COMBINING RELATIVITY AND QUANTUM MECHANICS:

SCHRÖDINGER'S INTERPRETATION OF $\phi$

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The incongruence between quantum theory and relativity theory is traced to the probability interpretation of the former. The classical continuum interpretation of \( \psi \) removes the difficulty. How quantum properties of matter and light, and in particular the radiative problems, like spontaneous emission and Lamb shift, may be accounted in a first quantized Maxwell-Dirac system is discussed.

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can only be done approximately for fairly heavy clocks and measuring rods. This would seem to restrict special relativity to the macroscopic domain, which is very unsatisfactory; moreover, special relativity does prove valid when applied to atoms and even to elementary particles, for instance, in considering the Doppler-effect of atoms or the time scale of decay of highly energetic mesons passing through the atmosphere (in the latter case the relativistic change of the time of decay is not a small effect, it means a factor of the order of 50 to 100).

Secondly, the current probability-interpretation of quantum mechanics consists in a prescription for computing the probability of finding a certain state of affairs at a given time. The said prescription has to make use (by integration over space) of the values of a certain function of the co-ordinates and the time at that time, i.e. of the simultaneous values of this function. In another Lorentz-frame - since the Lorentz transformation involves the co-ordinates and the time - two things happen; first the whole store of probability questions that the prescription answers is changed, because what is, e.g. a given region at a given time in one frame is no longer a given region at a given time in the new frame; secondly the store of 'simultaneous' values of that function (of the co-ordinates and the time) that the prescription uses for computing the probabilities is a different one in the new frame, different not just by a local transformation, but because the space-like cross-section (of the space-time manifold) on which the said function has to be considered is entirely different in the new frame - as different as two planes intersecting each other at a finite angle. This makes one doubtful whether the two sets of answers to the two sets of probability-questions are always fully equivalent, as they ought, of course, to be. However, I know that much thought has been devoted to this question in the last years, and also that I am not too well informed about it.

Dirac’s relativistic wave equation still stands out as the great success that has been scored in this whole subject. I believe that for Dirac’s equation the “probabilities of location” are consistent in all frames, and this as a consequence of the “equation of continuity” between the four quantities that correspond to the four-current and the spin, in any frame, be interpreted as probability of sojourn within an element of volume, and probabilities of passage through certain elements of surface within an element of time.”

Both problems Schrödinger mentions stem directly from the probability interpretation of quantum mechanics. There is an immediate asymmetry between the four-dimensionality of Minkowski geometry and Lorentz transformations on the one hand, and the three-dimensionality of the elements of the Hilbert space - the states of the system at a fixed time, on the other hand. The states are square integrable functions over a space-like surface at a given time \(t\). The probabilities are also always referred to a fixed time; we never add probability amplitudes at different times. Lorentz transformations produce one solution \(\psi(x,t)\) another solution \(\psi'(x',t')\), but they do not map states into states. There is no definite \(t\) such that states at \(t\) are obtained from those of at time \(t'\), independent of the evolution of the system, because \(t'\) is a function of \(x\). One has to make a new rearrangement, a new Hilbert space, reduction of wave packets, etc. with \(t'\) fixed [3]. Putting it another way, special relativity treats space and time on the same footing, quantum mechanics separates them; \(t\) is a parameter, \(x\) dynamic variables.

There would be no problem at all if the matter field \(\psi(x,t)\) is viewed as a classical continuous material medium or field, with \(|\psi|^2\) interpreted as a distribution of matter, or \(e|\psi|^2\) as a density of charge, as the square of the amplitude of classical wave is interpreted as intensity. This was envisaged originally by de Broglie and Schrödinger. We have then a deterministic evolution of the field hence can relate the values of the field \(\psi(x,t)\) for two different observers. Classical field theories are clearly relativistic. In quantum field theory too, we begin writing down covariant equations, Lagrangians, etc. as though they were classical, and then use the perturbative quantization afterwards.

For free particles there is a rigorous way of combining relativity with the Hilbert space formulation of quantum theory if we work in momentum space and do not ask about the measurement of position. This is done by the unitary irreducible representations of the Poincaré group in a Hilbert space, The states are labelled by momentum and spin, and transformed into a combination of such states for a different Lorentz observer. In fact, tests of relativity in particle physics always takes place in a Minkowski momentum space for the eigenvalues of energy and momentum operators. Note that in momentum space both energy and momentum are commuting operators, whereas in coordinate space time is a parameter but \(x\) an operator. The S-matrix theory also operates in the momentum space of a set of asymptotically free particles and correspondingly most experiments are done on asymptotic free particles. The description of localization of particles, i.e. their states in the coordinate space is not as straightforward. Schrödinger again was one of the first to study the localization property of the Dirac electron and introduced the notion of "zitterbewegung" showing an intricate internal structure of the electron associated with spin [4].
The space-time description is used to write down local interaction at each point \( x \) in a relativistic way. But then for interacting systems the reconciliation of quantum theory and relativity becomes difficult as can be seen from the fact that no finite closed theory has been devised so far incorporating (i) local interactions, (ii) relativity, and (iii) second quantized matter and electromagnetic fields (or other fields) with positive probabilities and unitarity. The open-ended perturbation theory makes use again of free particles in the intermediate states which we understand better.

II. REMARK ON THE INTRODUCTION OF A HILBERT SPACE OF STATES AND RELATIVITY

We have seen that the definition of a Hilbert space of states as function of space at a fixed time (or over a space-like surface) was one of the incongruences between quantum mechanics and relativity. The normalization of probabilities are also made at a fixed time. I shall now indicate a different procedure showing that the usual method is not as unique as one may think from the point of view of relativity. The probability space as a fixed time implies that we have instantaneous knowledge of probabilities at all distances, and we add up all these probabilities equal to one.

\[
\int dx^3 |\psi(x,t)|^2 = 1
\]

\( t = \text{fixed} \)

In operational relativity the knowledge of probability of particle being at a distance \( x \) comes to us with a signal velocity \( c \). Then \(|\psi(x,t - r/c)|^2\) represents the probability that the particle is at \( x \) as measured or seen at the origin. We may then equate to one the sum of measured probabilities for particles being at different locations, and obtain, instead of (1),

\[
\int dx |\psi(x,t - r/c)|^2 = 1
\]

\( t = \text{fixed} \)

This means that we consider wave functions on the backward light cone (instead of a space-like surface) and must use a measure on the light cone \( d\omega = 8\pi(1-x^2)^{-1/2}dx \). To my knowledge a quantum theory based on the normalization (2) has not been developed.

III. QUANTUM PROPERTIES IN A CLASSICAL FIELD THEORY

If we give \( \psi(x,t) \) a continuous medium interpretation, hence reconcile relativity and wave-mechanics, then our original dilemma appears in a new form: can we account for all the discrete, quantal properties of interactions of light and matter by a continuous classical field description?

Let us emphasize again that the coupled Maxwell-Dirac equations as classical field equations satisfy all the principles of relativity. They can be derived from an invariant action principle; the conservation laws are covariant and consistent with the equations of motion. Matter field and electromagnetic field exchange energy, act on each other in a self-consistent way; only the total energy-momentum tensor is conserved. These today very familiar notions were first formulated for wave mechanics by Gordon [5] and Schrödinger [6]. In such a system the Planck's constant \( \hbar \) enters into the Dirac (or Schrödinger) equation, hence we have the basic quantum relations \( E = \hbar \omega, \quad p = \hbar \mathbf{k} \), but the fields are not second quantized. The Cauchy problem is well-posed.

As far as discrete stationary states of bound systems (neglecting self-energies — see below) are concerned, the first quantized wave equation gives a complete answer. The discrete stationary solutions \( \psi_n \) with energies \( E_n \) are independent of the probability interpretation of the theory. One can also view \( E_n \) as the characteristic frequencies of an oscillating charge distribution. This kind of discreteness is familiar in classical physics. With the introduction of self-energies, however, this type of discreteness becomes approximate; in reality there are no sharp discrete levels of an atom.

Why did then the classical description fail and has been largely abandoned?

It is essentially due to its apparent inability to describe radiative processes, in particular spontaneous emission, Lamb shift, anomalous magnetic moment, and to account for pointlike and particle-like properties of electrons and photons (discrete counts). Further relativistic problems include pair production, vacuum polarization. Finally there is the problem of interpretation of many body configuration space wave function \( \psi(x_1,\ldots,x_n) \) from the point of view of classical fields. I shall now discuss some of the progress achieved in the solution of these problems.
IV. RADIATIVE PROCESSES IN CLASSICAL MAXWELL-DIRAC EQUATIONS

It is probably not well-known that Schrödinger tried to calculate what we now call the "Lamb-shift" already in 1926, just after his four papers on wave mechanics, at the end of his paper quoted above [6]. From the coupled Maxwell-Dirac (or Klein-Gordon) in the paper of Schrödinger) system it follows that the wave equation for the bound electron, for example, is not

\[ \frac{\partial^2 \psi}{\partial t^2} = -\frac{\hbar^2}{2m} \Delta \psi \]

but must be augmented by nonlinear radiative self-energy terms \( R(\psi) \)

\[ \frac{\partial^2 \psi}{\partial t^2} = -\frac{\hbar^2}{2m} \Delta \psi + R(\psi) \]

Classical radiation theory has such nonlinear self energy terms. The self energy terms are at first omitted in the standard quantum electrodynamics and then reintroduced via a quantized radiation field, photon by photon. One therefore works in the Hilbert space framework of the linear theory, Eq. (3).

The nonlinear equation (4) is obtained by eliminating \( A(x) \) from the coupled Maxwell-Dirac equations and has the explicit form

\[ \psi(x,t) = \sum_n \psi_n(x) e^{-iE_n t} \]

where \( \psi_n(x) \) and \( E_n \) have still to be determined. The insertion of (6) into (5) gives the set of equations

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If the self-energy terms on the right hand side of (7) are small, we see that the Fourier coefficients \( \psi_n(\vec{x}) \) are approximately the stationary wave functions of the atom. We have shown elsewhere in detail how to obtain all the standard QED-radiative results from this nonlinear Eq.(7) by iteration, such as Lamb-shift [7]-[8], spontaneous emission [9], vacuum polarization [10], anomalous magnetic moment, including the effects of cavities [11]. The normalization of \( \psi_n \) may be determined from the fact that the atom as a whole is neutral. In fact when properly regularized [10] the normalization of the nonlinear equation should be automatically determined. In these calculations the Hilbert space formalism is not used. In fact, the superposition principle does not strictly hold for Eq.(5), only approximatively. Moreover, we do not have strictly speaking exact discrete stationary states for a problem like H-atom with self energy except for the ground state. And this experimental fact is incorporated into the theory from the beginning. Furthermore, the atom responds to external radiation by the absorption and emission of resonance radiation which also follows from Eq.(5). All these calculations are in agreement with relativity.

The quantized nature of the radiation (photon) is reduced in the present approach to the quantized nature of the source, the atom. Depending on the nature of the source \( J_p \) and on the nature of the environment (expressed in its Green's function) we obtain different types of radiation with different statistical counting properties as observed experimentally. By properly adjusting the source one can produce coherent light, squeezed light, etc. of various degree.

For matter wave amplitude which historically caused considerable confusion, I shall use the symbol \( \psi \) for the probability amplitude.
The complete sharp discreteness of quantum theory is an idealisation. When all effects are included the result of observations are never discrete, but bumps embedded in a continuum.

V. PLANE WAVES VERSUS LOCALIZED INDIVIDUAL PARTICLES

We have seen that we can visualize the electron in an atom as a fairly localised (but not pointlike) charge distribution. How do we describe the free electron in a scattering experiment?

The free particle wave function has a plane wave solution which is usually attributed to a single particle. But this plane wave is surely not the matter distribution \( \psi \), but the probability amplitude \( \psi \), introduced above. It describes the behaviour of a single electron in repeated experiments with the usual interpretation and with the notion of a collapse of a wave function to a dot on the screen when one wants to apply it to a single event.

Plane wave description of particles can be used to account for statistical behaviour of repeated events and for regularities in repeated events, such as interference pattern. There is no need for a collapse of wave function as long as we do not wish to make any statement about any single event.

How can we describe a single dot on a screen representing a single event coming from a cathode a falling on a screen? It cannot be a wave packet, because wave packets do spread. Thus we must go beyond the standard Hilbert space framework of quantum mechanics to describe a single event. I believe a number of paradoxes and problems in the foundation of quantum theory are connected with the fact that we make a mental picture about what a single event should be. This includes the EPR-problem, for example.

We would like to suggest that single events should be described by the soliton-like solutions of the nonlinear Eq.(5) for \( \psi \), and not by the plane-wave solution \( \psi \) of the linear wave equation (the left hand side of (5)). Nonlinear equations are known to have solutions of the type [13]

\[
\psi = e^{i(ax-\omega t)} F(x) \tag{8}
\]

where \( F(x) \) is a localized function coming from the nonlinearity and the de Broglie phase is a solution of the linear equation. It is important to emphasize that the nonlinearity we are talking about is not an ad-hoc added nonlinearity, but one that is necessarily present due to the self-field of the charge particle.

VI. ANTIPARTICLES IN FIRST QUANTIZED DIRAC FIELD

We have seen how some of the quantized properties of electrons and photons can be accounted for by a classical matter field and by replacing the electromagnetic field by its matter sources. It remains the question of antiparticles and whether we have to quantized the matter field for elementary processes - for genuinely many body problems the second quantization is a very useful method.

Antiparticles first occurred after the Dirac equation, although it could have been introduced, as Dirac points out, right after special relativity, namely from the negative sign in front of the energy equation \( E = \sqrt{p^2 + m^2} \). Negative energy states in first quantized theory have to be interpreted with care, otherwise it leads to some unusual semantics, as "infinitely filled negative energy states" of the hole theory. There is however a consistent way of treating the antiparticles as positive energy states in a first quantized framework. This is the description in which electrons and positrons are two different states of the Dirac particle distinguished by the eigenvalue \( \mp \) of an internal quantum number \( n \), but both of positive energy. And such an internal quantum number is provided by the Dirac algebra [14]: The negative energy solutions of the equation \( (\gamma \cdot p - m)\psi = 0 \) are identical with the positive energy solution of the mass conjugate equation \( (\gamma \cdot p - m)\psi = 0, n = \pm 1 \). In other words, the factorization of the Klein-Gordon equation gives us two equations \( (\gamma \cdot p - m)\psi = 0 \) and we should consider both.

Now in the presence of an external field, the first equation is

\[
(\gamma(p-eA)-m)\psi = 0 \tag{9}
\]

The negative energy-momentum solutions of this equation corresponding to \( p = p, e = -e \) are the same as the positive energy momentum solutions of the mass conjugate equation also coupled minimally,

\[
(\gamma(p+eA)+m)\psi = 0 \tag{10}
\]
We may operate formally with negative energy solutions of (9), but the physical interpretation is given by (10). Thus the superselection rule between electrons and positrons is accounted for as they come from different Eqs. (9) and (10). In the relativistic Coulomb problem for example, it can be seen that the negative energy solutions of the electron are the positive energy solution of the equation with opposite mass and charge, hence can be interpreted as the scattering states of the positron in the Coulomb field.

This situation is described much more clearly in the 5-dimensional invariant formulation [15] of the Dirac equation:

\[ i\gamma^\mu \partial_\mu \psi(x,s) + \frac{1}{2m} \psi(x,s) = \alpha \]

(11)

where \( s \) is an invariant time. The two solutions \( \psi_1 \) and \( \psi_2 \) above correspond to the two "frequencies" in the invariant time coordinate

\[ \psi(x,s) = e^{+i\alpha} \psi_1(x). \]

(12)

In this formulation we can also understand the notion of pair-production or pair annihilation. A trajectory in the 5-dimensional \((x^\mu, s)\)-space can easily be constructed in such a way that its projection on the Minkowski space corresponds to a particle coming from infinity turning and going back to infinity. Although the curve in 5-dimensions is monotonic, its projection in the 4-space contains two branches at some point \( x^0 \), representing pair production.

That the concept of antiparticles is not the product of field quantization can also be seen in a classical model of the Dirac electron with spin [16]; it is again associated with the two "frequencies" in an invariant time coordinate and with spin.

Further properties of the classical field interpretation of the matter field has been discussed elsewhere [17].

REFERENCES