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ABSTRACT

The Inverse Scattering Method of Belinsky and Zakharov is used to investigate axially symmetric stationary vacuum soliton solutions in the five-dimensional representation of the Brans-Dicke-Jordan theory of gravitation, where the scalar field of the theory is an element of a five-dimensional metric. The resulting equations for the space-time metric are similar to those of solitons in general relativity, while the scalar field generated is the product of a simple function of the coordinates and an already known scalar field solution. Kerr-like solutions are considered that reduce, in the absence of rotation, to the five-dimensional form of the well-known Weyl-Levi Civita axially symmetric static vacuum solution. Hence these metrics can be interpreted as the exterior gravitational fields of rotating non-spherical configurations in the Brans-Dicke-Jordan theory. With a proper choice of the parameters in the solutions, the deviation from spherical symmetry will be due only to the angular momentum of the system, as it is with the Kerr metric in general relativity. An explicit exact solution is given.

I. INTRODUCTION

The Brans-Dicke-Jordan (BDJ) scalar tensor theory was first investigated in connection with Dirac's large number hypothesis, by Jordan 1) and, in connection with Mach's principle, by Brans and Dicke 2),3). The theory was originally expressed in a representation in which the local measurable value of the gravitational "constant", G, is a function of a scalar field 4,5). It can also be put in a form which is Einstein's general relativity with the scalar field of the theory acting as an additional external non-gravitational field 3). Another way of representing the BDJ theory is with a five-dimensional field equation 1),4),5), where the metric is independent of the fifth coordinate, and the elements $g_{\mu\nu}$, where $\mu = (0,1,2,3)$ is a space-time index, vanish. In vacuum, this is just a special case of the Klein-Jordan-Thiry 6) theory, where the $g_{\mu\nu}$ components are identified with an "electromagnetic" vector potential $A_{\mu}$. The Klein-Jordan-Thiry theory is in turn a generalization of the original Kaluza-Klein unified theory of gravity and electromagnetism, in which $g_{\mu\nu} = \text{constant}$.

In order to study the physical implications of a theory we must find solutions of its field equations. Fortunately, it has been possible to find exact solutions of relativistic gravitational theories, like general relativity and BDJ, when physically reasonable symmetries have been assumed. For instance, many astrophysical systems of interest are approximately stationary and axially symmetric, and can be very well described with metrics having these symmetries. The Kerr solution 7) is perhaps the most important known exact solution to the vacuum Einstein field equations. It represents the exterior gravitational field of stationary axially symmetric rotating systems, which in the absence of rotation reduces to the Schwarzschild spherically symmetric solution. That is, the Kerr solution does not possess deformations from spherical symmetry, other than those caused by the angular velocity.

There are also axially symmetric stationary solutions in the BDJ theory representing the exterior metric of rotating configurations 8), but they can only become spherically symmetric if the scalar field is a constant. On the other hand, there is a simple extension 9) of the Cosgrove-Tomimatsu-Sato solutions of general relativity 9) to the BDJ theory that allows the construction of solutions in which the system is spherically symmetric in the absence of rotation. However, the Cosgrove-Tomimatsu-Sato solutions involve, in general, transcendental functions.
In this paper the inverse scattering method of Belinsky and Zakharov (R2) is used to construct axially symmetric stationary vacuum solutions to the BDJ field equations. In particular, I shall consider solutions that reduce to the well-known spherically symmetric static BDJ vacuum metric in the absence of rotation, and therefore can be thought of as generalizations of the Kerr solution in the presence of a scalar field. This solution could be a starting point for the study of external gravitational fields of rotating stars in the BDJ theory. The analysis is carried out in a five-dimensional representation of the BDJ theory, where the application of the BZ technique is generalized in a straightforward way. Furthermore, this representation provides a basis for future possible applications of the BZ formalism to the problem of finding classical solutions of, Kaluza-Klein type, field theories of more than four dimensions. We point out, however, that the BZ formalism could be applied directly to a particular four-dimensional representation of the BDJ theory (the Einstein-scalar theory), since for an axially symmetric stationary metric in vacuum or with an electromagnetic field source, the scalar field decouples from the second-order field equations for the space-time metric, and therefore these equations are identical to those of general relativity.\[12\]

In fact, Belinsky studied exact solutions of the Einstein-scalar theory that describe the evolution of gravitational soliton waves against the background of Friedmann cosmological models. A very interesting feature of this work is that the energy-momentum tensor of the scalar field is now representing the matter field of a perfect fluid with an equation of state pressure = energy density.\[13\]

In the next section, we will briefly describe the Belinsky and Zakharov technique. Their notation \[11\] will be followed in its essentials. For the sake of generality, we will not, yet, restrict the number of dimensions considered, since the generalization only introduces trivial modifications to the BZ formulas. Hence, let us consider a m+2 dimensional metric that depends only on two co-ordinates \(p\) and \(Z\). The line element can be written in the Lewis form:\[15\]

\[ds^2 = g_{AB} dx^A dx^B = f(p,Z) \left[dp^2 + dZ^2\right] + g_{ab}(p,Z) dx^a dx^b;\]

where \(a, b = 1 - m\). \[2.1\]

Furthermore, the source-free Einstein equations in \(m+2\) dimensions admit the following co-ordinate condition:\[2.2\]

\[G_{AB} = R_{AB} - \frac{1}{2} g_{AB} R = 0.\]

With the metric (2.1) and the condition (2.3) the field equation (2.2) takes the form:\[2.4\]

\[(\rho g_{AB} g^{AB})_{,\rho} + (\rho g_{AB} g^{AB})_{,\Sigma} + g^{AB} \left(V_{,\rho} + V, \rho\right) = 0,\]

For the second co-ordinate condition to be satisfied, we have:\[2.3\]

\[det g = det g_{AB} = -\rho^2.\]

With the metric (2.1) and the condition (2.3) the field equation (2.2) takes the form:

\[(\rho g_{AB} g^{AB})_{,\rho} + (\rho g_{AB} g^{AB})_{,\Sigma} + g^{AB} \left(V_{,\rho} + V, \rho\right) = 0,\]

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II. THE INVERSE SCATTERING METHOD OF BELINSKY AND ZAKHAROV

This method allows the generation of large classes of new solutions from old, when the metric tensor depends only on two variables. Belinsky and Zakharov \[10,11\] developed and employed the method to obtain exterior solutions in general relativity. In particular, they have obtained the Kerr solution \[11\] starting from a flat space-time metric background. The technique can also be used in relativistic theories of more than four dimensions. This was done by Belinsky and Ruffini \[14\], in the framework of the five-dimensional Jordan-Brans-Dicke theory, to generate stationary axially symmetric solutions starting from a constant metric. In what follows we will briefly describe the Belinsky and Zakharov technique. Their notation \[11\] will be followed in its essentials. For the sake of generality, we will not, yet, restrict the number of dimensions considered, since the generalization only introduces trivial modifications to the BZ formulas. Hence, let us consider a m+2 dimensional metric that depends only on two co-ordinates \(p\) and \(Z\). The line element can be written in the Lewis form:\[15\]

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With the metric (2.1) and the condition (2.3) the field equation (2.2) takes the form:

\[(\rho g_{AB} g^{AB})_{,\rho} + (\rho g_{AB} g^{AB})_{,\Sigma} + g^{AB} \left(V_{,\rho} + V, \rho\right) = 0,\]
(\ln f)_\rho = -1/p + 1/4\rho \quad \text{Trace } [V^2 - V^2], \quad (2.5)

(\ln f)_\alpha = 1/2p \quad \text{Trace } UV, \quad (2.6)

where we used the conventional notation \( \frac{\partial}{\partial X} (\ ) \). Eq. (2.5) is the compatibility condition for the following system of linear equations:

\[ D_\alpha \psi = \left[ \frac{\phi}{\partial X} \right]_{X^2 + \rho^2} \frac{\partial}{\partial X} \psi = \left[ \frac{\partial V - \lambda \psi}{\lambda^2 + \rho^2} \right] \psi, \quad (2.7) \]

\[ D_\alpha \psi = \left[ \frac{\phi}{\partial X} + \frac{\lambda \psi}{\lambda} \right] \frac{\partial}{\partial X} \psi = \left[ \frac{\partial U + \lambda \psi}{\lambda^2 + \rho^2} \right] \psi, \quad (2.8) \]

where \( \lambda \) is a complex variable. Moreover, when \( \lambda = 0 \), Eqs. (2.7) and (2.8) are just

\[ \phi \psi_{\alpha} \psi^{-1} = V, \quad (2.9) \]
\[ \phi \psi_{\alpha} \psi^{-1} = U, \quad (2.10) \]

which imply that

\[ \psi(\lambda = 0) = g. \quad (2.11) \]

The solitonic solution \( \psi \) is given as a function of a particular solution \( \psi_0 \), corresponding to a given background metric \( \tilde{g} \), in the following way:

\[ \psi = \psi_0 + \sum_{k=1}^{n} \gamma_{\alpha} \rho_{\beta} \psi_{\gamma} \rho \psi_{\delta} \psi_{\epsilon}, \quad (2.12) \]

where

\[ (\gamma_{\alpha})_{\beta} \varepsilon = \sum_{k=1}^{n} \gamma_{\alpha} \rho_{\beta} \psi_{\gamma} \rho \psi_{\delta} \psi_{\epsilon}, \quad (2.13) \]

\[ \psi_0 \text{ arbitrary constants.} \]

\[ \psi_0^{-1}(k) \text{ is the inverse of } \psi_0 \text{ evaluated at} \lambda = \nu_k \pm Z \pm \sqrt{\nu_k - Z^2 + \rho^2}; \quad (2.15) \]

\[ \psi_0^{-1}(k) \text{ arbitrary constants and } \Gamma_{\alpha \beta}^{-1} \text{ are the elements of the inverse of the following matrix:} \]

\[ \Gamma_{\alpha \beta} = \frac{\psi_0^{-1}(k) \rho_{\alpha} \rho_{\beta}}{\nu_k - Z + \sqrt{\nu_k - Z^2 + \rho^2}}. \quad (2.16) \]

The metric \( g \) is then given by

\[ g = \psi(\lambda = 0) = \tilde{g} - \sum_{k=1}^{n} \gamma_{\alpha} \rho \psi_{\beta} \nu_k. \quad (2.17) \]

It also follows, from Eqs. (2.5) and (2.6) that

\[ e = C_\alpha \rho^n \prod_{k=1}^{n} (\nu_k)^2 - \nu_k^2 \prod_{k=1}^{n} (\nu_k^2 + \rho^2)^{-1} \text{det } \Gamma, \quad (2.18) \]

where \( f_0 \) is the solution corresponding to \( \tilde{g} \), and \( C_\alpha \) is a constant. Even though \( g \) satisfies Eq. (2.4), it is not a solution of the field equations (2.7), since now \( \text{det } g \) is not equal to \( \text{det } \tilde{g} = -\rho^2 \), but instead equal to

\[ \text{det } g = (-1)^n \prod_{k=1}^{n} (\rho/\nu_k)^2 - \nu_k^{-2} \prod_{k=1}^{n} (\nu_k^2 + \rho^2)^{-1} \text{det } \Gamma, \quad (2.19) \]

where \( n \) is taken to be an even number in order to preserve the signature of \( g \). However, the new metric

\[ g_{\alpha \beta} = \prod_{k=1}^{n} (\nu_k^2 + \rho^2)^{1/n} g = \tilde{g}. \quad (2.20) \]
is still a solution of Eq. (2.1) and, furthermore, satisfies the condition
\[ \det g_{\text{ph}} = -q^2. \]  
\[ (2.21) \]

The function \( f \) is also modified by the transformation (2.20).

The new function, \( f_{\text{ph}} \), is
\[ f_{\text{ph}} = f^{1/\alpha} g^{-2/\alpha} f, \]  
\[ (2.22) \]

where
\[ q^{-1} = \text{Const.} \rho^{-\frac{(n+1)^2}{2}} \prod_{k=1}^{(n+1)} (\nu_k)^{-1} \prod_{k=1}^{(n+2)} (\nu_k + \rho) \prod_{k=-\nu_k}^{\nu_k}. \]  
\[ (2.23) \]

The power \( 1/\alpha \) is the only explicit reference to the dimensionality of the geometry. The Belinsky and Zakharov formulas of Ref. 11 are recovered simply by putting \( \alpha = 2 \).

III. THE BDJ THEORY IN FIVE DIMENSIONS

The BDJ field equations are conventionally given in the form 2
\[ G_{uv} = R - \frac{1}{2} g_{uv} R = \frac{8\pi T_{uv}}{\phi} + \frac{\omega (\phi^{\alpha} \phi')_v}{\phi} + \frac{1}{2} g_{uv} \phi^{\alpha} \phi' + \frac{1}{6} (\phi^{\alpha} \phi' - 6 \phi' \phi^{\alpha}) \phi^{\alpha} \phi' g_{uv} g^{\alpha\beta}. \]  
\[ (3.1) \]

\[ \begin{align*}
\Box^2 \phi &= \phi^{\alpha} \phi' \\
&= \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left[ \sqrt{-g} g^{\alpha\beta} \partial_\mu \phi^{\alpha} \phi' \partial_\nu \phi^{\alpha} \phi' \right] \\
&= \frac{8\pi T_{uv}}{\phi^{\alpha} \phi'},
\end{align*} \]  
\[ (3.2) \]

where \( g_{uv} \) is the metric for the four-dimensional space-time, and \( \omega \) is the coupling constant for the scalar field \( \phi \). The comas and the semi-colons denote partial and covariant derivatives, respectively. The energy-momentum tensor, \( T_{uv} \), includes all non-gravitational fields and matter, and satisfies the conservation law
\[ T_{uv} = 0. \]  
\[ (3.3) \]

Units are chosen such that the gravitational constant and the speed of light are equal to one.

The theory has also been expressed in the more general form 3:
\[ \tilde{G}_{\mu\nu} = \frac{8\pi T_{\mu\nu}}{\phi} + \frac{(\nu + 3 + 3\alpha^2)}{2\alpha} \left[ \delta_{\mu\nu} \phi' \phi - \frac{1}{2} \tilde{g}_{\mu\nu} \phi^{\alpha} \phi' \phi^{\alpha} \phi' \right], \]  
\[ (3.4) \]

\[ \begin{align*}
\Box^2 \phi &= \frac{8\pi}{(2\nu + 3)} \phi' \\
&= \frac{1}{\phi^{\alpha} \phi'},
\end{align*} \]  
\[ (3.5) \]

where the new metric \( \tilde{g}_{\mu\nu} \) and the scalar field \( \tilde{\phi} \) are related to \( \phi \) and \( g_{\mu\nu} \) by the conformal transformation:
\[ \tilde{\phi} = \phi^2; \quad \alpha = \text{constant}, \]  
\[ (3.6) \]

\[ \tilde{g}_{\mu\nu} = \frac{1}{\alpha} g_{\mu\nu}. \]  
\[ (3.7) \]

The Einstein tensor \( \tilde{G}_{\mu\nu} \) and the covariant derivatives are built with the metric \( \tilde{g}_{\mu\nu} \) and the new energy momentum tensor \( T_{\mu\nu} \) is given in terms of \( T_{\mu\nu} \) by
\[ \tilde{T}_{\mu\nu} = \frac{T_{\mu\nu}}{\phi^{\alpha} \phi'}. \]  
\[ (3.8) \]

The above transformation, Eqs. (3.6) and (3.7), have been interpreted by Dicke 3 as a space-time dependent change in the units of measurement.

Note also that in vacuum, and if \( \nu = -3/2 \) the field equations (3.4) and (3.5) are conformally invariant, a result pointed out by Anderson 15.

In the limit \( \alpha \to 0 \) we obtain the field equations
\[ \tilde{G}_{\mu\nu} = 8\pi \tilde{T}_{\mu\nu} + \frac{(\nu + 3/2)}{\phi^2} \left[ \delta_{\mu\nu} \phi' \phi - \frac{1}{2} \tilde{g}_{\mu\nu} \phi^{\alpha} \phi' \phi^{\alpha} \phi' \right], \]  
\[ (3.9) \]

\[ \Box^2 \phi = \frac{8\pi}{2\nu + 3} \tilde{T}_{\mu\nu}, \]  
\[ (3.10) \]

where now the bars mean that the tensors are built with the metric
\[ \tilde{g}_{\mu\nu} = \phi^2 \tilde{g}_{\mu\nu}. \]  
\[ (3.11) \]
This representation, Eqs. (3.9) and (3.10), is just Einstein's equations with a scalar field, $\phi$, as external source; the so-called Einstein-scalar theory.

The case when

$$\alpha = \sqrt{\frac{2}\alpha + 3}$$

(3.12)

is of particular interest. In this situation, Eqs. (3.4) and (3.5) become

$$\ddot{\Phi}_{\nu} = -8\pi [1 - \alpha^{-1}] \ddot{\Phi}_{\mu}$$

$$\alpha = \sqrt{\frac{2\pi + 3}{3}}$$

(3.13)

(3.14)

It also follows from Eqs. (3.13) and (3.14) that

$$\ddot{\Phi}_{\mu} = -8\pi [1 - \alpha^{-1}] \ddot{\Phi}_{\mu}$$

(3.15)

Let us introduce the following five-dimensional metric:

$$\tilde{g}_{\mu\nu} = \Phi^2 \delta_{\mu\nu}$$

$$\tilde{g}_{ab} = \Phi^2 \delta_{ab}$$

$$\tilde{g}_{\mu\nu} = \Phi^2 \delta_{\mu\nu}$$

(3.16)

Note that $\tilde{g}_{ab}$ is "static" with respect to the additional fifth dimension $x^5$. It is straightforward to show that

$$\tilde{g}_{\mu\nu} = \tilde{g}_{\mu\nu} = \frac{1}{\Phi} \left[ \Phi_{\mu\nu} - \tilde{g}_{\mu\nu} \right]^2 \Phi$$

$$\tilde{g}_{ab} = \Phi^2 \delta_{ab}$$

$$\tilde{g}_{\mu\nu} = \Phi^2 \delta_{\mu\nu}$$

(3.17)

(3.18)

(3.19)

Therefore Eqs. (3.13), (3.14) and (3.15) can be put in the form

$$\tilde{g}_{\mu\nu} = 8\pi \frac{\ddot{\Phi}_{\mu\nu}}{\Phi}$$

(3.20)

$$\tilde{g}^{\mu\nu} = -4\pi [1 - \alpha^{-1}] \frac{\ddot{\Phi}_{\mu\nu}}{\Phi}$$

(3.21)

Finally, defining

$$\ddot{\Phi}_{\mu} = -\frac{\ddot{\Phi}_{\mu}}{\Phi}$$

(3.22)

and

$$\tilde{g}_{ab} = -\frac{1}{2} \frac{\ddot{\Phi}_{ab}}{\Phi}$$

(3.23)

we obtain the concise form

$$\tilde{g}_{AB} = 8\pi \ddot{T}_{AB}$$

(3.24)

The field equations (3.24) are the five-dimensional representation of the BDJ theory that is equivalent to Einstein's equations for a five-dimensional metric which is "static" with respect to the additional non-space-time dimension (BDJ-5 theory).

For a specific application of the five-dimensional BDJ theory in vacuum, where the effect of a scalar field on the cosmological singularity is studied, see Belinsky and Khalatnikov.

IV. APPLICATION OF THE BDJ METHOD TO THE BDJ-5 THEORY

We can apply the BDJ technique, described in Sec.II, to a five-dimensional vacuum metric $\tilde{g}_{AB}$, when this is a function of only two variables. Thus let us assume that the metric $\tilde{g}_{AB}$, Eq. (3.16) have the form (2.1):

$$ds^2 = \tilde{g}_{AB} dx^A dx^B = f[dx^2 + dz^2] + \tilde{g}_{ab} dx^a dx^b$$

(4.1)

where we made the identification
Then it follows that the metric (2.20) and (2.22), with \( m = 3 \):

\[
(g_{\rho\phi})_{ab} = \prod_{k=1}^{n} \left( \frac{\rho}{r} \right)^{2/3} g_{ab}(0,0) = \prod_{k=1}^{n} \left( \frac{\rho}{r} \right)^{2/3} \left( \sum_{k=1}^{n} \epsilon_{ab} \epsilon_{cd} \epsilon_{ef} \epsilon_{gh} \right).
\]

(4.13)

is a solution of the BMN-3 theory in vacuum, \( g_{ab} = 0 \), if the background metric, \( g_{ab} \), is also a solution, and, furthermore,

\[
\gamma_{\rho\phi} = \frac{1}{3} \rho \left( \frac{\rho}{r} \right)^{2/3} f,
\]

(4.14)

We now consider a sufficient condition for Eq. (4.5) to be valid.

We see that if

\[
\delta_{a}^{b} = 0, \quad k \leq q,
\]

(4.6)

\[
\delta_{a}^{b} = 0, \quad k > q; \quad a \neq 3,
\]

(4.7)

then the matrix

\[
\Gamma_{k} = \left( \frac{\epsilon_{ab} \epsilon_{cd} \epsilon_{ef} \epsilon_{gh}}{\epsilon_{kk} \epsilon_{bb} \epsilon_{dd} \epsilon_{ff} \epsilon_{gg} \epsilon_{hh}} \right),
\]

(4.15)

takes the block form

\[
\Gamma = \begin{bmatrix}
\Gamma_{1} & 0 \\
0 & \Gamma_{2}
\end{bmatrix},
\]

(4.8)

where

\[
\Gamma = \begin{bmatrix}
\Gamma_{1} & 0 \\
0 & \Gamma_{2}
\end{bmatrix},
\]

(4.9)

\[
\Gamma_{1} \Gamma_{2} \cdots \Gamma_{q}
\]

(4.10)

Ricci

Hence

\[
\Gamma^{-1} = \begin{bmatrix}
\Gamma_{1}^{-1} & 0 \\
0 & \Gamma_{2}^{-1}
\end{bmatrix},
\]

(4.11)

which implies that

\[
\Gamma_{k} = \left( \frac{\epsilon_{ab} \epsilon_{cd} \epsilon_{ef} \epsilon_{gh}}{\epsilon_{kk} \epsilon_{bb} \epsilon_{dd} \epsilon_{ff} \epsilon_{gg} \epsilon_{hh}} \right),
\]

(4.12)

or, equivalently,

\[
\delta_{a}^{b} = 0, \quad a \neq 3,
\]

(4.13)

Therefore, we will assume that \( \delta_{a}^{b} \) satisfies Eqs. (4.6) and (4.7).

We now explore further consequences of this assumption. We know that

\[
\delta_{a}^{b} = 0, \quad a \neq 3,
\]

(4.14)

Then the conditions (4.6) and (4.7) can be implemented if we choose

\[
\psi_{bc} = 0; \quad c \neq 3,
\]

(4.15)

and

-12-
Note that the assumption in Eq. (4.15) is consistent with Eqs. (2.7) and (2.8), if the matrix \( \tilde{g} \) satisfies Eq. (4.13). Hence, we will assume the validity of Eqs. (4.15), (4.16) and (4.17).

The metric (4.3) takes the form

\[
\tilde{g}_{\phi \phi} = \begin{pmatrix}
\tilde{g}_{11} & \tilde{g}_{12} & 0 \\
\tilde{g}_{21} & \tilde{g}_{22} & 0 \\
0 & 0 & \tilde{g}_{33}
\end{pmatrix}
\]

(4.18)

With the metric in the form (4.18), the field equations (2.4) become

\[
(\partial_{\phi} \tilde{g}^{-1}) \tilde{\omega} + (\partial_{\phi} \tilde{g}^{-1}) \tilde{\omega} = 0.
\]

(4.20)

Eq. (4.20) is equivalent to Laplace's equation with cylindrical symmetry. Eq. (4.19) has the same form that the second-order equation for a metric of the type Eq. (2.1) in general relativity. However, we see, from Eqs. (2.21) and (4.18), that

\[
\det \tilde{g} = \frac{\tilde{g}^{2}}{(\tilde{g}_{\phi \phi})_{33}}.
\]

(4.21)

while a corresponding 2x2 metric in general relativity satisfies

\[
\det g_{\text{GR}} = -\tilde{\phi}^{2}.
\]

(4.22)

Therefore, a solution \( g_{\text{GR}} \) in general relativity is not a solution in the BDJ-5 theory. Nevertheless, we can use \( g_{\text{GR}} \) to build a solution \( \tilde{g} \), in the following way:

\[
\tilde{g} = \frac{g_{\text{GR}}}{(g_{\phi \phi})_{33}}.
\]

(4.23)

Then we see that indeed \( \tilde{g} \), as given by the expression (4.23), is a solution of Eq. (4.19), if \( g_{\text{GR}} \) is also, since

\[
(\partial_{\phi} \tilde{g}^{-1}) \tilde{\omega} + (\partial_{\phi} \tilde{g}^{-1}) \tilde{\omega} = (\partial_{\phi} \tilde{g}^{-1}) \tilde{\omega} + (\partial_{\phi} \tilde{g}^{-1}) \tilde{\omega}.
\]

(4.24)

where we used Eq. (4.20). Furthermore, Eq. (4.23) implies that

\[
\det \tilde{g} = \frac{\det g_{\phi \phi}}{(g_{\phi \phi})_{33}} = \frac{-\tilde{\phi}^{2}}{(g_{\phi \phi})_{33}},
\]

(4.25)

and therefore

\[
\det g_{\phi \phi} = (g_{\phi \phi})_{33} \det \tilde{g} = -\tilde{\phi}^{2},
\]

(4.26)

as required by Eq. (2.3).

Summarizing, we have the following result. Given a solution, \( g_{\text{GR}} \), of the general relativity axially symmetric and stationary vacuum field equations, we get a corresponding solution in the BDJ-5 theory

\[
\tilde{g}_{\phi \phi} = \begin{pmatrix}
\tilde{g}_{\text{GR}} & (\tilde{g}_{\phi \phi})_{33} & 0 \\
(\tilde{g}_{\phi \phi})_{33} & 0 & (\tilde{g}_{\phi \phi})_{33} \\
0 & (\tilde{g}_{\phi \phi})_{33}
\end{pmatrix}
\]

(4.27)

where \( \text{ln}(\tilde{g}_{\phi \phi})_{33} \) is a solution of Laplace's equation (4.20). The equivalent result in the four-dimensional representation has been given by Sneddon and McIntosh [17]. It is also possible to generalize this conclusion to the case when electromagnetic field sources are present [12], [3].
Let us return to Eq. (4.3) and make use of the implications of conditions (4.15)-(4.17). We obtain

$$
(\eta_{ph})_{ab} = \left[ \eta_{ab} - \frac{1}{\eta_{kk}} \sum_{k,l=1}^{n-1} \eta_{kl} \eta_{kl} \right] .
$$

Thus we have

$$
(\eta_{ph})_{ab} = \left[ \eta_{ab} - \frac{1}{\eta_{kk}} \sum_{k,l=1}^{n-1} \eta_{kl} \eta_{kl} \right] ; \, a, b \neq 3 .
$$

Therefore, substituting Eq. (4.37) into Eq. (4.34), we get

$$
(\eta_{ph})_{33} = \Pi_{k,q} = \frac{\eta_{33}}{\eta_{kk}^2} \left( \frac{\eta_{33}}{\eta_{kk}} \right)^2 .
$$

From which we obtain

$$
(\eta_{kk}) = \frac{1}{\eta_{kk}} \left( \frac{1}{\eta_{kk}} \right) \left( \frac{1}{\eta_{kk}} \right) = (-1)^{n-q} \frac{\eta_{kk}}{\eta_{kk}},
$$

Thus we see that the expression for the scalar field (g u)-3-i involves in a very simple way the background 3, the poles w, and is independent of the matrix Q.

V. THE DIAGONAL FORM OF THE SOLUTION

In order to study the non-rotational limit of the solutions, we will consider in this section the case where \( r_{12} = r_{21} = 0 \). We can diagonalize the metric \( g_{ab} \) using the same procedure that led to the block form (4.36). That is, let us assume that \( g \) and \( \phi \) are diagonal and furthermore,

$$
C_k^k = 0; \, \text{if} \, k > n , \quad \text{and} \quad C_k^k = 0; \, \text{if} \, k < n \text{ or } k > q .
$$

On the other hand, we know the following general result for a non-singular matrix \( \eta_{kk} \):

$$
\frac{1}{\eta_{kk}^2} = \frac{1}{\eta_{kk}} \frac{1}{\eta_{kk}} + \frac{\rho^2}{\eta_{kk}^2} .
$$

where \( n \) and \( q \) have been chosen as even numbers, in order to preserve the signature of the metric. Thus, we see that the expression for the scalar field (g u)-3-i involves in a very simple way the background 3, the poles w, and is independent of the matrix Q.

Therefore, substituting Eq. (4.37) into Eq. (4.34), we get

$$
(\eta_{ph})_{33} = \Pi_{k,q} = \frac{\eta_{33}}{\eta_{kk}^2} \left( \frac{\eta_{33}}{\eta_{kk}} \right)^2 \left( \frac{\eta_{33}}{\eta_{kk}} \right)^2 .
$$

where \( n \) and \( q \) have been chosen as even numbers, in order to preserve the signature of the metric. Thus, we see that the expression for the scalar field (g u)-3-i involves in a very simple way the background 3, the poles w, and is independent of the matrix Q.
where \( s \leq q \), is a positive integer. Then it follows that

\[
\begin{align*}
N_1^k & = c^k \phi_1^{-1} = c^k \phi_2^{-1} = 0, \text{ if } k > s, \\
N_2^k & = c^k \phi_2^{-1} = 0, \text{ if } k \leq s, \text{ or } k > q,
\end{align*}
\]

and therefore

\[
\Gamma_{kk} = (\Gamma_{11})_{kk} + (\Gamma_{22})_{kk},
\]

where

\[
(\Gamma_{11})_{kk} = \begin{cases} 
0, & \text{if } k \text{ or } \ell > s \\
\Gamma_{1kk}, & \text{if } k \text{ and } \ell \leq s
\end{cases}
\]

and

\[
(\Gamma_{22})_{kk} = \begin{cases} 
0, & \text{if } (k \text{ or } \ell) \leq s, \text{ or } (k \text{ or } \ell) > q \\
\Gamma_{1kk}, & \text{if } s < (k \text{ and } \ell) \leq q
\end{cases}
\]

Hence, \( \Gamma \) takes the block form

\[
\begin{bmatrix}
\Gamma_{11} & 0 & 0 \\
0 & \Gamma_{22} & 0 \\
0 & 0 & \Gamma_{33}
\end{bmatrix},
\]

where we put

\[
\Gamma_2 = \Gamma_{33}.
\]

We find, from Eq.(5.29), that \( g_{ph} \) simplifies to

\[
(g_{ph})_{ab} = 0; \ a \neq b,
\]

Using the following result, shown in Appendix A
det \frac{1}{1 + A_k B_k} = \prod_{k \neq k'} \left( A_k - A_k \right) \left( B_k - B_k \right).

We see that, for m poles \( u_k \),
\[ \det \frac{1}{u_k^2 + \rho^2} = \prod_{k \neq k'} \left( u_k - u_k \right) \left( u_k + \rho^2 \right). \]

Therefore
\[ \det \Gamma_{11} = \prod_{k \neq k'} \left( \left( A_k - A_k \right) \left( B_k - B_k \right) \right) \left( u_k - u_k \right) \left( u_k + \rho^2 \right). \]

\[ \det \Gamma_{22} = \prod_{k \neq k'} \left( \left( A_k - A_k \right) \left( B_k - B_k \right) \right) \left( u_k - u_k \right) \left( u_k + \rho^2 \right). \]

\[ \det \Gamma_{33} = \prod_{k \neq k'} \left( \left( A_k - A_k \right) \left( B_k - B_k \right) \right) \left( u_k - u_k \right) \left( u_k + \rho^2 \right). \]

Substitution of Eqs. (5.19)-(5.21) in Eq. (5.13), and this in Eq. (4.13), gives

\[ f_{ph} = \sum_{k=1}^{n} f_{ph} \left( u_k - u_k \right) \sum_{k=1}^{n} \left( \left( A_k - A_k \right) \left( B_k - B_k \right) \right) \left( u_k - u_k \right) \left( u_k + \rho^2 \right). \]

VI. THE LIMIT \( \lim_{\rho \to 0} f_{ph} \)

It is important to study under what circumstances the solution \( f_{ph} \) reduces to the background metric \( g_{00} \). For example, it is useful to investigate the new solution, \( f_{ph} \), "near" the background when the physical interpretation of \( g_{00} \) is well known.

We can easily show that the diagonal element, Eqs. (4.18), (5.11) and (5.12), reduce to \( g_{00} \) if either one of the following two conditions are satisfied:

1) \( s = q - s = n - q \) and \( u_k = \frac{v_k}{v_k} + \rho^2 \), \( k \leq s = \frac{n}{2} \).

2) In each group of poles, \( 1 \leq k \leq s, s < k < q, q < k < n \), the poles come in pairs \( u_k, v_k \), such that
\[ u_k = v_k + \sqrt{(v_k)^2 + \rho^2}, \]
\[ v_k = u_k - \sqrt{(v_k)^2 + \rho^2}. \]

We see that the condition (i) implies the following results:

\[ \rho^2 = \rho^2, \quad k \neq k', \quad \sum_{k=1}^{n} \rho^2 = n, \quad \sum_{k=1}^{n} \rho^2 = n, \quad \sum_{k=1}^{n} \rho^2 = n. \]

Also from the assumption (ii) it follows that
\[ u_k + v_k = 2(w_k - \rho^2), \]
\[ u_k + v_k = 2(w_k - \rho^2), \]
and therefore, using Eq. (6.1),

\[ \sum_{k=1}^{n} \rho^2 = n, \quad \sum_{k=1}^{n} \rho^2 = n, \quad \sum_{k=1}^{n} \rho^2 = n = 1. \]

Hence, in both cases (i) and (iii), \( f_{ph} = g_{00} \), since the coefficients of \( g_{ab} \) in Eqs. (4.18), (5.11) and (5.12) are unity.
We also expect that, when $g_{ph} = \frac{\partial}{\partial t}$, then $f_{ph} = f_0$. In Appendix B, it is shown explicitly that, when $g_{ph}$ is diagonal, then for each one of the conditions (i) and (ii), indeed we have $f_{ph} = f_0$.

VII. SOLUTIONS WITH PSEUDO-EUCLIDEAN BACKGROUND

The simplest application of the Belinsky and Zakharov method to the BMJ-5 theory is the generation of solutions using a flat space-time metric with a constant scalar field as background:

\[
\hat{g} = \begin{pmatrix}
  0^2 & 0 & 0 \\
  0 & -1 & 0 \\
  0 & 0 & 1 \\
\end{pmatrix},
\]

where the constant scalar field, $\frac{\partial}{\partial t}$, has been normalised to unity. A particular solution $g_{ph}$, corresponding to the metric (7.1) is

\[
\psi_t(x) = \begin{pmatrix}
  -\lambda^2 & 0 & 0 \\
  0 & -1 & 0 \\
  0 & 0 & 1 \\
\end{pmatrix}; \quad \lambda^2 = 2^2 + 2\lambda - \beta^2.
\]

We can get new soliton solutions $g_{sp}$, $f_{sp}$ in the BMJ-5 theory in a straightforward way, using Eqs. (7.1)-(7.3) in Eqs. (4.29) and (4.36).

The problem of creating new soliton solutions in general relativity from a flat space-time metric has been discussed in detail by Belinsky and Zakharov [1]. In particular, they considered the two soliton case to build the Kerr metric. These solutions can readily be transformed into BMJ-5 solutions using the prescription in Eqs. (4.27). Hence an alternative way of generating solutions from flat space-time is to construct \( g \) in general relativity and then use Eq. (4.27), with the scalar field $f_{sp}$ given by

\[
\left( g_{sp} \right)_{33} = \left[ \frac{u}{v} - \frac{1}{v^2} \right]^{-\delta^2 - \nu}; \quad \delta = 1,
\]

or any other solution of Laplace's equation (4.20). For instance, the Kerr solution can be transformed to get a rotating solution in the BMJ theory. However, this solution does not become spherically symmetric in the absence of rotation, if the scalar field is non-constant, as we shall see below.

The Brans-Dicke-Jordan spherically symmetric static vacuum solution, $g_{sp}$, $f_{sp}$ takes the following form in the five-dimensional representation (Appendix C)

\[
\left( g_{sp} \right)_n = r^2 \left[ 1 - \frac{2\mathcal{R}}{r} \right]^{1 - \delta^2 - \nu} \sin^2 \theta,
\]

\[
\left( g_{sp} \right)_x = \left[ 1 - \frac{2\mathcal{R}}{r} \right]^{1 - \nu},
\]

\[
\left( g_{sp} \right)_y = \left[ 1 - \frac{2\mathcal{R}}{r} \right]^{1 - \nu},
\]

\[
f_{sp} = \left[ 1 - \frac{2\mathcal{R}}{r} \right]^{1 - \delta^2 - \nu} \left[ 1 - \frac{2\mathcal{R}}{r} + \frac{3}{r^2} \sin^2 \theta \right],
\]

where $\delta$ and $\nu$ are constants related by

\[
\delta^2 + 3\nu^2 = 1,
\]

and $\delta$ must be identified with the gravitational mass of the system (Appendix C). The spherical co-ordinates $r, \theta$, are related to the cylindrical co-ordinates $\rho, z$, in the following way:

\[
\rho = r \left[ 1 - \frac{2\mathcal{R}}{r} \right]^{1/2} \sin \theta,
\]

\[
z = (r - \delta) \cos \theta.
\]

We can re-express $g_{sp}$ in terms of $\rho$ and $z$:

\[
\left( g_{sp} \right)_{11} = \left[ \frac{\gamma^2}{\phi} \right]^{-\psi - \nu},
\]

\[
\left( g_{sp} \right)_{22} = \left[ \frac{\gamma^2}{\phi} \right]^{-\psi + \nu},
\]

\[
\left( g_{sp} \right)_{33} = \left[ \frac{\gamma^2}{\phi} \right]^{3\nu},
\]

where the functions $\psi$ and $\gamma$ are just the poles, with $\psi \equiv \frac{\partial}{\partial t} = -\delta$ and $+\delta$.
\[ u = -(6 + z) + \sqrt{(8+z)^2 + p^2} - (r-28)[1 - \cos \theta], \quad (7.15) \]

\[ v = (8 - z) - \sqrt{(8-z)^2 + p^2} - (r-28)[1 + \cos \theta], \quad (7.16) \]

On the other hand, according to Eq. (4.27) the scalar field \( \phi \) is given by

\[ \phi_{kerr}^+ = \begin{pmatrix} \phi_{ph} & 0 \\ 0 & \phi_{ph} \end{pmatrix}, \quad (7.17) \]

and since in the absence of rotation the Kerr solution becomes the Schwarzschild solution

\[ g_{sc} = \begin{pmatrix} r^2 \sin^2 \theta & 0 \\ 0 & -[1 - 28/r] \end{pmatrix}, \quad f_{sc} = \begin{pmatrix} 1 - \frac{288}{r} + \frac{28 \sin^2 \theta}{r} \\ 0 \end{pmatrix}, \quad (7.18) \]

then we have that, without rotation,

\[ g_{ph} = \begin{pmatrix} \phi_{ph} & 0 \\ 0 & \phi_{ph} \end{pmatrix}. \quad (7.19) \]

To compare the metric (7.5)-(7.7) with Eqs. (7.19) we must identify \( \phi_{ph} \) with \( 1 - 2\phi/r \). Thus we get

\[ g_{ph} = \begin{pmatrix} \phi_{ph} & 0 \\ 0 & \phi_{ph} \end{pmatrix}, \quad (7.20) \]

which is just like \( g_{ph} \), but with \( \phi = 1 \). Therefore the condition \( \delta^2 + 3\psi^2 = 1 \), can only be satisfied if \( \psi = 0 \), or equivalently when

\[ (\phi_{ph})_{33} = \text{constant}. \quad (7.21) \]

Conversely, if the scalar field \( \phi_{ph} \) is not a constant the solution (7.20) does not represent a spherically symmetric configuration.

More generally, if half of the poles, \( w_k \) (k odd) are chosen equal to \( a \) (or \( -\phi^2/b \)) and the other half equal to \( -a \) (or \( -\phi^2/b \)) then the diagonal solution, Eqs. (4.38), (5.11), (5.12), takes the form

\[ \delta_{ph} = \begin{pmatrix} \frac{-\phi}{\phi^2} & 0 \\ 0 & \frac{-\phi}{\phi^2} \end{pmatrix} \begin{pmatrix} 1 - \phi^2 \delta_0 & 0 \\ 0 & 1 - \phi^2 \delta_0 \end{pmatrix}, \quad (7.22) \]

\[ \delta_{ph} = \begin{pmatrix} \frac{-\phi}{\phi^2} & 0 \\ 0 & \frac{-\phi}{\phi^2} \end{pmatrix} \begin{pmatrix} 1 - \phi^2 \delta_0 & 0 \\ 0 & 1 - \phi^2 \delta_0 \end{pmatrix}, \quad (7.23) \]

\[ \delta_{ph} = \begin{pmatrix} \frac{-\phi}{\phi^2} & 0 \\ 0 & \frac{-\phi}{\phi^2} \end{pmatrix} \begin{pmatrix} 1 - \phi^2 \delta_0 & 0 \\ 0 & 1 - \phi^2 \delta_0 \end{pmatrix}, \quad (7.24) \]

which is similar to the functional form of \( g_{ph} \). Nevertheless, in order for \( g_{ph} \) to be equal to \( g_{ph} \) we must identify

\[ \delta + \psi = \frac{1}{2} (a - \frac{\phi}{\phi^2} \delta_0), \quad (7.25) \]

or, equivalently,

\[ \delta = \frac{1}{2} (a - \frac{\phi}{\phi^2} \delta_0), \quad (7.26) \]

but again, \( \delta^2 + 3\psi^2 \) cannot be unity unless \( \psi \) vanishes.

Thus it seems that to obtain the spherically symmetric solution as a diagonal limit of a soliton metric we must start with a non-flat metric \( g \). In particular, if we put \( g = g_{ph} \) it is possible to obtain the reduction \( g_{ph} \rightarrow g_{ph} \) when either of the conditions (i) or (ii), discussed in Sec. VI, are satisfied. Or, another possibility is to start with a static axially symmetric but not necessarily spherical background solution, and, instead of imposing the conditions (i) or (ii), require that the diagonal metric generated is equal to \( g_{ph} \). The study of solutions of this type, where \( g \) is given by the BDJ-5 generalization of the well-known Weyl-Levi Civita solution of general relativity, is discussed in the next section.
VIII. SOLUTIONS WITH WEYL-LEVI-CIVITA BACKGROUND

A solution of the BDJ-5 field equations that describe the exterior gravitational field of a static axially symmetric configuration is given by (Appx. C)

\[ s_{11} = \rho^2 \left[ 1 - \frac{2\rho}{r} \right]^{-1} \sin^2 \theta, \quad (8.1) \]
\[ s_{22} = \left[ 1 - \frac{2\rho}{r} \right] \delta_{ij}, \quad (8.2) \]
\[ s_{33} = \left[ 1 - \frac{2\rho}{r} \right]^{2\rho}, \quad (8.3) \]
\[ \psi = \left[ 1 - \frac{2\rho}{r} \right] \frac{\sigma^2 - \delta_{ij}}{\left[ 1 - \frac{2\rho}{r} \right] \sin^2 \theta}, \quad (8.4) \]

where

\[ \sigma^2 = \sigma^2 + 3\rho^2. \quad (8.5) \]

When \( \rho = 0 \), this solution is the well-known Weyl-Levi-Civita static metric.

We can verify that a particular solution of Eqs. (2.7) and (2.8) when \( g = g \) is

\[ \psi_{s1} = -\lambda \left[ \frac{(\lambda - \mu) (\lambda - \gamma)}{\lambda} \right]^{-\rho} \quad (8.6) \]
\[ \psi_{s2} = -\lambda \left[ \frac{(\lambda - \mu) (\lambda - \gamma)}{\lambda} \right]^{-\rho} \quad (8.7) \]
\[ \psi_{s3} = \left[ \frac{(\lambda - \mu) (\lambda - \gamma)}{\lambda} \right]^{2\rho} \quad (8.8) \]

The evaluation of \( \psi \) at \( \lambda = \mu \) gives

\[ \psi_{s1} (k) = -2\mu_k \psi_k \left[ \frac{(\mu_k - \mu) (\mu_k - \gamma)}{2\mu_k \mu_k} \right]^{1-\rho} \quad (8.9) \]
\[ \psi_{s2} (k) = \left[ \frac{(\mu_k - \mu) (\mu_k - \gamma)}{2\mu_k \mu_k} \right]^{1-\rho} \quad (8.10) \]

Note that

\[ \psi_{s1} (k) \rightarrow \psi_k \quad (8.11) \]

\[ \lim \lambda \rightarrow 0 \quad (8.12) \]

as required by Eq. (2.11). An alternative form for the solution can be obtained using the expression

\[ \lambda \psi = \lambda^2 + 2\lambda x - \rho^2 = (\lambda - \mu_k) (\lambda - \psi_0) \quad (8.13) \]

where the functions \( \mu_k \) and \( \psi_0 \) are poles with \( \lambda_k = 0 \):

\[ \mu_k = x + \sqrt{x^2 + \rho^2} \quad (8.14) \]
\[ \psi_0 = -x - \sqrt{x^2 + \rho^2} \quad (8.15) \]

It is worth noticing that, as follows from Eqs. (8.1)-(8.3),

\[ \lim \psi_{s1} \rightarrow \begin{bmatrix} 2^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (8.16) \]

and furthermore, using Eqs. (8.6)-(8.8) and (7.15), (7.16),

\[ \psi (\lambda) \rightarrow \begin{bmatrix} \psi_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (8.17) \]

Thus, when \( \rho = 0 \), the seed solutions \( \psi_k \), \( \psi_0 \) are the same as for the pseudo-Euclidean case (Sec. VII).

Let us consider the diagonal form, Eqs. (5.11), (5.12) and (4.38).

We see that a Weyl-Levi-Civita type solution can be obtained when the poles are chosen in the following way:
\[ u_k = \begin{cases} \frac{e^2}{v} = r[1 - \cos \theta], & k \text{ odd} \\ \frac{e^2}{v} = -r[1 + \cos \theta], & k \text{ even} \end{cases} \tag{8.18} \]

since, in this case,

\[ (g_{ph})_{11} = e^2 \left[ 1 - \frac{2\theta}{r} \right] -3\nu \tag{8.19} \]

\[ (g_{ph})_{22} = -\left[ 1 - \frac{2\theta}{r} \right] -3\nu \tag{8.20} \]

\[ (g_{ph})_{33} = \left[ 1 - \frac{2\theta}{r} \right] -2\nu \tag{8.21} \]

where

\[ \bar{\nu} = \nu + \frac{2}{3} \frac{\theta}{r} \tag{8.22} \]

In other words, if we start with \( g(\delta, \nu) \) as background, the new diagonal metric will be \( g(\bar{\delta}, \bar{\nu}) \). Furthermore, if the parameters \( \bar{\delta} \) and \( \bar{\nu} \) are chosen such that \( \bar{\delta}^2 = \delta^2 + 3\nu^2 = 1 \), we will obtain \( g(\bar{\delta}, \bar{\nu}) = g_{ph} \) and \( h(\bar{\delta}, \bar{\nu}) = h_{ph} \).

For these types of solutions, the functions \( \psi_k(\delta) \), Eqs.(8.9)-(8.11) take the form, for \( k \) odd

\[ \psi_{k11}^{(1)} = 2r(1 - \cos \theta)\left[ 2\left( 1 - \frac{\delta}{r} (1 - \cos \theta) \right) \right]^{-2\nu}, \tag{8.24} \]

\[ \psi_{k22}^{(1)} = -\left[ 2\left( 1 - \frac{\delta}{r} (1 - \cos \theta) \right) \right]^{-2\nu}, \tag{8.25} \]

\[ \psi_{k33}^{(1)} = \left[ 2\left( 1 - \frac{\delta}{r} (1 + \cos \theta) \right) \right]^{-2\nu}, \tag{8.26} \]

and for \( k \) even,

\[ \psi_{k11}^{(2)} = 2r(1 + \cos \theta)\left[ 2\left( 1 + \frac{\delta}{r} (1 - \cos \theta) \right) \right]^{-2\nu}, \tag{8.27} \]

\[ \psi_{k22}^{(2)} = -\left[ 2\left( 1 + \frac{\delta}{r} (1 - \cos \theta) \right) \right]^{-2\nu}, \tag{8.28} \]

\[ \psi_{k33}^{(2)} = \left[ 2\left( 1 + \frac{\delta}{r} (1 - \cos \theta) \right) \right]^{-2\nu}. \tag{8.29} \]

Thus we can construct generalizations of the Weyl-Levi Civita metric by substituting the above matrix \( \psi_0 \) and the metric (8.1)-(8.4) in Eqs.(4.29), (k,30) and (4.4). The case \( n = q = r = 2 \), is given explicitly below

\[ (g_{ph})_{11} = \frac{1}{2} \left[ 2\nu + 2\nu^2 \frac{\theta}{r} - 4 \left( \frac{\delta - \nu}{2} \right)^2 + a^2 \sin^2 \theta - \Delta \right] \frac{1}{\sin^2 \theta} \tag{8.30} \]

\[ (g_{ph})_{22} = \frac{1}{2} \left[ a - a^2 \sin^2 \theta \right] \tag{8.31} \]

\[ (g_{ph})_{33} = \left[ 1 - \frac{2\theta}{r} \right]^{-2\nu} \tag{8.32} \]

\[ (g_{ph})_{12} = \frac{2\theta}{r} \left[ a - b \cos \theta \right] - a \sin^2 \theta \left( a^2 - b^2 - R^2 \right) \tag{8.33} \]

\[ f_{ph} = \frac{2\theta}{r} \left( \delta, \nu \right) \tag{8.34} \]

where

\[ \delta = \delta - 1 \tag{8.35} \]

\[ \gamma = \frac{\delta - M_0}{2} + \frac{\delta + M_0}{2} \left[ 1 - \frac{2\theta}{r} \right] \sin^2 \theta \tag{8.36} \]

\[ M = \frac{M_0 - \delta}{2} + \frac{M_0 + \delta}{2} \left[ 1 - \frac{2\theta}{r} \right] \sin^2 \theta \tag{8.37} \]
The constants \( a_0, b_0, M_0, \) and \( \delta \) have been chosen such that \( \delta^2 - \delta = a - b^2 \), in order to obtain the Kerr-Nut solution when \( \delta = \delta = 0 \), as we shall see below.

The study of the physical interpretation of the above solution is under way, but we can already state the following results:

1) If \( \delta = 0 = 0 \), then \( \gamma = 0 = M = M_0 = a = 0 = b = b_0 \), and the solution becomes equivalent to the Kerr-Nut metric, since one can verify that the time coordinate transformation

\[
\tau = t + 2a \phi
\]

leads to the Boyer-Lindquist Kerr-Nut line element

\[
d\tau^2 = \frac{1}{\Omega} \left[ (\Delta - a^2 \sin^2 \theta) dt^2 - 4(\Omega b \cos \theta - a \sin \theta \Omega') dt d\phi + \Omega d\Omega' - a \sin \theta \Omega' d\phi^2 \right] + \left[ (a \sin \theta \cos \theta + 2b \cos \theta)^2 - a \sin^2 \theta (a^2 + b^2 + \Delta^2) \right] d\phi^2
\]

If \( b_0 = 0 \), we get the Kerr solution with mass \( M \) and geometric angular momentum \( M_0 \).

ii) If \( a_0 = b_0 = 0 \), we have \( (g_{ph})_{12} = 0 \), and therefore zero angular momentum. The metric (8.30)-(8.31) becomes a Weyl-Levi Civita solution

\[
(g_{ph})_{11} = r^2 \left[ 1 - \frac{2M_0}{r} \right] \frac{1}{\sin^2 \theta} \left( 1 - \frac{2M_0}{r} \right)^{2\delta - \delta \frac{2M_0}{r}}
\]

\[
(g_{ph})_{22} = -[1 - \frac{2M_0}{r}]^{2\delta - \delta \frac{2M_0}{r}}
\]

\[
(8.43)
\]

\[
(8.44)
\]

\[
(8.45)
\]

\[
(8.46)
\]

where \( \delta = \delta + 1 \), depending on the choice \( M_0 = \pm \delta \). If, furthermore, \( \delta^2 = \delta^2 + 3\delta^2 = 1 \), we have spherical symmetry.

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SUMMARY OF THE APPENDICES OF THIS WORK

In Appendices A-F we show the following:

**Appendix A**: Determinant
\[
\frac{1}{1 + A_k B_k} = \sum_{k>l} (A_k - A_l)(B_k - B_l) \left[ \prod_{k,l} (1 + A_k B_k) \right]^{-1}
\]

**Appendix B**: If \( g_{ph} \) is diagonal and any one of the conditions (i) or (ii), defined in Sec.VI, are satisfied then \( f_{ph} = f_0 \).

**Appendix C**: The Weyl Levi Civita metric in the BDJ-5 theory is derived.

**Appendix D**: Determinant \( \psi(\lambda) = \lambda G(\omega) \), where \( G(\omega) \) is a function only of \( \omega = (\lambda^2 + 2\omega \lambda - \omega^2) \lambda^{-1} \).

**Appendix E**: \( \prod_{k \neq k}^{2(n-1)} \frac{(\omega^2 + \mu_k \omega_k)(\mu_k - \omega_k)}{S(\omega_k - \omega)} \)

**Appendix F**: If \( g \) and \( \psi \) are diagonal, then \( \psi(\lambda) \psi^{-1} \psi^{(-2/\lambda)} = F(\omega) \), where \( F \) is an arbitrary function of \( \omega \).

APPENDIX A

In this appendix it is shown that
\[
\det \left[ \frac{1}{1 + A_k B_k} \right] = \prod_{k > l} (A_k - A_l)(B_k - B_l) \left[ \prod_{k,l} (1 + A_k B_k) \right]^{-1}.
\]

First note that
\[
\det \prod_{j \neq l} \left( 1 + A_k B_k \right) = \det \prod_{l \neq j} \left( 1 + A_k B_k \right)
\]

where we used the fact that \( \det [b_{ik}] = \prod b_{ik} \det a_{ik} \). Hence Eq. (A.1) is equivalent to
\[
\det \prod_{j \neq l} \left( 1 + A_k B_k \right) = \prod_{k \neq l} (A_k - A_l)(B_k - B_l).
\]

The proof of Eq. (A.3) will be by induction. Hence let us assume that Eq. (A.3) is valid for a \((n-1)\times(n-1)\) matrix:
\[
C_{k \ell}^{n-1} = \prod_{m \neq \ell} (1 + A_k B_m),
\]

\[
\det C_{k \ell}^{n-1} = \prod_{k \neq \ell} (A_k - A_l)(B_k - B_l).
\]

Then we have
\[
\prod_{k \neq \ell} (A_k - A_l)(B_k - B_l) = \prod_{k \neq \ell} (A_k - A_l)(B_k - B_l) \left[ \prod_{k,l} (1 + A_k B_k) \right]^{-1} (A_k - A_l)(B_k - B_l)
\]

\[
= \prod_{k \neq \ell} (A_k - A_l)(B_k - B_l) \det C_{k \ell}^{n-1},
\]

where we used Eq. (A.5). We know that

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Also, it is easy to show that

\[
\det C^{n-1}_{k k} = 1, 1, \ldots, 1, \frac{1}{2} \zeta, 2 \zeta, C^{n-1} C^{n-2} \ldots C^1 C^0 1_{n-1} 1_{n-1} 1_{n-1} 1_{n-1} = 1 \cdot 1 \cdot 1 \cdot 1.
\]

Also, it is easy to show that

\[
\frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} V_{n-1} = \frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} V_{n-1}.
\]

for any particular choice of \( n-1, i_k \). Consequently, using Eqs.(A.7) and (A.8),

\[
\begin{align*}
\frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n) &= \frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n) \\
- \frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n) &= \frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n) \\
- \frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n) &= \frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n) \\
- \frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n) &= \frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n)
\end{align*}
\]

The terms

\[
K_k = \left( A_n - A_k \right) (b_k - b_n) C^{n-1}_{k k} = \left( A_n - A_k \right) \frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n).
\]

are independent of \( i_k \). Hence any term on the right-hand side of Eq.(A.9) that contains a factor like \( K_{i_k} \) is symmetrical under permutations of \( i_k, i_{n-1}, \ldots, i_1 \). Therefore these terms do not contribute to the sum in Eq.(A.9). Thus, the only surviving terms are either independent of \( K_{i_k} \) or linear in \( K_{i_k} \). Consequently, Eq.(A.9) becomes

\[
\frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n) \frac{1}{1 \cdot 2 \cdot 3 \cdots (n-1)} = \frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n) \frac{1}{1 \cdot 2 \cdot 3 \cdots (n-1)}
\]

Using Eq.(A.10), we obtain after some rearrangement

\[
K_k \frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n) = \frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n) \frac{1}{1 \cdot 2 \cdot 3 \cdots (n-1)}.
\]

where we use the definitions

\[
\begin{align*}
K_{i_k} &= \frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n) \frac{1}{1 \cdot 2 \cdot 3 \cdots (n-1)} \\
K_{i_k} &= \frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n) \frac{1}{1 \cdot 2 \cdot 3 \cdots (n-1)}
\end{align*}
\]

Substituting Eq.(A.12) and using again the definitions (A.13)-(A.15) in Eq.(A.11) we get

\[
\frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n) \frac{1}{1 \cdot 2 \cdot 3 \cdots (n-1)} = \frac{1}{2} \begin{pmatrix} A_n - A_k \end{pmatrix} (b_k - b_n) \frac{1}{1 \cdot 2 \cdot 3 \cdots (n-1)}
\]

On the other hand, we know that
Eq. (A.17) can be re-expressed in the following form:

\[
\det c_{k}^{n} = \frac{\epsilon_{i_{1},i_{2},\ldots,i_{n}}}{n} \prod_{k} c_{k} \quad \quad (A.17)
\]

We see that

\[
\epsilon_{i_{1},i_{2},\ldots,i_{n}} \cdot \prod_{k} c_{k} = \prod_{k} c_{k} \quad \quad (A.18)
\]

Therefore, applying Eq. (A.19) to Eq. (A.18), we obtain

\[
\det c_{k}^{n} = \frac{\epsilon_{i_{1},i_{2},\ldots,i_{n}}}{n} \prod_{k} c_{k} \quad \quad (A.19)
\]

Comparing Eq. (A.20) with Eq. (A.16) we see that

\[
\det c_{k}^{n} \equiv \prod_{m=1}^{n} (1 + A_{k} B_{m}) = \prod_{k}^{n-1} (1 + A_{k} B_{m}) \quad \quad (A.21)
\]

or, equivalently, from (A.2)

\[
\det \left[ \frac{1}{1 + A_{k} B_{m}} \right] = \prod_{k}^{n} (1 + A_{k} B_{m}) \quad \quad (A.22)
\]

Since A.22 is valid for a 2 × 2 matrix, then we have shown that it is valid for all \( n \geq 2 \). Q.E.D. In particular, if \( A_{k} = B_{k} = \mu_{k} \), we have

\[
\det \left[ \frac{1}{1 + \mu_{k} B_{m}} \right] = \prod_{k}^{n} (1 + \mu_{k} B_{m}) \quad \quad (A.23)
\]
It is shown below that when $g_{ph}$ is diagonal, then for each one of the conditions (i) or (ii), in Sec. VI, we have

$$f_{ph} = \text{const. } f_0. \quad (B.1)$$

First consider the case (i). In this situation Eq. (5.22) becomes

$$f_{ph} = C_0^0 \left( \frac{1}{2} \right)^{\frac{n}{2}} f_0 \left( \prod_{k=1}^{n} \left( \frac{\psi_k}{\psi_k} \right) \right)^{\frac{n}{2}} \prod_{k=1}^{n} \left( M_k N_k M_k^T S_{11} \right), \quad (B.2)$$

We know that

$$\delta_{11} \delta_{22} \delta_{33} = \det \delta = -\rho^2. \quad (B.3)$$

and

$$\delta_{11} \delta_{22} \delta_{33} = \det \delta = -\rho^2. \quad (B.4)$$

In Appendix D it is shown that

$$\det \Psi (\lambda) = \lambda G(w), \quad (B.5)$$

where $G(w)$ is only a function of

$$w(\lambda) = \frac{\lambda^2 + 2\lambda \rho - \rho^2}{\lambda}. \quad (B.6)$$

Since, using Eq. (2.19),

$$w(\lambda_k) = \frac{\mu_k^2 + 2\mu_k \rho - \rho^2}{\mu_k} = 2\mu_k, \quad (B.7)$$

we have

$$\det \Psi (\lambda_k) = \text{const. } \mu_k. \quad (B.8)$$

Furthermore, we will also use the following relation, which is shown in Appendix E:

$$\prod_{k=1}^{n} (\mu_k - \mu_k) - \text{const. } \prod_{k=1}^{n} (\mu_k, \mu_k + \rho^2). \quad (B.9)$$

Substituting Eqs. (B.3), (B.4), (B.5) and (B.6) in Eq. (B.2) we obtain

$$f_{ph} = \frac{\text{const. } c^2 Q^2 f_0 \rho \delta^2 + 2\rho}{k \prod_{k=1}^{n} (\mu_k, \mu_k + \rho^2)}. \quad (B.10)$$

On the other hand, from Eq. (2.23), we have

$$Q^{-1} = \text{const. } \rho^{-2} \left( \prod_{k=1}^{n} \left( \frac{1}{2} \right) \right)^{\frac{n}{2}} \prod_{k=1}^{n} (\mu_k, \mu_k + \rho^2). \quad (B.11)$$

where we used Eq. (B.9). The last factor of Eq. (B.11) can be re-expressed in the following form:

$$\prod_{k=1}^{n} (\mu_k, \mu_k + \rho^2) = \prod_{k=1}^{n} (\mu_k, \mu_k + \rho^2), \quad (B.12)$$

where we use the condition (i). Thus we have, putting $n = 3s$.

$$Q^{-1} = \text{const. } \rho^{-2} \left( \prod_{k=1}^{n} \left( \frac{1}{2} \right) \right)^{\frac{n}{2}} \prod_{k=1}^{n} (\mu_k, \mu_k + \rho^2). \quad (B.13)$$

Substitution of Eq. (B.13) into Eq. (B.10) gives the desired result:

$$f_{ph} = \text{const. } f_0. \quad (B.14)$$
We now consider the condition (iii). Let us recall Eqs. (6.4) and (6.5)
\[ \mu_k \mu_k' = \rho^2, \]
\[ \mu_k + \mu_k' = 2(W_k - z). \]

Some consequences of Eqs. (6.4) and (6.5) will be derived below. First consider
\[ \prod_{k \neq \ell} (\mu_k - u_k) = \prod_{k \neq \ell} (\mu_k - u_k') \prod_{k \neq \ell} (\mu_k - u_k' - u_k) = \prod_{k \neq \ell} (\mu_k - u_k' - u_k), \]
(B.15)
but
\[ (u_k - u_k)(u_k - u_k') = \mu_k^2 - 2u_k(u_k - z) - \rho^2 \]
\[ = 2u_k(w_k - z) - 2u_k(w_k - z) = 2u_k(w_k - w_k'), \]
where we used Eqs. (6.4), (6.5) and (B.7). Consequently,
\[ \prod_{k \neq \ell} (u_k - u_k') = \prod_{k \neq \ell} (u_k - u_k') \prod_{k \neq \ell} (\mu_k - u_k') = \text{Const.} \prod_{k \neq \ell} \left( \frac{(\mu_k - u_k')^2}{\rho^2} \right) \prod_{k \neq \ell} \mu_k'. \]
(B.16)

Also it can easily be shown that
\[ \prod_{k \neq \ell} (\mu_k^2 + \rho^2) = \prod_{k \neq \ell} (\mu_k^2 + \rho^2)(\mu_k^2 + \rho^2) = \text{Const.} \prod_{k \neq \ell} \mu_k^2. \]
(B.17)

Therefore we have, for \( m \) poles \( \mu_k \), or equivalently \( \frac{m}{2} \) pairs \( \mu_k, \mu_k' \),
\[ \prod_{k \neq \ell} (\mu_k - u_k) = \text{Const.} \prod_{k \neq \ell} \left( \frac{\mu_k}{\rho^2} \right)^m = \text{Const.} \rho^{-2m}, \]
(B.18)
where we used Eq. (6.4)
This implies that

\[
\left[ \sum_{k=1}^{n} \left( H^k_0 \right)_{11} \right] \left[ \sum_{k,q} \left( H^k_0 \right)_{22} \right] \left[ \sum_{k,q} \left( H^k_0 \right)_{33} \right] = \text{Const.} \tag{B.26}
\]

Hence again

\[ f = \text{Const.} \]

\[ f = \text{ph.} \]

Q.E.D.

APPENDIX C

A solution of the Einstein scalar field equations (3.9) and (3.10) that represents the exterior gravitational field of a static axially symmetric configuration is

\[
\left( g_{E,S} \right)_{11} = r^2 \left( \frac{26}{r} \right)^{1-\delta} \sin^2 \theta , \tag{C.1}
\]

\[
\left( g_{E,S} \right)_{22} = 1 - \left( \frac{26}{r} \right)^{1-\delta} , \tag{C.2}
\]

\[
f_{E,S} = \frac{1 - \left( \frac{26}{r} \right)^{1-\delta}}{1 - \frac{26}{r} - \frac{r^2}{\sin^2 \theta} \sin^2 \theta} ; \quad \theta = \delta^2 + 3v^2 , \tag{C.3}
\]

\[
\phi = \left[ 1 - \left( \frac{26}{r} \right)^{1-\delta} \right] \sqrt{\frac{3v+3}{3}} . \tag{C.4}
\]

where \( \delta \) and \( v \) are constants, and the co-ordinates \( r, \theta \) are related to \( \varphi \) and \( \psi \) by Eqs. (7.10) and (7.11). If \( v = 0 \), the above equations transform into the well-known Weyl-Lovelock Civita metric 181,189 of general relativity. The total gravitational mass of the system in \( 6 \delta \), which follows from the requirement that the equations of motion become Newtonian in the limit \( r \to \infty \).

The case \( \delta^2 = 1 \) is of particular interest, since we get a spherically symmetric gravitational field. The line element takes the form

\[
ds^2 = \left[ 1 - \frac{26}{r} \right]_{\delta}^{\delta} dr^2 + \left[ 1 - \frac{26}{r} \right]_{\delta}^{\delta} dr^2 + r^2 \left[ ds^2 + \sin^2 \theta \, d\theta^2 \right] , \tag{C.5}
\]

where

\[
r^2 = r^2 \left[ 1 - \frac{26}{r} \right]_{\delta}^{1-\delta} . \tag{C.6}
\]

A solution of the BIAS field equation, \( \tilde{g} \), is built, from the metric (C.1)-(C.3) and the scalar field (C.4) using the transformations (3.16), (3.6), (3.7), together with Eq.(3.11).
or, in the canonical form (4.9),

\[ g^{ab} = \left[ 1 - \frac{2\theta}{r} \right]^{-2v} \mathbf{s}_{ab} ; \quad a, b = 1, 2, \quad (C.9) \]

\[ \mathbf{g}^{a} = \left[ 1 - \frac{2\theta}{r} \right]^{-2v} \mathbf{e}, \quad (C.10) \]

\[ f = \left[ 1 - \frac{2\theta}{r} \right]^{-2v} \mathbf{e} \cdot \mathbf{s} \quad . \quad (C.11) \]

APPENDIX D

Let us show that if \( \det \mathbf{g} = -\rho^2 \), then

\[ \det \psi(\lambda) = \lambda g(W), \quad (D.1) \]

where \( g(W) \) is a function only of \( W \), subject to the following condition:

\[ \lim \lambda \rightarrow 0 g(W) = -\rho^2 . \quad (D.2) \]

We know, from Eq.(2.8) that

\[ [D_2 g] \psi^{-1} = \frac{\partial W + \lambda W}{\lambda^2 + \rho^2} . \quad (D.3) \]

Consider the trace of the above equation

\[ \text{Trace} [D_2 g] \psi^{-1} = \frac{\rho^2 \text{Trace} U + \lambda \text{Trace} V}{\lambda^2 + \rho^2} = \frac{\rho^2 \text{Trace} g \cdot g^{-1} + \lambda \text{Trace} g \cdot g^{-1}}{\lambda^2 + \rho^2} . \quad (D.4) \]

Using the well-known result, valid for any non-singular matrix \( N \),

\[ \text{Trace} \left[ (\text{det} N)^{-1} \right] = \frac{d(\text{det} N)}{\text{det} N} , \quad (D.5) \]

we obtain, from Eq.(D.4),

\[ \frac{D_2 (\text{det} \psi)}{\text{det} \psi} = \frac{\rho^2 (\text{det} g) + \rho \lambda (\text{det} g) \cdot (\text{det} g)^{-1}}{\lambda^2 + \rho^2} = 2 \frac{\rho}{\lambda^2 + \rho^2} . \quad (D.6) \]

where we used
Similarly, from Eq.(2.7) it follows that

\[
\text{det } g = -\rho^2. \tag{D.7}
\]

On the other hand

\[
D_1(\ln \lambda) = \frac{2 - \lambda^2}{\lambda^2 + \rho^2} \ln \lambda = \frac{2 - \lambda^2}{\lambda^2 + \rho^2} \tag{D.9}
\]

Therefore

\[
\frac{D}{\lambda} \ln (\text{det } \lambda) = 0, \tag{D.11}
\]

\[
D_1 \ln (\text{det } \lambda) = 0. \tag{D.12}
\]

The solution to Eqs.(D.11) and (D.12) is

\[
\ddet \psi = \frac{\lambda^2 + 2\lambda \rho - \rho^2}{\lambda}. \tag{D.13}
\]

Furthermore, since \( \psi(\lambda = 0) = g \), we must have

\[
\ddet \psi = \ddet g = -\rho^2, \tag{D.14}
\]

and consequently

\[
\lambda \ddet \psi = -\rho^2. \tag{D.15}
\]
APPENDIX E

We will see below that

$$\prod_{k} \frac{u_k^{2(n-1)}}{u_k - u_k'} = \prod_{k \neq k'} \frac{(\rho^2 + \mu_k \mu_{k'}) (u_k - u_k')}{2(u_k - u_k')},$$

(E.1)

where $n$ is the number of poles, $u_k$. Thus consider two poles $u_k, u_k'$. They satisfy Eq.(2.15), which in turn implies

$$\frac{u_k^2 - 2u_k (u_k - z) - \rho^2}{u_k - u_k'} = 0,$$

(E.2)

$$\frac{u_k'^2 - 2u_k' (u_k - z) - \rho^2}{u_k - u_k'} = 0.$$  

(E.3)

The difference of these two equations gives, after some re-arrangement

$$u_k - z = \frac{\mu_k + u_k - \mu_k (u_k - u_k')}{u_k - u_k'},$$

(E.4)

Substitution of Eq.(E.4) in Eq.(E.2) yields

$$\rho^2 = -\mu_k u_k + \frac{2u_k u_k' (u_k - u_k')}{u_k - u_k'},$$

(E.5)

or, equivalently,

$$u_k' = \frac{(\rho^2 + \mu_k u_k) (u_k - u_k')}{2(u_k - u_k')},$$

(E.6)

Thus we have

$$\prod_{k} u_k' = \prod_{k} \frac{(\rho^2 + \mu_k u_k) (u_k - u_k')}{2(u_k - u_k')}.$$

(E.7)

APPENDIX F

We will prove that if $\psi$ is diagonal then

$$\psi(\lambda) = \psi\left(\frac{-\lambda^2}{\lambda + \rho^2}\right) = F(W),$$

(F.1)

where $F(W)$ is an arbitrary function of $W$. First it is shown that $D_1$ and $D_2$ are invariant under the following transformation:

$$\lambda = \frac{-\rho^2}{\lambda + \rho^2},$$

(F.2)

$$\rho = \rho,$$

(F.3)

$$z = \frac{z}{\lambda}.$$  

(F.4)

Hence we have

$$\frac{3}{3z} = \frac{3}{3z} + \frac{2\lambda}{\lambda} \frac{3}{3z} + \frac{3}{3z},$$

(F.5)

$$\frac{3}{3z} = \frac{3}{3z} + \frac{2\lambda}{\lambda} \frac{3}{3z} + \frac{3}{3z},$$

(F.6)

$$\frac{3}{3z} = \frac{3}{3z} + \frac{2\lambda}{\lambda} \frac{3}{3z} + \frac{3}{3z},$$

(F.7)

Then it follows that

$$D_1(\lambda) = \frac{3}{3z} - \frac{2\lambda}{\lambda^2 + \rho^2} \frac{3}{3z} = \frac{3}{3z} - \frac{2\lambda}{\lambda^2 + \rho^2} \frac{3}{3z} = D_1(\lambda),$$

(F.8)

$$D_2(\lambda) = \frac{3}{3z} + \frac{2\lambda}{\lambda^2 + \rho^2} \frac{3}{3z} + \left[ \frac{2\lambda}{3z} - \frac{3\rho}{\lambda^2 + \rho^2} \frac{\lambda^2}{\lambda^2 + \rho^2} \right] \frac{3}{3z},$$

(F.9)

The field equations(2.7) and (2.8) imply that
\[ [D_1(\lambda)\psi(\lambda)]^{-1}(\lambda) = \frac{\partial V - \lambda u}{\partial x^2 + \rho^2} = \frac{\partial V}{\partial x^2 + \rho^2} = \frac{\partial V}{\partial \rho} \]
\[ - [D_1(\lambda)\psi(\lambda)]^{-1}(\lambda) = \frac{\partial}{\partial x^2} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \lambda} \psi(\lambda) , \quad (F.10) \]

and similarly,
\[ [D_2(\lambda)\psi(\lambda)]^{-1}(\lambda) = \frac{\partial V + \lambda u}{\partial x^2 + \rho^2} = \frac{\partial V}{\partial x^2 + \rho^2} = \frac{\partial V}{\partial \rho} \]
\[ - [D_2(\lambda)\psi(\lambda)]^{-1}(\lambda) = \frac{\partial}{\partial x^2} \frac{\partial}{\partial \rho} \frac{\partial}{\partial \lambda} \psi(\lambda) , \quad (F.11) \]

Using Eqs. (F.8) and (F.9), in Eqs. (F.10) and (F.11), we obtain
\[ [D_1(\lambda)\psi(\lambda)]^{-1}(\lambda) + [D_1(\lambda)\psi(\lambda)]^{-1}(\lambda) = [D_1 g^{-1}(\lambda) = 0 , \quad (F.12) \]
\[ [D_2(\lambda)\psi(\lambda)]^{-1}(\lambda) + [D_2(\lambda)\psi(\lambda)]^{-1}(\lambda) = [D_2 g^{-1}(\lambda) = 0 . \quad (F.13) \]

If \( g \) and \( \psi \) are diagonal, then Eqs. (F.12) and (F.13) simplify to
\[ D_1(\lambda) \ln(\psi(\lambda) \psi(\lambda)^{-1}) = 0 , \quad (F.14) \]
\[ D_2(\lambda) \ln(\psi(\lambda) \psi(\lambda)^{-1}) = 0 . \quad (F.15) \]

The solution to Eqs. (F.14) and (F.15) is
\[ \psi(\lambda) \psi(\lambda)^{-1} = \psi + \frac{\partial^2}{\partial \lambda^2} \psi(\lambda) g^{-1} = \psi(g(W) , \quad (F.16) \]

where \( F(W) \) is an arbitrary function of
\[ W = \lambda^2 + 2\lambda \alpha - \rho^2 . \quad (F.17) \]