)\) of a reduced compact complex space \( Y \) and a proper subjective holomorphic \( \phi : Y \rightarrow X \) such that
   - \( A \subseteq X \) and \( \phi^{-1}(A) \subseteq Y \) are analytically rare and \( \phi : Y \rightarrow \phi^{-1}(A) \) biholomorphic.

2. Such \((Y, \phi)\) proper modification with \( Y \) non-singular is called a disingularization. \( A \) : Centre of the modification.

3. \( S \rightarrow X \) be a coherent sheaf, \( A = \{ x \in X \mid S \) is not locally free at \( x \} \). \( R(X, S) = R = \{ x \in X \mid x \) is regular at \( X \) and \( S \) is locally free at \( x \} \). Then \( A \) and \( X - R \) are both lower dimensional analytic subsets of \( X \). \( L(S) | R = L \) holomorphic vector bundle over the manifold \( R \) of rank \( r \) the rank of the locally free sheaf \( S/R \).

**Definition:** \( S \) is called semi positive (respectively semi-negative) if there exists a hermitian form \( h \) over \( L(S) \) such that the vector bundle \( L_R \) with the hermitian form \( h_R = h \mid L_R \) is semi-positive (respectively semi-negative) i.e. \( V \times \mathbb{C}^m \) neighbourhood of \( \mathbb{C}^m \) and an isomorphism \( \psi : L | U_x \rightarrow U \times \mathbb{C}^m \)
such that with respect to coordinate $z_1, \ldots, z_n$ centred at $x$ on $V$, we have

$$a) \quad (h_{ij}(x)) = \delta_{ij} \quad b) \quad d h_{ij}(x) = 0 \quad c) \quad \text{the hermitian form}$$

$$\sum_{\nu, \mu} j, i A^2 h_{ij}(x) d \mu \left( dz_\nu, \overline{dz_\mu} \right) \in \mathcal{O}_x$$

is semi-negative definite (respectively semi-positive definite).

**Theorem 21** (Grauert - Riemenschneider [16])

(a) Let $X$ be a Moishezon space, $S$ be a torsion-free coherent analytic sheaf over $X$. Then there exists a projective algebraic manifold $\tilde{X}$ and a proper modification $\tilde{T} : \tilde{X} \to X$ such that $\tilde{S} = \tilde{S} \otimes \mathcal{O}_{\tilde{X}}$ is locally free. If $S$ is quasipositive, so is $\tilde{S}$.

(b) Let $X$ be an $n$-dimensional Moishezon space; $S$ be a quasipositive torsion-free coherent sheaf over $X$. Then $H^j(X, S, \mathcal{O}_X) = 0$ for $j > 1$ where $\mathcal{K}(X)$ is the canonical sheaf of $X$.

(c) Let $M$ be a Moishezon space. Then $X$ admits a quasipositive torsion-free coherent sheaf of rank $1$.

Now we have the following problem:

**Grauert - Riemenschneider Conjecture (G-R conjecture) 22**: Let $X$ be a compact complex space which admits a quasipositive torsion-free coherent sheaf of rank $1$. Then $X$ is Moishezon.

We comment more on this later.

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Let $X$ be an irreducible complex space and $E$ be a holomorphic vector bundle of rank $r$, $E^* = \mathbb{C}^r$ be the dual bundle and $\mathbb{P}(E^*)$ be a principal $\mathbb{C}^r$-bundle and let $L(E) \to \mathbb{P}(E^*)$ be the associated line bundle. Then $L(S)$ is positive if and only if $L(S)$ is positive.

**Definition 23**: A holomorphic vector bundle $\mathbb{E} \rightarrow X$ is almost positive if there exists a $\mathbb{C}^r$-hermitian metric $\mathbb{h}$ on $L(S)$ and a dense open subset $R^0 \subset \mathbb{R} (\mathbb{P}(E))$ (set of regular points of $\mathbb{P}(E)$) such that the curvature form $\mathbb{K}_h$ is positive definite at all points of $R^0$.

**Definition 24**: $X$ be a complex space, $S$ be a coherent sheaf over $X$. By a monoidal transformation we mean a pair $(X, \phi)$ of a complex space $X$ and a proper modification $\phi : X \to X$ such that (i) the torsion-free preimage $S = \phi \otimes \mathcal{O}_X$ is locally free on $X$ (ii) if $\phi : Y \to X$ is any proper modification with (i), then $\exists$ holomorphic map $\psi : Y \to X$ such that $\phi = \psi \circ Y$.

(universal property)

Such monoidal transformations exist (Hironaka - Rossi[21]). Using this we can transfer the notion of almost positivity to a coherent sheaf.

**Definition 25**: We say coherent sheaf $S \rightarrow X$ is almost positive if the vector bundle $E = \mathbb{E} \otimes \phi$ is almost positive.
Remark: A vector bundle $E \to X$ is called Griffiths quasipositive ([17]) if there exists a hermitian metric $h$ on $E$ and a dense open set $R^0 \subset C^X$ such that $E|R^0$ is negative definite and then we can extend this to a coherent sheaf $S$ as in Definition 25 using a monoidal transformation.

Theorem 26 (Riemenschneider [113]): Every normal Moishezon space admits a torsion-free almost positive coherent sheaf of rank 1 which is positive outside an analytic subset $A$ of $X$ of codimension at least 2.

Theorem 27 Let $X$ be an irreducible normal compact complex space, $S$ be an almost positive coherent sheaf of rank 1 over $X$ which is positive outside an analytic subset $A$ of dimension zero. Then $X$ is Moishezon.

G-A Conjecture 28: An irreducible normal compact complex space $X$ is Moishezon if and only if there exists a torsion free almost positive coherent sheaf $S$ on $X$ of rank 1.

6. Primary positivity of a coherent sheaf:

Let $X$ be a reduced compact complex space, $S \to X$ be a coherent sheaf, $L(S)$ its associated linear fiber space and $L_R(S)$ its reduction and $A$ is the analytic set in $X$ over which $S$ is not locally free.

The primary comaparen of $L(S)$, denoted by $L^1(S)$, is by definition, the closure of $L^R(S)|X-A$ in $L_R(S)$.

Let $P = P(S) = L(S) - \{0\}/C^*$ and let $L$ be the tautological line bundle associated to the principal $C^*$-bundle $L(S) - \{0\} \to P$ and let $H = H^*_L$ be the dual bundle of $L$. Similarly, we can define $P_R$, $L_R$, $H_R$ associated to $L_R(S)$ and $P^1$, $L^1$, $H^1$ be those associated to $L^1(S)$.

Definition 29 (Rabinowitz [36]): (a) $S \to X$ is called positive (primary positive) if $L(S)$ carries a metric inducing a metric with positive curvature in $H^1 \to P$ (respectively $H^1 \to P^1$).

(b) $S$ is Finsler positive (primary F.p.) if $H_R \to P_R$ (respectively $H^1 \to P^1$) carries a metric with positive curvature.

(c) $S$ is weakly positive (primary weakly positive) if the zero section of $L(S)$ (respectively $L^1(S)$) is exceptional.

(d) $S$ is cohomologically positive (primary cohomologically positive) if for any coherent sheaf $T \to X$ there exists $r \in \mathbb{N}$ such that $H^k(X, S^r \otimes T) = 0$ (respectively $H^k(X, S^r \otimes T) = 0$) for all $\mu \gg \nu, k \gg 1$, where $S^r$ denotes the torsion free sheaf $S|\text{tor}(S)$ for any sheaf $S$ on $X$. 

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Proposition 30: Let $S \rightarrow X$ be a coherent sheaf. Then we have the following implications:

- $S$ is primary positive $\implies$ $S$ is primary positive
- $S$ is primary primary positive
- $S$ is primary weakly positive $\implies$ $S$ is primary weakly positive
- $S$ is primary cohomological positive $\implies$ $S$ is primary cohomological positive

Remark: $S$ is primary positive $\implies$ $S$ is Griffith's quasipositive $\implies$ $S$ is almost positive.

Theorem 31 (Rabinowitz): Let $X$ be a normal irreducible compact complex space. Then the following are equivalent (T F A S):

1. $X$ is Moishezon
2. $X$ carries a primary positive coherent sheaf
3. $X$ carries a primary weakly positive coherent sheaf
4. $X$ carries a primary Finsler positive coherent sheaf
5. $X$ carries a primary cohomological positive sheaf.

Analogous to the Grauert - Riemenschneider's result (cf. Theorem 21) we have

Theorem 32: Let $X$ be a normal irreducible compact complex space $S$ be a primary positive coherent sheaf of generic fiber dimension one. Then $H^k(X, S^k X(X)) = 0$ for all $k \geq 1$ and hence $X$ is Moishezon.

Remark: It is a natural problem whether every Moishezon space admits a weakly positive coherent sheaf $S$.

7. Finsler Structures:

Definition 33: By a complex Finsler structure $F$ on a complex manifold $X$ we mean a function $F$ on the tangent bundle $T(X)$ of $X$ such that

1. $F$ is $C^\infty$ outside the zero section of $T(X)$
2. $F(z, w) \geq 0$ and $= 0$ iff $w = 0$
3. $F(z, \lambda w) = \lambda F(z, w)$ for all $\lambda \in \mathbb{C}$

Similarly a holomorphic vector bundle $E$ over $X$ with a complex Finsler structure $F$ is called a Complex Finsler Vector Bundle.

Let $E$ be a vector bundle of rank $r$ over a complex manifold $X$ of complex dimension $n$. Let $P(E) = E - \{0\} / \mathbb{C} = \mathbb{C} - X / \mathbb{C}$ be the projective bundle of $E$ with associated holomorphic line bundle $L(E) \rightarrow P(E)$ and we have the canonical isomorphism of $L^* \cong P(E)$ with $E^* \cong X$. That is, $L^*$ can be obtained from $E$ by blowing up the zero section of $E$ to $P(E)$ and similarly $L$ can be obtained from $E$. Let $Z = (z_1, \ldots, z_n)$ be a local coordinate system in $X$ and $s = (s_1, s_2, \ldots, s_n)$ be a local holomorphic frame of $E$ and $w = (w_1, \ldots, w_n)$ be the dual
frame. Then \((Z, w) = (z_1, z_2, ..., z_n, w_1, ..., w_n)\) can be taken as a local coordinate system in \(E\). Let \(F\) be a Finsler structure on \(E\); write \(F_{ij} = \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j}\). Let \(E = \mathbb{P}^* E\) be the vector bundle over \(\mathbb{P}(E)\), where \(p : \mathbb{P}(E) \to X\). Then \(L\) is a subbundle of \(\mathbb{P}(E)\), \(p : E \to X\). Then set \(y_1 = w_1 \circ p\) and \((Z, w, y) = (z_1, ..., z_n, w_1, ..., w_n, y_1, ..., y_n)\) is a local coordinate system for \(E\). Set \(F(Z, w, y) = \sum F_{ij} (z, w) y_i y_j\).

Then \(F(z, w) = F(z, w, w)\). Note that when \(\det(F_{ij}) > 0\), \(F\) defines a hermitian structure in \(E\).

Definition 34: We say the Finsler structure \(F\) is convex if \(\det(F_{ij}) > 0\). Assume \(F\) is convex. Then \(\tilde{F} = \tilde{F}(z, w, y)\) defines a hermitian structure on \(E\). Let \(y = \frac{1}{F(z, w)} E R_{i j k l} \frac{\partial^2 F}{\partial z_i \partial \bar{z}_j} \frac{\partial^2 F}{\partial z_k \partial \bar{z}_l} dz^i \wedge d\bar{z}_j \wedge dz^k \wedge d\bar{z}_l\) be the associated curvature form, where \(R_{i j k l} = -F_{i j k l} \sum F_{k j \bar{z}_l} F_{i \bar{z}_k} \).

Definition 35: We say the convex Finsler structure \(F\) in \(E\) is positive (respectively negative) if the form \(y\) of \((X)\) is positive definite (respectively negative definite).

Note that since \(L \to \mathbb{P}(E) \to E \to X\), each Finsler structure \(F\) in \(E\) defines a hermitian structure \(h\) in \(L\) and vice versa.

Then we have the following relation between their corresponding curvatures.

Theorem 36: Let \(E\) be a holomorphic vector bundle over \(X\) with a convex \(F, S, F\). Then \(F\) has negative curvature iff the corresponding hermitian structure \(h\) of \(L\) has negative curvature in the usual sense.

Remark: The notion of a convex Finsler structure can be carried over to a linear fiber space (and coherent sheaf \(S\)) \(L(\cdot)\). Then we have

Theorem 37: Let \(E\) be a linear fiber space over \(X\). Then \(E\) carries a Finsler structure of negative curvature iff \(L_{R(\cdot)}\) carries a hermitian structure of negative curvature.

8. Cohomological positivity and Moishezon Spaces:

Let \(X\) be a compact complex space and \(\mathcal{M}_X\), for \(x \in X\), be the ideal sheaf of germs of holomorphic functions vanishing at \(x\).

Definition 38: A coherent sheaf \(E\) over \(X\) is said to be generated by its global sections if for each \(x \in X\), the natural map \(\Gamma(X, E) \to E_x/\mathcal{M}_X E_x\) is surjective.

Definition 39: (a) A line bundle \(L\) over \(X\) is called ample if for each coherent sheaf \(F\), there exists \(n_0 \in \mathbb{N}\) such that for all \(n \geq n_0\), \(F \otimes L^n\) is generated by its global sections.
A coherent sheaf $E$ over $X$ is called \textit{ample} if for each coherent sheaf $F$ there exists $n_0 \in \mathbb{N}$ such that for all $n > n_0$, $F \otimes \mathcal{S}^n(E)$ is generated by its global sections (here $\mathcal{S}^n(E)$ denotes the $n$th symmetric power of $E$).

**Proposition 40:** (Hartshorne[19]) Let $X$ be a reduced irreducible compact complex space and $L$ is a line bundle over $X$. Then $L$ is cohomologically positive $\iff$ $L$ is ample.

Let $E$ be a coherent sheaf over $X$. Let $L$ be the associated line bundle and $H = L^\vee$ be the dual bundle over $\mathbb{P}(E)$ as in § 6.

**Theorem 41:** Let $X, E, H$ be as above. Then the following are equivalent

1) $E$ is ample
2) $H$ is ample
3) $H$ is cohomologically positive
4) $E$ is cohomologically positive

**Definition 42:** Let $F$ be a coherent sheaf over $X$. We say $F$ separates points of $X$ if for any two distinct points $x, y \in X$, the natural map $\mathcal{O}(X, F) \rightarrow \mathcal{O}(x, F) \otimes \mathcal{O}(y, F)$ is surjective.

**Theorem 43** (Rama and Sitaramayya[39]): Let $f : X \rightarrow Y$ be a morphism of compact complex spaces and $F$ be a cohomological positive sheaf over $X$. Then there exists $n_0$ such that $f_* \mathcal{S}^n_0(F)$ is cohomological positive over $Y$ for all $n \geq 1$.

**Cor 44:** Let $X$ be a reduced compact complex space and $E$ be a coherent sheaf. Suppose $E$ is primary cohomological positive. Then there exists a coherent sheaf $F$ over $X$ which is also cohomologically positive.

**Theorem 45** (Rama and Sitaramayya[39]): Let $X$ be an irreducible reduced compact complex space. Then $X$ is Moishezon iff $X$ carries a cohomological positive coherent sheaf.

**Theorem 46** (Ancona[11], Rama & Sitaramayya[39]): Let $X$ be a reduced irreducible compact complex space. Then TFAE (the following are equivalent)

1) $X$ is Moishezon
2) $X$ admits a holomorphic positive coherent sheaf
3) $X$ carries a Finsler positive coherent sheaf
4) $X$ carries a weakly positive coherent sheaf.

**Theorem 47** (Rama & Sitaramayya[39]): An irreducible reduced compact complex space $X$ is Moishezon iff $X$ admits a coherent sheaf $F$ such that for each coherent sheaf $F$ there exists $n$ with $H^k(X, F \otimes \mathcal{S}^n) = 0$ for all $n \geq n$. 

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Remark: In this direction (---) the vanishing of the first cohomology group for each coherent sheaf $\mathcal{F}$ is much information.

A direction proof relating Moishezonness and Finsler positivity of a coherent sheaf over $X$ will be interesting geometrically, though seems difficult at present.

9. Almost positivity and Moishezon spaces: Recall the Grauert - Riemenschneider conjecture that a compact irreducible reduced complex space $X$ is Moishezon if $X$ admits a torsion free almost positive coherent sheaf of rank 1. This result is true when $\dim X = 2$ and Riemenschneider proved this conjecture with the extra condition $X$ is Kaehler.

Theorem 48 (Riemenschneider [29]): Let $X$ be a compact complex space and $L$ is a holomorphic line bundle over $X$ such that $L$ is semipositive everywhere and positive at least at one part $x \in X - S$ where $S$ is an analytic subset of $X$ such that $X - S$ carries a complete Kaehler metric. Then $X$ is Moishezon.

Theorem 49 (Schneider [24]): Let $X$ be a normal irreducible reduced compact complex space. Then $X$ is Moishezon iff $X$ admits a torsion free almost positive coherent sheaf $\mathcal{S}$ which is globally generated.

Remarks: i) The G-R conjecture is true under the extra condition that $S$ is globally generated.

ii) The argument in Schneider's proof is a special case of a result of Matsushima [28].

Theorem 50 (Matsushima): Let $X$ be an irreducible compact complex space and $E$ is a holomorphic vector bundle over $X$ of rank $q$ which is globally generated. Suppose at least one of the Chern numbers $c_k(E)$ of $E$ is non-zero. Then $X$ is Moishezon.

Remark: A special case of Matsushima's result for a class of complex hypersurfaces in complex tori was proved by Sitaramayya [47].

Now we can prove the following:

Theorem 51 (Ramaiah & Sitaramayya [40]): Let $X$ be a compact complex manifold, $L$ be an almost positive holomorphic line bundle over $X$. Let $D$ be a dense open set in $X$ over which $L$ is positive. Let $A = X - D$ be a subvariety admitting a Kaehler neighbourhood $\mathcal{V}$ in $X$ satisfying the geometric conditions $G_1$ and $G_2$. Then $X$ is Moishezon.
The Kaehler metric $g$ on $W$ is such that the hermitian form $\langle \mathcal{H}(\phi, \psi) + a(\phi, \psi) \rangle$ is positive definite on the space $\Lambda^q(W, E)$ of $\mathbb{C}$-valued $q$-forms on $W$ where $E = K(X) \otimes L'/W$ for large, $\mu = \mathcal{H}$ is the curvature operator of $g$.

There exist positive constants $\mu_0, \lambda$ such that the Levi form $\mathcal{L}$ of $W$ satisfies $\mathcal{L}(\phi, \psi) > \lambda |\phi|^2$ for all $\phi, \psi \in \Lambda^q(W, E)$ with $\mu > \mu_0$ where $\Lambda^q(W, E) = \mathcal{L}(\phi, \psi) > 0$ with $\mathcal{L}$ the adjoint of $\mathcal{L}$.

Remarks: The proof uses an integral formula and a technique due to Griffiths [17]. In fact the existence of such a neighborhood $W$ of $A$ is a technical facility to the effective use of Griffith's integral formula.

2) If $\dim A = 1$, such a Kaehler neighborhood $W$ exists and conditions $G_1$ and $G_2$ are automatically satisfied.

3) This theorem can be further generalized similar to some deep results of Grothendieck on cohomology: vanishing (cf. Rama [41]).

4) Recently Siu [47] proved the G O conjecture when measure of $A$ is zero and Siu announced recently that the G O conjecture is completely proved by him. More precisely, we have

**Theorem 51 (Siu):** (Stronger vision of G O Conjecture):

Let $X$ be a compact complex manifold which admits a hermitian holomorphic line bundle $L$ whose curvature is positive definite everywhere and positive definite at some point.

Then $X$ is Moishezon.

II. GEOMETRY OF CONVEX SPACES

10. PRELIMINARIES ON STEIN SPACES

**Definition 53:** Let $X$ be a complex space of complex dimension $n$ and let $K$ be any compact subset of $X$. They by the holomorphic convex hull $K_X$ of $K$ we mean $K_X = \{ z \in X | \sup_{K} |f(z)| \leq \sup_{K} |f(z)| \}$.

(a) $X$ is called holomorphically convex if $K_X$ is compact for compact $X$ of $X$.

(b) $X$ is called Stein if (i) $X$ is holomorphically convex

(ii) for $z \in X$, $\exists$ holomorphic functions $f_1, f_2, \ldots, f_n$ such that $(f_1, \ldots, f_n)$ is a complex analytic coordinate system of $X$ at $z$.

**Definition 54:** Let $U$ be open subset of $\mathbb{C}^n$, $u, v: \mathbb{R}^n \rightarrow (-\infty, 0)$ be a function of class $C^2$.

(a) we say $u$ is plurisubharmonic (strongly plurisubharmonic) if the Levi form $L(u) = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} w_j w_k \geq 0$ (positive
semi definite) at each point $z \in \mathbb{C}^n$ (respectively, $> 0$

b) $\mathcal{L}$ is called pseudo convex (respectively strongly pseudo convex) if there exists a pluri subharmonic function (respectively strongly pluri subharmonic) function $u$ on $\mathcal{L}$ such that

$$\mathcal{L}_c = \{ z \in \mathbb{C}^n \mid u(z) < C \} \subset \mathcal{L}$$

for each $c \in \mathbb{R}$.

c) Let $X$ be a complex manifold space of complex dimension $n$.

Then $X$ is called pseudo convex (strongly pseudo convex) if

$$\mathcal{L}(X) = \{ x \in X \mid \text{the Levi form } \mathcal{L}(x) \text{ is positive semi definite for each } x \in X \}$$

is positive definite for all $x \in X - K$, for some compact $K \subset X$.

Such $\phi$ is called a strongly pluri subharmonic exhaustion function on $X$.

Theorem 55 (Grauert): A complex manifold $X$ is Stein iff $X$ admits a $C^2$ strongly plurisubharmonic exhaustion function.

Remark: R. Narasimhan [33] improved this for $C^0$ exhaustion function on Norguet - Siu [34] further relaxed this for $X$ to be Stein.

Theorem 56 (Narasimhan [33], Bishop [7], Wiegmans [59]):

Every Stein space $X$ of complex dimension $n$ can be holomorphically embedded with $\mathbb{C}^N$ for $N$ sufficiently large (for the manifold case $N = 2n + 1$).

The above theorem gives embedding of Stein spaces (non-compact case) in complex number spaces. On the other hand, we have for the compact situation.

Theorem 57 (Kodaira): Let $M$ be a compact complex manifold.

Then $M$ admits a projective embedding $M \to \mathbb{P}^N$, $\mathbb{C}$ iff $M$ admits a positive holomorphic line bundle $L$.

Theorem 58 (Grauert): Let $M$ be a compact complex space.

Then $M$ admits a holomorphic embedding into some projective space $\mathbb{P}^N$ if $M$ admits a positive holomorphic vector bundle then $M$ can be embedded holomorphically into certain Grassmannian.

Remark 59: In fact Grauert proved a compact complex space $X$ admitting a holomorphic vector bundle $E$ which is weakly negative (in that its zero section is exceptional) can be embedded holomorphically into a complex Grassmannian.

11. Generalities on Convexity

Recall that Grauert's definition [15] of $X$ being strongly pseudo convex implies the existence of a maximal
compact analytic subvariety $S$ outside which the Levi form is positive definite. Thus in the Grauert sense a strongly pseudo convex space is a data $(X, S)$ where $S$ is called the exceptional set.

**Definition 60**: Let $D$ be an open subset of $\mathbb{C}^n$, $\psi : D \to \mathbb{R}$ be a $C^\infty$ function. $\psi$ is called strongly $q$-pseudo convex if the hermitian form $L(\psi) = \sum_{i,j=1}^{n} \frac{\partial^2 \psi}{\partial z_i \partial \bar{z}_j} u_i u_j$ has at least $(n-q)$ positive eigen values at each point of $D$ where $z_1, ..., z_n$ are the coordinates of $\mathbb{C}^n$ and $q > 0$.

**Definition 61**: Let $X$ be a complex space, $\psi : X \to \mathbb{R}$ be a $C^\infty$ function. $\psi$ is called strongly $q$-pseudo-convex if for each $x_0 \in X$ there exist an open neighbourhood $U_{x_0}$ of $x_0$, a biholomorphic map $\pi$ of $U_{x_0}$ onto an analytic subset of an open set $D$ of some $\mathbb{C}^n$ and a strongly $q$-pseudo convex function $\psi$ on $D$ such that $\pi(x) = x$ and the closure of $\{ x \in X \mid \psi(x) < c \}$ is $x \in X \mid \psi(x) < c$ for $\forall c < \Psi$.

**Definition 62**: A complex space $X$ is called $p$-convex if there exists a $C^\infty$ map $\phi : X \to (-\infty, \infty)$ where $\text{dom} \phi$ is $(-\infty, \infty)$ such that

1. $\{ x \in X \mid \phi(x) < c \}$ is compact for each $c \in (-\infty, \infty)$
2. for some $b^1 \in (-\infty, b)$, $\phi$ is strongly $p$-pseudo convex on $\{ x \mid \phi(x) > b \}$

**Definition 63**: Let $X$ be a complex analytic space and $S$ be a compact analytic subvariety of $X$. $S$ is called an exceptional set if $\text{dim} S > 0$ for each $x \in S$ and there exists a $C^\infty$ map $\psi : X \to Y$ inducing a biholomorphism $X - S \cong Y - T$, where $T$ is a finite set of $Y$ and $\psi \in C^\infty_Y$.

**Definition 64**: Let $X$ be a complex analytic space with exceptional set $S$ i.e. the data $(X, S, Y, T, \psi)$ is given. Then $X$ is called $1$-convex space if $\psi$ is Stein space.
Remarks (a) This means that 1-convex spaces are obtained topologically "by attaching or welding some compact analytic spaces to Stein spaces".

(b) Recall the following definitions due to Narasimhan-Fornaess:

A complex space $X$ is said to be "obtained from a Stein space by blowing up finitely many points" if (1) $X$ is a compact analytic subset $S \subset X$ with $\dim X > 0$ for each $x \in S$.

(2) $X$ is a Stein space $Y$, a finite set $A \subset Y$ and a proper holomorphic surjective map $q : X \to Y$ inducing a biholomorphic $X - S \cong Y - A$ and satisfying $q_* C_X \cong q_* C_Y : S$ is exceptional set and $y$ is the Remmert reduction of $X$.

Thus these 1-convex spaces are objects of study for both algebraic and analytic geometers.

Definition 65: A 1-convex space $X$ is said to be embeddable if $X$ can be realized as a closed analytic subvariety of some $\mathbb{C}^N \times \mathbb{P}^m$.

Example 66.1 Consider the "blow up" of $\mathbb{C}^2$ at the origin. That is, consider $\mathbb{C}^2 \times \mathbb{P}^1$ with points $(z,1)$ where $z = (z_1, z_2)$ and $l = (1,1)$. Then $X = \{(z,1) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid z \neq 1 \text{ i.e. } z_1 = z_2 = 1\}$ is a closed submanifold of $\mathbb{C}^2 \times \mathbb{P}^1$ of complex dimension 2. Let $\pi : X \to \mathbb{C}^2$ be the natural projection i.e. $\pi (z,1) = z_1$. Then $(X, \pi, \pi^{-1}(0) \cong \mathbb{P}^1)$ is called the blow up of $\mathbb{C}^2$ at the origin.

Here $S = \pi^{-1}(0) = \mathbb{P}^1$ is the exceptional set and $(X,S)$ is a 1-convex space since $Y = \mathbb{C}^2$ is Stein. In fact, note that $(X,S)$ is embeddable into $\mathbb{C}^2 \times \mathbb{P}^1$ and $S$ is projective algebraic.

Since the theory of strongly pseudoconvex spaces is like the Kodaira theory (in the sense of Grothendieck) the following result is natural:

Theorem 67 (to, Kazama, Watanabe[12]): Let $X$ be a 1-convex manifold $L$ be a holomorphic line bundle on $X$ which is positive on $X$. Then $X$ is embeddable 1-convex manifold.

Since 1-Convex spaces are finite point blowups of Stein spaces which are embeddable in some $\mathbb{C}^N$ we have:

Theorem 68 (Vo Van Tan[57]): Let $(X,S)$ be a 1-convex space. Suppose there exists a holomorphic line bundle $L$ on $X$ such that $L|S$ is positive. Then $X$ is embeddable.

Remark: Note that if $(X,S)$ is an embeddable 1-convex space then $S$ is necessarily projective algebraic and $X$ is necessarily Kahler as $X \to \mathbb{C}^N \times \mathbb{P}^1$ for some $N$ and $l$.

The following questions naturally arise:
1. Characterize the exceptional sets \( S \) of 1-convex spaces \((X, S)\) ?

2. Characterize the structure of the exceptional sets \( S \) of embeddable 1-convex spaces \((X, S)\) ?

3. Does there exist a 1-convex space \((X, S)\) with \( S \) not projective algebraic ?

4. Does there exist a non-Kaehlerian 1-convex manifold ?

5. Let \((X, S)\) be a 1-convex space with \( S \) projective algebraic then is \( X \) embeddable ?

6. Let \( S \) be a Moishezon space. Then does there exist a 1-convex space \( X \) admitting \( S \) as its exceptional subvariety ?

No answer these questions as the situation has become very clear now.

Moishezon \([30] \) conjectured the following in 1974.

**Conjecture 60 (Moishezon):** Let \((X, S)\) be a 1-convex space. Then the exceptional subvariety \( S \) of \( X \) is Moishezon.

On the otherhand for embeddable 1-convex spaces \((X, S)\) we have

**Theorem 69:** Let \( S \) be a compact irreducible analytic space.

Then \( S \) can be realized as an exceptional set of some embeddable 1-convex space \((X, S)\) iff \( S \) is projective algebraic.

Using Hironaka's analytic version of Chow's lemma \([21]\) we got

**Theorem 70:** Let \( S \) be a compact irreducible analytic space.

Then \( S \) is an exceptional set of some 1-convex space \((X, S)\) if and only if \( S \) is Moishezon.

**Remark:** Using the earlier constructed classes of complex manifolds which are Moishezon but not Kaehler (cf. Theorem 1.6) and using above theorem one can construct 1-convex spaces \((X, S)\) which is not embeddable and which is not Kaehler. Thus if we denote by \( E_{c_1} \) and \( C_c \) the classes of embeddable 1-convex spaces and 1-convex spaces respectively then \( E_{c_1} \subset C_c \).

As already noted, the notions of 1-convexity of Vo Van Tan and finite point "blowing ups of Stein Spaces" of Karasihahan-Porn brave are the same, though they arose in different contexts.

However, the Grauert's notion of strongly pseudo convexity which is defined in terms of positive eigen values of the Hei form, is to be understood in terms of 1-convexity. Fortunately, the link is given by

**Theorem 71 (Conjecture of Karasihahan and Pornaesen 1980):** Let \( X \) be a complex space carrying a strongly plurisubharmonic exhaustion function \( \Phi : X \rightarrow (-\infty, \infty) \). Then \( X \) is holomorphic.
convex and $X$ is obtainable from a Stein space $Y$ as a finite point blow up. This conjecture gives $X$ is strongly pseudo convex in Grauert's sense implies $X$ is $1$-convex in the Vo Van Tan sense.

This conjecture is proved by Coltiou \[10\]

In the other direction we have

**Theorem 1.2 (Coltiou and Mihalache)** Let $(X,S)$ be a $1$-convex space in the sense of Vo Van Tan. Then $X$ carries a strongly plurisubharmonic exhaustion function $\varphi : X \to [\mathbb{-\infty, \infty})$. More over $\varphi$ can be chosen to be $-\infty$ exactly on the exceptional set $S$.

**Remark:** Hence the notions $X$ is strongly pseudo convex in the sense of Grauert and $X$ is a "finite point blow up of a Stein space" (Narasimhan - Fornaess) and $X$ is a $1$-convex space in the sense of Vo Van Tan are essentially the same.

Putting together all these we get the following:

**Theorem 7.3 (Characterization of $1$-convex spaces):** Let $X$ be an analytic space. Then the following are equivalent (TFAE):

a) $X$ is a $1$-convex space.

b) $\dim H^q(X, \mathcal{E}) < \infty$ for any $q > 0$ and each coherent sheaf $\mathcal{E}$ on $X$ (Grauert - Anderotti, 1962).

c) $X$ carries a continuous exhaustion function which is strongly plurisubharmonic outside a compact set $S$.

d) $X$ carries a strongly plurisubharmonic exhaustion function $\psi : X \to [\mathbb{-\infty, \infty})$. 
12. \((p,q)\) - Convex - Concave Spaces:

First we note that compact complex spaces can be regarded as pseudo concave spaces and so in view of Kodaira - Grauert type of results we expect on pseudo concave manifolds (respectively spaces) an embedding theory only under extraconditions such as admitting a geometric object like line bundle, vector bundle or coherent sheaf with certain notion of positivity. On the other hand, Stein spaces can be regarded as strongly pseudo convex spaces and since cohomology vanishing is free (without any condition) for Stein spaces in view of Cartan's theorem B we could get an embedding theorem for Stein spaces into \(\mathbb{C}^n\) without any extra conditions. Thus the classes of all complex spaces of \((p,q)\) - convex - concave type lie as intermediate ones between Stein and compact complex spaces. Thus it is a natural question to study embedding or immersion type of theorems for \((p,q)\) - convex - concave spaces \(X\) which in turn comes to proving cohomology vanishing theorem for such \(X\). It is a general philosophy that embedding theorems follow as a consequence of cohomology vanishing. Thus we have the following:

**Theorem 75** (Andreotti - Siu) Let \(X\) be a normal strongly \(1\)-pseudo concave space \(X\) having all its non compact irreducible components of dimension \(\geq 3\). Then \(X\) admits a projective embedding if \(X\) carries a line bundle \(L\) satisfying a certain positivity condition.

Recall that a complex space \(X\) is called \((p,q)\) - Convex - Concave with respect to a pair \((a_i^1, a_i^1)\) if there exists a proper \(C^2\) function \(\Phi : X \to (a_i^1, a_i^1) \times \mathbb{R}\) and \(a_i^1 \in \mathbb{R} \cup \{a_i^1\}\) such that there exist \(a_i^1, b_i^1 \in \mathbb{R}\) with \(a_i^1 < a_i^1 < b_i^1 < b_i^1\) for which \(\Phi\) is strongly \(p\)-pseudo convex on \(B_{a_i^1}^1 = \{x \in X : \Phi(x) > a_i^1\}\) and strongly \(q\)-pseudo concave on \(B_{b_i^1}^1 = \{x \in X : \Phi(x) < b_i^1\}\) and \(a_i^1 < b_i^1 < b_i^1\).

Now we have

**Theorem 76** (A. Silva [46]) Let \(X\) be a strongly \(1\)-pseudo convex manifold. Then \(X\) admits a projective embedding if \(X\) carries a line bundle satisfying a positivity condition.

The corresponding result for complex spaces is as follows

**Theorem 75** (Andreotti - Siu) Let \(X\) be a normal strongly \(1\)-pseudo concave space \(X\) having all its non compact irreducible components of dimension \(\geq 3\). Then \(X\) admits a projective embedding if \(X\) carries a line bundle \(L\) satisfying a certain positivity condition.
Let us recall Grauert's definition of p-pseudo convexity.

Let $X$ be a complex space of bounded complex dimension $n$. A function $\psi$ on $X$ is said to be strongly p-pseudo convex if for each $x \in X$ there exist an analytic isomorphism $\tau$ of an open neighbourhood $V_x$ of $x$ onto an analytic subset of $U$ of $X$ with coordinates $(z_1, \ldots, z_n)$ and a real valued $C^2$-function $\varphi$ on $U$ such that $\varphi - \Re \psi \circ \tau$ and the hermitian matrix $(\partial^2 \varphi / \partial z_j \partial z_k)$ has at least $(n-p+1)$ positive eigenvalues at every point of $U$.

For $p = 1$ case we have the following:

**Theorem 77 (Grauert)**: Let $X$ be a strongly 1-convex space. Then there exists a certain Stein space $Y$, a finite set of points $\{y_1, \ldots, y_k\}$ of $Y$ and a proper subjective holomorphic map $f : X \to Y$ such that $U = \{y_1, \ldots, y_k\}$ is a maximal compact analytic subset of $X$ and $f : X - U \to Y - \{y_1, \ldots, y_k\}$ is an analytic isomorphism.

**Definition 78**: Let $L = (L, X, H^1)$ be a holomorphic line bundle. We say $L$ is negative if $\exists$ a hermitian metric $h$ along the fibers of $L$ such that the real valued function on $L$ given by the square length of a vector in that metric is differentiable and strongly 1-convex outside the zero section of $L$.

**Theorem 79** (Andreotti - Grauert, Ramis 1973): Let $X$ be a $(p, q)$-convex-concave space and $\mathcal{F}$ is a coherent sheaf on $X$. Then the $C$-vector spaces $H^r(X, \mathcal{F})$ have finite dimension for $p \leq r < \text{prof}(\mathcal{F}) - q - 1$. However, for strongly 1-convex spaces we have a stronger result, namely

**Theorem 80**: Let $(X, \mathcal{L})$ be a strongly 1-convex space, $L = (L, X, H^1)$ be a holomorphic line bundle such that $L | \mathcal{L}$ is positive. Then for each coherent sheaf $\mathcal{F}$ on $X$, there exists $k_0$ such that for each $k > k_0$, $H^r(X, \mathcal{F} \boxtimes (L^k)) = 0$ for $r \geq 1$ (and hence by a result of Lieberman - Rossi [27], $L$ is strongly 1-convex.)

**Definition 81**: Let $X$ be a strongly $(p, q)$-convex-concave space. Let $L = (L, X, H^1)$ be a holomorphic line bundle. We say $L$ is $(p, q)$-positive, if the zero section of $L$ has a trinular neighbourhood $T$ which is strongly $(p, q)$-convex-concave space.
Remark. In fact this notion is introduced by Grauert for vector bundles as weak negativity in a special case.

Then we have

**Theorem 62.** Let $X$ be a strongly $(p,q)$-convex-concave space and $L = \bigoplus_i L_i$ be a $(p,q)$-positive line bundle. Then for each coherent sheaf $\mathcal{F}$ on $X$ there exists $k_0 \in \mathbb{N}$ such that for each $k > k_0$, $H^r(X, \mathcal{F} \otimes (L^k)) = 0$ for $p \leq r \leq \text{prof}(\mathcal{F}) - q - 1$.

We now recall the definition of profondeur of a coherent sheaf.

Let $R$ be a noetherian ring and $M$ be $R$-module of finite type and let $\mathfrak{a}$ be an ideal in $R$. Then $f_1, \ldots, f_q \in M$ is called a regular $M$-sequence if $f_1$ is not a zero divisor of $M/\sum_{j=1}^{q-1} f_j M$ where $f_1 \neq 0$ and $f_j M = 0$.

**Definition.** The maximum length of regular $M$-sequence is called the profondeur of $M$ with respect to $\mathfrak{a}$, denoted by $\text{profondeur}(M)_{\mathfrak{a}}$.

Take $R = \mathcal{O}_X$ and $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Let $A \subset X$ be a subvariety and let $\mathfrak{a} = \mathcal{I}(A) \subset \mathcal{O}_X$ be the ideal sheaf defined by $A$. Then for each $x \in A$ define

$$\text{profondeur}_{A,x}(\mathcal{F}) = \begin{cases} \infty & \text{if } \mathcal{I}_x \subset \mathcal{I} \text{ is a regular } \mathcal{O}_{\mathcal{I}} \text{-sequence for } \mathcal{I} = \mathcal{I}(A) \text{ and } \mathfrak{a} \text{ modulo } \mathcal{I} \text{ is regular } \mathcal{O}_{\mathcal{I}} \text{-sequence for } \mathcal{I} = \mathcal{I}(A) \times \mathcal{O}_{\mathcal{I}} \text{ modulo } \mathcal{I} \text{ \footnote{This footnote will be automatically removed by the output.}}. \\
\text{prof } \mathcal{F} & \text{otherwise}
\end{cases}$$

Then define $\text{prof } \mathcal{F} = \inf_{x \in A} \{ \text{profondeur}_{A,x}(\mathcal{F}) \}$.

By taking $A = X$ we get the definition of $\text{prof } \mathcal{F}$.

Consider a resolution of $\mathcal{F}$ as a syzygy

$$0 \rightarrow \mathcal{O}_X \rightarrow \cdots \rightarrow \mathcal{P}_1 \rightarrow \mathcal{P}_0 \rightarrow \mathcal{F} \rightarrow 0$$

In view of Jordan-Holder theorem the notion of minimal length for such resolutions of $\mathcal{F}$ is independent of the resolutions of $\mathcal{F}$, called the homological dimension of $\mathcal{F}$ over $\mathcal{O}_X = \text{prof } (\mathcal{F})$.

For details refer Grothendieck [15].

Using this cohomology vanishing theorem we get

**Theorem 83.** Let $X$ be a normal $(1,1)$-convex-concave complex space and $L = \bigoplus_i L_i$ be a $(1,1)$-positive holomorphic line bundle. Suppose all the non-compact irreducible components of $X$ have dimension $> 3$. Then
X can be immersed into an analytic subset of $\mathbb{C}^n \times \mathbb{P}^1$.

**Remark** Except the $(1,1)$-mixed case, the embedding or immersion of general $(p,q)$-type complex space $X$ into an analytic subset of a suitable ambient space is an open problem.

This problem requires the following ingredients:

1) First understand a $p$-convex space $X$ or more generally a $(p,q)$-type space as a "good" morphism or as a decent fibration over a "well geometrically understood" space $Y$.

2) $X$ is called cohomologically $q$-convex of dimension $\dim_X^q < \infty$ for all $\mathcal{F} \in \text{coh}(X)$ and for all $v \geq q$.

3) Prove a cohomology vanishing theorem for such $X^p,q$.

4) Imbed or immerse such $X \rightarrow Y \times \mathbb{P}^1$ for suitable $Y$.

**Geometry of holomorphic convex spaces.**

First note that $1$-convex space is automatically a holomorphically convex space admitting a "good fibration".

**Definition.** Let $q \geq 1$ be an integer. A complex space $X$ is said to be cohomologically $q$-convex if $H^r(X,\mathcal{F}) = 0$ for all $\mathcal{F} \in \text{coh}(X)$ for $v \geq q$.

**Remark.** S. Nakano [31] studied cohomologically $1$-complete spaces (manifolds) which he called as weakly $1$-complete spaces and proved a vanishing theorem for them. By a known [30] sufficient condition for Moishezonness of a compact complex space cohomologically $1$-complete is a good notion. Moreover for $q = 1$, $1$-convexity and cohomologically $1$-convexity coincide by an earlier result (cf. [5]). Grauert in his studies of a strongly pseudoconvex spaces $X$ asserted the existence of a maximal compact analytic subset $\mathcal{A}$ of $X$ called the exceptional set of $X$.

In analogy we have

**Theorem.** (Vo Van Tan [57]) (a) If $X$ is cohomologically $q$-complete then $X$ has no compact analytic subvariety of complex dimension $\geq q$.

(b) If $X$ is cohomologically $q$-convex then $X$ admits only finite many compact irreducible components of complex dimension $\geq q$.

**Definition.** A compact analytic subvariety $S$ in $X$ is said to be $q$-maximal if (a) for any compact irreducible analytic subvariety $T \subseteq X$, with dimension $T > q$ then $T \cup S$

(b) $\dim \mathcal{E}_x^q S_x > 0$ for all $x \in S$.

In view of Grauert's result, it is natural to ask
Problem If $X$ is a cohomologically $q$-convex space then does $X$ admit a $q$-maximal compact analytic subvariety $S \subset X$?

Remarks
1) For $q = 1$ this is the case (Grauert 1962) and Narasimhan 1962.
2) For $q = n = \dim_X X$ this is o.k.
3) Note in the proof of Narasimhan, $X, 1$-convex implies $X$ is holomorphic convex and hence holomorphic convex is used to get the exceptional set $A$. The difficulty is for $q > 1$. It should be noted that in general, cohomologically $q$-convexity does not imply holomorphic convexity as $X = \mathbb{C}^2 - \{0\}$ is cohomologically 2-convex but is not holomorphic convex.

On the other hand it is not known given $q > 1$, whether a holomorphic convex space $X$ is cohomologically $q$-convex.

Now we consider the category of holomorphic convex spaces.

Problem Let $X$ be a holomorphic convex space. Suppose $X$ is cohomologically $q$-convex, does $X$ admit a $q$-maximal compact analytic subvariety $S$ with $1 < q = \dim_X X$?

Recall (a) a complex space $X$ is holomorphically convex if for any infinite discrete set $\mathcal{D} \subset X$, there exists a holomorphic function $f$ on $X$ for which $f(z)$ is unbounded.

b) Let $X$ be a holomorphic convex space. Then there exists a Remmert - Stein reduction $Y$ of $X$, that is, i) a Stein space $Y$ (ii) a proper surjective holomorphic map $\pi : X \to Y$ with all its fibers connected. (iii) $\Gamma(Y, \mathcal{O}_Y) \cong \Gamma(X, \mathcal{O}_X)$

Let $S = \{ x \in X \mid x$ is not isolated in its fiber $\pi^{-1}(\pi(x)) \}$ be the degenerate set. Then $S$ is analytic subvariety in $X$ (Remmert).

Examples a) Stein spaces, compact complex spaces are holomorphically convex spaces.

b) Let $X$ be a holomorphic convex space, $Y$ be its Remmert - Stein reduction and $S$ be its degenerate set.

   a) Let $X$ be the "blow up" of $\mathbb{C}^n$ at the origin. Then $Y = \mathbb{C}^n$ and $S = 0$ where $m = n - 1$ and $\pi : X \to Y$.

   b) Let $\mathcal{D}$ be an infinite discrete set in $\mathbb{C}^n$ and let $X$ be the non-ideal transform $q^n$ with centre $\mathcal{D}$. Then $Y = \mathbb{C}^n$ and $S = \mathbb{P}^{n-1}$.

   c) Let $\mathcal{D}$ be a submanifold of $\mathbb{C}^n$ with $r < n-1$ and let $X$ be the non-ideal transform of $\mathbb{C}^n$ with centre $\mathcal{D}$. Then $Y = \mathbb{C}^n$ and $S = \mathbb{C}^r \times \mathbb{P}^{m}$ with $m = n - r - 1$. 
v) Let $X = \mathbb{C}^n \times \mathbb{P}^m$. Then $Y = \mathbb{C}^n$ and $S = X \setminus \mathbb{C}^n \times \mathbb{F}_m$.

The structure of a holomorphic convex cohomologically $q$-convex space is given by

**Theorem 89** Let $X$ be a holomorphic convex space. Suppose $X$ is cohomologically $q$-convex then $X$ admits a $q$-maximal compact analytic subvariety $S$ with $1 < q < n = \text{dim}_X$.

The problem is to classify holomorphically convex spaces $X$.

That is, to assertian the algebraic properties (cohomological) of $X$ are completely characterized by the analytic geometric properties (the existence of compact analytic subvarieties).

In this connection, note that a cohomologically $q$-complete space $X$ admits no compact analytic subvariety of complex dimension $> q$.

On the other hand, the converse is false, namely,

Let $X = \mathbb{C}^2 \setminus \{0\}$ which has no compact analytic subvariety of complex dimension $> 1$, whereas $X$ is not cohomologically 1-complete since $\text{dim}_q H(X, \mathcal{O}_X) = \infty$.

However, in the presence of holomorphic convexity this is the case.

**Theorem 90** Let $X$ be a holomorphic convex space having no compact analytic subvariety of $C - \text{dim} ~ 1$. Then $X$ is cohomologically 1-complete.

More generally we have

**Theorem 91** Let $X$ be a holomorphic convex space having no compact analytic subvariety of $C - \text{dim} ~ q$. Then $X$ is cohomologically $q$-complete with $1 < q < n = \text{dim}_X$.

Similar results hold for cohomologically 1-convex (respectively $q$-convexity).

**Theorem 92** (H. Narasimhan) If $X$ is a holomorphic convex space admitting a 1-maximal compact analytic subvariety then $X$ is cohomologically 1-convex.

**Theorem 93** (Vo Van Tan) Let $X$ be a holomorphic convex space. Suppose $X$ admits a $q$-maximal compact analytic subvariety $S$ with $1 < q < n = \text{dim}_q X$. Then $X$ is cohomologically $q$-convex.

For $q = n$ we have

**Theorem 94** (Siu [51]) Let $X$ be a complex space with $\text{dim}_q X = n$. If $X$ admits an $n$-maximal compact analytic subvariety $S$ then $X$ is cohomologically $n$-convex. If $S = 0$ then $X$ is cohomologically $n$-complete.

Theorems 90–94 give the classification of holomorphic convex spaces.
Definition 95  Let $X$ be a complex space. By the cohomological dimension of $X$ we mean $\text{cd} (X) = \text{the smallest integer } v > -1$ such that $H^i (X, \mathcal{O}_X^*) = 0$ for all $i > v$ and all $\mathcal{O}_X^* \in \text{Coh} (X)$. By the finitude dimension of $X$ we mean $\text{fd} (X) = \text{the smallest } v > 0$ such that $\dim H^i (X, \mathcal{O}_X^*) < \infty$ for all $i > v$ and all $\mathcal{O}_X^* \in \text{Coh} (X)$.

Remark  For an irreducible analytic space $X$ of complex dimension $n$, $X$ is compact $\iff$ $\text{cd} (X) = n$ ($\iff$ $\text{fd} (X) = 0$).

Now we would like to discuss the relation between q-convexity and q-completeness for holomorphic convex spaces. In other words, let $X$ be analytic space having no compact analytic subvariety of $C$-dimension $> q$. (a) If $X$ is $q$-convex, is $X$ $q$-complete? (b) If $X$ is cohomologically $q$-convex, is $X$ cohomologically $q$-complete?

Note that the existence of compact analytic subvarieties of complex dimension $> q$, is an obstruction for going to q-completeness (respectively cohomologically q-completeness) from q-convexity (respectively cohomologically q-convexity).

For $q = 1$ the answer is affirmative (R. Narasimhan). However, for $q > 1$, the answer is in the negative.

Example 96  Let $X = \mathbb{P}^n - Y$ where $Y$ is any non-singular compact complex $2$-dimensional submanifold in $\mathbb{P}^n$. Then $X$ is $2$-convex, but by dimension argument $X$ cannot have any compact analytic subvarieties of $C$-dim $> 2$.

Suppose $X$ is cohomologically $2$-complete. Then by a result of Serron and Villani we have $H^i_0 (X, \mathcal{O}) = 0$. Then by Poincaré duality $H^2_c (X, \mathcal{O}) \cong H^0_0 (X, \mathcal{O}) = 0$ since $\dim X = 8$.

Then from the following exact sequence

$\cdots \rightarrow H^2 (\mathbb{P}^n, \mathcal{O}) \rightarrow H^2 (Y, \mathcal{O}) \rightarrow H^2_c (X, \mathcal{O}) \rightarrow \cdots$  

and so $H^1 (Y, \mathcal{O}) = 0$.

But we know there exist non-singular $Y$ in $\mathbb{P}^n$ with $b_2 (Y) \neq 0$.

However, in the presence of holomorphic convexity we have

Theorem 97  Let $X$ be a holomorphic convex space of complex dimension $n$. Suppose $X$ is cohomologically $q$-convex with its $q$-maximal compact analytic subvariety $S$ ($1 < q < n$).
If \( \dim X < p \) then \( X \) is cohomologically \( q \)-complete (for \( p \geq q \)).

**Remark** For \( q = n \), Theorem 94 gives a positive answer to our problem (b). For (a) even for holomorphic convex spaces, the difficulty lies in the singularity set of \( X \).

The relation between \( q \)-convexity and cohomologically \( q \)-convexity (respectively completeness) is given by the following.

**Theorem 98** (Andreotti - Grauert) Let \( X \) be an analytic space
(a) If \( X \) is \( q \)-complete then \( X \) is cohomologically \( q \)-complete
(b) If \( X \) is \( q \)-convex then \( X \) is cohomologically \( q \)-convex.

**Remarks 99**

1) We have the following
   Conjecture (Grauert 1959) Let \( X \) be an-dimensional non compact complex space, then \( X \) is \( q \)-convex for some \( q \leq n \).

2) for \( q = 1 \) the converse of theorem 98 holds (Serre, Narasimhan)

3) The problem is whether the converse of Theorem 98 holds.
   for \( q > 1 \) for (a) general complex spaces (b) holomorphic convex spaces.

4) We have the following special case.
   Let \( X \) be a 1-convex space and hence \( X \) is holomorphic convex.
   Let \( S \) be its 1-maximal compact analytic subvariety. Then \( X \) is cohomologically \( q \)-complete where \( q = \dim S + 1 \).

Note that if 3 (b) above has a positive answer then we have a nice characterization of holomorphic convex spaces.

i) by their function theoretic properties (\( q \)-convex functions)
ii) by their algebraic properties (sheaf cohomology)
iii) by their geometric properties (existence of higher dimensional compact analytic subvarieties).

5) Let \( X \) be a given \( C \)-analytic space with \( \dim X = n \)
   (a) If \( X \) has only finitely many compact \( n \)-irreducible components
       then is \( X \) \( n \)-convex ?
   (b) If \( X \) has no compact \( n \)-dimensional irreducible component
       then is \( X \) \( n \)-complete ?

Again for \( n = 1 \) 5) has positive answer.
14. Appendix on the Grauert - Riemenschneider Conjecture

The Grauert - Riemenschneider conjecture states

**Conjecture A.1** Let $X$ be a compact irreducible reduced complex space of complex dimension $n$ admitting an almost positive torsion free coherent sheaf of rank 1. Then $X$ is Moishezon.

**Remark A.2** Grauert stated this for quasi positive coherent sheaves.

This conjecture reduces to the following special case

**Conjecture A.3** Let $M$ be a compact complex manifold which admits a hermitian holomorphic line bundle $L$ whose curvature form $\Omega_L$ is positive definite on a dense subset $G$ of $M$. Then $M$ is Moishezon.

This is because by a theorem of Moishezon - Hironaka - Rossi there exists a desingularization $\tilde{X}$ of $X$ such that the pull back of the coherent sheaf $S$ to $\tilde{X}$ will be locally free.

We have the following special cases known:

(a) $M$ is Kaehler and the curvature form $\Omega_L$ of $L$ is positive at some point or dimension of $M$ is $2$.

(b) The set $A$ of points in $M$ where the curvature form fails to be positive in contained in an analytic subvariety of dimensions 0 or 1 and in higher dimensions suitable curvature conditions on a Kaehler neighbourhood $W$ of $A$ are needed [cf. 49].

(c) Siu [49] proved a cohomology vanishing theorem and hence the $L^1$ conjecture with extra conditions on the eigen values of $\Omega_L$.

(d) Schneider [44] proved this under the extra condition that $L$ is globally generated by sandwiching the algebraic dimension and Kodaira embedding dimension using Siegel's theorem.

Recently Siu proved a special case of this conjecture and succeeded in modifying this argument to the general case. (announced and yet to be written up). In what follows we attempt to outline the proof of this conjecture.

First we consider the following

**Theorem (Siu [49]) A.5** Let $M$ be a compact complex manifold and $L$ be a hermitian holomorphic line bundle over $M$ whose curvature form $\Omega_L$ is positive semidefinite every where and is strictly positive outside a set of measure zero. Then $M$ is Moishezon.

The basic idea of the proof is to produce sufficiently many holomorphic sections of the line bundle, $L$ and to obtain sufficiently many meromorphic functions on $M$ to make $M$ Moishezon.
For this we get a suitable Stein cover $\mathcal{B}_{2}^{n}$ for $M$ and then
Schwarz Lemma for $L^{2}$-sections of bundle $L^k$ and then we use Leray
isomorphism between the space of $L$-valued harmonic forms on $M$ and
the space of Ganguly with coefficients in $L$ to study the dimension
of the cohomology groups and the technical and crucial part of
the proof is the following estimation on the dimension of the
cohomology groups, namely

**Theorem A.6**  Let $M$ be a compact complex manifold of complex
dimension $n$ and $L$ be a hermitian holomorphic line bundle over $M$ such that
$
abla_{L}$ is positive semi-definite on $M$ and is strictly positive outside
a set of measure zero. Then for each $C > 0$, for each $q \in \mathbb{N}$ there
exists a positive integer $k_{1}$ depending on $C$ such that $\dim H^{q}(M, L^{k})$
\[ \leq \frac{C}{k} \text{ for } k > k_{1}. \]

We comment on the proof of this theorem later and first sketch the
proof of Theorem A.6.

**Proof:** We have $C_{1}(L)^{n} > 0$ since $\nabla_{L}$ is positive semi-definite
everywhere on $M$ and $\nabla_{L} > 0$ at some point and so by Riemann - Koch
theorem

\[ \begin{align*}
\tilde{v}^{n} & \quad (-1)^{q} \dim H^{q}(M, L^{k}) = \exp \left( k C_{1}(L)^{n} \right) \cdot \left[ M \right] \\
\end{align*} \]

where $\left[ M \right]$ is the fundamental class of $M$ and $r_{j}$ are the
class roots of $\mathcal{I}(M)$, we get

\[ \begin{align*}
\tilde{v}^{n} & \quad (-1)^{q} \dim H^{q}(M, L^{k}) \geq \frac{1}{4n} C_{2}(L)^{n} k^{n} \\
\end{align*} \]

In theorem A.7 take $\mathcal{E} = \frac{1}{4n} C_{2}(L)^{n} k^{n}$ for $k$ sufficiently large
and get $\dim H^{q}(M, L^{k}) \geq \frac{1}{4n} C_{2}(L)^{n} k^{n}$
for all $q \gg 1$ and $k$ sufficiently large and so
\[ \dim H^{q}(M, L^{k}) \geq \frac{C_{1}(L)^{n}}{4n} k^{n} \quad k >> 0 \quad (A.8) \]

Now the idea is to move away from the base locus to get a local
embedding of $M$ to make it Moishezon.
Let $Z_{K} = \{ z \in M \mid Y(z) = 0 \text{ for all } r \in \prod (M, L^{k}) \}$ for
each $k$ and let $Z = \bigcap_{k=1}^{\infty} Z_{K}$. Then $Z \cap K_{1} \in \mathbb{N}$ such that
$Z = Z_{K_{1}}$. Let $L = L^k_{\alpha}$. For a point $z_{\alpha} \in M - Z$. 

Note that for each $k$, $\nu < k < \infty$ we have the map $\phi_{k} : M - Z \to \mathbb{P}^{N}$. 

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defined by choosing a basis for $\mathfrak{g}(M, LK)$. Let $L_k$ denote the level set $L_k(Z_0)$ of $\Phi_k$ at $Z_0$ and let $\bigcap_{k=0}^{\infty} L_k(Z_0) = B_{Z_0}$ be called a level set of $\bigcap_{k=1}^{\infty} \mathfrak{g}(M, LK)$ at $Z_0$.

Let $d$ be the minimum of the complex dimensions of the branches of all the level sets of $\mathfrak{g}(M, LK)$ over $M-Z$.

Now we assume $M$ is not Moishezon and get a contradiction to (A.8).

If $M$ is not Moishezon, none of $\Phi_k$ is local embedding and hence $d > 0$. Over $M$ by a finite number of open unit balls $B_j$ ($1 \leq j \leq m$), meet in a coordinate patch, so that (i) $F$ is trivial on some open neighbourhood of $B_j$.

(ii) the centre $a_j$ of each $B_j$ is outside $Z$ and (i) each $E_j$ is a regular point of a level set $E_j$ of $\bigcup_{k=1}^{\infty} \mathfrak{g}(M, LK)$ where $\dim_{\mathbb{C}} E_j$ at $a_j$ is $d$.

So assume each $E_j$ is the level set of $\mathfrak{g}(M, LK)$. For every positive integer $k$, the rank of $\Phi_k$ at $a_j$ is maximum. Choose a positive number $r < 1$ such that $|B_j(a_j, r) \cap \xi|$ still cover $n$. Let $\| \cdot \|$ denote the pointwise norm of a section.

of $L_k$ there exists $c > 1$ independent of $k$ (obtained from the hermitian metric of $L_k B_j$) such that for every holomorphic section $s$ of $L_k$ over $B_j$ with $s(a_j) = 0$ with order $1$ we have

$$
\sup_{p \in B_j} \| s(p) \| \leq c^k r^J \sup_{q \in B_j} \| s(q) \| \quad (A.9)
$$

$$(B_j' \text{ is a suitable sub ball})$$

Let $h_k = \dim_{\mathbb{C}} \mathfrak{g}(M, LK)$.

Claim: $h_k \leq m \left( n-d + k \frac{\log c}{\log 1/r} + 1 \right)$.

Suppose this claim is false. Let $l = \left( \frac{k \log c}{\log 1/r} + 1 \right)$.

Then $h_k > m \left( n-d + l \right)$ (A.10) and $c^k r^J < 1$ (A.11).

Since the rank of $\Phi_k$ at each $a_j$ is equal to its maximum rank $n-d$ over $M-Z$, by (A.10) there exists a section $s \in \mathfrak{g}(M, LK)$ with $s \neq 0$ such that $s$ vanishes to order $1$ at each $a_j$ ($1 \leq j \leq m$).

Then using (A.9) and (A.11) we get

$$
\sup_{p \in \xi} \| s(p) \| < \sup_{p \in \xi} \| s(q) \|
$$

a contradiction. Hence our above claim is true.
Clearly this claim contradicts A.8.

Now we sketch the idea of proof of Theorem A.7.

Give \( M \) a hermitian metric and then represent elements of \( H^k(M, L^k) \) by \( L^k \)-valued harmonic forms on \( M \). Then using \( L^2 \)-estimates of \( \tilde{\phi} \) we get a linear map from the space of harmonic forms to the space of cocycles with coefficients or values in \( L^2 \). This is the usual Leray type isomorphism:

\[ \phi : L^2 \rightarrow \tilde{\phi} \].

Then we use the familiar basic idea of a function of some order vanishing at sufficiently many points to a suitably high order must be the zero function only. For this purpose take a lattice of points \( \Lambda \) with distance \( \frac{1}{4k} \) in a small neighbourhood \( \mathcal{W} \) of \( A = M - \Omega \). Note that we have measure of \( A \) is zero. Then using the Bochner-Kodaira technique for a compact hermitian manifold and the Schwarz Lemma we can show that any cocycle coming from a harmonic form via the above linear map, vanishes at all points of the lattice \( \Lambda \) to a suitable higher order \( l \) (fixed) must vanish identically and hence the dimension of \( H^k(M, L^k) \) is dominated by a fixed constant times the number of lattice points which is proportional to \( \text{Vol}\mathcal{W} \times k^n \). But measure of \( M - \Omega \) is zero and so we can make \( \text{Vol}\mathcal{W} \) as small as we want and so choose \( \delta > 0 \) smaller than any prescribed positive number by taking \( k \) sufficiently large. This completes the outline of proof of A.7.

Now we come to the general case. We have

**Theorem A.** (G.R conjecture, strong version (Siu))

Let \( M \) be a compact complex manifold of complex dimension \( n \) and \( L \) be a hermitian holomorphic line bundle over \( M \) such that its curvature form \( \omega_L \) is positive semi-definite everywhere and positive definite at least at one point. Then \( M \) is Moishezon.

**Remark.** This clearly implies the Grauert-Niemenschneider conjecture.

Now we outline the proof of strong version.

Let \( C = \{ p \in M \mid \omega_L(p) \text{ is positive definite} \} \) clearly \( C \) nonempty by hypothesis.

Let \( \lambda > 0 \) be arbitrary.

Let \( \mathcal{K} = \{ p \in M \mid \omega_L(p) = \delta \phi \text{ has its smallest positive eigen value does not exceed } \lambda \} \), where \( \phi \) is the plurisubharmonic function-log \( h \). Clearly \( \mathcal{K} \) contains \( A = M - G \).

The idea is to introduce a lattice suitably slicing in coordinate directions and modify the above argument.
Let 0 be any point of $\mathcal{H}$. Choose a coordinate polydisk $D$ with coordinates $z_1, \ldots, z_n$ centered at 0 and choose a global trivialization of $L$ over $D$ such that for some constant $c > 0$:

$$c \sum_{i=2}^{n} |z_i(p_1) - z_i(p_2)|^2 \leq |\phi(p_1) - \phi(p_2)|$$

for $p_1, p_2 \in D$.

Moreover, both $c$ and the polyradius of $D$ can be chosen to be the same for all points $0$ of $\mathcal{H}$. Cover $\mathcal{H}$ by a finite number of such coordinate polydisks so that for some constant $m$ depending only on $n$, no more than $m$ of them intersect. Then introduce a lattice $L$ so that they are $\frac{1}{\sqrt{m}}$ apart along the $z_1$ direction and are $\frac{1}{\sqrt{n}}$ apart along the directions $z_2, \ldots, z_n$. Now the total number of lattice points is no more than a constant times $\text{Vol}(\mathcal{H}) \cdot n \cdot k^n$. Now choose $\lambda$ sufficiently small and so for any given $\varepsilon > 0$ and for $q > 1$ we get as before

$$\dim H^q(M, L^k) \leq C \varepsilon^k$$

for each $q \geq 1$ for $k \gg 0$ and so $\dim H^0(M, L^k) \geq C k^n$ for $k \gg 0$, and so we are back to the special case.

Now we give a non-compact version of G-R-Conjecture.

Recall the definition of a strongly pseudo convex domain.

A relatively compact domain $\mathcal{O}$ with smooth boundary in a complex manifold $\mathcal{M}$ is strongly pseudo convex if $\mathcal{O}$ is defined by a some $C^\infty$ function $\psi$ by $\psi < 0$ near its boundary, with a non-zero gradient such that the complex Hessian of $\psi$ as a hermitian form is positive definite.

**Theorem (Grauer)** Let $\mathcal{M}$ be a compact complex manifold, $L$ be a hermitian holomorphic line bundle. The set

$$\mathcal{O} = \{ v \in L \mid \|v\| < 1 \}$$

is strongly pseudo convex if and only if $L$ is positive.

**G-R Conjecture (Non Compact version)** Let $\mathcal{O}$ be a relatively compact open subset of a complex manifold such that the boundary $\partial \mathcal{O}$ of $\mathcal{O}$ is a strongly pseudo convex at every point and is strictly pseudo convex at some point $p_0$. Then there exists a holomorphic function $f$ on $\mathcal{O}$ such that $f \to \infty$ along some sequence $(z_n) \subset \mathcal{O}$ as $z_n \to p_0$ as $n \to \infty$.

We have the following:

**The Eigenvalue Conjecture** Let $\mathcal{M}$ be a compact complex manifold and $L$ a hermitian holomorphic line bundle over $\mathcal{M}$ with $\mathcal{O}$.

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positive semi-definite everywhere and positive definite at some point. Then \( \lambda \left( K^1 L^k \right) > 0 \) where \( \lambda (M, L^k) \) is the smallest positive eigenvalue of the Laplacian \( \nabla^2 \) on the Hilbert space of all global \( L^2 \) sections of \( L^k \) over \( M \). This conjecture will definitely be true because the positivity of \( L^k \) increases with \( k \) and the function \( \lambda (\cdot, L^k) \) increases with more positivity of \( L^k \).

One can show that the eigenvalue conjecture implies the G-R conjecture strong version.

Remarks 1) The method of above proof gives the following cohomology vanishing theorem.

**Theorem** Let \( M \) be a compact complex manifold of complex dimension \( n \) and \( L \) be a hermitian holomorphic line bundle over \( M \) with \( L^k \) positive semi-definite everywhere and positive outside a set of measure zero. Then \( H^q (M, L^k) = 0 \) for \( q > 1 \).

2) The method of proof also gives a method of proving the existence of holomorphic sections of a holomorphic line bundle \( L^k \) over \( M \) which has curvature of mixed sign. However this method has its limitations in view of the application of the Hirzebruch - Riemann - Roch theorem.

This completes the solution of the conjecture of Siu which will soon be written up by him.

Thus having understood the Geometry of Moishezon spaces and more generally of \( q \)-convex spaces \( X \), it is natural to study the distinction of complex space being Moishezon or algebraic. As pointed earlier in this article for 2-dimensional complex manifolds no distinction between Moishezonness and algebraicity. However for compact complex spaces \( X \) even in dimension 2 they are distinct and also in case of manifolds of complex dimension 3. These famous examples of Grauert - Riemann - schneider and Hironaka - Moishezon will be discussed in a subsequent article in which we study the obstruction for a Moishezon space to be algebraic and its relation to the singularity theory.

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