CONVEXITY FOR SURFACES OVER TRIANGLES

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For the functions defined on triangle and having continuous partial derivatives up to the second order, a necessary and sufficient condition for their convexity over the triangle is presented. Then a simple proof of the convexity for the Bernstein triangular polynomials, a theorem due to Chang and Davis, is given.
It is obvious that the surface $z=P(t)$ is convex on $T$ iff the curve (4) is convex for any $P_0, P_1 \in T$. Let $C^2(T)$ be the collection of all functions $z=P(t)$ continuous partial derivatives up to the second order on $T$. If $F \in C^2(T)$ then the convexity of the curve (4) can be checked only by elementary Calculus. Upon taking derivatives of the first and the second order with respect to $t$ in both sides of (4), we obtain

\[ \frac{dz}{dt} = \frac{\partial P}{\partial t} \]

and

\[ \frac{d^2z}{dt^2} = \frac{\partial^2 P}{\partial t^2} + \frac{\partial P}{\partial t} \frac{\partial^2 P}{\partial t \partial v} + \frac{\partial^2 P}{\partial t \partial w} \]

in which $H(P)$ denotes the Hessian of $F$, namely

\[ H(P) = \begin{pmatrix} \frac{\partial^2 F}{\partial u^2} & \frac{\partial^2 F}{\partial u \partial v} & \frac{\partial^2 F}{\partial u \partial w} \\ \frac{\partial^2 F}{\partial v \partial u} & \frac{\partial^2 F}{\partial v^2} & \frac{\partial^2 F}{\partial v \partial w} \\ \frac{\partial^2 F}{\partial w \partial u} & \frac{\partial^2 F}{\partial w \partial v} & \frac{\partial^2 F}{\partial w^2} \end{pmatrix} \]

and each partial derivative of $F$ is taken regardless of $(1), (2), (3)$, as if $u, v$ and $w$ are independent variables. Writing $\xi = u - u_0, \eta = v - v_0, \zeta = w - w_0$, note that

\[ \xi, \eta, \zeta \neq 0. \]

Since the curve (4) is convex iff $\frac{d^2z}{dt^2} \geq 0$ for $t \in [0,1]$, we arrive immediately at

**Theorem 1.** Function $F \in C^2(T)$ is convex on the triangle $T$ iff

\[ (\xi, \eta, \zeta)H(P)(\xi, \eta, \zeta) \geq 0 \]

holds for any $P \in T$ and for any $(\xi, \eta, \zeta)$ such that $\xi, \eta, \zeta \neq 0$.

Under assumption (7), the quadratic form in (8) can be reduced to the following sum of squares (see [4]):

\[ \Delta_\xi \Delta_\eta \xi \eta \Delta_\zeta \zeta \]

in which

\[ \Delta_\xi = \frac{\partial^2 P}{\partial u^2}, \quad \Delta_\eta = \frac{\partial^2 P}{\partial v^2}, \quad \Delta_\zeta = \frac{\partial^2 P}{\partial w^2} \]

Substituting $\xi, \eta, \zeta$ into (9), we get

\[ (\xi, \eta, \zeta)H(P)(\xi, \eta, \zeta) = (\Delta_\xi \Delta_\eta \xi \eta + \Delta_\xi \Delta_\zeta \xi \zeta + \Delta_\eta \Delta_\zeta \eta \zeta) \]

Since $\xi$ and $\eta$ in the right-hand side of (10) are independent, we see that the quadratic form is nonnegative iff inequalities

\[ \Delta_\xi \Delta_\eta \geq 0, \quad \Delta_\xi \Delta_\zeta \geq 0, \quad \Delta_\eta \Delta_\zeta \geq 0 \]

hold for every point $P$ in the triangle $T$. Putting these inequalities into symmetric form, we obtain

**Theorem 2.** Function $F$ is convex on the triangle $T$ iff the following inequalities

\[ \Delta_\xi \Delta_\eta \geq 0, \quad \Delta_\xi \Delta_\zeta \geq 0, \quad \Delta_\eta \Delta_\zeta \geq 0 \]

hold for every point $P$ in $T$.

Next, it is easy to verify

**Corollary.** A sufficient condition for the convexity of $F$ is that inequalities

\[ \Delta_\xi \geq 0, \quad \Delta_\eta \geq 0, \quad \Delta_\zeta \geq 0 \]

hold for every point $P$ in $T$.

3. Application

Let $f(P)$ be a function defined on the triangle $T$. The $n$th Bernstein polynomial associated with $f$ is defined by
\[ B^0(f;P) := \sum_{i+j+k=n} f(i/n, j/n, k/n) J^0_{i,j,k}(P) \]  
(13)

where

\[ J^0_{i,j,k}(P) := \frac{n!}{i! j! k!} \left( \begin{array}{c} n \\ i, j, k \end{array} \right), \quad i+j+k=n. \]  
(14)

Write \( f_{i,j,k} := f(i/n, j/n, k/n) \) for brevity. Upon using three formal shifting operators

\[ E_{i,j,k} := f_{i+1,j,k}, \quad E_{2i,j,k} := f_{i,j+k+1}, \quad E_{3i,j,k} := f_{i,j,k+1}, \]

where \( i+j+k=n-1 \), the Bernstein polynomial (13) can be rewritten in a very neat form

\[ B^n(f;P) = (uE_1 + vE_2 + wE_3)^n f_{0,0,0}, \]  
(15)

Chang and Davis proved in [2] the following

**Theorem 3.** If

\[ \begin{align*}
\sum_{k=1}^{n} & (E_k - E_{k-1}) f_{i,j,k} \\
\sum_{k=1}^{n} & (E_k - 2E_{k-1} + E_{k-2}) f_{i,j,k} \\
\sum_{k=1}^{n} & (E_k - 3E_{k-1} + 3E_{k-2} - E_{k-3}) f_{i,j,k}
\end{align*} \]  
(16)

hold for \( i+j+k=n-2 \), then \( B^n(f;P) \) is convex on \( T \).

Several proofs for the Theorem 3 have appeared (see, for example, [1], [17], [4]), here we present another proof based upon the previous results of this paper.

It is very easy to show that

\[ \frac{\partial}{\partial u} B^n(f;P) = n(uE_1 + vE_2 + wE_3)^{n-1} E_1 f_{0,0,0}, \]  
\[ \frac{\partial}{\partial v} B^n(f;P) = n(uE_1 + vE_2 + wE_3)^{n-1} E_2 f_{0,0,0}, \]  
\[ \frac{\partial}{\partial w} B^n(f;P) = n(uE_1 + vE_2 + wE_3)^{n-1} E_3 f_{0,0,0}, \]

By the first inequality of (16) and the fact \( J_{0,0,0} \geq 0 \), we conclude that \( \Delta_v \geq 0 \) for any point \( P \) in \( T \). The second and the third inequalities of (16) imply that \( \Delta_w \geq 0 \) and \( \Delta_w \geq 0 \) respectively. By the corollary of last section the convexity for \( B^n(f;P) \) is then established.

\[ \frac{\partial^2}{\partial u^2} B^n(f;P) = n(n-1)(uE_1 + vE_2 + wE_3)^{n-2} E_1^2 f_{0,0,0}, \]  
\[ \frac{\partial^2}{\partial v^2} B^n(f;P) = n(n-1)(uE_1 + vE_2 + wE_3)^{n-2} E_2^2 f_{0,0,0}, \]

\[ \frac{\partial^2}{\partial w^2} B^n(f;P) = n(n-1)(uE_1 + vE_2 + wE_3)^{n-2} E_3^2 f_{0,0,0}, \]

similar expressions for other partial derivatives of the second order.

Hence we have

\[ \frac{\partial^2}{\partial u \partial v} B^n(f;P) = n(n-1)(uE_1 + vE_2 + wE_3)^{n-2} (E_1 E_2 f_{0,0,0}, \]}

or equivalently

\[ \Delta_u = n(n-1) \sum_{i+j+k=n-2} (E_k - E_{k-1}) f_{i,j,k} \]  
(16)

By the first inequality of (16) and the fact \( J_{0,0,0} \geq 0 \) for any \( P \) in \( T \), we conclude that \( \Delta_u \geq 0 \) for any point \( P \) in \( T \). The second and the third inequalities of (16) imply that \( \Delta_v \geq 0 \) and \( \Delta_w \geq 0 \) respectively. By the corollary of last section the convexity for \( B^n(f;P) \) is then established.

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REFERENCES


