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HYPERBOLIC MANIFOLDS AS VACUUM SOLUTIONS IN KALUZA-KLEIN THEORIES *

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ABSTRACT

The relevance of compact hyperbolic manifolds in the context of Kaluza-Klein theories is discussed. Examples of spontaneous compactification on hyperbolic manifolds including d dimensional ($d \geqslant 8$) Einstein-Yang-Mills gravity and 11-dimensional supergravity are considered. Some mathematical facts about hyperbolic manifolds essential for the physical content of the theory are briefly summarized. Non-linear σ-models based on hyperbolic manifolds are discussed.

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1. INTRODUCTION

Recently, a remarkable success was achieved in the construction of all interactions in the superstring approach in ten-dimensional space-time [1]. To explain the observable picture of particles in the four-dimensional world it is important to understand the spontaneous compactification of the ten-dimensional theory (for a review of spontaneous compactification see Refs.[2]). Usually it is assumed that the high-dimensional world is a product of the form $M^d = M^4 \times B^{d-4}$, where B is a compact manifold or a noncompact one with a finite volume [3,4] , moreover B is usually assumed to be a Riemannian manifold of positive curvature. Riemannian manifolds of negative curvature were not considered as internal spaces. Apparently, the reason for this lack of attention is that any complete manifold of positive curvature is a compact one, while the simplest hyperbolic manifold namely the Lobachevski space is a non-compact one and has an infinite volume (a realization of the Lobachevski space is the mass-shell hyperboloid in Minkowski space). However, compact manifolds of negative curvature do exist and there is a remarkable mathematical theory of compact manifolds of constant negative curvature (hyperbolic manifolds). Another reason that hyperbolic manifolds were not considered in Kaluza-Klein theories is that the vacuum configurations for the matter fields discussed in the literature lead only to manifolds of non-negative curvature.*)

In this paper it will be shown that manifolds of negative curvature do appear in Kaluza-Klein theories under some rather general conditions. The relevance of compact hyperbolic manifolds in the context of Kaluza-Klein theories was recently pointed out in Ref.[5]. Here we shall consider examples of spontaneous compactification on hyperbolic manifolds including eleven-dimensional supergravity and 7-dimensional gauged supergravity. As it was mentioned in Ref.[6] the hyperbolic manifolds appear as a rule as a solution of the Einstein equations in the Kaluza-Klein theories with extra compactified time-like variables. The hypothesis of the existence of extra time-like variables was suggested in Ref.[7]. About the hyperbolic manifold in the ten-dimensional effective theory of superstrig see Ref.[6].

The observable particles in Kaluza-Klein theories in the leading approximation are massless particles living in a four-dimensional world and their number is determined by the number of zero-modes of invariant operators like the Hodge de Rham operator, Dirac operator, etc. on the manifold B.

*) A very interesting mechanism of dimensional reduction induced by non-linear σ-model based on the hyperbolic coset space $H^2 = SU(1,1)/U(1)$ was considered by Gell-Mann and Zweibach [4]. The σ-model can curl up to two spatial dimensions into a non-compact positively curved surface of finite area, the "teardrop".

From this point of view compact hyperbolic manifolds have very attractive properties. As it is well known, the Dirac operator on manifold with positive curvature has no zero modes for the fields with spin equal 1/2 (Lichnerowicz's theorem), so the corresponding particles are non-observable [2]. For hyperbolic manifolds this difficulty is overcome.

In the papers of Wetterich and Nicolai[3] and Gell-Mann and Zweibach[4] it was suggested to use a non-compact manifold of positive curvature with finite volume as internal space B. Let us mention that non-compact hyperbolic manifolds with finite volume do exist. For example, the 2-manifold of the form H^2/Γ have a finite volume if and only if Γ is the Fuchsian group of the first type [7].

Our paper is organized as follows. In Sec. II we point out some general conditions under which hyperbolic manifolds may appear in the Kaluza-Klein theories. In Sec. III the geometrical construction of the compact hyperbolic manifolds is summarized. In this section we concentrate on those properties of the manifolds which are essential to determine the physical content of the theories. In Sec. IV particular examples of higher-dimensional theories which lead to compactification on compact hyperbolic manifolds are presented. We end up with a brief discussion of non-linear σ-models on hyperbolic manifolds (Sec. V).

II. STRUCTURES OF EXTRA SPACE AND SPACE-TIME

The general mechanism of the appearance of hyperbolic manifold as a solution of the Einstein equations can be understood in the following way. Let us assume that the stress tensor for matter field T_{MN} for a vacuum solution has the form

$$T_{\hat\mu\hat\nu} = \gamma_1 g_{\hat\mu\hat\nu}, \quad T_{ab} = \gamma_2 g_{ab}, \quad T_{\hat\mu a} = 0, \tag{1}$$

where $\hat\mu$, $\hat\nu$ = 0,1,2,3,5,...r; a,b = 1,...d-r. This is a very strong restriction, but various ways of obtaining these equations have been found. Among them the Freund-Rubin ansatz, monopole and instanton mechanism, embedding the Yang-Mills connection into spin connection, some particular mechanism for gravity coupled to non-linear σ-model, [2-5].

In these cases the classical vacuum is a product of two Einstein spaces, $M^d = M^{d-r} \times B^r$,

$$R_{\hat\mu\hat\nu} = C_1 g_{\hat\mu\hat\nu}, \quad R_{ab} = C_2 g_{ab} \tag{2}$$

with

$$C_1 = \frac{1}{d-2}\left[r(\gamma_1-\gamma_2)-2\gamma_1\right], \quad C_2 = \frac{1}{d-2}\left[(d-r)(\gamma_2-\gamma_1)-2\gamma_2\right] \tag{3}$$

Conditions (1) usually hold when the matter fields are non-zero only on B^r.

Assuming that M^{d-r} is anti-de-Sitter (adS)-like or that the stress tensor satisfies the weak energy condition (see[9]) one obtains that B has to be an Einstein space of positive curvature ($c_2 > 0$). Thus the question arises how a hyperbolic space can appear as an internal space, if under the above physically reasonable conditions we always get that the space B^r ought to be an Einstein space of positive curvature. The answer is that the following possibility exists. We consider M^{d-r} in the form of a product of two Einstein spaces (a four-dimensional space and a (d-r-4) dimensional one): $M^{d-r} = M^4 \times \mathcal{N}^{d-r-4}$ instead of $M^{d-r} = adS^{d-r}$. In the case of $M^{d-r} = M^4 \times \mathcal{N}^{d-r-4}$ we obtain that $\mathcal{N}^{d-r-4}$ should be an Einstein space of negative curvature. So, it is possible to get a hyperbolic space as a portion of the internal space.

It is essential to point out that in this case extra dimensions are consistent with but do not imply compactification. One can believe that the contribution to the functional integral from non-compact hyperbolic manifolds are suppressed.

Another possibility of appearance of hyperbolic manifolds in the Kaluza-Klein theories arises as it was mentioned in the Introduction when one introduces extra time-like variables.

III. HYPERBOLIC MANIFOLDS

In this section we shall present a brief review of some mathematical facts about compact hyperbolic manifolds (for more details the reader is referred to [9-20]). The spaces of constant Riemannian curvature K such that K > 0, K < 0 and K = 0 are locally the Riemann spherical space, Lobachevski space H^n, and Euclidean space E^n, respectively. Let us recall the definition of hyperbolic space H^n,

$$H^n = \{x \in M^{n+1}, \ -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = -1, \ x_1 > 0\}$$

M^{n+1} is Minkowski space with the signature $(- + \dots +)$. More precisely, the following theorem (Killing, Hopf) is true. A complete connected Riemannian n-manifold of constant curvature K and of dimension $n \geq 2$ is isometric to a quotient S^n/Γ with $\Gamma \subset O(n+1)$ if $K > 0$, E^n/Γ with $\Gamma \subset E(n)$ if $K=0$, H^n/Γ with $\Gamma \subset O(1,n)$ if $K < 0$ where Γ acts freely (i.e. without fixed points) and properly discontinuously (i.e. Γ is discrete) [11,12].

Let us start with the more simpler case of two-manifolds. Any two-dimensional manifold in R^3 can be locally described by the equation $z = f(x,y)$. Then the curvature is given in the formula

$$K = \frac{f_{xx} f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)}$$

The positivity of the curvature means that for any point the surface is locally on one side of the tangent plane (Fig. 1a). If the curvature is negative then the surface lies part on one side and part on the other side of the tangent place in the neighbourhood of a point (Fig. 1b).

One cannot visualise the hyperbolic plane H^2 in the same way that one visualises the sphere S^2 because H^2 cannot be isometrically embedded in E^3 but locally it is isometric to a surface in E_3 with a saddle point at every point. An example is the one sheet hyperboloid, shown in Fig. 1c. This surface clearly has a saddle point at every point. However, this surface is not simply connected. Let us mention that the two-dimensional torus also cannot be isometrically embedded in E^3. Indeed, taking a square sheet of paper and gluing the two opposite sides (Fig.2) we immediately notice that the opposite circles A and B in Fig. 2 cannot be glued unless our square is from ductile material.

As a model of H^2 we consider the upper half-plane $R_+^2 = \{(x,y) \in R^2, \ y > 0\}$ and the metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$. To understand the geometry of H^2 we need to know which curves on H^2 are geodesics. Every geodesic in H^2 is a vertical straight line or a semi-circle centred on the x-axis. As for E^2, any pair of points in H^2 lies on a unique geodesic and two distinct geodesics meet in at most one point. The essential difference form E^2 is that given a geodesic $\mathcal{M}$ in H^2 and a point x not on $\mathcal{M}$, there are infinitely many geodesics through x which do not meet $\mathcal{M}$ (Fig.3). While for given geodesic $\mathcal{B}$ in E^2 and a point y not on $\mathcal{B}$ there is only one geodesic through y which does not meet $\mathcal{B}$.

Let us consider triangles in H^2 with geodesic edges. The area of a triangle with angles α, β, γ is $\pi - (\alpha + \beta + \gamma)$. (As compared to the Euclidean situation, where specifying the three angles of a triangle in E^2 one cannot determine its area. For triangles in S^2 the area is $(\alpha + \beta + \gamma) - \pi$).

There are two kinds of isometry of H^2. One is a reflection of R_+^2 with respect to a vertical straight line and the second is inversion of R_+^2 in a semicircle of a radius r centred at a point x_0 on the x-axis. It is possible to prove that the full isometry group of H^2 is generated by these isometries and it is a three parameter group[11,12]. To specialize any isometric transformation of H^2 it is sufficient to define the transformation of arbitrary two points in R_+^2 (this transformation is described by 3 parameters since the distance between the points is conserved).

Let us describe the simplest example of compact hyperbolic 2-manifold. Let X be a regular octagon in H^2 with all angles $\pi/4$ as shown in Fig.4. The distance between the points A and B is the same as the distance between D and C, so we can specialize the isometry α of H^2 by the requirement that

$$\alpha(A) = D, \ \alpha(B) = C$$

and we of course have $\alpha(AB) = DC$. Let β, γ and δ be the isometry of H^2 such that

$$\beta(BC) = ED, \ \gamma(EF) = HG, \ \delta(FG) = AH$$

(see Fig.4).

It is easy to prove that the group of isometries Γ of H^2 generated by α, β, γ and δ is discrete and has X as a fundamental region. (For any group G acting on a space M, one defines a fundamental region for G to be a closed subset x of M such that: 1) $\bigcup_{g \in G} gX = M$; 2) $\mathring{X} \cap g\mathring{X} = \emptyset$ for all non-trivial elements g in G, where $\mathring{X}$ is the interior X)(See [8,10,12]). The quotient H^2/Γ can be obtained from X by identifying the pairs of points identified by α, β, γ and δ. If Γ is a discrete group of isometries of H^2 and acts freely on H^2 then H^2/Γ inherits a natural metric such that the projection $H^2 \to H^2/\Gamma$ is local isometry. The construction of compact hyperbolic manifold H^2/Γ is analogous to common construction of a 2-torus as a quotient E^2/Γ. Taking a square in E^2 and considering the isometries of E^2 such that $\alpha(AB) = DC$, $\beta(AD) = BC$, see Fig. 5)) this group of isometries has the square as a fundamental region, and the quotient E^2/Γ can be obtained from the square by identifying the pairs of opposite sides of the square.

In Fig.6 we try to show the abstract gluing of the sides of the octagon. Fig.6a presents the gluing which corresponds to identification of the sides after the isometric transformations α and δ. The identification of sides HG and FE shown in Fig.6b and identification of sides EF with BC afterwards lead to a sphere with two handles.This surface cannot be isometrically embedded in E^3. Indeed, trying to glue the sides of the octagon made of solid sheet (i.e. the distance between the points are defined by E^3 geometry) we immediately notice that this is impossible. We ought to change the distance between internal points of the octagon, i.e. the geometry, in order to glue the sides as indicated in Fig. 4.

So, by the construction described above we get the standard picture for the closed orientable surface $\mathcal{M}^2$ of genus 2 and by the construction we have a hyperbolic geometry on the suface.

To obtain the surface of genus n we ought to consider some special identification of the sides of a regular 4n-polygon.

From Fig.7a one can see that the sum of the interior angles of the triangle on the $\mathcal{M}^2$ is less than π and it decreases as the area of the triangle grows. If we cut out a piece of the hyperbolic surface and then try to embed it isometrically in E^2, we find that it is impossible without overlaps (Fig.7b). If we will do the same for the sphere we find that we must cut the interior of the circle along its radius as in Fig.7c.

Let us mention that if G does not act freely, then H^2/Γ still inherits a natural metric, but now there are singular points [12].

It is still known [11,12] that in dimension two there is a very close relationship between geometry and topology. Each closed surface admits a metric of constant curvature. There are three cases depending on the Euler number of the surface.

- There are only two closed surface with positive Euler number, namely the 2-sphere S^2 and the real projective plane P^2. P^2 inherits from S^2 a metric of constant positive curvature.
- There are only two closed surfaces with zero Euler number, namely the torus T^2 and Klein bottle and they both admit a metric of constant and zero curvature.
- All the other closed surfaces have negative Euler number and admit metric of constant negative curvature.

A few facts about n-manifolds ($n \geqslant 3$). The set of compact hyperbolic n-manifolds ($n \geqslant 3$) with any given volume is finite. The Mostow rigidity theorem [19] says that any invariant of the geometry of a hyperbolic n-manifold ($n \geqslant 3$) is actually an invariant of its homotopic type. This fact was interpreted in [5,6] as the uniqueness of the vacuum solution.

An example of a compact hyperbolic 3-manifold is the hyperbolic dodecahedral space [13]. To describe this manifold let us define the Seifert-Weber dodecahedral manifold. The best way to understand this manifold is to imagine in E^3 a regular dodecahedron (which has 12 faces each of them being a regular pentagon) and then imagine that its faces are glued abstractly in the following way. Each face is glued to its opposite counterpart after counterclockwise rotation by $3/5\ \pi$ about the axis perpendicular to its surface (Fig.8). Like the torus such a manifold has no boundary or edge at all. The identification that gives rise to the Seifert-Weber space is that all 20 vertices of the dodecahedron must meet at a point. The angle at each vertex must therefore shrink until all 20 angles are small enough to fit together around a single point. This shrinking of angles, keeping the regularity of the dodecahedron, is possible if we will consider the dodecahedron in Lobachevski 3-space. So, Seifert-Weber space can be given a locally hyperbolic geometry if the dodecahedron that generates it is a regular dodecahedron in hyperbolic space H^3.

Let us mention, that it is possible to get the spherical 3-manifold from another gluing of the sides of dodecahedron. This manifold is called the Poincaré manifold and it can also be obtained if each pentagon is glued to its counterpart after counterclockwise rotation by $1/5\ \pi$. This abstruct gluing that leads to the Poincare manifold specifies that four vertices of the dodecahedron must fit around a point. So, the angles must be increased, what can be done if it is considered in a space of positive curvature.

IV. MODELS

We now turn to the description of examples of theory admitting compactification on hyperbolic manifolds.

1. Let us consider the coupling of d-dimensional gravity ($d \gtrsim 8$) and Yang-Mills system with the Lagrangian

$$\mathcal{L} = e\left(R - \frac{1}{4g^2} F_{MN}^{2}\right)$$

where

$$F_{MN} = \partial_M A_N - \partial_N A_M + [A_M, A_N]$$

and where the metric g_{MN} has a signature $(- + + + + \dots +)$, $e = \det|g_{MN}|$. In this case we may obtain a solution of Einstein equations of the form $adS^4 \times \Omega^{d-r-4} \times S^r$ $(r \geq 2)$, where Ω^{d-r-4} $(d-r-4 \geq 2)$ is a compact hyperbolic manifold. To obtain this solution we set the gauge field in S^r space equal to the spin connection, suitably embedded in the gauge group [21].

If we consider $d = 8$ and use monopole ansatz we get the solution $adS^4 \times \Omega^2 \times S^2$. The first Betti number $b_1(\Omega^2)$ is equal to $2n$, where n is the genus of the surface Ω^2. So, one gets that d-dimensional vector field generates $b_1(\Omega^2)\, b_0(S^2) + b_0(\Omega^2)\, b_1(S^2) = 2n$ massless scalars in 4-dimensional space-time.

In the case $d \geq 9$ one can obtain the solution in the form $adS^4 \times \Omega^3 \times S^r, r \geq 2$. If we take Ω^3 equal to the hyperbolic space of the dodecahedron described in the previous section, then the first Betti number $b_1(\Omega^3)$ is equal to zero and one does not get 4-dimensional massless scalars.

2. Let us now show that hyperbolic manifold can also occur as a vacuum solution of the eleven-dimensional supergravity. The bosonic equations of motion have the form [2]

$$R_{MN} - \frac{1}{2} g_{MN} R = \frac{1}{3}\left(H_{MPQL} H_N{}^{PQL} - \frac{1}{8} g_{MN} H^2\right)$$

$$\nabla_M H^{MPQL} = -\frac{1}{576}\, \varepsilon^{M_1 \dots M_8 PQL} H_{M_1 \dots M_4} H_{M_5 \dots M_8}$$

This theory admits the compactification $M^{11} = adS^7 \times S^4$ as well as $M^{11} = adS^4 \times \Omega^3 \times S^4$, where the Freund-Rubin ansatz with the following non-zero components $H_{abcd} = h\sqrt{g(y)}\,\varepsilon_{abcd}$, $a,b,c,d = 8,9,10,11$ has been

used. Ω^3 is a compact hyperbolic manifold. Standard Kaluza-Klein ansatz provides the appearance of $SO(5)$ gauge fields corresponding to the isometry group of S^4. Compact hyperbolic manifolds according to Bochner's theorem do not have Killing vector fields at all.

3. Let us consider gauged N=2, d=7 supergravity [22]. The bosonic sector of this theory is described by the metric g_{MN}, field strengths F_{MNPQ} and F^{ij}_{MN} and a scalar field σ. The equations of motion have the form

$$R_{MN} = -5(\partial_M \ln\sigma)(\partial_N \ln\sigma) - \sigma^2\left(F^{ij}_{NQ} F_N{}^{Q,ij} - \frac{1}{10} g_{MN} F^{ij}_{PQ} F^{PQ,ij}\right)$$
$$- \frac{1}{6}\sigma^{-4}\left(F_{MPQL} F_N{}^{PQL} - \frac{3}{20} g_{MN} F_{PQLR} F^{PQLR}\right) - \frac{2}{5} V(\sigma)\, g_{MN},$$

$$-5\Box \ln\sigma = \frac{1}{12}\sigma^{-4} F_{PQLR} F^{PQLR} - \frac{1}{2}\sigma^{2} F^{ij}_{PQ} F^{PQ,ij} - \sigma\frac{\partial V}{\partial\sigma},$$

$$\partial_M\left(e\,\sigma^{-4} F^{MQ_1Q_2Q_3}\right) = \varepsilon^{Q_1Q_2Q_3MNRS}\left(\frac{ih}{3} F_{MNPS} + \frac{i}{4\sqrt{2}} F^{ij}_{MN} F^{ij}_{RS}\right),$$

$$\mathcal{D}_M\left(\sigma^2 F^{MN}\right) = -\frac{i}{24\sqrt{2}}\, \varepsilon^{NQ_1\dots Q_6} F_{Q_1\dots Q_4} F_{Q_5Q_6},$$

where $V(\sigma) = -\frac{1}{4}\sigma^2(g^2 + 16\sqrt{2}\, gh\, \sigma^5 - 64 h^2 \sigma^{10})$ g is the gauge coupling constant and h is the topological mass term parameter, we use the notations of Refs.[22,23]. As it was noticed in Ref.[23] demanding the vacuum solution in the form of a product of a maximally symmetric space-time (labelled by $\mu,\nu = 0,1,2,3$) with a three-dimensional internal space (labelled by $a,b = 5,6,7$) and setting the gauge field in the 3-space equal to the spin connection one gets the Einstein's equations

$$R_{\mu\nu} = C_1 g_{\mu\nu}, \qquad R_{ab} = C_2 g_{ab}$$

with negative constant C_2. In Ref. [23] this is interpreted as the appearance of non-compact Einstein internal spaces. However, the possibility exists to take for this model the vacuum solution to be in the form $adS^4 \times \Omega^3$ where Ω^3 is a compact hyperbolic manifold.

To conclude this section, we would like to note that many interesting problems remain open. It is important to consider the problem of stability of the solution in the form $adS^4 \times \mathcal{M}^r$ both on the classical level as well on the quantum one. To investigate the classical stability it is essential to know the spectrum of the Lichnerowicz operator and Hodge de Rham operator on compact hyperbolic manifolds (about the spectrum of the Hodge de Rham operator on hyperbolic manifolds (see Refs.[24]). It is also interesting to study the fermion sector for models admitting compactification on compact hyperbolic manifolds. For this purpose one should know the spectrum of the Dirac operator on hyperbolic manifolds.

V. NON-LINEAR σ-MODELS ON HYPERBOLIC MANIFOLDS

Here we mention some arguments pointing out that an essential difference between non-linear σ-models on compact hyperbolic manifolds and on manifolds of non negative curvature may exist. Non-linear σ-models on compact hyperbolic manifolds are theories of maps from space-time into a compact Riemannian manifold $\mathcal{M}^n$ of constant negative curvature. Non-compact σ-models arise naturally in extended supergravity theories [26] and non-compact σ-models on Lobachevski space was considered in Ref.[27]. As it was argued in Ref.[27], the non-compact σ-models are infra-red free and do not exhibit the dynamical generation of a mass gap.

Let us consider 1-dimensional σ-model. The Lagrangian has the form

$$\mathcal{L} = g_{ab}(\varphi)\,\dot{\varphi}^a \dot{\varphi}^b$$

where

$$\dot{\varphi}^a = \frac{d\varphi^a(t)}{dt}$$

In this case the equation of motion is actually the equation for a geodesic on the manifold

$$\frac{d^2\varphi^a}{dt^2} + \Gamma^a_{bc}\,\dot{\varphi}^b \dot{\varphi}^c = 0$$

There is a well known remarkable property of the geodesic flow on compact hyperbolic manifolds. Namely, the exponential instability of the geodesic leads to a stochastic behaviour of the corresponding geodesical flow [25].

It would be interesting to study the two-dimensional non-linear σ-model which is indeed the theory of free string. Since the dynamics of one degree of freedom living on a hyperbolic manifold has no common feature with the dynamics of a particle on flat space or on spherical manifold we believe that the string dynamics on compact hyperbolic manifolds might be extremely different from the one on flat space or on spherical space.

Let us note that for two-dimensional non-compact $O(2,1)$ σ-model there exist the Lax linear representation, instantons and higher conserved currents, which can be constructed in a similar way as for $O(3)$ σ-model. This is true locally also for σ-models on compact hyperbolic manifold. But in the last case we lose the integrability and we deal with stochastical behaviour.

To perform the quantization of the σ-model based on the hyperbolic manifold H^n/Γ, we must investigate the functional integral

$$\int \exp\left[i\int \langle \partial_\mu \varphi, \partial_\mu \varphi \rangle dx\right] \delta(\langle \varphi, \varphi \rangle + 1)\, \mathcal{D}\varphi(x),$$

where $\langle\phi,\phi\rangle = -\phi_0^2 + \phi_1^2 + \ldots + \phi_n^2$ and the field ϕ is restricted to the fundamental domain of the group Γ with the corresponding boundary conditions (see Sec.III). The work in this direction is now in progress.

In conclusion we would like to note that it seems that the beautiful mathematical theory of hyperbolic manifolds may be very fruitful in the quantum field theory.

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FIGURE CAPTIONS

Fig. 1a. Two-dimensional surface with positive curvature.

Fig. 1b. Two-dimensional surface with negative curvature.

Fig. 1c. The not simply connected surface with saddle point at every point.

Fig. 2a. The gluing of the two opposite sides of the square.

Fig. 2b. The gluing of the two opposite circles of the cylinder.

Fig. 3. The upper half-plane R^2_+.

Fig. 4. The regular octagon in H^2.

Fig. 5. The torus.

Fig. 6a. The gluing which corresponds to indentification of the side AB with side CD and HA with GF.

Fig. 6b. The gluing of the sides HG and FE.

Fig. 7a. The triangles on the $\mathcal{H}^2$.

Fig. 7b. The piece of the hyperbolic surface $\mathcal{H}^2$ which we try to embed isometrically in E^2.

Fig. 7c. The piece of the sphere which we try to embed isometrically in E^2.

Fig. 8. The Seifert-Weber dodecahedral manifold. Identically marked sides of the dodecahedral are identified.

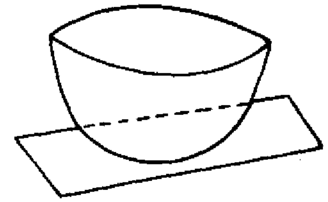


Fig. 1 a

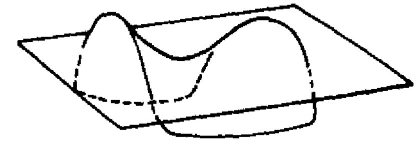


Fig. 1 b

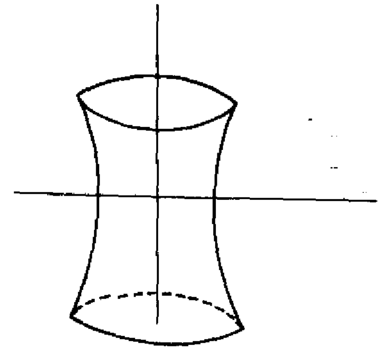


Fig. 1 c

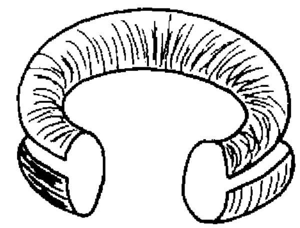
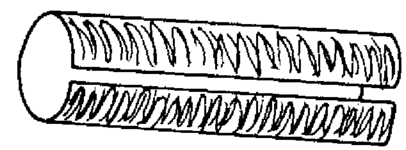


Fig. 2 b



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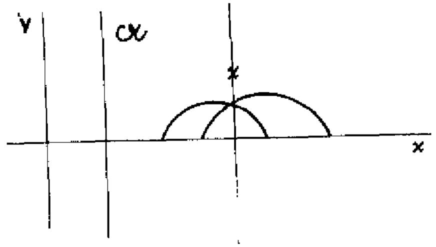
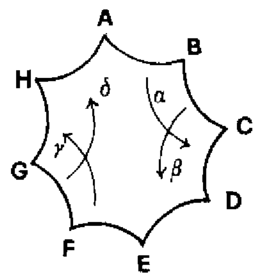


Fig. 3



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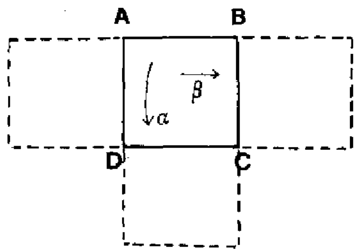

Fig. 5

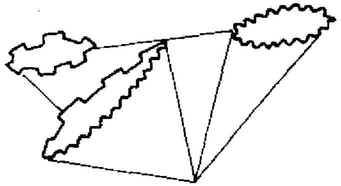

Fig. 6 a

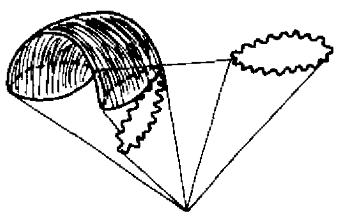

Fig. 6 b

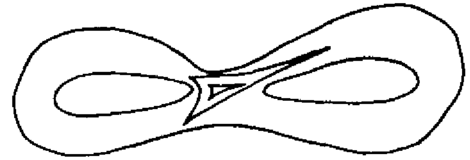

Fig. 7 a

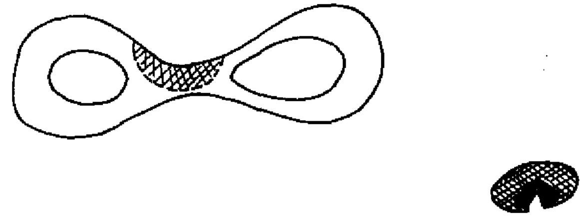

Fig. 7 b

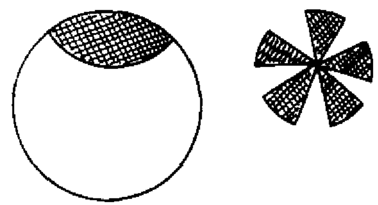

Fig. 7 c

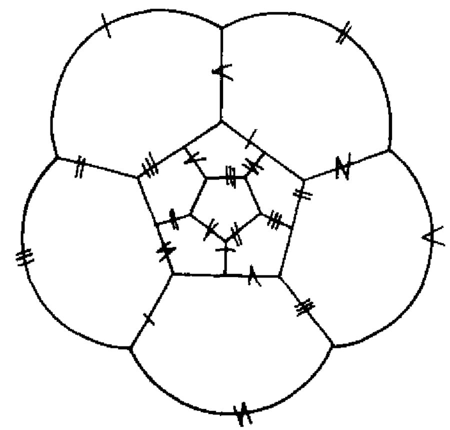

Fig. 8