NATURALLY ORDERED ABUNDANT SEMIGROUPS

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**ABSTRACT**

An investigation of ordered abundant semi-groups $S$ is made on which $E^*$ and $L^*$ are regular and the order of $S$ extends the natural order on the set $E$ of idempotents of $S$, where $E$ generates a regular subsemi-group. Structure theorems are obtained when $S$ contains a greatest idempotent $u$ such that $uSu$ is a type A monoid and when $S$ has a multiplicative type A transversal such that every 6-class in $S$ contains a maximum element.

**Introduction**

It is well known that on any semigroup $S$, the set of idempotents $E$ is partially ordered by $\preceq$ where:

\[ e \preceq f \text{ if and only if } ef = e = fe \quad (e, f \in E) \]

This order is called the natural order on $E$. A partial order $\preceq$ on $S$ is a natural partial order if it extends $\preceq$ in the sense that

\[ e \preceq f \implies e \preceq f \quad (e, f \in E) \]

Recently, Mitsch [19] formulated a natural partial order on any semigroup $S$. It has been noted (see, i.e. [17] or [20]) that this order is not, in general, compatible with multiplication on $S$.

A naturally partially ordered semigroup $(S, \preceq)$ is naturally ordered if the order $\preceq$ is compatible on both sides with multiplication on $S$. Most of the results in the naturally partially ordered semigroups were obtained by concentrating on naturally ordered semigroups. The several classes of regular semigroups investigated by Blyth [1, 2, 3, 4] are shown [17] to be naturally ordered and contains an idempotent $u$ maximum with respect to the imposed order. The class of regular semigroups $S$ which possess a normal medial idempotent studied by Blyth and McFadden [7] is the class of naturally ordered regular semigroups with a maximum idempotent [18]. Blyth and McFadden [5] characterized the naturally ordered regular semigroups which contain a greatest idempotent. So there is substantial literature on the naturally ordered regular semigroups.

Naturally ordered abundant semigroups were first investigated by Lawson [16] where he proved for a particular natural order relation on an idempotent-connected abundant semigroup $S$ whose set of idempotents generates a regular subsemigroup to be compatible if and only if $S$ is locally type A. The subject of this paper lies in this field. The main objective is to extend Blyth and McFadden's results [5, 6] on certain classes of naturally ordered regular semigroups to certain classes of naturally ordered abundant semigroups. It has been achieved by a similar approach as that one in the regular case. The main result of section 2 is to prove that $S$ is a naturally ordered abundant semigroup whose set of idempotents generates a regular subsemigroup with a greatest idempotent $u$ and the $E^*$ and $L^*$ are regular on $S$, then $S$ is isomorphic to a spined overlap product of $Su$ and $Us$. In section 3 we impose on $S$ the condition that $uSu$ to be type A to characterize in this case $uS$ (similarly $Su$) as a quasi-direct product of a naturally ordered band with a greatest element that is a left (similarly right) identity and a naturally ordered type A monoid, from which we deduce the structure of $S$. In the final section we describe the structure of a class of naturally ordered abundant semigroups with a multiplicative type A transversal.
1. Preliminaries

We will use, whenever possible, the notations of [13] without comment. For the definition of type A semigroups, we refer the reader to [12] and to [15] for the other undefined terms.

From [5] and [17] we recall the following corollary.

**COROLLARY 1.1** If \( S \) is a naturally ordered semigroup with set \( E \) of idempotents and has a greatest idempotent \( u \), then (for all \( e \in E \)),

\[ eue = e. \]

Let \( S \) be a naturally ordered abundant semigroup whose set \( E \) of idempotents generates a regular subsemigroup \( \langle E \rangle \) with a greatest idempotent \( u \). As a consequence of corollary 1.1, (for all \( x \in E \)),

\[ x = xx^* = xx^*ux^*xux^* \quad \text{and} \quad x = x^*ux \]

for any \( x^* \), \( x^+ \), that is,

**COROLLARY 1.2** (For all \( x \in S \)), \( xx^* = xx^*ux^*xux^* \).

**LEMMA 1.3** (For all \( x \in S \)); \( x L^* ux \) and \( x R^* xu \).

**Proof:** Since for any \( s, t \in S \),

\[ uxs = uxt \Rightarrow x^* uxs = x^* uxt \Rightarrow xs = xt \Rightarrow uxs = uxt \]

Then \( x L^* ux \). Similarly \( x R^* xu \)

It follows from Lemma 1.3 that (for all \( x, y \in S \)) \( x L^* uy \) implies \( ux L^* uy \) as well as \( x R^* yu \) implies \( xu R^* yu \). In particular we have:

**COROLLARY 1.4** (For all \( x \in S \)), \( ux L^* ux \) and \( x^* u R^* xu \) for any \( x^* \), \( x^+ \).

It is immediate from Lemma 1.3 and Corollary 1.4 to have:

**COROLLARY 1.5** (For all \( x \in S \)), \( ux L^* x \) and \( x^* u R^* x \)

Now since \( \langle E \rangle \) is a regular subsemigroup and thus the regular elements of \( S \) generate a regular subsemigroup [14, Result 2]. We conclude from [5] the following result.

**THEOREM 1.6** If \( e \) and \( f \) are elements in the subsemigroup \( \langle E \rangle \), then

(i) \( e \) is the greatest idempotent of \( L^e \)

(ii) \( e \) is if and only if \( u = ue \)

(iii) \( e \) is the greatest idempotent of \( R^e \)

(iv) \( f \) is if and only if \( f = eu \).

Moreover, we have from [5],

**COROLLARY 1.7** (i) \( eue = e \) for any \( e \in \langle E \rangle \).

(ii) \( \langle E \rangle uE = uE \) is a normal subband.

(iii) \( uE = uE \) is a normal subband.

(iv) \( uE = uE \) is a semilattice.

It follows from theorem 1.6 (or [17, Lemma 3.1]) that every \( L^e \)-class \( (R^e \)-class) in \( S \) contains a unique idempotent from \( uE(uE) \). Since for any idempotent \( e \) in \( E \), \( eSe \) is an abundant subsemigroup of \( S \) [10]. Then we conclude that \( uSu \) is an adequate subsemigroup of \( S \) whose semilattice of idempotents is \( uE \).
2. A Spined Overlap Product

In what follows \((S, \preceq)\) denotes a naturally ordered abundant semigroup whose set \(E\) of idempotents generates a regular subsemigroup \(\langle E \rangle\) with a greatest idempotent \(u\).

**PROPOSITION 2.1** The cartesian product \(S_u \times uS\) with a multiplication defined by the rule;

\[
(xu, ua)(yu, ub) = (xuayu, uayub*);
\]

is a semigroup.

**Proof:** Since for any elements \(a\) and \(b\) of \(S; \) a \(\preceq u\) \(b \preceq ub\) (Lemma 1.3), then;

\[
ua \preceq ub \Rightarrow ua \preceq ub\;
\]

and the multiplication is well defined.

To verify the associativity, we note for any \(a, y, b, c\) in \(S\) that \(a \preceq ay\), which implies \(ay \preceq ayu\), and thus \(ayu \preceq (ayu)\preceq b\).

Therefore \((ayu)\preceq (ayu)\preceq b\) in \(\langle E \rangle\). Hence, \(u(ayu)\preceq u(ayu)\preceq b\). Now since \(u(ayu)\preceq u(ayu)\preceq yu\), then we get

\[
xuayu(ayu)\preceq zu = xuayu(ayu)\preceq b = xuayub\preceq zu
\]

and the first components of the two triple products

\[
[(xu, ua)(yu, ub)][(zu, uc)] = (xuayu, uayub*)(zu, uc)
\]

and

\[
(xu, ua)((yu, ub)(zu, uc)) = (xu, ua)(yu, ub)zu = (xuayu, uayub*zu)
\]

coincide.

By a similar argument, the second components coincide, and the associativity holds.

**PROPOSITION 2.2** The following conditions are equivalent:

(1) \(S_u \times uS\) is an ordered semigroup under the cartesian ordering.

(2) For all \(a, b\) in \(S\); \(a \preceq b\) implies \(a*u \preceq ub\) for any \(a, b\).

(3) For all \(a, b\) in \(S\); \(a \preceq b\) implies \(a*u \preceq b\) for some \(a, b\).

**Proof:** We show that (1) \(\leftrightarrow\) (2) \(\leftrightarrow\) (3).

(1) \(\leftrightarrow\) (2); If (1) holds, and \(a \preceq b\) in \(S\), then in particular;

\[
(u, ua) \preceq (u, ub)
\]

and;

\[
(u, ua) = (u, u)(u, ub) \preceq (u, ub)
\]

which implies \(ua \preceq ub\) and (2) holds.

Since \(ua \preceq u\) \(a \preceq ub\) (Lemma 1.3), then if (2) holds, we get \(ua \preceq ub\) implies \(ua \preceq ub\) and consequently (1) follows.

(2) \(\leftrightarrow\) (3); Since a \(\preceq ub\) and b \(\preceq ub\) for any \(a, b\) in \(S\) (corollary 1.5), then immediately (3) follows from (2).

If (3) holds and \(a \preceq b\) in \(S\), then \(a \preceq ub\) for some \(a, b\) thus \(ua \preceq ub\) and now (2) holds by theorem 1.6.

If the semigroup \(S\) satisfies one (hence all) of the conditions of proposition 2.2, then the relation \(L^*\) is said to be regular on \(S\).

When \(S_u \times uS\) is equipped with the multiplication described in proposition 2.1, we use the terminology of [5] to call the resulting semigroup: \(L^*\)-overlap product of \(S_u\) and \(uS\). The dual of \(L^*\)-overlap product is \(M^*-overlap\) product of \(S_u\) and \(uS\) whose multiplication is defined by the rule:

\[
(xu, ua)(yu, ub) = (xuayu, uayub^+);
\]

It is an ordered semigroup with respect to the cartesian ordering if and only if for all \(x\) and \(y\) in \(S; \) \(x \preceq y\) implies \(xu \preceq yu\) for any \(x, y\) and the dual of proposition 2.2 holds. In this case, the relation \(M^*\) is said to be regular on \(S\).

Consider now the subset of \(S_u \times uS\) given by:

\[
S_u \mid \times \mid uS = \{ xu, ux; \ x \in S \}
\]

**PROPOSITION 2.3** If \(L^*(M^*)\) is regular on \(S\), then \(S_u \mid \times \mid uS\) is an ordered subsemigroup of the \(L^*(M^*)\)-overlap product of \(S_u\) and \(uS\).

**Proof:** For the \(L^*\)-overlap product, we have;

\[
(xu, ux)(yu, uy) = (xuayu, uayuy^*) = (xyu, xyu)\]

By corollary 1.2

For the \(M^*\)-overlap product, we have;

\[
(xu, ux)(yu, uy) = (xu, uxu, uxyuy^*) = (xu, uxy, uxyu)\]

By corollary 1.2

We note from the proof of proposition 2.3 that products in \(S_u \mid \times \mid uS\) are the same whichever multiplication is chosen on \(S_u \times uS\). For this reason we follow [5] to call \(S_u \mid \times \mid uS\) a spined overlap product of \(S_u\) and \(uS\). Now establish the following theorem:

**THEOREM 2.4** If \(L^*\) and \(M^*\) are regular on \(S\), then \(S\) is isomorphic to \(S_u \mid \times \mid uS\).

**Proof:** Let \(L^*\) and \(M^*\) be regular on \(S\) and define;

\[
\theta: S \rightarrow S_u \mid \times \mid uS \text{ by } \theta(a)(xu, ux)(x, u) \in S
\]

Clearly, \(\theta\) is an isomorphism surjective map and if \(xu \preceq yu\) for some \(x, y\) in \(S\), then \(xu \preceq yu\) which implies by the regularity of \(M^*\) and corollary 1.4 that \(x \preceq uy\). Similarly we have \(ux \preceq uy\) thus by the regularity of \(L^*; \) \(ux \preceq uy\).

Therefore:

\[
x \preceq uy\]

and \(y\) is an order isomorphism. It is easy to see that \(\theta\) is a semigroup homomorphism. Hence it is an isomorphism of ordered semigroups.
Let us assume that $L_*$ and $R_*$ are regular on $S$. Theorem 2.4 shows that the structure of $S$ is determined by that of the subsemigroups $S_u$ and $uS$. We now discuss $uS$, a similar discussion holds for $S_u$.

Since for any element $x$ of $S$, $uxL*ux$ and $uxR*ux$, we can see that $uS$ is an abundant subsemigroup of $S$ with a greatest idempotent $u$ which is also a left identity. Moreover, $uS$ is quasi-adequate, for its set of idempotents is $uS^+u$ and this is a subsemigroup of $uS$. In fact, $uS$ is a normal band (corollary 1.7) and thus the equivalence relation $\Delta$ defined on $uS$ by the rule:

$$(a,b) \in \Delta \text{ if and only if } a = ubf \text{ for some } e \in E(b^+) \text{, } f \in E(b^-)$$

is a congruence and consequently, is the minimum adequate good congruence on $uS$. We refer the reader to [11] for further details.

Now consider the natural homomorphism $\pi: uS \to uS/\Delta$ and define $\pi: uS \to uS$ by $(x, y) \in \Delta \mapsto xu, (xu, y)$ for any $x, y$ in $S$:

$$\pi(x, y) = xu = yu \text{ for some } e \in E(y), f \in E(y^*).$$

Then $\pi$ is a map.

From the fact that $u$ is a left identity in $uS$, it follows that $\pi$ is a homomorphism. Moreover, $\pi$ is good (see [10] for definition), to see this, let $x, y$ be in $uS$. Then we have:

$$x \Delta y \mapsto xu = yu \mapsto xu = yu \text{ (applying } \pi)$$

and

$$x \Delta y \mapsto xu = yu \mapsto xu = yu \text{ (applying } \pi)$$

Hence $\pi$ is good.

Now, since for any $x$ in $uS$;

$x = xu^*$ and $x^*E(xu^*) = E((xu)^*)$

then $(x, xu) \in \Delta$ and $\pi(x, xu) = xu$.

Thus, $uS$ is a split quasi-adequate semigroup as defined in [8]. Notice that:

$\pi = \{x, xu \in uS\}$

which is an adequate semigroup whose semilattice of idempotents is $UE = uS^*$, say. If $uS$ is a type A semigroup then for any element $a$ of $uS$, there is a bijection:

$$\alpha: aE^* \to aE^*$$

such that $ea = a(e\alpha_a)$ for any $e \in E^*$.

$(a, a\alpha_a)$ (see [10] or [12]). Take $\theta$ to be the restriction of $\alpha$ to $a\theta(a)$ and extend $\theta$ to $\theta: uS \to uS$ by defining $e\theta_a$ for any $e \in E^*$.
3. A Quasi-Direct Product

The main objective of this section is to extend the structure given in [5] for the naturally ordered orthodox semigroups with a greatest idempotent that is a left identity to the quasi-adequate case. For this objective we need the following theorem.

THEOREM 3.1: Let R be a naturally ordered band with a greatest element u that is a left identity and let E be the structure semilattice of R. Then R is a strong semilattice (EIRJ,;fey) of right zero semigroups R's each with a greatest element; every f is isometric and preserves these greatest elements.

Retain the hypotheses of theorem 3.1. It has been noted that xu is the greatest element in R for any x in R. The set of the greatest elements of R's is a semilattice isomorphic to E. For notation convenience we identify these isomorphic semilattices by E. Let S be a naturally ordered type A monoid whose semilattice of idempotents coincides with the structure semilattice E of R.

PROPOSITION 3.2: There is a unique homomorphism \( \varphi \) from S into Hom (R,E) described by \( x \mapsto x^u \) satisfying the following condition: (A) for every element y in S, the restriction of \( \varphi \) to \( R^\lambda \) maps every element of \( R^\lambda \) to the greatest element of \( R \) (A). Moreover, \( \varphi \) is isotone if and only if \( \varphi \) is isotone.

Proof: Define \( \varphi : R \to E \) by \( \varphi (a) = au \) for any \( a \) in R. It is clear that \( \varphi \) is a homomorphism sending each element \( a \) in \( R^\lambda \) to the greatest element of \( R \). For any \( x \) in R, define \( \varphi \) by \( \varphi (a) = (aux)^u = (ax)^u \). Clearly, \( \varphi \) is a map. Also for any \( au, bu \) in \( E \),

\[
(au, bu) \mapsto (aux, baux)^u = (ax)^u
\]

and \( \varphi \in \text{Hom}(E, E) \). By theorem 3.1, \( \varphi \) is isotone. Put \( \varphi \) in \( \text{Hom}(E, E) \), and \( \varphi \) is isotone. Moreover, for any \( y \) in S and \( x \) in \( R \),

\[
t \varphi \varphi (ty) = \varphi \varphi (ty)^u = (xy)^u = (xy)^u
\]

and \( \varphi \varphi \) is the greatest element of \( R \). Hence \( \varphi \) satisfies condition (A). The uniqueness of \( \varphi \) follows from the fact that if \( \varphi : S \to \text{Hom}(E, E) \) is a homomorphism such that \( \varphi \) satisfies condition (A), then for any \( y \) in S and \( x \) in \( E \), we have by condition (A) that;

\[
\varphi (xy) = \varphi(y^u) = y^u
\]

Suppose now that \( \varphi \) is isotone and \( x \leq y \) in S, then;

\[
(ux)^u = (uy)^u \leq (ux)^u
\]

But \( u \neq x, x \neq y \) (lemma 1.1), therefore \( (ux)^u \leq (uy)^u \) and \( (ux)^u \neq (uy)^u \), and since every two \( L \)-related idempotents in \( R \) are equal (theorem 1.8(iii)).

\[xu = (ux)xu \leq (uy)xu \]

which implies \( xu \neq yx \) and \( xu \neq xu \). Since \( x \leq yx \), then \( xu \neq xu \) and \( xu \) is regular on \( S \) by proportion 2.2.

Conversely, if \( x \neq xu \) then \( x \neq xy \) in \( S \), then for any \( y \) in \( S \), \( xu \neq xy \) in \( S \). Thus \( (ux)^u \neq (xy)^u \) and for any \( t \in R \),

\[
t \varphi (txy) = t \varphi (xy) = t \varphi (xy)
\]

that is, \( \varphi \) is isotone. Hence \( \varphi \) is isotone.

We now proceed to establish a construction for naturally ordered quasi-adequate semigroups \( Q \) with a greatest idempotent \( u \) that is a left identity. Let \( R \) be a naturally ordered type A monoid whose semilattice of idempotents coincides with the structure semilattice \( E \) of \( R \).

Let \( Q = \{ (R, S) \in \text{Hom}(R, E) \} \) and endo \( Q \) with the cartesian order and law of composition;

\[
(a, x) \circ (b, y) = (a \varphi (b), xy)
\]

where \( \varphi \) is as defined in the proof of proposition 3.2. Since for any \( x \) and \( y \) in \( S \); \( x \varphi (y) = x \varphi (xy) \). Then by condition (A) we have for any \( (a, x) \) and \( (b, y) \) in \( Q \); and the multiplication in \( Q \) is well-defined. Due to the fact that each \( \varphi \) is a homomorphism, it is a straightforward matter to verify that \( Q \) is a semigroup.

The following sequence of results provides considerably more information about \( Q \).

LEMMA 3.3: The set of idempotents of \( Q \) forms a band isomorphic to \( R \).

Proof: Since for any element \( (a, x) \) in \( Q \);

\[
(a, x) \circ (a, x) = (a \varphi (a), x) = (x \varphi (a), x)
\]

then clearly \( (a, x) \) is an idempotent in \( Q \). If and only if \( x \) is an idempotent in \( S \). Therefore, the set of idempotents of \( Q \) is \( \{ (a, x) \in Q : x \varphi (a) = x \} \) which is obviously a band. Now consider the map from \( R \) into the band of \( Q \) which is described by \( s \to (s, su) \). It is readily an injective map and for any idempotent \( (a, x) \) in \( Q \), \( x \varphi (a) = x \), that is \( x \) is an idempotent in \( S \). Therefore, the set of idempotents of \( Q \) is \( \{ (a, x) \in Q : x \varphi (a) = x \} \) and since \( a \neq x, x \neq y \) in \( E \), \( x \neq y \) and hence \( s \to (s, su) \to (a, su) \to (a, x) \). Therefore the described
map is a bijection and thus it is an isomorphism for:

\[(a, au) \cdot (b, bu) = ((abu) \cdot ub, abu) \]

and for any elements \((b, y), (c, z)\) in Q we have:

\[(a, x) \cdot (b, y) = (a, x\cdot y) = (b, y \cdot x) = (c, z \cdot x)\]

Then \((a, x) \cdot (x', x) = (a, x' \cdot x) = (x' \cdot x, u, x) = (a, x)\)

and for any elements \((b, y), (c, z)\) in Q we have:

\[(b, y) \cdot (x, z) = (c, z) \cdot (a, x) = (b, y) \cdot (x, z) = (b, y) \cdot (x, z) = (c, z) \cdot (a, x)\]

Then \((x', x) \in (a, x)\), that is, every L*-class in Q contains an idempotent. For an idempotent in each L*-class, take an element \((a, x)\) in Q and consider the idempotent \((x', x)\) in Q. Since

\[(x', x') = (x' \cdot x, u, x) = (x' \cdot x, u, x) = (x' \cdot x, u, x) = (x' \cdot x, u, x)\]

Then \((x', x') \in (a, x)\), hence Q is an abundant semigroup.

Proof: Let \((a, x)\) be an element of Q and consider the idempotent \((a, x)\) in Q. Since

\[(a, x) \cdot (x', x) = (x', x) \cdot (a, x) = (a, x) = (a, x)\]

and for any elements \((b, y), (c, z)\) in Q we have:

\[(a, x) \cdot (b, y) = (a, x \cdot y) = (b, y \cdot x) = (c, z \cdot x)\]

Then \((a, x) \cdot (x', x) = (a, x' \cdot x) = (x' \cdot x, u, x) = (a, x)\)

and for any elements \((b, y), (c, z)\) in Q we have:

\[(b, y) \cdot (x, z) = (c, z) \cdot (a, x) = (b, y) \cdot (x, z) = (b, y) \cdot (x, z) = (c, z) \cdot (a, x)\]

Then \((x', x') \in (a, x)\), hence Q is an abundant semigroup.

The following corollary is an immediate consequence of Lemma 3.3 and Lemma 3.4

COROLLARY 3.5 Q is a quasi-adequate semigroup.

It follows from the natural order on R and S, and from the fact the \(L^*\) is isomorphism for the left compatibility partial order on Q. For the left compatibility we need \(L^*\) to be isomorphism, that is, \(L^*\) to be regular on S. In this case, Q is a compatible ordered quasi-adequate semigroup. To see that order \(L^*\) is in fact also a natural order, let \((a, e)\) and \((b, f)\) be idempotent in Q such that

\[(a, e) \cdot (b, f) = (a, e) = (b, f) \cdot (a, e)\]

which is equivalent to

\[(a, e) \cdot (b, f) = (a, e) = (b, f) \cdot (a, e)\]

whence \(e \cdot f = f \cdot e\), that is, \(e \cdot f = f \cdot e\) since \(e \cdot f = f \cdot e\) in R. Therefore \(e \cdot f = f \cdot e\). Hence \((a, e) \leq (b, f)\) if and only if \(e \cdot f = f \cdot e\). Since \(e \cdot f = f \cdot e\) if and only if \(e \cdot f = f \cdot e\), then \((a, e) \leq (b, f)\) if and only if \(e \cdot f = f \cdot e\). But \(e \cdot f = f \cdot e\) if and only if \(e \cdot f = f \cdot e\). Hence \((a, e) \leq (b, f)\) if and only if \(e \cdot f = f \cdot e\)

Moreover, if I is the identity element of S and i is the greatest element in R (i = I in the identification of E), then \((i, I)\) is the greatest element in Q. We note that for any element \((a, x)\) in Q:

\[(a, x) \cdot (b, f) = (a, x) = (b, f) \cdot (a, x)\]

Then \((a, x) \cdot (x', x) = (a, x) = (x', x) \cdot (a, x)\)

and for any elements \((b, y), (c, z)\) in Q we have:

\[(b, y) \cdot (x, z) = (c, z) \cdot (a, x) = (b, y) \cdot (x, z) = (b, y) \cdot (x, z) = (c, z) \cdot (a, x)\]

Then \((x', x') \in (a, x)\), hence Q is an abundant semigroup.

Hence Q is an abundant semigroup.

The following corollary is an immediate consequence of Lemma 3.3 and Lemma 3.4

COROLLARY 3.5 Q is a quasi-adequate semigroup.
((a,x),(b,y)) \in \mathcal{S} if and only if x = y.

Proof: Let (a,x) and (b,y) be elements of Q such that ((a,x),
(b,y)) \in \mathcal{S}, that is, (a,x) = (V,e)(b,y)(w,f) where (V,e)E((b,y)-
E((y,*) and (w,f)E((b,y)*)E((y,*) (see the proof of Lemma 3.4).
Therefore eE((y,*), fE((y,*). But S is adequate. Hence e = y, f = y.
Now the second component of the triple multiplication is ef = y = y = y.

Conversely, let (a,x), (b,y) be elements in Q such that x = y.

Then in particular R = R* and

(b,y) (a,x*) = (b, yx*) = (b, y)

= ((yx*)ux,y)
= yx*a,x
= (a,x)

It is routine to check that (a,x*)E((b,y)) = E((b,y)) and thus ((a,x), (b,y)) \in \mathcal{S}.

As a direct consequence of Lemma 3.7 we have:

COROLLARY 3.8 \mathcal{S} is a congruence on Q.

Now it follows from corollary 3.8 and [11, proposition 2.6] that \mathcal{S} is the minimum adequate congruence on Q. Define \mathcal{Q}/\mathcal{S} by (a,x)\equiv(b,y)iff (a,x)(b,y) \in \mathcal{S} and thus ((a,x), (b,y)) \in \mathcal{S}.

Moreover, for any (a,x) in Q

(a,x)(1,1) = (a \equiv 1, 1, 1) \equiv (x, 1)

Therefore;

\mathcal{Q}(1,1) = \{(x, 1, x) \in S \}

Clearly, \mathcal{Q}(1,1) is isomorphic to S and thus we have the following result:

PROPOSITION 3.9 Any two of the three semigroups; \mathcal{Q}/\mathcal{S}, S and \mathcal{Q}(1,1) are isomorphic.

Summing up we have the following theorem:

THEOREM 3.10 Let R be a naturally ordered band with a greatest element u that is a left identity. Let S be a naturally ordered type A monoid whose semilattice of idempotents coincides with the structure semilattice E of R and that E* is regular on S. Then Q = Q(R, S) is a naturally ordered quasi-adequate semigroup with a greatest idempotent (1,1) that is a left identity and E* is regular on Q such that (1,1)Q is a type A monoid isomorphic to S. If we add to the hypothesis of theorem 3.10 that S is also inverse, then Q(R, S) is an orthodox semigroup as in [5], but if S is not inverse such as a cancellative monoid which is not a group, then Q(R, S) is no longer a regular semigroup.

There is, of course a dual result to theorem 3.10 which can be stated as follows:

THEOREM 3.11 Let L be a naturally ordered band with a greatest element u that is a right identity. Let S be a naturally ordered type A monoid whose semilattice of idempotents coincides with the structure semilattice E of L and that E* is regular on S. Let \mathcal{Q} = Q(L, S) be the set of \mathcal{L} \times \{1, x \} x \{1, y \} x \{1, z \} and endo Q with the cartesian order and the composition;

(a,x)(b,y) = (a \times b) x \{1, y \}

Then Q is a naturally ordered quasi-adequate semigroup with a greatest idempotent (1,1) that is a right identity and E* is regular on Q such that (1,1)Q is a type A monoid isomorphic to S.

We call - as in [5] - Q(R, S) is the quasi-direct product of R and S and Q(L, S) is the quasi direct product of L and S. For a converse of theorem 3.10 we now show that every naturally quasi-adequate semigroup S with a greatest idempotent u that is a left identity such that Su is a type A monoid is algebraically such a quasi-direct product.

THEOREM 3.12 Let S be a naturally ordered quasi-adequate semigroup with a greatest idempotent u that is a left identity such that Su is type A. Then if E is the band of idempotents of S, the mapping \cdot \cdot \cdot from S into Q(E, Su) described by x* = (x*, xu) is an algebraic isomorphism which is an order isomorphism if and only if E* is regular on S.

Proof: It is clear that Su is a naturally ordered type A monoid whose semilattice of idempotents is Eu. Moreover, E* \subset E is a semilattice structure of right zero semigroups whose semilattice is Eu (theorem 3.1). We can therefore construct the quasi-direct product Q = Q(E, Su) and (x*, x) for any x in S; x* \in E* x u and xu \in (x*, x) \in Su. By recognizing the left identity u, it is easy to see that \cdot \cdot \cdot is a map. Moreover, if x and y are in S such that (x*, xu) = (y*, yu), then x* = x = y and x \in E is one-to-one. To see also \cdot \cdot \cdot is a homomorphism, let x, y be in S and notice that;

\cdot \cdot \cdot x y u = (x*, xu) (y*, yu)
= (x* y, yuy)
= (xuyuy)
= ((xuyuy)*)
= ((xuyuy)*)
= ((xuyuy)*)
= ((xuyuy)*)
= ((xuyuy)*)
= ((xuyuy)*)
= ((xuyuy)*)

For surjectivity, let (a,x) be in Q(E, Su), that is, a \in E \times R*. Then au = x* = x* (x* E Eu). Whence;
Hence $\phi$ is an algebraic isomorphism.

Finally, we note that $\omega$ is described by $(a,x) \mapsto xa$ and so $\phi$ is clearly isotone. Thus $\Omega$ is an order isomorphism if and only if $\phi$ is isotone which is clearly the case if and only if $L^*$ is regular on $S$.

The dual of theorem 3.12 which is a converse of theorem 3.11 can be stated as follows:

**Theorem 3.13** Let $S$ be a naturally ordered quasi-adequate semigroup with a greatest idempotent $u$ that is a right identity such that $uS$ is a type A monoid. Then if $E$ is the band of idempotents of $S$, the mapping $\Omega: S \rightarrow Q(E,uS)$ described by $x \mapsto (x,ux)$ is an algebraic isomorphism which is an order isomorphism if and only if $L^*$ is regular on $S$.

In conclusion, let $S$ be a naturally ordered abundant semigroup whose set $E$ of idempotents generates a regular subsemigroup with a greatest idempotent $u$ such that $uS$ is a type A monoid and that $L^*$ and $E^*$ are regular on $S$. Then, by theorem 3.4, $S$ is isomorphic to the spined overlap product $Su|x|uS$. As a direct application of theorem 3.13, $Su$ is isomorphic to the quasi-direct product $Q(E,uS)$ and from theorem 3.12 we have $uS$ is isomorphic to the quasi-direct product $Q(E,uS)$. Thus, by theorems 3.11, 3.10, the following theorem is evident.

**Theorem 3.14** $S$ is isomorphic to a spined overlap product $Q_x|Q_y$ where $Q_x(Q(L,T))$ is a quasi-direct product of an ordered left normal band $L$ with a greatest element and a naturally ordered type A monoid $T$. $Q_y(Q(R,T))$ is a quasi-direct product of an ordered right normal band $R$ with a greatest element and a naturally ordered type A monoid $T$.

There is a structure theorem for the class of regular semigroups with a multiplicative inverse transversal (see [6]) which has been extended to the class of abundant semigroups in the following form:

**Theorem 4.1** Let $(E,S,x)$ be an idempotent-generated regular semigroup with a multiplicative semilattice transversal $E^*$ and let $S$ be a type A semigroup whose semilattice of idempotents is (isomorphic to) $E^*$. Then

$$W = W(E,S,x) = \{(h, a, b) \in E \times E ; g(a, b) \mapsto (h, a, b) \}$$

with the multiplication

$$(g, a, b)(v, b, w) = (g \ast (h v), a \ast b, h \ast b)$$

is an abundant semigroup whose set of idempotents generates a regular subsemigroup and $W$ contains a multiplicative type A transversal.

Conversely, any such semigroup can be constructed in this way.

We refer the reader to [9] for the notation and further details.

The ordered analogue of the regular structure theorem is given in (6, section 6) by describing any naturally ordered regular semigroup $S$ satisfying the following condition: Every element in $S$ has a maximum inverse, $\Omega$ and $\Lambda$ are regular on $S$ and $S$ contains a multiplicative inverse transversal whose semilattice of idempotents is the semilattice transversal of $E(S)$.

The aim of this section is to extend this description to the abundant case by giving the ordered analogue of theorem 4.1.

To proceed towards this aim, suppose $(E,S)$ is a naturally ordered idempotent-generated regular semigroup in which each element $x$ contains a maximum element $x^*$. Here as in [9] two elements $x$ and $y$ are $\Omega$-related if and only if there are idempotents $e$ and $f$ such that

$$x = eyf; \text{ where } e \in y^*, f \in \Omega^* \text{ for some } y^*, y.$$ Suppose further that $E^* = \{x^*, x \in (E,S)\}$ is a multiplicative skeleton of $(E,S)$ (see [6]). Let $S$ be a naturally ordered type A semigroup whose semilattice of idempotents is $E^*$. Then we have the abundant semigroup $W'(E,S,x)$ whose set $E(W')$ of idempotents generates a regular subsemigroup $E(W')$ and $W$ contains a multiplicative type A transversal

$$W' = \{(x^*, x^*) ; x \in S\}$$

which is isomorphic to $S$ whose set of idempotents is the semilattice.
In order to find out more about the structure of $W$ in the order case, it seems necessary to impose condition (s) which relate the order structure of $W$ more closely to the algebraic structure of $W$. It has been done by imposing the condition that $R_*$ and $L_*$ are regular on $S$ in the sense that $a \leq b$ in $S$ implies $a^* \leq b$ and $a^* > b$ respectively. This coincides with the definition of section 2. In general, for an ordered abundant semigroup $V$ in which for every element $x$ in $V$, $x \leq a^x = \max\{a \in S | a \leq x\}$, we have that $a^* \leq b$ and $a^* > b$. Hence $L_*$ is regular on $S$.

To see how $W$ can be effected by imposing the condition that $R_*$ and $L_*$ are regular on $S$, we need to define the order $\leq$ on $T$ as in [6] by the rule:

$$(\forall a, b \in E^*) : a \leq b \iff (\forall x \in E^*) : x \leq a^x \leq b^x$$

**LEMMA 4.2** The following conditions are equivalent:

(1) $L_*(M^*)$ is regular on $S$,

(2) For all $a, b, c, d \in S$, $a \leq b$ implies $a \leq c \leq d \leq b$.

**Proof:** Suppose $L_*$ is regular on $S$ and $a \leq b$ in $S$. Then for each $x$ in $E^*$, $a \leq b$ and by the regularity of $L^*$,

$$x_a = (x_a)_{b} = (x_a a)^* = (x_b b)^* = (x_b b)^* = x_b$$

Hence $a \leq b$.

Similarly, for any $a, b \in S$, we have $a \leq b$ as a result of the regularity of $M^*$.

Conversely, suppose that $\forall x \in E^*$ whenever $a \leq b$. Then $a \leq b$ in $S$ implies $a \leq b$, in particular, $a \leq b \iff a \leq b$. But $a \leq b$ is $\iff (a \leq b)^*$. Therefore, $(a \leq b)^* \leq b$. It follows, since $S$ is naturally ordered, that $(a \leq b)^* \leq a$ and $a \leq b$. Hence $L_*^*$ is regular on $S$.

By a similar argument: $a \leq b \iff a \leq b$ whenever $a \leq b$. In $S$, leads to the regularity of $L^*$ on $S$.

**THEOREM 4.3** If $L^*$ and $R^*$ are regular on $S$, then $W \leq (E, S, \alpha)$ is, with the cartesian ordering, a naturally ordered abundant semigroup in which every $\alpha$-class contains a maximum element and such that both $L^*$ and $R^*$ are regular on $W$.

**Proof:** Let $(g, a, h), (v, b, w)$ be in $W$ with $(g, a, h) \leq (v, b, w)$. Given $(x, c, y) \in W$. Notice that:

$$(g, a, h)(x, c, y) = (g, h)(x, c, y)_{a} = axc, (hx)_{b, y}$$

$$(v, b, w)(x, c, y) = (v, w)(x, c, y)_{b} = bxw, (wx)^{y}_{b, c}$$

Since $g < v, h < w$, then $h w \leq x w$ and thus, since $a \leq b$, by the regularity of $L^*$ and $R^*$ and Lemma 4.2 we get:

$$g(hx)_{a} \leq (v, w)_{b} \leq (v w)^{y}_{a, b}$$

Hence $(g, a, h) < (v, b, w)$ in $W$. It follows from [9, proposition 3.5] that $W \leq (E, S, \alpha)$ is a semilattice.

**THEOREM 4.4** If $L^*$ and $R^*$ are regular on $S$, then $W \leq (E, S, \alpha)$ is a naturally ordered abundant semigroup in which every $\alpha$-class contains a maximum element and such that both $L^*$ and $R^*$ are regular on $W$.

**Proof:** Let $(g, a, h), (v, b, w)$ be in $W$ with $(g, a, h) \leq (v, b, w)$. Given $(x, c, y) \in W$. Notice that:

$$(g, a, h)(x, c, y) = (g, h)(x, c, y)_{a} = axc, (hx)_{b, y}$$

$$(v, b, w)(x, c, y) = (v, w)(x, c, y)_{b} = bxw, (wx)^{y}_{b, c}$$

Since $g < v, h < w$, then $h w \leq x w$ and thus, since $a \leq b$, by the regularity of $L^*$ and $R^*$ and Lemma 4.2 we get:

$$g(hx)_{a} \leq (v, w)_{b} \leq (v w)^{y}_{a, b}$$

Hence $(g, a, h) < (v, b, w)$ in $W$. It follows from [9, proposition 3.5] that $W \leq (E, S, \alpha)$ is a semilattice. In fact, $(a, a, a)$ is the maximum element in $(a, a, a)$. For, if the element $(v, b, w)$ in $W$ such that $(v, b, w) \leq (a, a, a)$, that is, $(v, b, w) \leq (a, a, a)$, then $(v, b, w) \leq (a, a, a)$, that is, $(v, b, w) \leq (a, a, a)$. Also, from the fact that $v < b$, we get:

$$v \leq b \leq v$$

Similarly, $h \leq w$ so $W$ is naturally ordered.

For any element $(g, a, h) \in W$, we know from [9] that $(a, a, a)$ is in $W$ and $(g, a, h) \leq (a, a, a)$. In fact, $(a, a, a)$ is the maximum element in $(a, a, a)$. For, if the element $(v, b, w)$ in $W$ such that $(v, b, w) \leq (a, a, a)$, that is, $(v, b, w) \leq (a, a, a)$, then $(v, b, w) \leq (a, a, a)$, that is, $(v, b, w) \leq (a, a, a)$. Also, from the fact that $v < b$, we get:

$$v \leq b \leq v$$

Similarly, $h \leq w$ so $W$ is naturally ordered.

From [9, Lemma 2.3] and the proof of proposition 3.3 in [9], we may have $(a, a, a)^* > (a, a, a)$ and $(a, a, a)^* > (a, a, a)$ where $(e, f, g)$ and $(i, k, j)$ are in $[8, w])$. That is, $c \leq e^*$ such that $(e, f, g)(a, a, a)$ and $(i, k, j)(a, a, a)*$. Now by the characterization of $L^*$ and $R^*$ in $[9]$, it follows from $(e, f, g)(a, a, a)$ and $(i, k, j)(a, a, a)*$. Similarly, $L^*$. Also we have; $e \leq * \iff * \leq e$ which implies $e^* = e$ and $j \leq * \iff * \leq j$ which implies $j^* = j$. We can see that:

$$(v, b, w) = (e(t x)^{y}_{a, b})(e(t a)^{y}_{a, b})(v, b, w)$$

$$(e(t a)^{y}_{a, b})(v, b, w)$$

The proof continues with further conditions and implications, but the main points include the natural order and the regularity conditions on $L^*$ and $R^*$.
Notice that \( v = e \) and \( e^*a^* \) which implies \((e, a^*)f \) where \( a^* \in E^* \). Therefore \( v \leq e \). Similarly \( w \leq e \).

Hence

\[(v, b, w) \leq (e, a^*), (v, a, b), (e, a^*, b) \]

Now recall for any \((g, a, h)\) in \( W \), that

\[(g, a, h) = (g, a^*, a^*)(a^*, a^*, h) \]

that is, \( e(g, a, h) = (g, a^*, a^*) \), \( f(g, a, h) = (a^*, a^*, h) \).

If \((g, a, h) < (v, b, w) \) in \( W \); that is, \( e \leq e \), \( a \leq b \), \( h \leq w \), then by the regularity of \( R^* \) and \( L^* \) on \( S \), we get \( a \leq b \) and \( a^* \leq b^* \), that is

\[ e(g, a, h) = (v, b, w) \quad \text{and} \quad f(g, a, h) = (v, b, w) \]

It remains now to prove the converse of theorem 4.3 which is the ordered analogue of the converse part of theorem 4.1.

**THEOREM 4.4** Let \( S \) be a naturally ordered abundant semigroup with set \( E \) of idempotents generates a regular subsemigroup \( \langle E \rangle \) and for any \( x \in S \); \( x \leq x \) contains a maximum element \( x^* \). Let \( R^* \) and \( L^* \) be regular on \( S \). Suppose that \( S^* = \{ x^* : x \in S \} \) is a multiplicative type \( A \) transversal of \( S \) whose set of idempotents \( E^* \) is the corresponding semilattice transversal of \( \langle E \rangle \). Then there is an ordered semigroup isomorphism from \( S \) onto \( W = W(E, S^*, x^*) \).

**Proof:** As in the proof of [9, theorem 4.2], the mapping \( \theta : S \rightarrow W(W(E, S^*, x^*)) \) given by \( x \rightarrow (e(x, x^*), f(x)) \) for each \( x \in S \) is an algebraic isomorphism. Its inverse \( \theta^{-1} \) is given by \( (g, a, h) = (g, a^* \oplus a \oplus a^*) \). Since \( W \) has the cartesian ordering, \( \theta^{-1} \) is obviously isotone. To show that \( \theta \) is isotone, suppose that \( x \leq y \) in \( S \). Let \( x = e(x, x^*), y = e(y, y^*) \); then \( e \leq e \) and \( f \leq f \) (\( R^* \) and \( L^* \) are regular on \( S \)). From \( x = e(x, x^*) \) and \( f = f(x, x^*) \), we have:

\[ x^* = x^* x^* \leq x^* x^* = x^* x^* = x^* x^* \leq x^* x^* = x^* x^* \leq x^* x^* \quad (x \leq y) \]

Since

\[ x^* y^* = y^* x^* = x^* y^* \]

and \( S \) is naturally ordered, then \( x^* y^* \leq y^* x^* \leq y^* x^* \).

Hence \((e, x, f) \leq (e, y, f)\)

and \( \theta \) is isotone.

**ACKNOWLEDGMENTS**

The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency, and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.


