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REMARKS ON THE GAUGE FIXING PROBLEMS OF THE SPINNING STRING IN NEVEU-SCHWARZ-RAMOND FORMALISM

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ABSTRACT

We discuss the problems of gauge fixing for the spinning string in the Neveu-Schwarz-Ramond formalism. We demonstrate that the usual Teichmüller parameters have fermionic counterparts. For a surface of genius g the complex dimension of the fermionic Teichmüller variables is $2g-2$ if $g \geq 2$.

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INTRODUCTION

The old superstring is contained in the NSR model of the spinning string [1]. The problem of quantizing the superstring is essentially the problem of fixing the fermionic gauge, and this is relatively straightforward for the old string. The aim of the present note is to discuss the problem.

For the new superstring of Green and Schwarz [2] the relevant group is the Siegel [3] transformations and this remains problematic.

The old superstring is just two-dimensional supergravity coupled to a set of 10 scalar supermultiplets. In component notation we have

$$\text{gravity multiplet} \begin{cases} e_\alpha{}^a & \text{2-bein} \qquad a,\alpha = 1,2 \\ \chi_a & \text{gravitino} \end{cases}$$

$$\text{matter multiplets} \begin{cases} X^\mu & \text{Bose} \qquad \mu = 1,2\ldots,10 \\ \psi^\mu & \text{Fermi} \end{cases}$$

As given by Deser and Zumino [4] the signature is Lorentzian. For use in path integrals we need a Euclidean version. In the Lorentzian version the gravitino field χ_α is real in the sense that its two chiral components are real. In the Euclidean version the chiral components are complex but comprise a conjugate pair. The same is true of the matter fermions, ψ^μ, e.g.

$$\psi^\mu = \begin{pmatrix} \psi_1^\mu \\ \psi_2^\mu \end{pmatrix} , \qquad \psi_1^{\mu *} = \begin{cases} \psi_1^\mu, & \text{Lorentz} \\ \psi_2^\mu, & \text{Euclid} \end{cases}$$

These distinctions are important when considering the possible bilinears of the theory.

The Euclidean Lagrangian is the

$$\mathcal{L} = e\left[\frac{1}{2} g^{\alpha\beta}\partial_\alpha X \partial_\beta X + \frac{i}{2}\psi^\dagger \gamma^\alpha \partial_\alpha \psi \right.$$
$$\left. \chi_\alpha^\dagger \gamma^\beta \gamma^\alpha \left(i\partial_\beta X + \frac{1}{4}\chi_\beta \psi^\dagger \right)\gamma_5 \psi \right] . \quad (1)$$

where $g^{\alpha\beta}$ is related in the usual way to $e_\alpha{}^a$, and $e = \det e_\alpha{}^a$. The connection term in

$$\nabla_\alpha \psi = \partial_\alpha \psi + \frac{i}{2}\omega_\alpha \gamma_5 \psi \quad (2a)$$

does not contribute because

$$\psi^\dagger \gamma^\alpha \gamma_5 \psi = 0 . \quad (2b)$$

According to Deser and Zumino the local supersymmetry takes the form

$$\delta X^\mu = i\bar{\varepsilon}\psi^\mu \quad (3a)$$

$$\delta \psi^\mu = \left(\partial_\alpha X^\mu + i\bar{\psi}^\mu \chi_\alpha \right)\gamma^\alpha \varepsilon \quad (3b)$$

$$\delta e_\alpha{}^a = -2i\,\bar{\varepsilon}\gamma^a \chi_\alpha \quad (3c)$$

$$\delta \chi_\alpha = -\nabla_\alpha \varepsilon \quad (3d)$$

In addition there is a conformal (Weyl) supersymmetry

$$\delta\chi_\alpha = \gamma_\alpha \eta . \quad (4)$$

At the classical level these symmetries are strong enough to gauge away the interactions. Quantum corrections, however, may be subject to anomalies. Indeed, the consistent quantizing of the spinning string hinges on showing that $e_\alpha{}^a$ can be reduced to $\delta_\alpha{}^a$ and χ_α to 0: delivering thereby a free flat (in the sense of two dimensions) theory of X^μ and ψ^μ.

Amplitudes should be represented by path integrals of the type

$$\int (dX\, d\psi\, de\, d\chi)\, \exp(-S) \times \text{wave functions} . \quad (5)$$

The first problem is to define these integrals by fixing the gauges in a sensible way. In particular, one must factor out the volumes of the local symmetry groups. Polyakov achieved this at the tree level using the diffeomorphisms to reduce $g_{\alpha\beta}$ to the form

$$g_{\alpha\beta} = e^{2\varphi}\delta_{\alpha\beta} \quad (6)$$

and local supersymmetry to reduce χ_α to

$$\chi_\alpha = \gamma_\alpha \rho . \quad (7)$$

The conformal variables, φ and ρ, couple only through the anomalies (which disappear in the critical dimension) and through off-shell extrapolations of the wave functions. On shell amplitudes, in the critical number of dimensions, $D = 10$, should be independent of φ and ρ. Integration over φ and ρ then yields irrelevant volume factors.

The Polyakov integration[5)] of $g_{\alpha\beta}$ over manifolds of spherical topology was generalized by Polchinski[6)] to the case of toroidal topology (1 loop). We propose to do the same for χ_α. The problem solved by Polchinski was not trivial in that the integral over toroidal manifolds, factored by diffeomorphisms and Weyl scale transformations reduces to a finite dimensional integral over Teichmuller parameters. In other words, the gauge fixing procedure now involves a significant topological component. In the fermionic case this seems to imply anticommuting Teichmuller parameters.

The symbol $\int(dg)$ is intended to denote integration over two-*dimensional Riemannian manifolds*, i.e. *over an abstract space* $\mathcal{M}$ whose points correspond to the two-dimensional manifolds. Any two points of $\mathcal{M}$ connected by diffeomorphism and rescaling are "equivalent". In this sense all points of $\mathcal{M}$ are equivalent to those in a finite dimensional subset. This subset can always be chosen such that the metric tensor g_{ab} is completely standardized, different points in the subset being distinguished only by the range of the two-dimensional co-ordinates. (Such a subtle distinction of inequivalent points in the space $\mathcal{M}$ is completely missed by the old technique of gauge fixing which considered only the form of $g_{\alpha\beta}$ locally. Global distinctions were ignored.)

THE BOSONIC MEASURE

The idea of Polyakov[5] is to make the space, $\mathcal{M}$, of metric tensors on the 2-space M, into a "Riemannian function space" by inventing a metric tensor for it. This tensor could be thought of as an operator, G, acting on the tangent space of $\mathcal{M}$. It is defined by

$$(\delta g, G\,\delta g) = \frac{1}{2}\int_M d^2\sigma \sqrt{g}\left(g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma} + C\,g^{\alpha\beta}g^{\gamma\delta}\right)\delta g_{\alpha\beta}\,\delta g_{\gamma\delta}\,, \tag{8}$$

where the integral extends over a closed 2-manifold, M. This functional is supposed to represent the $(\text{distance})^2$ between two neighbouring points, $g_{\alpha\beta}$ and $g_{\alpha\beta} + \delta g_{\alpha\beta}$, in the space of metrics, $\mathcal{M}$.

If G is a decent operator then it should have a determinant, Det G. With Det G it should be possible to define the Riemann-Lebesque measure on $\mathcal{M}$, viz.

$$d\mu = \prod_\sigma d g_{\alpha\beta}(\sigma)\;\left|\operatorname{Det} G\right|^{1/2}. \tag{9}$$

To make sense of this it will be necessary to define a co-ordinate system on $\mathcal{M}$. Most useful would be co-ordinates that can be easily fixed by means of the general co-ordinate and Weyl invariances. For example, one should be able to write

$$g_{\alpha\beta}(\sigma) = \left|\det\frac{\partial\sigma}{\partial\hat{\sigma}}\right|\;\frac{\partial\hat{\sigma}^\gamma}{\partial\sigma^\alpha}\frac{\partial\hat{\sigma}^\delta}{\partial\sigma^\beta}\;e^{2\varphi(\hat{\sigma})}\;\hat{g}_{\gamma\delta}(\hat{\sigma};\tau), \tag{10}$$

where the form of $\hat{g}$ is standardized. With genus ≥ 2 it is possible to represent M as an irregular polygon in the hyperbolic plane,

$$ds^2 = \frac{2\,dz\,d\bar{z}}{(1-|z|^2)^2}. \tag{11}$$

Opposite sides are to be identified. The positions of the 4g corners must be specified in terms of the 3g-3 complex moduli τ. Then by means of an appropriate reparametrization one ought to be able to map the 4g corners into regular positions, say

$$\exp\left(i\,\frac{2\pi n}{4g}\right), \qquad n = 1, 2, \ldots, 4g. \tag{12}$$

In these co-ordinates the metric tensor will depend explicitly on the moduli. It will be a well defined function of the new co-ordinates, $\hat{\sigma}^\alpha$, and the moduli, τ^i. The expression (10) seems to give a complete parametrization of the space of metrics, $\mathcal{M}$. The points in this function space are characterized by three functions, σ^α and φ, plus 3g-3 numbers, τ^i:

$$\{g_{\alpha\beta}\} \sim \{\sigma^\alpha, \varphi, \tau^i\}. \tag{13}$$

No generality is lost by supposing that $\det \hat{g} = 1$, since a scale factor can be absorbed in the definition of φ.

The utility of a parametrization like (10) lies in the fact that the functions σ^α and φ represent gauge degrees of freedom and so can be gauged away. Nothing of physical significance can depend on σ^α and φ. We shall use the Faddeev-Popov procedure to fix them as follows:

$$\sigma^\alpha(\hat{\sigma}) = \hat{\sigma}^\alpha \quad \text{and} \quad \varphi(\sigma) = 0. \tag{14}$$

But first it is necessary to obtain det G in the neighbourhood of the points (14). The first step is to expand (10) in the neighbourhood of (14):

$$g_{\alpha\beta}(\sigma) = \hat{g}_{\alpha\beta}(\sigma) + 2\varphi(\sigma)\,\hat{g}_{\alpha\beta}(\sigma) + \left(\xi^\gamma{}_{,\alpha}(\sigma)\,\hat{g}_{\gamma\beta}(\sigma) + \xi^\gamma{}_{,\beta}\,\hat{g}_{\gamma\alpha} - \xi^\gamma{}_{,\gamma}\,\hat{g}_{\alpha\beta} + \xi^\gamma\,\hat{g}_{\alpha\beta,\gamma}\right) + \ldots \tag{15}$$

where $\varphi(\sigma)$ and $\xi^\alpha(\sigma) = \hat{\sigma}^\alpha - \sigma^\alpha$ are small quantities.

The next step is to take the first differential of this formula, i.e. to find the response of $g_{\alpha\beta}$ to infinitesimal variations in the co-ordinates φ, ξ^α and τ^i

$$
\begin{aligned}
\delta g_{\alpha\beta} = {} & \delta\tau^i \left(\hat{g}_{\alpha\beta,i} + 2\varphi\, \hat{g}_{\alpha\beta,i} + \xi^\gamma{}_{,\alpha}\, \hat{g}_{\gamma\beta,i} + \cdots \right) \\
& + 2\delta\varphi\, \hat{g}_{\alpha\beta} \\
& + \left(\delta\xi^\gamma{}_{,\alpha}\, \hat{g}_{\gamma\beta} + \delta\xi^\gamma{}_{,\beta}\, \hat{g}_{\gamma\alpha} - \delta\xi^\gamma{}_{,\gamma}\, \hat{g}_{\alpha\beta} + \delta\xi^\gamma\, \hat{g}_{\alpha\beta,\gamma} \right)
\end{aligned}
\tag{16}
$$

+ higher orders in φ and ξ .

This expression is to be substituted into the distance functional (8) to establish the form of G in the new co-ordinates. We simplify the computation by restricting to the points $\varphi = \xi = 0$ (which is all that we need) and writing (16) in the compact form,

$$
\delta g_{\alpha\beta} = \delta\tau^i\, \hat{g}_{\alpha\beta,i} + 2\delta\varphi\, \hat{g}_{\alpha\beta} + \left(\nabla_\alpha \delta\xi_\beta + \nabla_\beta \delta\xi_\alpha - \hat{g}_{\alpha\beta} \nabla_\gamma \delta\xi^\gamma \right) .
\tag{17}
$$

The distance functional reduces to

$$
\begin{aligned}
(\delta g, G\, \delta g) = {} & G_{ij}\, \delta\tau^i \delta\tau^j \\
& + 2 \int d^2\sigma\; \delta\tau^i\, G_{i\alpha}\, \delta\xi^\alpha \\
& + \int d^2\sigma\; \delta\xi^\alpha\, G_{\alpha\beta}\, \delta\xi^\beta \\
& + 2(1+C) \int d^2\sigma\; \delta\varphi\, \delta\varphi
\end{aligned}
\tag{18}
$$

where

$$
G_{ij} = - \int d^2\sigma\; \hat{g}_{\alpha\beta,i}\, \hat{g}^{\alpha\beta}{}_{,j}
\tag{19a}
$$

$$
G_{i\alpha} = - 2 \nabla^\beta\, \hat{g}_{\alpha\beta,i}
\tag{19b}
$$

$$
G_{\alpha\beta} = - 2 \left(\hat{g}_{\alpha\beta} \nabla^2 - [\nabla_\alpha, \nabla_\beta] \right) .
\tag{19c}
$$

In these expressions (19c) coincides with Polyakov's formula[5] and (19a) with Polchinski's one-loop improvement. The mixing term (19b) becomes relevant at two or more loops and is implicit in Moore and Nelson.[7]

The computation of $|\mathrm{Det}\, G|^{1/2}$ can be formalized as a Gaussian functional integral,

$$
\begin{aligned}
|\mathrm{Det}\, G|^{-1/2} &= \int (d\delta\xi)(d\delta\varphi)\, d\delta\tau\, \exp\left[-\tfrac{1}{2} (\delta g, G\, \delta g) \right] \\
&= \int (d\delta\xi)\, d\delta\tau\, \exp\Big[-\tfrac{1}{2} G_{ij}\, \delta\tau^i \delta\tau^j \\
&\qquad - \int d^2\sigma\; \delta\tau^i\, G_{i\alpha}\, \delta\xi^\alpha \\
&\qquad - \tfrac{1}{2} \int d^2\sigma\; \delta\xi^\alpha\, G_{\alpha\beta}\, \delta\xi^\beta \Big] \\
&= |\det H_{ij}|^{-1/2}\; |\mathrm{Det}\, G_{\alpha\beta}|^{-1/2} .
\end{aligned}
\tag{20}
$$

where the 6g-6 (real)-dimensional matrix H_{ij} is given by

$$
H_{ij} = G_{ij} - \int d^2\sigma\, d^2\sigma'\; G_{i\alpha}(\sigma)\, G^{\alpha\beta}(\sigma,\sigma')\, G_{\beta j}(\sigma') .
\tag{21}
$$

This expression results from a shift in the integration variable $\delta\xi^\alpha$. The correlation function $G^{\alpha\beta}$ is defined by

$$
G_{\alpha\beta}\; G^{\beta\gamma}(\sigma,\sigma') = \delta_\alpha^\gamma\; \delta_2(\sigma-\sigma')
\tag{22}
$$

(which is supposed to be unambiguous for $g \geqslant 2$, since $G_{\alpha\beta}$ has no zero-modes, i.e. conformal Killing vectors).

It is easy to show that the Faddeev-Popov factor associated with the gauge choice (14) is trivial. Hence the final expression for the measure is

$$
\int d\mu = \int \prod_1^{6g-6} d\tau^i\; |\mathrm{Det}\, G_{\alpha\beta}|^{1/2}\; |\det H_{ij}|^{1/2}
\tag{23}
$$

i.e. a finite-dimensional integral. The range remains to be specified.

The integrand of (23) is supposed to be invariant with respect to the so-called modular group. This is a discrete group with infinite number of elements which acts on the space of τ^i . It corresponds to an ambiguity in the specification of $\hat{g}(\hat{\sigma},\tau)$. There is supposed to be an infinite set of points, τ , for which $\hat{g}$ represents the same Riemann surface. For such points, connected by a modular transformation, the corresponding 2-spaces are connected by a conformal transformation.

In the case of genus = 1, the modular group is SL(2,Z), i.e. the set of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{with integer } a,b,\dots \text{ and } ad-bc = 1 \ .$$

It has two independent generators.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{giving} \quad \tau \to \tau + 1$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \tau \to -\frac{1}{\tau} \ .$$

In general, the action of this group is defined by

$$\tau \to \frac{a\tau + b}{c\tau + d} \ .$$

The <u>fundamental region</u> to which the integral over τ should be restricted is always taken to be

$$\mathrm{Im}\,\tau > 0, \qquad |\tau| > 1 \ , \qquad |\mathrm{Re}\,\tau| < \frac{1}{2} \ .$$

For genus ≥ 2 one does not know what the modular group should be.

In addition to the integral over $\mathcal{M}$, there will be one over $\mathcal{X}$ the space of mappings $M \to R^D$, i.e. the space-time co-ordinates, $X^\mu(\sigma)$. The metric on $\mathcal{X}$ can be taken to be

$$(X, G_X X) = \int_M d^2\sigma \ \sqrt{g} \ X^\mu(\sigma)\, X^\mu(\sigma) \ . \tag{24}$$

We need to evaluate

$$\int (dX) \ \exp\left[-\frac{T}{2}\int_M d^2\sigma \ \sqrt{g}\ g^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\mu\right] =$$

$$= \int (dX) \ \exp\left[-\frac{T}{2}\int_M d^2\sigma \ \sqrt{g}\ X^\mu \left(-\nabla^2 X^\mu\right)\right] \ , \tag{25}$$

where ∇^2 is the Laplacian on scalars. Because of the zero mode this integral diverges. In the special gauge discussed above where

$$\int_M d^2\hat{\sigma}\ \sqrt{\hat{g}} = \text{constant}$$

we should be able to separate the zero-mode, x^μ, and write (25) as

$$\int d^D x \ \left|\,\mathrm{Det}'(-\nabla^2)\right|^{-D/2} \ , \tag{26}$$

where Det' means "product of non-zero eigenvalues" .

FERMIONIC MEASURE

We apply a prescription analogous to Eq.(17) to the fermionic variables χ_α. Again the tangent space has three components

$$\delta\chi_\alpha = \gamma_\alpha \delta\rho + \nabla_\alpha \delta\lambda + \delta\nu_\alpha \quad , \tag{27}$$

where $\delta\nu_\alpha$ is constrained by

$$\gamma_\alpha \delta\nu^\alpha = \nabla_\alpha \delta\nu^\alpha = 0 \quad . \tag{28}$$

The space of solutions of these equations is finite dimensional and we may write

$$\delta\nu^\alpha(\sigma) = \sum_i \zeta_i^\alpha(\sigma)\delta\nu^i \, , \tag{29}$$

where the ζ^α_i represent spinor differentials associated with the manifold M.

Analogous to the conformal vectors would be the constant spinors κ,

$$\nabla_\alpha \kappa = 0 \qquad\qquad \kappa = \sum_r \delta k_r \, \kappa_r \quad . \tag{30}$$

The same point χ_α is then represented by λ and $\lambda+\kappa$. One must therefore constrain the integral over λ to be orthogonal to such constant spinors. This could lead to a significant factor,

$$\int (d\,\delta'\lambda) \exp\left(-\tfrac{1}{2}\|\delta'\lambda\|^2\right) = \left|\det \tilde{Q}\right|^{-1/2} \quad . \tag{31}$$

Here the norms are defined by

$$\|\delta\lambda\|^2 = \int_M d^2\sigma \sqrt{g}\; \delta\lambda^\dagger \gamma_5 \delta\lambda$$

$$= \sum_{rs} \delta k_r \delta k_s \, \tilde{Q}_{rs} + \|\delta'\lambda\|^2 \tag{32a}$$

$$\tilde{Q}_{rs} = \int_M d^2\sigma \sqrt{g}\; \kappa_r^\dagger \gamma_5 \kappa_s \quad . \tag{32b}$$

where κ_r, $r = 1,2,..$ is a basis for the covariantly constant spinors (these are c-numbers, not Grassmann. Hence $\tilde{Q}_{rs} = \tilde{Q}_{sr}$).

Analogous to the Teichmuller metric H_{ij}, there will be a fermionic

$$\tilde{H}_{ij} = \int_M d^2\sigma \sqrt{g}\; g_{\alpha\beta}\, \nu_i^{\alpha\dagger} \gamma_5 \nu_j^\beta \tag{33}$$

which is antisymmetric.

Putting these together, we write

$$(d\chi) = (d\rho)(d\lambda)\, d\nu \;\; K(\rho,\nu) \; , \tag{34a}$$

where ν_i are the Grassmann-Teichmüller parameters and K is a Jacobian. Taking the Polchinski prescription

$$(d\,\delta\chi) = (d\delta\rho)(d\delta'\lambda)\, d\delta\nu \;\; K(\rho,\nu) \tag{34b}$$

we compute K as follows;

$$1 = \int (d\delta\chi) \exp\left(-\tfrac{1}{2}\|\delta\chi\|^2\right)$$

$$= K(\rho,\nu) \int (d\delta\rho)(d\delta'\lambda)\, d\delta\nu \;\times$$

$$\times \exp\left(-\tfrac{1}{2}\int d^2\sigma\sqrt{g}\, g^{\alpha\beta}\left(\delta\rho^\dagger\gamma_\alpha + \nabla_\alpha\delta'\lambda^\dagger + \delta\nu_\alpha^\dagger\right)\gamma_5\left(\gamma_\beta\delta\rho + \nabla_\beta\delta'\lambda + \delta\nu_\beta\right)\right)$$

$$= K(\rho,\nu)\int (d\delta\rho)(d\delta'\lambda)\, d\delta\nu \;\times$$

$$\times \exp\Big(-\tfrac{1}{2}\int d^2\sigma\sqrt{g}\Big(-2\,\delta\rho^\dagger\gamma_5\delta\rho - \left(\delta\rho^\dagger\gamma_5 \nabla\!\!\!/\, \delta'\lambda + h.c.\right) - \delta'\lambda^\dagger\gamma_5\nabla^2\delta'\lambda\Big)$$

$$- \tfrac{1}{2}\sum_{ij}\delta\nu_i \tilde{H}_{ij}\,\delta\nu_j \Big) \tag{35}$$

Acting on spinors we have [8]

$$\nabla^2 - \tfrac{1}{2}\nabla\!\!\!/^{\,2} = \tfrac{1}{2}\left(\nabla^2 + \frac{R}{4}\right) . \tag{36}$$

It remains to effect integration over the fermionic coordinates ψ^μ. With the norm

$$\|\delta\psi\|^2 = \int_M d^2\sigma \sqrt{g}\ \psi^{\mu +} \gamma_5 \psi^{\mu} \tag{37}$$

one should separate the zero mode ψ_0

$$\not{\nabla}\psi_0 = 0 \quad . \tag{38}$$

With the usual definitions [6)]

$$\begin{aligned} I &= \int (d\delta\psi)\, e^{-\frac{1}{2}\|\delta\psi\|^2} \\ &= \int (d\delta\psi_0)(d\delta'\psi)\, e^{-\frac{1}{2}\int d^2\sigma \sqrt{g}\ \psi_0^{+}\gamma_5\psi_0 - \frac{1}{2}\|\delta'\psi\|^2} \\ &= \left(\int d^2\sigma \sqrt{g}\right)^{\nu} \int (d\delta'\psi)\, e^{-\frac{1}{2}\|\delta'\psi\|^2} \quad . \end{aligned} \tag{39}$$

To summarize, the amplitudes are represented by path integrals of the form

$$\int (dX\, d\psi\, de\, d\chi)\, e^{-S} \times \text{wave functions}$$

$$= \int d\bar{x}\, d\psi_0\, (dX)'(d\psi)'(d\varphi\, d\xi)'\, dc\, (d\rho\, d\lambda)'\, d\nu \ \times$$

$$\times \left|\mathrm{Det}'(\bar{g}_{\alpha\beta}\nabla^2 + [\nabla_\alpha \nabla_\beta])\right|^{1/2} \left|\mathrm{Det}'(\nabla^2 - \tfrac{1}{2}\not{\nabla}^2)\right|^{-1/2} \left|\det H\right|^{1/2}$$

$$\times \left(\det\tilde{H} / \det\tilde{G}\right)^{-1/2} \exp\left[-\frac{1}{2}\int_M d^2\sigma \left(\bar{g}^{\alpha\beta}\partial_\alpha X \partial_\beta X + \psi^{+}\not{\nabla}\psi + \dots\right)\right] (w.f.), \tag{40}$$

where χ_α is understood to be expressed in terms of $\rho, \lambda, \dots$ etc. Superconformal invariance now permits φ, ξ, ρ and λ to be fixed conveniently. It is not clear yet whether there is some non-trivial dependence of the fermionic parameters ν.

TENSORS AND SPINORS IN A COMPLEX BASIS

The metric can always be brought to isothermal form

$$\begin{aligned} ds^2 &= e^{2\varphi}\left((d\sigma^1)^2 + (d\sigma^2)^2\right) \\ &= 2 e^{2\varphi}\, dz\, d\bar{z} \quad . \end{aligned} \tag{41}$$

where

$$z = \frac{\sigma^1 + i\sigma^2}{\sqrt{2}} \quad , \quad \bar{z} = \frac{\sigma^1 - i\sigma^2}{\sqrt{2}} \quad . \tag{42}$$

Transformations which preserve the isothermal form are analytic in z

$$z \to z' = f(z) \quad , \quad \partial_{\bar{z}} f(z) = 0 \tag{43a}$$

$$e^{2\varphi} \to e^{2\varphi'} = e^{2\varphi} \left|\frac{dz}{dz'}\right|^2 \tag{43b}$$

i.e.

$$\varphi'(z') = \varphi(z) + \ln\left|\frac{dz}{dz'}\right| \quad . \tag{44}$$

Irreducible tensors have only one component in this basis. They are classified by a pair of weights (p,q). Thus, if

$$\psi \to \psi' = \left(\frac{dz'}{dz}\right)^p \left(\frac{d\bar{z}'}{d\bar{z}}\right)^q \psi \tag{45}$$

then ψ is a tensor of type (p,q). In particular

$$dz \sim (1,0) \quad \text{and} \quad d\bar{z} \sim (0,1)$$

$$g_{z\bar{z}} = e^{2\varphi} \sim (-1,-1) \quad .$$

The covariant derivatives ∇_z and $\nabla_{\bar z}$ are defined by

$$\nabla_z \psi = e^{-2p\varphi}\,\partial_z\left(e^{2p\varphi}\psi\right) \qquad (46a)$$

$$\nabla_{\bar z} \psi = e^{-2q\varphi}\,\partial_{\bar z}\left(e^{2q\varphi}\psi\right) . \qquad (46b)$$

These are covariant tensors of type $(p-1,q)$ and $(p,q-1)$, respectively. They are covariant because $e^{2p\varphi}\psi$ is of type $(0,q-p)$ so that $\partial_z(e^{2p\varphi}\psi)$ is covariant of type $(-1,q-p)$, etc.

Definitions can be extended to spinors by introducing isothermal frames

$$e^+ = e^{\varphi}\,dz \quad \text{and} \quad e^- = e^{\varphi}\,d\bar z \qquad (47a)$$

so that $ds^2 = 2e^+e^-$. Frame rotations generally imply

$$e^+ \to e^+ e^{i\alpha} , \quad e^- \to e^- e^{-i\alpha} , \qquad (47b)$$

where $\alpha(z)$ is real. The isothermal form is maintained under analytic reparametrization if the frame rotations are fixed appropriately. Thus one wants

$$e^{\varphi'(z')}\,dz' = e^{\varphi(z)}\,dz\, e^{i\alpha(z)} , \qquad (48a)$$

which implies

$$\begin{aligned} i\alpha(z) &= \varphi'(z') - \varphi(z) + \ln\frac{dz'}{dz} \\ &= \ln\frac{dz'}{dz}\Big/\left|\frac{dz'}{dz}\right| \\ &= i\arg\frac{dz'}{dz} . \end{aligned} \qquad (48b)$$

A typical spinor ψ would transform according to

$$\psi(z) \to \psi'(z) = e^{\frac{i}{2}\alpha(z)}\psi(z) \qquad (49)$$

under frame rotations and as a scalar (say) under reparametrization. In the isothermal frames therefore, its response to analytic reparametrizations would be

$$\begin{aligned} \psi(z) \to \psi'(z') &= e^{\frac{i}{2}\arg\frac{dz'}{dz}}\psi(z) \\ &= \left(\frac{dz'}{dz}\right)^{1/4}\left(\overline{\frac{dz'}{dz}}\right)^{-1/4}\psi(z) \end{aligned}$$

i.e. it is a tensor of type $(1/4,-1/4)$. The important feature is that $p-q = 1/2$. The defining character is

$$\begin{aligned} p - q &= \text{integer for tensors} \\ &= \tfrac{1}{2}\ \text{integer for spinors} . \end{aligned}$$

An important property concerns complex conjugation

$$\text{if} \quad \psi \sim (p,q) \quad \text{then} \quad \psi^* \sim (q,p) .$$

Consider now the tensors $\zeta_{\alpha\beta}$ subject to

$$g^{\alpha\beta}\zeta_{\alpha\beta} = 0 \quad \text{and} \quad \nabla^{\alpha}\zeta_{\alpha\beta} = 0 .$$

The first of these implies $\zeta_{z\bar z} = 0$. The second then gives

$$\begin{aligned} 0 &= \nabla^z \zeta_{zz} \\ &= g^{z\bar z}\nabla_{\bar z}\zeta_{zz} \\ &= e^{-2\varphi}\partial_{\bar z}\zeta_{zz} \end{aligned}$$

and the conjugate relation for $\zeta_{\bar z\bar z} = \zeta_{zz}^*$,

$$\partial_z\zeta_{\bar z\bar z} = 0 .$$

Thus, ζ_{zz} is an analytic function of z (quadratic differential).

The fermionic Teichmuller modes must satisfy

$$\gamma_\alpha \nu^\alpha = 0 \qquad \text{and} \qquad \nabla_\alpha \nu^\alpha = 0 .$$

It is convenient to introduce frame notation here,

$$\gamma_\alpha \nu^\alpha = \gamma_+ \nu^+ + \gamma_- \nu^- \qquad \gamma_\pm = \frac{\gamma_1 \mp i\gamma_2}{\sqrt{2}}$$

$$= \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} \nu_1^+ \\ \nu_2^+ \end{pmatrix} + \begin{pmatrix} 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \nu_1^- \\ \nu_2^- \end{pmatrix}$$

$$= \sqrt{2}\begin{pmatrix} \nu_2^- \\ \nu_1^+ \end{pmatrix}$$

$$= 0$$

i.e.

$$\nu^+ = \begin{pmatrix} 0 \\ \nu_2^+ \end{pmatrix} , \qquad \nu^- = \begin{pmatrix} \nu_1^- \\ 0 \end{pmatrix} .$$

Because $\nu^\pm = \frac{1}{\sqrt{2}}(\nu^1 \pm i\nu^2)$, where ν^1 and ν^2 are Majorana it follows that $\nu^- = B\nu^{+*}$, $B = \gamma_1$. Hence *)

$$\nu_1^- = \nu_2^{+*} \sim \left(-\tfrac{3}{4}, \tfrac{3}{4}\right) .$$

---

*) It is necessary to fix a sign convection. Since under tangent space rotations

$$e^+ \to e^+ e^{i\alpha} \qquad \sim \left(\tfrac{1}{2}, -\tfrac{1}{2}\right)$$

we must have

$$\psi^\dagger \gamma^+ \psi \to \psi^\dagger \gamma^+ \psi \, e^{i\alpha}$$

i.e.

$$\psi_1^\dagger \psi_2 \to \psi_1^\dagger \psi_2 \, e^{i\alpha} \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

i.e.

$$\psi \to e^{-\frac{i}{2}\alpha\gamma_5} \psi .$$

This means that

$$\psi_1 \sim \left(-\tfrac{1}{4}, \tfrac{1}{4}\right) .$$

$$\psi_2 \sim \left(\tfrac{1}{4}, -\tfrac{1}{4}\right) .$$

In the above example, therefore we must interpret the transformation

$$e^{i\alpha} = \left(\frac{dz'}{dz}\right)^{1/2} \left(\frac{d\bar{z}'}{d\bar{z}}\right)^{-1/2} .$$

to imply

$$\nu_1^+ \sim \left(-\tfrac{1}{4}+\tfrac{1}{2}, \tfrac{1}{4}-\tfrac{1}{2}\right) = \left(\tfrac{1}{4}, -\tfrac{1}{4}\right)$$

$$\nu_2^+ \sim \left(\tfrac{1}{4}+\tfrac{1}{2}, -\tfrac{1}{4}-\tfrac{1}{2}\right) = \left(\tfrac{3}{4}, -\tfrac{3}{4}\right) .$$

The condition $\gamma_\alpha \nu^\alpha = 0$ eliminates the $(\frac{1}{4}, -\frac{1}{4})$ piece and leaves the irreducible piece,

$$\nu^+ = \begin{pmatrix} 0 \\ \nu_2^+ \end{pmatrix} \sim \left(\tfrac{3}{4}, -\tfrac{3}{4}\right)$$

$$\nu^- = \begin{pmatrix} \nu_1^- \\ 0 \end{pmatrix} \sim \left(-\tfrac{3}{4}, \tfrac{3}{4}\right) .$$

Consider now the other condition, $\nabla_\alpha \nu^\alpha = 0$. It breaks into two components

$$\nabla_\alpha \nu^\alpha = \nabla_+ \nu^+ + \nabla_- \nu^- = \begin{pmatrix} \nabla_- \nu_1^- \\ \nabla_+ \nu_2^+ \end{pmatrix} .$$

According to the general formula

$$\nabla_1 \nu_2^+ = e^{-\varphi} \nabla_z \nu_2^+$$

$$= e^{-\varphi} e^{-\frac{3}{2}\varphi} \partial_z \left(e^{\frac{3}{2}\varphi} \nu_2^+\right)$$

$$= e^{-\frac{5}{2}\varphi} \partial_z \left(e^{\frac{3}{2}\varphi} \nu_2^+\right)$$

which has weight $(\frac{1}{4}, -\frac{1}{4})$ as it should.

Hence the Grassmann analogues of quadratic differentials must be of the form

$$\nu_2^+ \sim e^{-\frac{3}{2}\varphi} \times \text{analytic function of } \bar{z} .$$

The normalization condition would be

$$\| \nu \|^2 = \int_M d^2\sigma \sqrt{g}\, g_{\alpha\beta}\, \nu^{\alpha +} \gamma_5\, \nu^{\beta}$$

$$= \int dz d\bar{z}\; e^{2\varphi} \left(\nu^{+\dagger} \gamma_5 \nu^{+} + \nu^{-\dagger} \gamma_5 \nu^{-} \right)$$

$$= -2 \int dz d\bar{z}\; e^{2\varphi}\; \nu_z^{+\dagger} \nu_z^{+}$$

$$< \infty$$

CONCLUSION

In these notes we dealt with the issue of gauge fixing for the spinning string in the Neveu-Schwarz-Ramond formalism. We demonstrated that the gravitino field χ_α has a Grassmannian-Teichmüller component which cannot be gauged away by superconformal transformations. These variables are the fermionic generalizations of the bosonic Teichmüller variables. Using the Riemann-Roch theorem it can be shown that for a closed surface of genius g the bosonic Teichmüller variables span a space of $3g-3$ complex dimensions.[9] The analogous number for the fermionic variables is derived similarly and in $2g-2$ if $g \geq 2$. For $g = 0$ it is zero and for $g = 1$ it is 1.

To proceed further we need to know the detailed structure of the integrands of the amplitudes in particular their dependence on τ and ν.

Recently [10], there has been some progress in calculating the bosonic determinants entering our Eq.(40). These calculations use the Selberg trace formula. Using the same formula Gava et al. [11] have shown that the occurrence of the divergences in the bosonic partition function can be related to shrinking to zero of the hyperbolic geodesics on the world sheet. However, a useful parametrization of the space of moduli is still lacking. This problem is perhaps more challenging for the fermionic moduli.

One important problem not dealt with in this note is the GSO[1] projection on supersymmetric subspace of all possible states of NSR model. This is essentially the problem of summing over the appropriate G-parity states in Polyakov's path integral. To achieve this Witten [12] has suggested to sum over all possible 2^{2g} different spin structures of a surface of genus g.

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After this work was completed we received Refs.13 and 14, where similar ideas are discussed.

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