ALGEBRAIC TREATMENT
OF SECOND POSCHL-TELLER, MORSE-ROSEN
AND ECKART EQUATIONS

A.O. Barut
Akira Inomata
and
Raj Wilson

1987 MIRAMARE-TRIESTE
I. INTRODUCTION

In the preceding paper we proposed (Barut, Inomata and Wilson 1987a) a new method of algebraization of physical equations. In this method the given equation is expressed in terms of invariant bilinear forms of ladder operators of certain Lie group matrix elements. These ladder operators, which may be constructed from Infeld-Hull-Miller factorizations, close under a Lie algebra. From the enveloping algebra we obtain the required energy spectrum and the unitary representations of the corresponding Lie group give the exact normalized solutions of the equation. We used this method to study the first Pöschl-Teller equation for diatomic molecules and in this paper we discuss the second Pöschl-Teller equation, the Morse-Rosen equation for polyatomic molecules and the Eckart equation for electron penetration barrier. Algebraization of these three equations involves the algebra of non compact Lie group \( S_0(2,1) \) and its unitary representations (Bargmann 1947; Dulle and Stein 1966; Barut and Pronsdal 1965; Toller 1965; Barut and Phillips 1968; Mukunda 1969; Lindblad and Haged 1970; Rühl 1970; Riedenhorn and Louck 1981). In each case we obtain the correct energy spectrum and the eigen solutions. We also point out that the action of the ladder operators we construct is similar to the action of the covariant differentiation operator ("cov" operator) of the Bondi-Metzner-Sachs group (Newman and Penrose 1966; Goldberg et al. 1967). For the second Pöschl-Teller equation only the special case \( k = 0 \) was solved before. We treat both the bound states and scattering solutions.

II. THE SECOND PÖSCHL-TELLER EQUATION

The second Pöschl-Teller equation is

\[
\left\{ \frac{\partial^2}{\partial \tau^2} - \frac{\alpha^*}{\sinh^2 \alpha \tau} \left[ \frac{1}{\sinh^2 \alpha \tau} - \frac{\lambda (\lambda + 1)}{\cosh^2 \alpha \tau} \right] + \frac{2\mu E}{\hbar^2} \right\} \psi(r) = 0 ,
\]

\( \tau \in [0, \infty) \)
We may assume $\lambda > \kappa$, for if $\lambda < \kappa$, we can change $\lambda \rightarrow -\lambda - 1$, because the equation remains unchanged under $x \rightarrow -x + i$ and $\lambda \rightarrow -\lambda - 1$. Furthermore, the mappings $[i + i, \lambda = \lambda - 1, E = -E]$ or $[a + i, \lambda = \lambda - 1]$ provide the analytical continuation to the first Poschl-Teller equation. We introduce a change of parameters: $\xi = -m - \frac{1}{2}, \lambda = m - \frac{1}{2}, \beta = 2m\pi$ and Eq. (2.1) becomes

$$\left\{ \frac{\partial^2}{\partial \xi^2} - \frac{1}{4} \left[ \frac{(m+g+i\xi)(m+g+i\beta)}{\sinh^2 \beta/2} - \frac{(m-g-i\xi)(m-g-i\beta)}{\cosh^2 \beta/2} \right] + \Lambda \right\} \psi_\xi(r) = 0$$

$$\Lambda = ME/2\hbar \chi^2$$

(2.2)

Following Infeld-Hull-Miller factorization of type A, we define operators $M^+, M^-, M_3$ acting in same space of functions to be determined

$$M^+\psi_{m,g} = e^{i\chi/2} \left[ \frac{\hbar}{2} \left( m+\frac{1}{2} \right) \coth \frac{\beta}{2} \right] \psi_{m+1,g}$$

$$M^-\psi_{m,g} = e^{-i\chi/2} \left[ \frac{\hbar}{2} \left( m-\frac{1}{2} \right) \coth \frac{\beta}{2} \right] \psi_{m-1,g}$$

$$M_3\psi_{m,g} = -i\frac{\partial}{\partial \chi} \psi_{m,g} = m \psi_{m,g}; \quad \alpha \in [0, \pi).$$

(2.3)

The operators $M^+, M^-, M_3$ close under $SU(1,1)$ algebra (Bargmann 1947; Barut and Fronsdal 1965) satisfying the relations:

$$[M^+, M^-] = -2M_3$$

$$[M^+, M_3] = \mp M^.$$
We now define
\[ -i \frac{\partial}{\partial x} \Phi_{m,g} = g \Phi_{m,g} \]
and obtain the $SO(2,1)$ generators
\[ M^+ = M_1 \pm i M_2 \]
and the $SU(2,1)$ generators
\[ M_1 = -i \cos \alpha \cosh \beta \frac{\partial}{\partial x} - i \sin \alpha \frac{\partial}{\partial \beta} + \frac{i}{2} \sin \alpha \cosh \beta \]
\[ M_2 = -i \sin \alpha \cosh \beta \frac{\partial}{\partial x} + i \cos \alpha \frac{\partial}{\partial \beta} - \frac{i}{2} \cos \alpha \cosh \beta \]
\[ M_3 = -i \frac{\partial}{\partial \beta} \]
\[ [M_1, M_2] = i M_3, \quad [M_3, M_1] = i M_2, \quad [M_2, M_3] = i M_1. \]

As in I we shall now introduce the second set of ladder operators. We go back to Eq. (2.2) which can also be obtained from (2.1) by interchanging $\alpha$ and $\beta$ such that $\alpha = -\alpha - \beta$, $\beta = -\alpha + \beta$. Again, factorization of type A leads to the ladder operators
\[ G^+ \Phi_{m,g} = -e^{i \frac{\partial}{\partial x} \frac{1}{2} \beta + \frac{1}{2} (g + m + \frac{1}{2}) \tanh \beta \frac{\partial}{\partial \beta} + \frac{i}{2} (g - m - \frac{1}{2}) \frac{\partial}{\partial \beta} \Phi_{m,g} \]
\[ = [\Lambda + (g + m + \frac{1}{2}) \tanh \frac{\beta}{2}] \Phi_{m,g+1} \]

As expected the above commutation relations differ from those of (2.7) by an extra negative sign. From (2.7) and (2.9) we conclude that
\[ [M_i, G_j] = 0, \quad i, j = 1, 2, 3. \]

The operators $G^+, G^-, G_3$ form another $SU(1,1)$ algebra and the Casimir product leads to the same energy spectrum (2.4) and the same algebraic form (2.5). We define $G_i = G_i - i \Theta_i$ and obtain the generators $G_i, i = 1, 2, 3$,
\[ G_1 = i \cosh \beta \frac{\partial}{\partial x} - i \sinh \beta \frac{\partial}{\partial \beta} - \frac{i}{2} \cosh \beta \]
\[ G_2 = -i \sinh \beta \frac{\partial}{\partial x} - i \cosh \beta \frac{\partial}{\partial \beta} - \frac{i}{2} \sinh \beta \]
\[ G_3 = -i \frac{\partial}{\partial \beta} \]
\[ [G_1, G_2] = i G_3, \quad [G_3, G_4] = -i G_2, \quad [G_2, G_3] = -i G_1. \]

As we now define
\[ 2.6 \]
and obtain the $SO(2,1)$ generators
\[ 2.7 \]
Thus the solutions of Eq. (2.1) are the eigenfunctions satisfying

\[ Q_{\frac{1}{2}(\ell - 1)} \psi(r) = \ell(\ell - 1) \psi(r) ; \quad \psi = e^{\imath \theta} e^{i \xi x} \eta_{m_g} \]

\[ M_3 \psi_{m_g} = m \psi_{m_g} \]

\[ G_3 \psi_{m_g} = \gamma \psi_{m_g} \]

From (2.3) and (2.8) we obtain the following recurrence relations:

\[ \left( \frac{\ell + m \cosh \beta}{\sinh \beta} \right) \psi_{m_g}^{(n)} = \frac{1}{2} \left[ (\ell + m)(\ell + m + 1) \right] \psi_{m_g}^{(n-1)} + \frac{1}{2} \left[ (\ell + m - 1)(\ell + m) \right] \psi_{m_g}^{(n+1)} \]

\[ \left( \frac{m + \gamma \cosh \beta}{\sinh \beta} \right) \psi_{m_g}^{(n)} = -\frac{1}{2} \left[ (\ell + m)(\ell + m - 1) \right] \psi_{m_g}^{(n+1)} - \frac{1}{2} \left[ (\ell + m - 1)(\ell + m) \right] \psi_{m_g}^{(n-1)} \]

\[ \left( \frac{\beta + \gamma}{2} \coth \beta \right) \psi_{m_g}^{(n)} = \frac{1}{2} \left[ (\ell + m)(\ell + m - 1) \right] \psi_{m_g}^{(n+1)} + \frac{1}{2} \left[ (\ell + m - 1)(\ell + m) \right] \psi_{m_g}^{(n-1)} \]

\[ = \frac{1}{2} \left[ (\ell + m)(\ell + m - 1) \right] \psi_{m_g}^{(n+1)} + \frac{1}{2} \left[ (\ell + m - 1)(\ell + m) \right] \psi_{m_g}^{(n-1)} \]

In this case, in contrast to I, because we have a non-compact group, the crucial observation again is to compare (2.12) with the recurrence relations for the Bargmann function given by (Schneider and Wilson 1979)

\[ \left( \frac{n - n' \cosh \beta}{\sinh \beta} \right) V(n) = \frac{1}{2} \left[ (\ell + n)(\ell + n') \right] \frac{1}{2} \left[ (\ell - n)(\ell - n') \right] V(n) + \frac{1}{2} \left[ (\ell + n - 1)(\ell + n') \right] \frac{1}{2} \left[ (\ell - n - 1)(\ell - n') \right] V(n) \]

\[ \left( \frac{n' - n \cosh \beta}{\sinh \beta} \right) V(n') = -\frac{1}{2} \left[ (\ell + n')(\ell + n) \right] \frac{1}{2} \left[ (\ell - n')(\ell - n) \right] V(n') - \frac{1}{2} \left[ (\ell + n - 1)(\ell + n') \right] \frac{1}{2} \left[ (\ell - n - 1)(\ell - n') \right] V(n') \]

\[ \frac{\beta}{2} \frac{\ell}{n} V(n) = \frac{1}{2} \left[ (\ell + n)(\ell + n') \right] \frac{1}{2} \left[ (\ell - n)(\ell - n') \right] V(n) - \frac{1}{2} \left[ (\ell + n - 1)(\ell + n') \right] \frac{1}{2} \left[ (\ell - n - 1)(\ell - n') \right] V(n) \]

The comparison of (2.12) and (2.13) gives the exact normalized solution:

\[ \psi_{m_g}^{(n)}(\beta) = \sqrt{2^{\ell - 1} \sqrt{2^{\ell - 1} \sinh \beta}} \frac{1}{m_g} V\left( -\ell \cosh \beta \right) \]

\[ \int \psi_{m_g}^{(n)}(\beta) \psi_{m_g}'(\beta) d\beta = \delta_{m_g m_g} \delta_{\ell \ell} \]

\[ \int V(n) V(n') \sinh \theta d\theta = \frac{1}{\sqrt{2\xi - 1}} \delta_{n n'} \delta_{\ell \ell} \]

(2.14)
In order to express the solution in an explicit form we use the following (Barut and Wilson 1976)

\[
V(\theta) = (-1)^{\eta-n} V(\theta) = (-1)^{\eta-n} V(\theta)
\]

and in terms of the original variables: \( r, e, x, \delta = \eta, r; m = (x - 1) \)

we obtain, after using Euler's identity for \( F \) functions, the final form

\[
\frac{V(\theta)}{m, n} = \left[ \frac{(m - 1)}{(x - 1)} \left( \frac{m + 1}{x + 1} \right) \right] \frac{1}{2} \left( \sinh \frac{\theta}{2} \right) \left( \cosh \frac{\theta}{2} \right) \left( \sinh \frac{\theta}{2} \right)
\]

This is one of the standard solutions, the related second standard solution is obtained by replacing the \( r \)-dependent part by

\[
\left( \sinh \alpha_r \right)^{\lambda} \left( \cosh \alpha_r \right)^{\lambda+1} \frac{1}{2} F_1 \left[ \left( \frac{\lambda - 1}{2} \right), \left( \frac{\lambda + 1}{2} \right); \frac{3}{2} - \kappa, \frac{3}{2} - \kappa; - \sinh^2 \alpha_r \right]
\]

Thus we have obtained the exact solution to (2.1) for finite number of bound states satisfying the discrete spectrum (2.1). For the special case \( \kappa = 0 \) our result is in agreement with the earlier result using non-algebraic methods (Flugge 1971), however the latter gives only the \( r \)-dependent part of the solution without the appropriate coefficients.

Historically the equations similar to (2.1) were studied by Weyl (1916). According to Weyl's criterion the equation has eigensolutions if \( \alpha \neq \gamma \neq \beta \), where \( \alpha = 1 + \left[ \frac{1}{3} \kappa + \frac{1}{2} \right] + \left[ \frac{1}{3} \kappa - \frac{1}{2} \right] \), \( \gamma = 1 + \left[ \frac{1}{3} \kappa + \frac{1}{2} \right] \) and \( \beta = 1 + \left[ \frac{1}{3} \kappa - \frac{1}{2} \right] \). And it has discrete eigenspectrum as given by (2.14) if \( \gamma - \frac{2}{3} < 0 \Rightarrow 1 + \left[ \frac{1}{3} \kappa + \frac{1}{2} \right] - \left[ \frac{1}{3} \kappa - \frac{1}{2} \right] < 0 \) and the eigenfunctions are square integrable as in (2.14). Furthermore Eq.(2.1) has also a continuous spectrum if \( \gamma - \frac{2}{3} < 0 \Rightarrow 1 + \left[ \frac{1}{3} \kappa + \frac{1}{2} \right] - \left[ \frac{1}{3} \kappa - \frac{1}{2} \right] > 0 \).

This can be immediately obtained by using the continuous principal representation of \( SU(1,1); C_0^1 g = \xi = \frac{1}{2} + it, \xi \in (-\infty, 0) \) or \( t = \frac{1}{2} - it, t \in (0, + \infty) \); \( m, g = 0, \pm 1, \pm 2, ... \) and \( C_0^1 g \) with \( (m, g) = \pm \frac{1}{2}, \pm \frac{3}{2}, ... \) The \( C_0^1 g \) and \( C_0^1 g \) representations of Bargmann (1947) respectively correspond to \( U^0(g, \frac{1}{2} + it) \) and \( U^0(g, \frac{1}{2} + it) \) representations of Kunze and Stein (1960). From (2.14) we obtain the continuous energy spectrum

\[
E_c = 2\alpha_1^2 + \frac{1}{2} x t
\]

The continuum solutions which are the scattering states as obtained from (2.16) are not square integrable. However, it is known (Bargmann 1947; Kunze and Stein 1960; Barut and Phillips 1966; Mukunda 1970; Lindblad and Nagel 1970; Minn 1970) that for a fixed pair \( (m, g) \) or \( (\kappa, \lambda) \) the linear combinations of solutions of the type (2.16) and (2.17) are dense in the Hilbert space of square integral functions over the half-line \( r \in [0, + \infty] \).
Thus from (2.16) and (2.11) and using Kummer's identity for \( \text{F}_1 \) functions we obtain \( \{a = it\} \).

\[
\psi(\xi) = A(i \sinh \xi \lambda) \left[ \frac{K}{2} + \frac{\xi}{2} \right] \left( \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right), \left( \frac{1}{2} + \frac{\lambda + K - 1}{2} - it \right)
\]

\[
+ B(i \sinh \xi \lambda) \left[ \frac{K}{2} - \frac{\xi}{2} \right] \left( \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right), \left( \frac{1}{2} + \frac{\lambda + K - 1}{2} - it \right)
\]

Where

\[
A = \left[ \frac{2 \left[ \frac{\eta}{2} \left( \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right) \right]^{\frac{1}{2}} \pi \left( \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right) + \pi \left( \frac{1}{2} + \frac{\lambda + K - 1}{2} - it \right) \right] \left( \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right)
\]

\[
B = \left[ \frac{2 \left[ \frac{\eta}{2} \left( \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right) \right]^{\frac{1}{2}} \pi \left( \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right) + \pi \left( \frac{1}{2} + \frac{\lambda + K - 1}{2} - it \right) \right] \left( \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right)
\]

\[
\left( \frac{3}{2} - K \right) = - \sinh \frac{2}{K} \xi \lambda
\]

where

\[
A = \left[ \frac{2 \left[ \frac{\eta}{2} \left( \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right) \right]^{\frac{1}{2}} \pi \left( \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right) + \pi \left( \frac{1}{2} + \frac{\lambda + K - 1}{2} - it \right) \right] \left( \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right)
\]

\[
B = \left[ \frac{2 \left[ \frac{\eta}{2} \left( \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right) \right]^{\frac{1}{2}} \pi \left( \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right) + \pi \left( \frac{1}{2} + \frac{\lambda + K - 1}{2} - it \right) \right] \left( \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right)
\]

\[
\left( \frac{3}{2} - K \right) = - \sinh \frac{2}{K} \xi \lambda
\]

Solutions obtained in (2.19) and (2.20) are not square integrable and they are essentially obtained from suitable linear combinations of \((\sinh \xi \lambda)^{1/2} (\sinh \xi \lambda)^{-1/2}\) with \(k = \frac{1}{2} + a\) and \(l = \frac{1}{2} - a; a = it\). Since the Bargmann functions are asymptotically exponential it is possible to obtain square integrable functions by forming asymptotic "wave packets". To obtain this conveniently we express (2.19) or (2.20) as follows:

\[
\psi(\xi) = A \left( \frac{\xi}{2} \right)^{\frac{3}{2} \pi} \left[ \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right] \left( \frac{1}{2} + \frac{\lambda + K - 1}{2} - it \right)
\]

\[
+ B \left( \frac{\xi}{2} \right)^{\frac{3}{2} - \pi} \left[ \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right] \left( \frac{1}{2} + \frac{\lambda + K - 1}{2} - it \right)
\]

This solution can be expressed in another form as obtained by Bargmann as follows:

\[
\psi(\xi) = A \left( \frac{\xi}{2} \right)^{\frac{3}{2} \pi} \left[ \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right] \left( \frac{1}{2} + \frac{\lambda + K - 1}{2} - it \right)
\]

\[
+ B \left( \frac{\xi}{2} \right)^{\frac{3}{2} - \pi} \left[ \frac{1}{2} + \frac{\lambda - K + 1}{2} + it \right] \left( \frac{1}{2} + \frac{\lambda + K - 1}{2} - it \right)
\]

\[
\left( \frac{3}{2} - K \right) = - \sinh \frac{2}{K} \xi \lambda
\]
where $A$ and $B$ are some arbitrary constants depending on $\kappa$, $\lambda$, and $\rho$, and to be fixed by physical requirements or by asymptotic square integrability of $\psi(r)$. In fact several $\mu$-functions as in (2.19) or (2.20) are hidden inside $A$ and $B$; after all at the end they will become irrelevant. Since $\lim_{r \to \pm \infty} \tanh \kappa r = 1$, we can simplify the $\mu$-functions in (2.21) by using Gauss's theorem:

\[ \lim_{r \to \pm \infty} \psi(r) = A_\pm e^{\frac{i}{\kappa} k r} + B_\pm e^{-\frac{i}{\kappa} k r}, \]

we obtain from (2.21) \[ r^{-2} \frac{d}{dr} \left[ \psi(r) \right] = \frac{\Gamma(c) \Gamma(c-a-b)}{\Gamma(c-a) \Gamma(c-b)}, \quad \Re(c-a-b) > 0, \]

we now normalize $\psi_j(r)$ such that $A = 1$ and $B = 0$. This fixes $A$ and $B$, and consequently, after some straightforward computations, we obtain the reflection and transmission coefficients as,

\[ \Re(c-a-b) > 0, \]

\[ \lim_{r \to \pm \infty} \psi(r) = A_\pm e^{\frac{i}{\kappa} k r} + B_\pm e^{-\frac{i}{\kappa} k r}, \]

we can easily see that (2.22) satisfy the unitarity of the scattering matrix which is a $U(2)$ matrix:

\[ |A_+|^2 + |B_+|^2 = |A_-|^2 + |B_-|^2, \]

and the quasi-unitarity of the transfer matrix which is a $SU(1,1)$ matrix:

\[ |A_+|^2 |B_+|^2 = |A_-|^2 |B_-|^2, \]

provided $\kappa = 0, \pm 1, \pm 2, \ldots$. This is not surprising since asymptotically the $g$-terms in (2.3) and the $m$-terms in (2.6) disappear and the presence of one of the two potential terms in (2.1) becomes irrelevant.

For the case $\kappa = 0, \pm 1, \pm 2, \ldots$, we get,

\[ |R| = \frac{\sin^2 \lambda}{\sin^2 \lambda + \sin^2 \left( \pi \frac{k}{d_0} \right)}, \]

\[ |T| = \frac{\sinh^2 \left( \pi \frac{k}{d_0} \right)}{\sin^2 \lambda + \sinh^2 \left( \pi \frac{k}{d_0} \right)}, \]

\[ (2.22) \]

\[ (2.23) \]
which are in agreement with standard results (Flugge 1971).

III. MORSE-RÖSCH-EQUATION AND ECKART-EQUATION

The Morse-Rösch equation for polyatomic molecules is

\[
\left\{ \frac{\partial^2}{\partial x^2} + \left( \frac{2M U_0}{\hbar^2} - \frac{2MB_0}{\hbar^2} \right) \tanh x - \frac{2M E}{\hbar^2} \right\} \Phi(x) = 0,
\]

\[x \in [0, \infty) \quad (3.1)\]

This equation can be algebraized by using the factorization (Xiong 1976) either of type A or of type E. The algebraic ladder operators from the factorization of type A raise or lower the eigenvalues (quantum numbers) of the self-adjoint operators which are the elements of a Lie algebra, as we have seen in the case of first and second Pöschl-Teller equations. But those operators from the factorization of type E raise or lower the eigenvalues of the invariant operators (Casimir operators) in the enveloping algebra of a Lie algebra. We show these equivalent algebraizations for Eq. (3.1).

First, in (3.1) we use the following re-parametrizations and substitutions

\[\frac{2M E}{\hbar^2 \alpha_i^2} = -p^2, \quad \frac{2M U_0}{\hbar^2 \alpha_i^2} = -\Delta_i / 4, \quad \frac{2MB_0}{\hbar^2 \alpha_i^2} = -2p^2, \quad \alpha_i x + \frac{i \pi}{2} \rightarrow ln \tanh (x/2), \]

\[\Phi(x) \rightarrow \left( \frac{i}{\sinh^2 x} \right)^{\frac{N}{2}} \Phi(z). \]

Consequently (3.1) becomes,

\[
\left\{ \frac{\partial^2}{\partial z^2} - \frac{1}{\sinh^2 z} \cdot \left[ (p+q)(p-q) + q^2 + 2pq \coth z \right] \right\} \Phi(z) = 0
\]

or

\[
\left\{ \frac{\partial^2}{\partial z^2} - \frac{4}{\sinh^2 \frac{z}{2}} \cdot \left[ (p-q+4)(p-q-2) + (p-q-2)(p-q+4) \right] \right\} \Phi(z) = 0
\]

This equation is identical to the second Pöschl-Teller Eq. (3.2) with the following identifications:

<table>
<thead>
<tr>
<th>Pöschl-Teller</th>
<th>Morse-Rösch</th>
</tr>
</thead>
<tbody>
<tr>
<td>m \rightarrow p</td>
<td>\Phi(z)</td>
</tr>
<tr>
<td>a \rightarrow q</td>
<td>\Phi(z)</td>
</tr>
<tr>
<td>A \rightarrow \Delta</td>
<td>\Phi(z)</td>
</tr>
<tr>
<td>B \rightarrow \alpha_i^2</td>
<td>\Phi(z)</td>
</tr>
</tbody>
</table>

Thus the algebraization by factorization of type A can be carried out as before. From Eqs. (2.4) and (3.2) we obtain the energy spectrum:

\[E_n = -\left[ \frac{k^2 \alpha_i^2}{8M} \left( \frac{M a^2}{k^2 \alpha_i^2} \right)^2 + \frac{2MB_0}{k^2 \alpha_i^2} \left( \frac{1}{B_0^2} \right) \right], \quad \Phi(z) = 0 \quad (3.2)\]

\[D_n = 2p = 2(s-n) \quad (s - \left[ MB_0/k^2 \alpha_i^2 \right]) \geq n = 0, 1, 2, \ldots \]

\[S = k^2 a^2 = \frac{1}{2} + \frac{1}{2} \left( t + 8M U_0/k^2 \alpha_i^2 \right)^2 \]

\[\Phi(z) = 0 \quad (3.3)\]
Furthermore, from (3.7) and (2.14) we obtain the solution to (4.1) as:

\[ \Phi(x) = (2S+1)^{1/2} \sqrt{\frac{S}{S+1}} \cdot \left( \frac{-1}{1 - i} \right) \]

\[ \sinh x = i \cosh x, \quad \cosh x = - \tanh x \]

\[ \int \Phi^*(x) \Phi(x) \cosh x \, dx = 1, \]

where \( \cosh x \, dx \) is the normalized Haar measure induced by the Bargmann functions. Using (2.15) we obtain \( \Phi(x) \) explicitly as,

\[ \Phi(x) = \left[ \frac{2^{2n-2S} \left( S+q, n \right) \left( -S-q, n \right)}{2^{2n-2S} \left( S+q, n \right) \left( -S-q, n \right)} \right]^{1/2} \]

\[ q(x) = (\cosh x)^{-S+1} \left[ -n_r, 2S+1-n_r, S-n_r+1 \right] \frac{e^{i\alpha x}}{e^{e^{x} + e^{-x}}} \]

(3.6)

Our results (3.7) and (3.7) are in agreement with earlier calculations (Weyl 1910) (the extra term \( \left( \frac{p^2 - q^2}{p} \right) \) of Nieto's normalization does not appear in our normalization due to the Haar measure (3.6)). Our algebraization using the factorization type A is somewhat artificial (Miller 1968) (in fact according to the Infeld-Hull classification the parameter \( q \) is artificially introduced to achieve factorization type A). The complex substitution: \( ax + \frac{i}{2} \tanh \left( \frac{S}{2} \right) \) is similar to Weyl's unitarity trick.

The scattering solutions of (3.1) is obtained by taking \( p \to i \nu_{q} \) and \( q \to -i \nu_{q} \) \( \left( \nu_{q}, \nu_{q} \right) \in \left[-\infty, \infty \right) \) so that the continuous energy spectrum from (3.5) is given by

\[ E_{\nu_{q}} = \frac{\hbar \epsilon^{2}}{2M} \nu_{q}^2 + \frac{\hbar \epsilon^{2}}{2M} \nu_{q}^2 + \frac{\hbar \epsilon^{2}}{2M} \nu_{q}^2 \]

(4.8)

The algebraization as in (2.3) - (2.11) further implies that we consider \( D_{q} \) discrete series unitary representations of \( SU(2,1) \) on a continuous basis, where a non-compact operator is diagonal, that is, \( D_{q} \) are similar to \( \Phi \) in (2.7) and \( \Phi_{q} \) are similar to \( \Phi \) in (2.9).

\[ P_{2} \psi_{p, \nu_{q}} = \nu_{p} \psi_{p, \nu_{q}}, \quad Q_{2} \psi_{p, \nu_{q}} = \nu_{q} \psi_{p, \nu_{q}} \]

and the corresponding matrix element we consider is the matrix element of \( \exp(-i\nu_{q} \alpha_{p}) \) or \( \exp(-i\nu_{q} \alpha_{p}) \) between the respective basis. These matrix elements have already been obtained (Barut and Phillips 1968; Lindblad and Nagol 1970). We give below the solutions which are normalized to reproduce the correct asymptotic behaviour:

\[ \Phi_{p}^{(x)}(\xi) = \frac{\nu_{q}^{(x)}(\nu_{p} - \nu_{q})}{2\pi} \left\{ \frac{(\xi_{p} - \xi_{q})}{\xi_{p} + \xi_{q}} \right\} \left[ \frac{\pi}{\pi S + 1} \right] \left[ \frac{\nu_{q}^{(x)}(\nu_{p} - \nu_{q})}{\nu_{p}^{S+1} + \nu_{q}^{S+1}} \right] \]

(3.7)

The scattering solutions of (3.1) is obtained by taking \( p \to i \nu_{q} \) and \( q \to -i \nu_{q} \) \( \left( \nu_{q}, \nu_{q} \right) \in \left[-\infty, \infty \right) \) so that the continuous energy spectrum from (3.5) is given by

\[ E_{\nu_{q}} = \frac{\hbar \epsilon^{2}}{2M} \nu_{q}^2 + \frac{\hbar \epsilon^{2}}{2M} \nu_{q}^2 + \frac{\hbar \epsilon^{2}}{2M} \nu_{q}^2 \]

(3.8)
where we have made use of the representation function obtained by (Hult 1970).

Before we algebraize the Morse-Rosen equation using the factorization of type $E$, we discuss the closely related Eckart equation (Eckart 1940) which was used to describe the penetration of a potential barrier by electrons.

The Eckart equation reads,

$$\frac{d^2 f}{dx^2} + \frac{2 \text{MA}}{\hbar^2} \frac{1}{1 - \frac{Z}{W}} f(x) = 0,$$

with \( Z = -e \) and \( A, B, \ell \) constants, \( B > 0 \).

(3.10)

We can rewrite (3.10) as

$$\frac{d^2 f}{dx^2} - \frac{MB}{2\hbar^2} \left( \frac{1}{\cos^2(\frac{\pi x}{\ell})} \right) - \frac{MA}{\hbar^2} \tanh(\frac{\pi x}{\ell}) + \frac{M}{\hbar^2} (2W-A) f(x) = 0,$$

(3.11)

which is similar to (3.1). This means that in the Eckart equation (3.11) we can use the similar substitutions and parametrizations as in (3.2)

\[
\left( \frac{2MW}{\hbar^2} - \frac{MA}{\hbar^2} \right) \frac{\ell^2}{\ell^2} = -a^2 - b^2
\]

\[
\frac{MB}{2\hbar^2} \frac{\ell^2}{\ell^2} = \Delta^4 + \frac{1}{4}; \quad \frac{MA}{\hbar^2} \frac{\ell^2}{\ell^2} = -2a^2b
\]

\[
\frac{\pi x}{\ell} + \frac{\pi Z}{2} \leftrightarrow \ln \tanh \left( \frac{Z}{2\ell} \right)
\]

\[
f(x) \leftrightarrow \left( \frac{i}{\sinh x} \right) \tilde{g}(z)
\]

(3.12)

Consequently we obtain an equation similar to (3.3).

$$\left\{ \frac{d^2 \tilde{g}}{dz^2} - \frac{1}{4} \left( \frac{(a+b) \partial_z \tilde{y}}{(a+b) \tilde{y}} - \frac{(a-b) \partial_z \tilde{y}}{(a-b) \tilde{y}} \right) + \Delta^4 \right\} \tilde{g}(z) = 0.$$

(3.13)

This means that we can repeat the algebraization exactly identical to the one we developed for the second Pöschl-Teller equation and from (2.8) we obtain the important condition

$$\frac{MB}{2\hbar^2} \frac{\ell^2}{\ell^2} - \frac{1}{4} = \Delta^4 = -\left( \ell - \frac{Z}{2} \right)^2.$$

(3.14)

In order to maximize the potential barrier Eckart assumes (see Fig.1 of Eckart 1930) \( B > 0 \Rightarrow \Delta^4 \rightarrow -\frac{1}{2} \). For \( \Delta^4 = -\frac{1}{2} \Rightarrow Z = 0 \), there may exist a few bound states given by \( \Delta^4 \) discrete series representations of \( SO(2,1) \).

However the existence of such bound states may be removed by taking \( \Delta^4 \) sufficiently large. Thus we are led to the continuous principal series representations \( C^\Delta_\ell \) of \( SO(2,1) \); \( \Delta^4 = \frac{1}{2} \geq \ell \in (0,\infty) \); for \( \delta = 0 \), \( \langle a, b \rangle = 0, \pm 1, \pm 2, \ldots \) or for \( \delta = \frac{1}{2}, \langle a, b \rangle = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots \). Now, from the energy relation given in (3.12) we see that there exist two possibilities. First, for \( W < A/2 \) there exist asymptotically plane wave solutions (pulses) with fixed discrete energy. However, the second possibility, for which \( W > A/2 \), is very interesting. In this case in (3.12) we use the analytic continuation \( a \rightarrow iv_a, b \rightarrow iv_b (v_a, v_b) \in (-m, m) \). The continuous energy spectrum is given by

$$W = \frac{\pi^2}{2M\ell^2} \frac{\Delta^4}{\lambda_a^2} + \frac{MA}{\ell^2} \frac{\lambda_b^2}{\lambda_a^2} + \frac{A}{2}.$$

(3.15)

Thus we consider the continuous principal series representations \( C^\Delta_\ell \) on a continuous basis \( \{ A_\lambda \} \) are similar to \( H \) in (2.7) and \( \tilde{H} \) are similar to \( C^\Delta_1 \) in (2.9).
$f(z) = \psi \theta f(\xi)$,
$B \, f_{\xi}(\xi) = B \, f_{\xi}(\xi)$,

where $f(z)$ are notation-wise similar to $\varphi_{0,0}(4)$ in (2.11). The spectrum of $A_x, A^2$ is the real line with multiplicity two — there exists two eigenstates for each eigenvalue. This is because there exists an outer automorphism (parity) of the Lie algebra $\{A_x, A^2\}$, $\{B \}$, where

$$\{A_x, A^2, -A^2\} = \{-A_x, -A^2\}$$

$$\{B, -A^2\} = \{-B, -A^2\}$$

In the case of the continuous principal series $C$ (and also for the supplementary series) this automorphism can be realized explicitly by

$$A = P \, A \, P^{-1}, \quad P \, f_{\xi, \nu} = e^{i \psi \frac{\pi}{A}} \, f_{\nu, \nu}$$

$$B = P \, B \, P^{-1}, \quad P \, f_{\xi, \nu} = e^{i \psi \frac{\pi}{A}} \, f_{\nu, \nu}$$

Therefore $P^2 = 1$ and the eigenvalues of $P$ are $\pm 1$ which we denote as $\epsilon_c, \epsilon_s$. As $[P, A_x] = [P, B] = 0$, $P$ and $A_x$ and $B$ can be diagonalized simultaneously.

The matrix element we consider in this case is the matrix elements of $\exp(-iA \, z)$ or $\exp(-iB \, z)$ between the two continuous bases of eigenvalues $\nu$, $\nu$ with respective multiplicity $\epsilon_c$ and $\epsilon_s$. These representation functions are already known (Barut and Phillips 1968). Thus from (3.6), (3.9) and (3.12) we obtain the solutions which are as usual normalized to reproduce the correct asymptotic behaviour:

$$\psi \left( \frac{1}{2} + it + i \nu \right) \left( \frac{1}{2} - it - i \nu \right)$$

$$\left. \exp \left( -i \theta (\nu + t) \right) \left( \frac{1}{2} + it + i \nu \right) \right| \left. \exp \left( -i \theta (\nu - t) \right) \left( \frac{1}{2} - it - i \nu \right) \right.$$
above, the second quadratic Casimir product \( Q'' \) vanishes while the first quadratic Casimir product \( Q' \) is proportional to \( Q_p \) and \( Q_q \). The invariant product \( Q' \) may be constructed (similar to \( Q_p \) and \( Q_q \)) in terms of bilinear forms of the \( l \)-raising and \( l \)-lowering ladder operators provided these operators form an algebra. We will see below that they do not close under a commutation relation and we will see that these bilinear products are elements in the enveloping algebra of \( \text{SU}(2,1) \times \text{SO}(2,1) \times \text{SO}(2,2) \).

We use the parametrizations (3.2) in (3.1) and from factorization of type \( W \) we obtain the ladder operators:

\[
L_+ \Phi_L(x) = \left[ -\frac{2}{\alpha_x} + l \alpha_x \tan h x \alpha_x - \frac{b q_x}{\alpha_x} \right] \Phi_L(x)
\]

\[
= \frac{\alpha_x}{\ell} \left[ (l - p)(l + p)(l - q)(l + q) \right]^{1/2} \Phi_L(x)
\]

\[
L_- \Phi_L(x) = \left[ \frac{2}{\alpha_x} + (l - 1) \alpha_x \tan h x \alpha_x - \frac{b q_x}{\alpha_x} \right] \Phi_L(x)
\]

\[
= \frac{\alpha_x}{(l - 1)} \left[ (l - 1 - p)(l + p)(l - 1 - q)(l + q) \right]^{1/2} \Phi_L(l - 1)
\]

\[
\begin{align*}
[L_+, L_-] &= \frac{\alpha_x}{\ell} \left[ (l - p)^2 + \frac{b^2 q^2}{(l - 1)^2} \right] - \left( l^2 + \frac{b^2 q^2}{\ell^2} \right)
\end{align*}
\]

(3.17)

We see that \( L_+ \) do not close under an algebra. In order to see the implication of the \( \text{SO}(2,1) \times \text{SO}(2,1) \) group structure, we use the substitution (3.2) in \( x \) variables and obtain,

\[
\begin{align*}
L_+ \Phi_L(x) &= \left[ \text{sinh} \frac{\alpha_x}{\ell} + \frac{b q_x}{\ell} \right] \Phi_L(x) \\
&= \frac{1}{\ell} \left[ (l - p)(l + p)(l - q)(l + q) \right]^{1/2} \Phi_L(x)
\end{align*}
\]

(3.18)

The recurrence relations for Bargmann functions from \( \text{SO}(2,1) \times \text{SO}(2,1) \) are given by (Schneider and Wilson 1979; Barut and Wilson 1976; Basu and Wolf 1983)

\[
\begin{align*}
\left[ \text{sinh} \frac{\alpha_x}{\ell} + \frac{b q_x}{\ell} \right] \Phi_{n+1}(x) &= \frac{1}{\ell} \left[ (l - n)(l + n) \right]^{1/2} \Phi_n(x) \\
\left[ \text{sinh} \frac{\alpha_x}{\ell} - \frac{b q_x}{\ell} \right] \Phi_{n-1}(x) &= \frac{1}{\ell} \left[ (l - n)(l + n) \right]^{1/2} \Phi_{n+1}(x)
\end{align*}
\]

(3.19)

On comparison we find that \( \Phi_L(x) \) is given by the Bargmann function \( V_{n+1}(x) \) as obtained earlier in (3.6). The energy spectrum is certainly given by the parametrizations (3.2) and furthermore the algebraization using factorization of type \( F \) for the continuum part of (3.1) and for the Eckart equation follows immediately as (3.1) and (3.11) are identical except the coefficients.

IV. DISCUSSION

As in I, the \( \text{SU}(2,1) \times \text{SU}(2,1) \) algebras, we found Eqs.(2.3) and (2.8), describe a fixed energy states of a family of systems with quantized coupling constants \( \kappa \) and \( \lambda \), as some kind of a periodic table of elements. The energy range is finite, determined by the range \( \kappa \) from \( \k_\text{min} \) to \( \k_\text{max} \) or \( n = 0 \) to \( n = \frac{\lambda - \kappa}{2} \), (see Eq.(2.4)). In addition we can change energy, or \( L \), for fixed coupling constants \( \kappa \) and \( \lambda \) as the type \( K \) factorization shows, Eqs.(3.17) and (3.18).
Again we see that our family of systems can be embedded in an SU(2,1) (Barut et al., 1971). For the first Poschl-Teller equation the range of discrete energy is infinite but we have a finite family of systems, while for the second Poschl-Teller equation, we have an infinity of systems, but a finite number of discrete energy levels. In other words, the role of the subgroups is interchanged, SU(1,1) ⊗ SU(1,1) versus SU(2) ⊗ SU(2). This accounts for the analytic continuation between the two cases.
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