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Sonia Berger

Roberto Hojman

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Jorge Santamarina

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Sonia Berger

Instituto de Física, Universidad Austral de Chile, Casilla 567, Valdivia
Chile

Roberto Hojman

Departamento de Física, Facultad de Ciencias, Universidad de Chile,
Casilla 653, Santiago, Chile
Centro de Física El Trauco, Fanor Velasco 41D, Santiago, Chile.

and

Jorge Santamarina**

International Centre for Theoretical Physics, Trieste, Italy.

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** Permanent address: Instituto de Física, Universidad Austral de Chile, Casilla 567, Valdivia, Chile.

ABSTRACT

The gravitational field equations for a spherical symmetric perfect fluid are completely solved. The general analytical solution obtained depends on an arbitrary function of the radial coordinate. As an illustration example of the proposed procedure the exterior Schwarzschild solution is regained.

In this article we derive the whole set of exact spherically symmetric solutions of Einstein gravitational field equations (with cosmological constant) when a perfect fluid is assumed to be the source of the gravitational field.

An extension of the solution to plane and hyperbolic symmetries can easily to obtained and our previous result |1| for the (exact solution of the) plane symmetric case is regained.

In what follows we pose the mathematical problem and then we mimic the procedure used in our recent work |1| to reduce the problem to mere quadratures.

The line element considered here is

$$ds^2 = g^2(x)\,dt^2 - dx^2 - r^2(x)\left[d\theta^2 + \sin^2\theta\, d\varphi^2\right] \tag{1}$$

which represents the most general static line element admitting spherical transformations.

Einstein field equations with cosmological constant are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} \tag{2}$$

with

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{3}$$

In Ref. 1 we proved that if the right-hand side of Eq. (2) models a perfect fluid with pressure p and energy density ρ

$$T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu - p\, g_{\mu\nu} \tag{4}$$

and whose flow lines are tangent to the unit vector $u^\mu = \frac{1}{g}\delta^\mu_0$, the change of variable

$$\tilde{p} = p - \Lambda$$
$$\tilde{\rho} = \rho + \Lambda \tag{5}$$

transforms Eq. (2) into an equivalent system with $\Lambda = 0$. Therefore, twirls can now be dropped keeping in mind that the $\Lambda \neq 0$ case is already included. The field equations (2) for the metric given by (1) are (see for instance Ref. 1)

$$\frac{2r''}{r} + \frac{r'^2}{r^2} - \frac{1}{r^2} = -\rho$$

$$\frac{r'^2}{r^2} + \frac{2r'g'}{rg} - \frac{1}{r^2} = p \tag{6}$$

$$\frac{r''}{r} + \frac{g''}{g} + \frac{r'g'}{rg} = p$$

and the equation of hydrostatic support (Bianchi identities) is

$$(\rho + p)\frac{g'}{g} + p' = 0 \tag{7}$$

where the prime denotes x differentiation.

The system of equations can be re-accommodated if the first of Eq. (6) is multiplied by $r^2 r'$ and integrated over x. Then

$$r'^2 = 1 - \frac{2m(r)}{r} \tag{8}$$

where

$$\frac{dm}{dr} = \frac{1}{2}\rho r^2 \tag{9}$$

Replacing Eqs. (7) and (8) in the second Eq. (5) one obtains

$$(r-2m)\left(\frac{1}{r^3}-\frac{2}{r^2}\frac{dp}{dr}\frac{1}{\rho+p}\right)=p+\frac{1}{r^2} \tag{10}$$

Defining

$$G \equiv -\frac{(r-2m)}{p+\frac{1}{r^2}} \tag{11}$$

Eq. (10) becomes

$$r^3G\,(G-r^3)\frac{dp}{dr}+r^3\,(G+r^3)\left(\frac{dG}{dr}+r^2\right)p+(G+r^3)\left(r^3+r\frac{dG}{dr}-2G\right)=0 \tag{12}$$

Eq. (12) can be integrated at once for $p(r)$ if $G(r)$ is a given function, in fact

$$p(r)=\exp\left[\int\frac{(G+r^3)\left(\frac{dG}{dr}+r^2\right)}{G\,(r^3-G)}\right]\Bigg\{p_0+\int\frac{(G+r^3)\left(r^3+r\frac{dG}{dr}-2G\right)}{r^3G\,(G-r^3)}\exp\left[-\int\frac{(G+r^3)\left(\frac{dG}{dr}+r^2\right)}{G\,(r^3-G)}dr\right]dr \tag{13}$$

where p_0 is an integration constant.

The function $\rho(r)$ can be obtained with the help of Eq. (1) and the definition of G

$$\rho=\frac{1}{r^2}\left(G\frac{dp}{dr}+\frac{dG}{dr}p-\frac{2G}{r^3}+\frac{\frac{dG}{dr}}{r^2}+1\right) \tag{14}$$

where $p(r)$ is given by Eq. (13). The metric coefficient g can be found by direct integration of Eq. (7) and then using Eq. (10) and the

definition (11)

$$g^2(r)=g_0^2\exp\left(-2\int\frac{dp/dr}{\rho+p}dr\right)=\frac{1}{r}\,e^{-\int\frac{r^2}{G}dr} \tag{15}$$

To complete the integration we can recover the link between the metric coefficient r and the original variable x. From Eq. (7)

$$x=\int\frac{dr}{\sqrt{1-\frac{2m}{r}}} \tag{16}$$

with $m(r)$ constructed from Eq. (11).

As we have remarked in our previous work the crucial step to get the general solutions (13), (14), (15) and (16) is the definition (11) of the arbitrary function $G(r)$. It is worthwhile to emphasize that any choice of $G(r)$ does provide a solution to field equations (6), (7) and no spurious solutions are introduced anywhere as it can be straight-forwardly shown by replacing p, ρ, g and r given, respectively, by expressions (13), (14), (15) and (16) in the original field equations. It is rather simple and illustrative to see how the presciption works.

By using the plane symmetric version of the above depicted procedure |1|, the most general function G that produces a γ law equation of state was found |2|. Also, the associated fluid pressure, energy density and metric coefficients were explicitly written down. In particular, after a change of variables, the solutions given by Tabensky and Taub |3|, and by Teixeira, Nolk and Som |4| were re-encountered.

In the present case, choose for instance

$$G(r)=-r^2\,(r-2M) \tag{17}$$

(where M is a constant) and

$$p_0=0 \tag{18}$$

Under such conditions

$$r^3 + r\frac{dG}{dr} - 2G = 0 \tag{19}$$

and consequently (see Eq. (13))

$$p(r) = 0 \tag{20}$$

Eqs. (14), (19) and (20) imply

$$\rho(r) = 0 \tag{21}$$

Also, comparing Eq. (11) with Eq. (17)

$$m(r) = M \tag{22}$$

Thus, Eq. (16) can be written as

$$dx^2 = \frac{dr^2}{1 - \frac{2M}{r}} \tag{23}$$

Finally, from Eq. (15)

$$g^2 = \left(1 - \frac{2M}{r}\right) \tag{24}$$

As it can be readily recognized, Eqs. (20),(21),(23) and (24) represent the well known exterior Schwarzschild solution.

Any other solution can be cast in the above scheme as well.

In a forthcoming article |5| the function G leading to interior Schwarzschild |6|, Tolman |7|, other known |8| and some new explicit solutions will be exhibited.

The time dependent extension of the proposed method is currently under analysis |9|.

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