IC/85/51

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

LOCALLY SUPERSYMMETRIC σ-MODEL WITH WESS-ZUMINO TERM IN TWO DIMENSIONS AND CRITICAL DIMENSIONS FOR STRINGS

E. Bergshoeff

S. Randjbar-Daemi

Abdus Salam

H. Sarmadi

and

E. Sezgin

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1985 MIRAMARE-TRIESTE

IC/85/51

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

LOCALLY SUPERSYMMETRIC σ-MODEL WITH WESS-ZUMINO TERM IN TWO DIMENSIONS AND CRITICAL DIMENSIONS FOR STRINGS *

E. Bergshoeff
International Centre for Theoretical Physics, Trieste, Italy,

S. Randjbar-Daemi **
Institute for Theoretical Physics, Sidlerstr.5, CH-3012 Bern, Switzerland,

Abdus Salam
International Centre for Theoretical Physics, Trieste, Italy,
and
Imperial College, London, England,

H. Sarmadi
International Centre for Theoretical Physics, Trieste, Italy,
and
International School of Advanced Studies, Trieste, Italy,

and

E. Sezgin
International Centre for Theoretical Physics, Trieste, Italy.

MIRAMARE - TRIESTE
May 1985

* To be submitted for publication.
** Supported by Swiss National Funds.

ABSTRACT

We construct an $N = 1$ locally supersymmetric σ-model with a Wess-Zumino term coupled to supergravity in two dimensions. If one takes the σ-model manifold to be the product of d-dimensional Minkowski space M^d and a group manifold G, and if the radius of G is quantized in appropriate units of the string tension, then the model describes a Neveu-Schwarz-Ramond (NSR)-type string moving on $M^d \times G$. (Our model generalizes earlier work of Refs.[1,2] which do not contain a Wess-Zumino term and that of Refs.[5],[6] which is not locally supersymmetric.) The zweibein and the gravitino field equations yield constraints which generalize those of the NSR model to the case of a non-Abelian group manifold. In particular, the fermionic constraint contains a new term trilinear in the fermionic fields. We quantize the theory in the light-cone gauge and derive the critical dimensions. We compute the mass spectrum of a closed string moving on $M_d \times G$ and for the bosonic case we show that it coincides with that of the string which compactifies on r-tori where r is the rank of a simply laced G. We also show that massless fermions do not arise for non-Abelian G for the spinning string.

I. [illegible]

Some time ago, Deser and Zumino [1] and Brink, di Vecchia and Howe [2] constructed the coupling of ten scalar supersymmetric multiplets to $d = 2$ supergravity. They showed that the theory is conformally invariant and describes the $d = 10$ Neveu-Schwarz-Ramond [3] string model.[15]. From the $d = 2$ point of view this theory is a locally supersymmetric σ-model in which the scalar manifold is a $d = 10$ flat Minkowski space-time, $M_{10} = ISO(9,1)/SO(9,1)$. In this paper we consider the generalization of this model in which the scalars parametrize an arbitrary Riemannian manifold. In such a generalization, a Wess-Zumino term [4,5,6] is coupled to $d = 2$ supergravity. This model furnishes a covariant description of a string moving in curved space, provided that this space is $M_d \times G$ where M_d is d-dimensional Minkowski space-time and G is a compact group manifold. The characteristic size of G is quantized in units of the string tension [7,8b], while restrictions on d arise, due to the requirement of Lorentz invariance in M_d, of the quantized theory.

These results are relevant for the reduction of the critical dimensions in which the string theory can be consistently quantized. Recently, in Refs.[7] and [8b] it was shown that the critical dimension for the bosonic string is given by

$$d = 26 - \frac{d_G}{1 + \frac{c_A}{2|k|}} \qquad (\text{bosonic string}), \tag{1.1}$$

where d_G is the dimension of the group G, k is an integer *) and c_A is the eigenvalue of the second Casimir operator of G in the adjoint representation (see Eq.(4.8)). Extending this result, in Ref.[7] it was conjectured that the critical dimension for the fermionic string is given by

$$d = 10 - \frac{2}{3} \cdot \frac{d_G}{1 + \frac{c_A}{2|k|}} - \frac{1}{3} d_G \qquad \begin{pmatrix}\text{spinning} \\ \text{string}\end{pmatrix}. \tag{1.2}$$

In this paper, starting from our locally supersymmetric action (see Eq.(2.1)) we quantize the theory in the light cone gauge, rederive (1.1) and verify (1.2).

The case of $k = 1$ is special, since, as was shown by Witten [9], in this case the scalars of the non-linear σ-model become equivalent to free fermions. For $k = 1$, the solution to (1.1) includes $G = SU(n)$, $SO(2n)$, E_6/Z_3, E_7/Z_2, E_8 or any product of these groups with

---

*) More precisely k is an integer for $SU(N)$, $SO(N)$, $SP(n)$, E_6/Z_3, E_7/Z_2 and E_8, 0 mod 3 for F_4 and 0 mod 4 for G_2.

$$r = 26 - d \qquad (\text{bosonic string})^{*)} \tag{1.3}$$

where r is the rank of G. **)

As we shall see, one relevant example of (a semi-simple) G is ***)

$$d = 6 \ , \qquad G = E_8 \times E_8 \times SU(3) \times SU(2) \times U(1) \qquad \left(\begin{matrix}\text{bosonic}\\ \text{string}\end{matrix}\right) \tag{1.4}$$

For $k = 1$, assuming that G is simple, the unique solution to (1.2) is ****)

$$\begin{aligned} d &= 8 \ , & G &= SO(3) \\ d &= 6 \ , & G &= SU(3) \\ d &= 5 \ , & G &= SO(5) \\ d &= 3 \ , & G &= SU(4) \end{aligned} \qquad \left(\begin{matrix}\text{spinning}\\ \text{string}\end{matrix}\right) . \tag{1.5}$$

(Note that $d = 4$ is not included for any simple G.) Possible applications of these solutions will be discussed in the conclusions.

This paper is organized as follows. In Sec.II we construct the $N = 1$ locally supersymmetric action with a Wess-Zumino term. In Sec.III we derive the field equations and constraints following from the action. In Sec.IV we quantize the system and derive the critical dimension formula by the requirement of Lorentz invariance in M_d. In Sec.V we discuss the spectrum of a closed string moving on $M_d \times G$. Finally in Sec.VI we discuss some of the open problems.

---

*) For $k > 1$, some of the solutions to (1.1) are: $d = 7$, E_7 ($k = 3$); $d = 8$, $SU(5)$ ($k = 15$); $d = 6$, $SU(7)$ ($k = 5$).

**) For a complete discussion see Goddard, Nahm and Olive (and other papers) in Ref. [8a]. (It has recently been shown [10] that for arbitrary k the $O(N)$ σ-model is equivalent to free particles obeying parastatistics of order k. Regarding the question of parastatistics in strings see also Ref.[11] where it is shown that $d = 2 + \frac{8}{q}$ for a free fermionic theory exhibiting parastatistics of order q. For $q = 2$ this gives $d = 6$).

***) Note that here the $E_8 \times E_8$ refers to a 496 dimensional group manifold, and not to a 16-dimensional torus which arises in the heterotic string models [12]. Other rank 20 groups which might be relevant to the bosonic string in $d = 6$ are: $E_8 \times E_8 \times SU(3) \times SU(2) \times SU(2)$ and $SU(3) \times [U(1)]^{18}$.

****) For $k > 1$, the unique solution (for simple G) is: $d = 8$, $SU(2)$ ($k = 2$); $d = 4$, $SO(5)$ ($k = 2$), $SU(3)$ ($k = 5$); $d = 2$, $SO(5)$ ($k = 7$).

II. A LOCALLY SUPERSYMMETRIC ACTION WITH THE WESS-ZUMINO TERM

We construct the action of an $N = 1$ locally supersymmetric σ-model coupled to supergravity in two dimensions, where the scalars of the σ-model parametrize an arbitrary Riemannian manifold M. $N = 1$ supergravity in two dimensions contains a zweibein e_μ^a and a gravitino ψ_μ. As is well known, in two dimensions these fields do not describe physical degrees of freedom. Nevertheless they play an important role in that their field equations yield constraints on the scalars and spinors of the σ-model.

The full action reads as follows:

$$\begin{aligned} e^{-1}\mathcal{L} = \frac{1}{2\pi\alpha'} \Big[& -\tfrac{1}{2} g^{\mu\nu} \partial_\mu \phi^i \partial_\nu \phi^j g_{ij} - \tfrac{i}{2} \bar\chi^i \gamma^\mu \left(\partial_\mu \chi^j + \Gamma^j_{k\ell} \partial_\mu \phi^k \chi^\ell \right) g_{ij} \\ & + \bar\psi_\mu \gamma^\nu \gamma^\mu \chi^i \partial_\nu \phi^j g_{ij} - \tfrac{1}{4} \bar\psi_\mu \gamma^\nu \gamma^\mu \psi_\nu \bar\chi^i \chi^j g_{ij} \\ & - \tfrac{1}{12} R_{ijk\ell} \bar\chi^i \chi^k \bar\chi^j \chi^\ell \\ & - e^{-1} \frac{k}{8\pi} \varepsilon^{\mu\nu} \alpha_{ij} \partial_\mu \phi^i \partial_\nu \phi^j - \frac{ik}{16\pi} \omega_{ijk} \bar\chi^i \gamma^\mu \gamma_5 \chi^j \partial_\mu \phi^k \\ & + \frac{k}{64\pi} \omega_{i\ell j;k} \bar\chi^i \chi^k \bar\chi^j \gamma_5 \chi^\ell - \frac{k^2}{512\pi^2} g^{mn} \omega_{ikm} \omega_{j\ell n} \bar\chi^i \gamma_5 \chi^k \bar\chi^j \gamma_5 \chi^\ell \\ & + \frac{ik}{48\pi} \omega_{ijk} \bar\psi_\mu \gamma^\nu \gamma^\mu \chi^i \bar\chi^j \gamma_\nu \gamma_5 \chi^k \Big] \ , \quad \mu = 0,1 \quad i = 1, \ldots, \dim M . \end{aligned} \tag{2.1}$$

and is invariant under the following transformations:

$$\begin{aligned} \delta e_\mu^a &= 2i \bar\varepsilon \gamma^a \psi_\mu - \Lambda e_\mu^a \\ \delta \psi_\mu &= \left(\partial_\mu - \tfrac{1}{4} \omega_\mu{}^{ab} \gamma_{ab} \right) \varepsilon + i \gamma_\mu \eta - \tfrac{1}{2} \Lambda \psi_\mu \\ \delta \phi^i &= - \bar\varepsilon \chi^i \\ \delta \chi^i &= - i \gamma^\mu \left(\partial_\mu \phi^i + \bar\psi_\mu \chi^i \right) \varepsilon - \Gamma^i_{jk} \delta\phi^j \chi^k + \frac{k}{16\pi} g^{i\ell} \omega_{\ell jk} \left(\bar\chi^j \gamma_5 \chi^k \right) \varepsilon , \\ & \quad + \tfrac{1}{2} \Lambda \chi^i \end{aligned} \tag{2.2}$$

where $\varepsilon(\sigma,\tau)$, $\eta(\sigma,\tau)$ and $\Lambda(\sigma,\tau)$ are the supersymmetry, conformal supersymmetry and Weyl scale transformation parameters, respectively. In (2.1) and (2.2) we have used the following definitions and conventions. The scalars ϕ^i ($i = 1,\ldots \dim M$) parametrize a Riemannian manifold, M, with metric $g_{ij}(\phi)$. The spinors χ^i and ψ_μ are two-component Majorana *). The parameter α' is the coupling constant of the σ-model and the coefficient k in front of the Wess-Zumino term is defined such that the path integral corresponding to (2.1) is well defined [9] (and last reference in [6]). Thus k is an integer (see footnote p.2). Our Riemann tensor is defined as $R^i_{jk\ell} = \partial_\ell \Gamma^i_{jk} + \ldots$ Note that $\gamma^\mu = \gamma^a e^\mu_a$ ($\mu, a = 0,1$). The 3-form ω_{ijk} is closed and is the curl of a second-rank antisymmetric tensor $\alpha_{ij}(\phi) = -\alpha_{ji}(\phi)$, which is a function of the scalars ϕ:

$$\omega_{ijk} = \partial_i \alpha_{jk} + \partial_k \alpha_{ij} + \partial_j \alpha_{ki} \quad . \tag{2.3}$$

The spin connection, ω_μ^{ab}, contains the contorsion tensor $K_{\mu ab} = 2i\bar{\psi}_a \gamma^\mu \psi_b$. The term in $\delta\chi^i$ containing the Christoffel symbol Γ^i_{jk} has been added so that when it is taken to the left-hand side ($\delta\chi^i + \Gamma^i_{jk}\,\delta\phi^j\chi^k$), it transforms as a vector on M, as it should since the remaining terms on the right-hand side have the same transformation property. Finally, the notation ; is standard for Riemannian covariant derivatives.

In order to derive (2.1) and (2.2) we proceed in two steps. First we generalize the action and transformation rules given in Refs.[1] and [2] to the case of an arbitrary Riemannian manifold, M, with metric $g_{ij}(\phi)$, as follows:

$$\begin{aligned} e^{-1}\mathcal{L}_1 = \frac{1}{2\pi\alpha'} \Big[& -\tfrac{1}{2} g^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^j g_{ij} - \tfrac{i}{2}\bar{\chi}^i\gamma^\mu(\partial_\mu\chi^j + \Gamma^j_{k\ell}\partial_\mu\phi^k\chi^\ell) g_{ij} \\ & + \bar{\psi}_\mu\gamma^\nu\gamma^\mu\chi^i\,\partial_\nu\phi^j g_{ij} - \tfrac{1}{4}\bar{\psi}_\mu\gamma^\nu\gamma^\mu\psi_\nu\,\bar{\chi}^i\chi^j g_{ij} \\ & - \tfrac{1}{12} R_{ijk\ell}\,\bar{\chi}^i\chi^k\,\bar{\chi}^j\chi^\ell \Big] \; . \end{aligned} \tag{2.4}$$

---

*) We use the following conventions. Our metric is $\eta_{ab} = \mathrm{diag}(-1,+1)$. The $d = 2$ gamma matrices γ^a ($a = 0,1$) are $\gamma^0 = \sigma_2$, $\gamma^1 = i\sigma_1$ and $\gamma_5 = -\sigma_3$. We take $\varepsilon^{01} = -1$, $\varepsilon_{01} = +1$ and $\bar{\chi} = \chi^T\gamma^0$. In two dimensions the following Fierz relations hold: $\psi\bar{\lambda} = -\frac{1}{2}(\bar{\lambda}\psi + (\bar{\lambda}\gamma_5\psi)\gamma_5 - (\bar{\lambda}\gamma_\mu\psi)\gamma_\mu)$ and $\psi(\bar{\lambda}\chi) = -\lambda(\bar{\chi}\psi) - \chi(\bar{\psi}\lambda)$. Some further useful relations are: $\gamma_\mu\gamma_\nu = -g_{\mu\nu} + e^{-1}\varepsilon_{\mu\nu}\gamma_5$ ($e = \det e^a_\mu$) and $\varepsilon^{\mu\nu}\gamma_\nu = \gamma^\mu\gamma_5$.

and

$$\begin{aligned} \delta e^a_\mu &= 2i\,\bar{\varepsilon}\gamma^a\psi_\mu - \Lambda e^a_\mu \\ \delta\psi_\mu &= (\partial_\mu - \tfrac{1}{4}\omega_\mu^{ab}\gamma_{ab})\varepsilon + i\gamma_\mu\eta - \tfrac{1}{2}\Lambda\psi_\mu \\ \delta\phi^i &= -\bar{\varepsilon}\chi^i \\ \delta\chi^i &= -i\gamma^\mu(\partial_\mu\phi^i + \bar{\psi}_\mu\chi^i)\varepsilon - \Gamma^i_{jk}\,\delta\phi^j\chi^k + \tfrac{1}{2}\Lambda\chi^i \; . \end{aligned} \tag{2.5}$$

Remarkably we find that without any further modifications the covariantized action (2.4) is already fully invariant under the local supersymmetry and scale transformations given in (2.5). In fact a large class of variations are those which arise in either the model of Refs.1 and 2 ($g_{\mu\nu} \neq \eta_{\mu\nu}$, $g_{ij} = \eta_{ij}$) provided that in the variation of the action the derivatives are covariantized with respect to M (e.g. $\partial_\mu\chi^i \to \partial_\mu\chi^i + \Gamma^i_{jk}\partial_\mu\phi^j\chi^k$) or the globally supersymmetric σ-model [13] ($g_{\mu\nu} = \eta_{\mu\nu}$, $g_{ij} \neq \eta_{ij}$). Therefore one only has to check the cancellation of the new variations which do not fall into these two classes. Most cancellations are trivial, the only non-trivial ones being those arising from the variation of the zweibein and χ^i in the $R\chi^4$ term. These variations give a vanishing result:

$$\begin{aligned} & -\tfrac{i}{6}\bar{\varepsilon}\gamma^\lambda\psi_\lambda\, R_{ijk\ell}\,\bar{\chi}^i\chi^k\bar{\chi}^j\chi^\ell + \tfrac{1}{6} R_{ijk\ell}\,\bar{\chi}^i\chi^k\bar{\chi}^j(i\gamma^\nu\varepsilon\,\bar{\psi}_\nu\chi^\ell) \\ & + \tfrac{1}{6} R_{ijk\ell}\,\bar{\chi}^i\chi^k\chi^\ell(i\gamma^\nu\varepsilon\,\bar{\psi}_\nu\chi^j) = 0 \quad . \end{aligned} \tag{2.6}$$

To prove this, one must Fierz rearrange χ^j and χ^ℓ in the second and third term and use the fact that

$$R_{ijk\ell}\,\bar{\chi}^i\chi^k\bar{\chi}^j\gamma_5\chi^\ell = R_{ijk\ell}\,\bar{\chi}^i\chi^k\bar{\chi}^j\gamma_\mu\chi^\ell = 0 \quad . \tag{2.7}$$

This identity is easily proven by noting that $R_{ijk\ell}$ is symmetric, whereas the χ terms are antisymmetric in the pair interchange $(ij) \leftrightarrow (k\ell)$. We thus conclude that the action given in (2.4) is invariant under (2.5).

We now consider the extension of (2.4), (2.5) by adding a Wess-Zumino term in a locally supersymmetric manner. A globally supersymmetric Wess-Zumino term has been constructed in Refs.[5,6].In the case of local supersymmetry we leave the transformation rules (2.5) intact except for $\delta\chi^i$ which we modify to read

$$\delta\chi^i = -i\gamma^\mu(\partial_\mu\phi^i + \bar{\psi}_\mu\chi^i)\varepsilon - \Gamma^i_{jk}\,\delta\phi^j\chi^k + \frac{k}{16\pi}g^{i\ell}\omega_{\ell jk}\,(\bar{\chi}^j\gamma_5\chi^k)\,\varepsilon \quad . \tag{2.8}$$

Note that the k-dependent term in $\delta\chi^i$ is precisely the one which occurs in the globally supersymmetric model. To obtain an action which is invariant under the local supersymmetries (2.5), (2.8), we first covariantize the globally supersymmetric Wess-Zumino action [5,6] with respect to the two-dimensional space-time. This yields

$$e^{-1}\mathcal{L}_2 = -e^{-1}\frac{k}{8\pi}\varepsilon^{\mu\nu}\alpha_{ij}\,\partial_\mu\phi^i\partial_\nu\phi^j - \frac{ik}{16\pi}\omega_{ijk}\,\bar{\chi}^i\gamma^a\gamma^5\chi^j\,\partial_\mu\phi^k e^\mu_a$$
$$+\frac{k}{64\pi}\omega_{i\ell j;k}\,\bar{\chi}^i\chi^k\,\bar{\chi}^j\gamma_5\chi^\ell - \frac{k^2}{512\pi^2}g^{mn}\omega_{ikm}\omega_{j\ell n}\,\bar{\chi}^i\gamma_5\chi^k\bar{\chi}^j\gamma_5\chi^\ell \,. \tag{2.9}$$

We now consider the new variations in $\mathcal{L}_1 + \mathcal{L}_2$, given in (2.4) and (2.9), which do not arise in the globally supersymmetric Wess-Zumino term of Refs.[5]and[6]. One class of variations are the terms proportional to ω^2 coming from the variation of the determinant e and of χ^i in the $\omega^2\chi^4$ term. These variations are

$$-\frac{1}{2\pi\alpha'}\Big[\frac{k^2}{512\pi^2}\omega_{ikm}\omega_{j\ell}{}^m\,\bar{\chi}^i\gamma_5\chi^k\,\bar{\chi}^j\gamma_5\chi^\ell\,(2i\,\bar{\varepsilon}\gamma^\lambda\psi_\lambda)$$
$$+\frac{k^2}{128\pi^2}\omega_{ikm}\omega_{j\ell}{}^m\,\bar{\chi}^i\gamma_5\chi^k\,\bar{\chi}^j\gamma_5\,(-i\gamma^\nu\varepsilon\,\bar{\psi}_\nu\chi^\ell)\Big] \quad . \tag{2.10}$$

The Fierz rearrangement of $\chi^\ell\bar{\chi}^j$ in the second term yields two terms one of which cancels the first term. The final result is

$$\frac{-ik^2}{512\alpha'\pi^3}\omega_{ikm}\omega_{j\ell}{}^m\,\bar{\chi}^i\gamma_5\chi^k\bar{\chi}^j\gamma_\lambda\chi^\ell\,\bar{\varepsilon}\gamma^\nu\gamma_5\gamma^\lambda\psi_\nu \quad . \tag{2.11}$$

In order to cancel this we add to the action the following new term:

$$e^{-1}\mathcal{L}_3 = \frac{ik}{96\pi^2\alpha'}\omega_{ijk}\,\bar{\psi}_\mu\gamma^\nu\gamma^\mu\chi^i\,\bar{\chi}^j\gamma_\nu\gamma_5\chi^k \quad . \tag{2.12}$$

One can easily show by appropriate Fierz rearrangements that all the χ's in (2.12) contribute with the same weight to $\delta\mathcal{L}_3$. In particular, the ω-dependent variation of the χ's in (2.12) gives

$$3\times\frac{ik}{96\pi^2\alpha'}\Big(\omega_{ijk}\,\bar{\psi}_\mu\gamma^\nu\gamma^\mu\Big)\Big(\frac{k}{16\pi}\,\omega_{imn}\,\bar{\chi}^m\gamma_5\chi^n\,\,\bar{\chi}^j\gamma_\nu\gamma_5\chi^k\,\varepsilon\Big), \tag{2.13}$$

which exactly cancels (2.11).

Remarkably we now find that with the addition of $\mathcal{L}_3$ given in (2.12) the full Lagrangian $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3$ given in (2.4), (2.9), (2.11) or (2.1) is invariant under the local supersymmetries (2.5), (2.8) or (2.2) without any further modifications. We have checked this by a straight-forward but tedious calculation which will not be reproduced.

III. FIELD EQUATIONS AND CONSTRAINTS

We now study the field equations which follow from the action, (2.1). As mentioned in the introduction the scalar manifold is taken to be $M_d \times G$. We shall see that for a particular relation among the parameters of the theory (see Eq.(3.7')), the field equations for ϕ^i and χ^i will be completely integrable. The field equations for e^a_μ and ψ_μ lead to constraints on ϕ^i and χ^i which generalize those of the Neveu-Schwarz-Ramond model. These constraints will play an important role in determining the critical dimension for strings in curved space, as we shall show in the next section.

We first choose co-ordinates appropriate to the product structure $M_d \times G$;

$$\phi^i = \begin{pmatrix} x^\alpha \\ y^I \end{pmatrix} \quad , \quad \chi^i = \begin{pmatrix} \lambda^\alpha \\ \chi^I \end{pmatrix} \qquad \begin{array}{l} \alpha = 0,1,\ldots,d-1 \\ I = 1,\ldots,d_G \end{array} \; . \tag{3.1a}$$

Correspondingly

$$g_{ij} = \begin{pmatrix} \eta_{\alpha\beta} & 0 \\ 0 & g_{IJ}(y) \end{pmatrix} \quad , \quad \alpha_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2\pi\alpha'}{R^2}\,\alpha_{IJ}(y) \end{pmatrix} \; , \tag{3.1b}$$

where $\eta_{\alpha\beta}$ is the usual metric in M_d and g_{IJ} is the $G \times G$ invariant metric on G. Note that we have introduced an additional parameter R, which is associated with the characteristic size of the compact group manifold G. In terms of the co-ordinates x^α and y^I the Lagrangian (2.1) splits into two parts which are separately invariant. One part, $\mathcal{L}(M_d)$, depends only on x^α and has exactly the same form as the Lagrangian given in Refs.[1] and [2]

$$e^{-1}\mathcal{L}(M_d) = \frac{1}{2\pi\alpha'}\Big[-\tfrac{1}{2} g^{\mu\nu}\partial_\mu x^\alpha \partial_\nu x^\beta \eta_{\alpha\beta} - \tfrac{i}{2}\bar\lambda^\alpha \gamma^\mu \partial_\mu \lambda^\beta \eta_{\alpha\beta}$$
$$+ \bar\psi_\mu \gamma^\nu \gamma^\mu \lambda^\alpha (\partial_\nu x^\beta)\eta_{\alpha\beta} - \frac{1}{4}\bar\psi_\mu \gamma^\nu\gamma^\mu \psi_\nu \bar\lambda^\alpha \lambda^\beta \eta_{\alpha\beta}\Big] \; . \tag{3.2}$$

The other part $\mathcal{L}(G)$ depends only on the group co-ordinates y^I. On a group manifold G we can take ω_{IJK} to be

$$\omega_{IJK} = \frac{1}{R} f_{abc} L_I^a L_J^b L_K^c = \frac{-1}{R} f_{abc} R_I^a R_J^b R_K^c = \frac{1}{R} f_{IJK} \; , \tag{3.3}$$

where $f_{ab}^{\;\;c}$ are the structure constants of the group G and $L_I^a(\phi)$ are the left invariant basis elements on the group manifold G, which are defined by

$$g^{-1}\partial_\mu g = \frac{1}{R} L_I^a T_a \partial_\mu y^I \; , \qquad g\partial_\mu g^{-1} = \frac{-1}{R} R_I^a T_a \partial_\mu y^I \; . \tag{3.4}$$

Here g is a group element and T^a are the antihermitian generators of the Lie algebra of G. From (3.3) it follows that ω_{IJK} is an invariant 3-form on G and hence that α_{IJ} is an invariant 2-form (up to a total derivative).

Using (3.3) and the following relations [14,13]

$$R_{IJKL} = \frac{-1}{4R^2} f_{IJ}^{\;\;M} f_{KLM} \tag{3.5a}$$

$$R_{IJKL}\,\bar\chi^I \chi^K \bar\chi^J \chi^L = 3\, R_{IJKL}\,\bar\chi^I \gamma_5 \chi^K \bar\chi^J \gamma_5 \chi^L \tag{3.5b}$$

the Lagrangian $\mathcal{L}(G)$ now reads

$$e^{-1}\mathcal{L}(G) = \frac{-1}{4\pi\alpha'}\partial_\mu y^I \partial_\nu y^J \Big(g^{\mu\nu} g_{IJ} + \frac{k\alpha'}{2R^2} e^{-1}\varepsilon^{\mu\nu}\alpha_{IJ}\Big)$$
$$- \frac{i}{4\pi\alpha'}\bar\chi^I \gamma^\mu \Big[\partial_\mu \chi^J + \Big(\Gamma^J_{KL} - \frac{k\alpha'}{4R^3}\gamma_5 f_{KL}^{\;\;J}\Big)\partial_\mu y^K \chi^L\Big] g_{IJ}$$
$$+ \frac{1}{32\pi\alpha' R^2} f_{IJ}^{\;\;M} f_{KLM}\Big(1 - \frac{\alpha'^2 k^2}{4R^4}\Big)\bar\chi^I \gamma_5 \chi^K \bar\chi^J \gamma_5 \chi^L$$
$$+ \frac{1}{2\pi\alpha'}\bar\psi_\mu \gamma^\nu \gamma^\mu \chi^I \partial_\nu y^J g_{IJ} - \frac{1}{8\pi\alpha'}\bar\psi_\mu \gamma^\nu\gamma^\mu \psi_\nu \bar\chi^I \chi^J g_{IJ}$$
$$+ \frac{ik}{48\pi R^3} f_{IJK}\bar\psi_\mu \gamma^\nu \gamma^\mu \chi^I \bar\chi^J \gamma_\nu \gamma_5 \chi^K \; . \tag{3.6}$$

We see that for

$$R^2 = \frac{|k|\alpha'}{2} \tag{3.7}$$

the quartic χ^4 terms in (3.6) cancel. Without loss of generality we shall always take k to be positive. Using (3.7), in the superconformal gauge

$$e_\mu^{\;a} = f(\sigma,\tau)\,\delta_\mu^{\;a} \quad , \quad \psi_\mu = \gamma_\mu \lambda(\sigma,\tau) \tag{3.7'}$$

the field equations following from (3.2) and (3.6) are

$$A_1 = \eta^{\mu\nu}\partial_\mu\partial_\nu x^\alpha = 0 \quad , \qquad A_2 = \gamma^\mu\partial_\mu\lambda^\alpha = 0 \quad , \tag{3.8a,b}$$

$$A_3 = \eta^{\mu\nu}\partial_\mu\partial_\nu y^I + \partial_\mu y^J\partial_\nu y^K\left(g^{\mu\nu}\Gamma^I_{JK} - \frac{1}{2R}\varepsilon^{\mu\nu} f_{JK}{}^I\right) = 0 \ , \tag{3.8c}$$

$$A_4 = \gamma^\mu\partial_\mu\chi^I + \gamma^\mu\left(\Gamma^I_{JK} - \frac{1}{2R} f_{JK}{}^I\gamma_5\right)\partial_\mu y^J\ \chi^K = 0 \ . \tag{3.8d}$$

$$A_5 = \gamma^\nu\gamma^\mu\lambda_\alpha\partial_\nu x^\alpha + \gamma^\nu\gamma^\mu\chi^I\partial_\nu y^J\ g_{IJ}$$
$$+ \frac{i}{12R} f_{IJK}\gamma^\nu\gamma^\mu\chi^I\bar\chi^J\gamma_\nu\gamma_5\chi^K = 0 \ , \tag{3.8e}$$

$$A_6 = \eta_{\alpha\beta}\left(-\partial_\mu x^\alpha\partial_\nu x^\beta + \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\partial_\rho x^\alpha\partial_\sigma x^\beta\right) - \frac{i}{2}\bar\lambda^\alpha\gamma_\nu\partial_\mu\lambda_\alpha$$
$$+ g_{IJ}\left(-\partial_\mu y^I\partial_\nu y^J + \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\partial_\rho y^I\partial_\sigma y^J\right) - \frac{i}{2}\bar\chi^I\gamma_\nu\partial_\mu\chi^J g_{IJ}$$
$$- \frac{i}{2}\bar\chi^I\gamma_\nu\left(\Gamma^J_{MN} - \frac{1}{2R}\gamma_5 f_{MN}{}^J\right)\partial_\mu y^M\chi^N\ g_{IJ} = 0 \ . \tag{3.8f}$$

Note that $A_1 = A_2 = A_3 = A_4 = 0$ are the physical field equations, while $A_5 = A_6 = 0$ are constraint equations. The field equation $A_3 = 0$ corresponds to the variation $\delta I/\delta y^I - \Gamma^K_{IJ}(\delta I/\delta\chi^K)\chi^J = 0$ to ensure covariance. This variation gives the result (3.8c) plus a term which vanishes upon use of the χ^I field equation. Note that the resulting equation, $A_3 = 0$, does not depend on $\dot\chi^I$. Also in (3.8f) the χ^I field equation has been used.

It is convenient to rewrite (3.8c) in terms of the bosonic part of the currents associated with the right and left translations on the group manifold. These currents are defined by

$$J^{\mu a}_R = \frac{1}{2}\left(\eta^{\mu\nu} + \varepsilon^{\mu\nu}\right)\partial_\nu y^I\ L_I^a \tag{3.9a}$$

$$J^{\mu a}_L = \frac{1}{2}\left(\eta^{\mu\nu} - \varepsilon^{\mu\nu}\right)\partial_\nu y^I\ R_I^a \ . \tag{3.9b}$$

Furthermore, the following relations will be useful in simplifying the field equations [14]

$$g_{IJ} = L_I^a L_J^a = R_I^a R_J^a$$

$$\Gamma^I_{JK} = -L_K^a\partial_J L_a^I + \frac{1}{2R} f_{JK}{}^I = -R_K^a\partial_J R_a^I - \frac{1}{2R} f_{JK}{}^I$$

$$\partial_I L_J^a - \partial_J L_I^a + \frac{1}{R} f_{IJ}{}^K L_K^a = 0 = \partial_I R_J^a - \partial_J R_I^a - \frac{1}{R} f_{IJ}{}^K R_K^a \tag{3.10}$$

Using (3.9) and (3.10), we can rewrite (3.8c) as

$$\partial_\mu J^{\mu a}_R = 0 \quad , \qquad \partial_\mu J^{\mu a}_L = 0 \tag{3.11a,b}$$

To simplify $A_4 = A_5 = A_6 = 0$ we now define the components of χ^I in the left and right invariant basis elements as follows:

$$\chi^a = \chi^I L_I^a \quad , \qquad \tilde\chi^a = \chi^I R_I^a \tag{3.12}$$

It is not difficult to show that the supersymmetry variation of $J^{\mu a}_R$ contains $\chi^a_L = \frac{1}{2}(1+\gamma_5)\chi^a$ and that of $J_L^{\mu a}$ contains $\tilde\chi^a_R = \frac{1}{2}(1-\gamma_5)\tilde\chi^a$. Thus it is natural to express equations $A_4 = A_5 = A_6 = 0$ in terms of χ^a_L and $\tilde\chi^a_R$. Again using (3.9) and (3.10), from (3.8d,e,f) it now follows that

$$\gamma^\mu\partial_\mu\chi^a_L = 0 \quad , \qquad \gamma^\mu\partial_\mu\tilde\chi^a_R = 0 \tag{3.13a,b}$$

$$\left(\eta^{\mu\nu} + \varepsilon^{\mu\nu}\right)\left(\lambda_{L\alpha}\partial_\nu x^\alpha + \chi^a_L J^a_{\nu R} - \frac{i}{12R} f_{abc}\chi^a_L\bar\chi^b_L\gamma_\nu\chi^c_L\right) = 0 \tag{3.14a}$$

$$\left(\eta^{\mu\nu} - \varepsilon^{\mu\nu}\right)\left(\lambda_{R\alpha}\partial_\nu x^\alpha + \tilde\chi^a_R J^a_{\nu L} - \frac{i}{12R} f_{abc}\tilde\chi^a_R\bar{\tilde\chi}^b_R\gamma_\nu\tilde\chi^c_R\right) = 0 \tag{3.14b}$$

$$\eta_{\alpha\beta}\left(-\partial_\mu x^\alpha\partial_\nu x^\beta + \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\partial_\rho x^\alpha\partial_\sigma x^\beta\right) - \frac{i}{2}\bar\lambda^\alpha_L\gamma_\nu\partial_\mu\lambda_{L\alpha}$$
$$- \frac{i}{2}\bar\lambda^\alpha_R\gamma_\nu\partial_\mu\lambda_{R\alpha} + \eta_{\alpha\beta}\left(-\partial_\mu y^I\partial_\nu y^J + \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}\partial_\rho y^I\partial_\sigma y^J\right)$$
$$- \frac{i}{2}\bar\chi^a_L\gamma_\nu\partial_\mu\chi^a_L - \frac{i}{2}\bar{\tilde\chi}^a_R\gamma_\nu\partial_\mu\tilde\chi^a_R = 0 \ . \tag{3.15}$$

Note that the last terms in (3.8d,f) have dropped out, due to the fact that we have expressed $\chi^{I}_{L,R}$ in the appropriate basis.

In summary, the field equations following from the action (3.2), (3.6) are given by (3.8a,b), (3.11a,b) and (3.13a,b), while the constraints are given by (3.14) and (3.15). In particular, note the presence of the χ^3 terms in (3.14) which are necessary for the closure of the super-Virasoro algebra. We will use these equations as a starting point for the quantization of the system in the next section.

IV. THE CRITICAL DIMENSION FORMULA FOR STRINGS MOVING ON $M^d \times G$

We now solve the free field equations given in (3.8), (3.11) and (3.13). Next we quantize the system subject to the constraints given in (3.14) and (3.15). The requirement of Lorentz invariance of the quantized theory in M_d puts restrictions on the dimensions d and d_G of M_d and the group manifold G, respectively. These restrictions have recently been obtained for the bosonic string in Refs.[7] and [8b] and conjectured for the fermionic string in Ref.[7]. In this section we shall restrict ourselves to closed strings. The case of open string can be treated similarly.

The solutions to the field equations are given by (setting $2\alpha' = 1$) *)

$$x^{\alpha}(\tau,\sigma) = q^{\alpha} + p^{\alpha}\tau + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\left(\alpha^{\alpha}_{n}e^{-2in(\tau+\sigma)} + \tilde{\alpha}^{\alpha}_{n}e^{-2in(\tau-\sigma)}\right) \tag{4.1a}$$

$$J^{a}_{+R} = \sqrt{2}\sum_{n=-\infty}^{\infty}\beta^{a}_{n}e^{-2in(\tau+\sigma)} \tag{4.1b}$$

$$J^{a}_{-L} = \sqrt{2}\sum_{n=-\infty}^{\infty}\tilde{\beta}^{a}_{n}e^{-2in(\tau-\sigma)} \tag{4.1c}$$

*) The light cone co-ordinates on the string world sheet are defined by $\xi^{\pm} = \frac{1}{\sqrt{2}}(\tau \pm \sigma)$, and those on M_d by $x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^{d-1})$. In this section the index i refers to transverse directions on M_d.

$$\lambda^{\alpha(1)} = \sum_{n=-\infty}^{\infty} d^{\alpha}_{n}e^{-2in(\tau-\sigma)} \tag{4.1d}$$

$$\lambda^{\alpha(2)} = \sum_{n=-\infty}^{\infty}\tilde{d}^{\alpha}_{n}e^{-2in(\tau+\sigma)} \tag{4.1e}$$

$$\chi^{a(1)} = \sum_{n=-\infty}^{\infty} S^{a}_{n}e^{-2in(\tau-\sigma)} \tag{4.1f}$$

$$\chi^{a(2)} = \sum_{n=-\infty}^{\infty}\tilde{S}^{a}_{n}e^{-2in(\tau+\sigma)} , \tag{4.1g}$$

where $\lambda^{\alpha(1)}$ $(\lambda^{\alpha(2)})$ is the single non-vanishing component of λ^{α}_{L} (λ^{α}_{R}). In (4.1d) and (4.1e) the sums are over integers (half integers) if a periodic (antiperiodic) boundary condition for λ is chosen, and similarly for (4.1f) and (4.1g). To eliminate the gauge degrees of freedom, we use the light cone gauge defined by

$$x^{+} = p^{+}\tau , \quad \lambda^{+(1)} = 0 , \quad \lambda^{+(2)} = 0 . \tag{4.2}$$

In this gauge, substituting (4.1) into the constraint equations (3.14) and ++ and -- projections of (3.15), we solve for the Fourier components of x^- and λ^-, respectively, as follows:

$$\begin{aligned}\alpha^{-}_{n} = \frac{2}{p^+}\Bigg(&\frac{1}{2}\sum_{m=-\infty}^{\infty}\alpha^{i}_{n-m}\alpha^{i}_{m} + \frac{1}{2}\sum_{m=-\infty}^{\infty}\beta^{a}_{n-m}\beta^{a}_{m} \\ &+\frac{1}{2}\sum_{m=-\infty}^{\infty}\left(m-\frac{n}{2}\right)d^{i}_{n-m}d^{i}_{m} \\ &+\frac{1}{2}\sum_{m=-\infty}^{\infty}\left(m-\frac{n}{2}\right)S^{a}_{n-m}S^{a}_{m}\Bigg) \equiv \frac{2}{p^+}L_n\end{aligned}$$

$$\tilde{\alpha}^{-}_{n} = \frac{2}{p^+}\tilde{L}_n , \quad p^- = 2\alpha^-_0 = 2\tilde{\alpha}^-_0 , \quad p^i = 2\alpha^i_0 = 2\tilde{\alpha}^i_0 \tag{4.3a}$$

$$\begin{aligned}d^{-}_{n} = \frac{2}{p^+}\Bigg(&\sum_{m=-\infty}^{\infty}d^{i}_{n-m}\alpha^{i}_{m} + \sum_{m=-\infty}^{\infty}S^{a}_{n-m}\beta^{a}_{m} \\ &-\frac{1}{6\sqrt{k}}\,i f_{abc}\sum_{\ell,m=-\infty}^{\infty}S^{a}_{n-m-\ell}S^{b}_{\ell}S^{c}_{m}\Bigg) \equiv \frac{2}{p^+}F_n\end{aligned} \tag{4.3b}$$

$$\tilde{d}^{-}_{n} = \frac{2}{p^+}\tilde{F}_n ,$$

where $\tilde{L}_n$ and $\tilde{F}_n$ are the same as L_n and F_n with all the oscillators replaced by the ones with tilda. We now quantize this system and compute the central extension[7,8a] in the commutator algebra of L_n and F_n. Demanding the closure of the Lorentz algebra requires a special value of this central extension which in turn determines the critical dimension. The quantization proceeds by imposing the following (anti) commutator relations

$$[q^i, p^j] = i\,\delta^{ij} \quad , \qquad [\alpha^i_m, \alpha^j_n] = m\,\delta_{m+n,0}\,\delta^{ij} \tag{4.4a}$$

$$[\beta^a_m, \beta^b_n] = \frac{-i}{\sqrt{k}} f^{abc} \beta^c_{m+n} + n\,\delta_{m+n,0}\,\delta^{ab} \tag{4.4b}$$

$$\{d^i_m, d^j_n\} = \delta^{ij}\,\delta_{m+n,0} \tag{4.4c}$$

$$\{S^a_m, S^b_n\} = \delta^{ab}\,\delta_{m+n,0} \quad . \tag{4.4d}$$

Similar relations hold for tilda oscillators, and the latter commute with those without tilda. In deriving (4.4b) we have used the result of Witten [9] which gives the commutator $[J^a_{+L}, J^b_{+L}]$.

We now consider the commutator algebra of L_n and F_n given in (4.3). First we note that these satisfy the following (anti) Poisson brackets:

$$[L_m, L_n]_P = (m-n)\,L_{m+n}$$

$$[F_m, L_n]_P = (m-\tfrac{1}{2}n)\,F_{m+n}$$

$$\{F_m, F_n\}_P = 2\,L_{m+n} \tag{4.5}$$

and similar expressions for $\tilde{L}_m$ and $\tilde{F}_m$.

These brackets define a super Virasoro algebra with no central extension. In the quantum case one must take care of operator ordering. With the normalization of the individual terms given as in (4.3) one finds, however, that the (anti) commutator algebra of L_n and F_n does not close. By demanding closure, we find that the terms in L_m and F_m have to be

normalized differently than in (4.3). The resulting (quantum) expressions are

$$L_n = \frac{1}{2}\sum_{m=-\infty}^{\infty} :\alpha^i_{n-m}\alpha^i_m: + \frac{1}{2}\sum_{m=-\infty}^{\infty}(m-\tfrac{1}{2}n):d^i_{n-m}d^i_m:$$

$$+ \frac{1}{2(1+\frac{c_A}{2k})}\sum_{m=-\infty}^{\infty} :\beta^a_{n-m}\beta^a_m: + \frac{1}{2}\sum_{m=-\infty}^{\infty}(m-\tfrac{1}{2}n):S^a_{n-m}S^a_m:$$

$$+ \left(\varepsilon(d-2) + \varepsilon' d_G\right)\delta_{n,0} \tag{4.6a}$$

$$F_n = \sum_{m=-\infty}^{\infty} :d^i_{n-m}\alpha^i_m: + \frac{1}{\sqrt{1+\frac{c_A}{2k}}}\left[\sum_{m=-\infty}^{\infty} :\beta^a_{n-m}S^a_m:\right.$$

$$\left. - \frac{i}{6\sqrt{k}} f^{abc}\sum_{l,m=-\infty}^{\infty} :S^a_{n-l-m}S^b_m S^c_l:\right]$$

$$p^- = \frac{4}{p^+}(L_0-\alpha_0) = \frac{4}{p^+}(\tilde{L}_0-\alpha_0) \quad , \tag{4.6b}$$

where

$$f_{acd}\,f_b{}^{cd} = C_A\,\delta_{ab}$$

$$\varepsilon = \begin{cases} 0 & \text{Antiperiodic b.c. for } \lambda \\ 1/16 & \text{Periodic b.c. for } \lambda \end{cases} \tag{4.7}$$

$$\varepsilon' = \begin{cases} 0 & \text{Antiperiodic b.c for } \chi \\ 1/16 & \text{Periodic b.c. for } \chi \end{cases} \quad .$$

In (4.6a) in order to discuss all the boundary conditions simultaneously we have added a constant term to L_0. The value of c_A for Lie groups is given by

G	SU(n)	SO(2n+1)	Sp(n)	SO(2n)	G_2	F_4	E_6	E_7	E_8
c_A	2n	4n-2	2n+2	4n-4	8	18	24	36	60

(4.8)

One can now show that L_n and F_n satisfy the super Virasoro algebra [16]

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m (m^2-1) \delta_{m+n,0} \tag{4.9a}$$

$$\{F_m, F_n\} = 2 L_{m+n} + \frac{1}{3} c \left(m^2 - \frac{1}{4}\right) \delta_{m+n,0} \tag{4.9b}$$

$$[F_m, L_n] = \left(m - \frac{1}{2} n\right) F_{m+n} \quad , \tag{4.9c}$$

where c is the central extension which is given by

$$c = \frac{3}{2}(d-2) + \frac{d_G}{1+\frac{C_A}{2k}} + \frac{1}{2} d_G \quad . \tag{4.10}$$

We now require Lorentz invariance in M_d. The only non-trivial commutator is $[M^{i-}, M^{j-}] = 0$. We find that this commutator holds provided that $\alpha_0 = \frac{1}{2}$ and $c = 12$. Thus follows the following critical dimension formula:

$$d = 10 - \frac{2}{3} \frac{d_G}{1+\frac{C_A}{2k}} - \frac{1}{3} d_G \quad (\text{spinning string}) \quad . \tag{4.11}$$

In the case of the purely bosonic string where d^i_m and S^a_m are absent (and thus $F_m = 0$), the only surviving commutator is given by (4.9a) with central extension c which now reads

$$c = d-2 + \frac{d_G}{1+\frac{C_A}{2k}} \quad (\text{bosonic string}) \quad . \tag{4.12}$$

The validity of $[M^{i-}, M^{j-}] = 0$ now requires that $\alpha_0 = 1$ and $c = 24$ which implies the following critical dimension formula:

$$d = 26 - \frac{d_G}{1+\frac{C_A}{2k}} \quad (\text{bosonic string}) \quad . \tag{4.13}$$

Finally, we quote the quantum mass formula

$$M^2 = 8 (L_0 - \alpha_0) - p^i p^i \quad . \tag{4.14}$$

For the case of semisimple group $G = G_1 \times G_2 \times \cdots \times G_p$ the formulae (4.11) and (4.13) are replaced by

$$d = 10 - \frac{2}{3} \sum_{i=1}^{p} \frac{d_G^{(i)}}{1+\frac{C_A^{(i)}}{2k^{(i)}}} - \frac{1}{3} \sum_{i=1}^{p} d_G^{(i)} \quad \left(\begin{array}{c}\text{spinning}\\ \text{string}\end{array}\right) \tag{4.15}$$

$$d = 26 - \sum_{i=1}^{p} \frac{d_G^{(i)}}{1+\frac{C_A^{(i)}}{2k^{(i)}}} \quad (\text{bosonic string}) \quad . \tag{4.16}$$

V. MASS SPECTRUM

In this section we study the mass spectrum of a bosonic closed string moving on $(\text{Minkowski})_d \times G$, which will illustrate both gravitational as well as Yang-Mills degrees of freedom. We shall comment on the spinning string case at the end of this section.

The quantum mass operator in $(\text{Minkowski})_d$ of the closed bosonic string is given by (see Eq.(4.14))

$$\frac{1}{4} \alpha' M^2 = N_d + N_G - 1 \quad , \tag{5.1}$$

where

$$N_d = \sum_{i=1}^{d-2} \sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n \tag{5.2}$$

$$N_G = \frac{1}{2\left(1+\frac{C_A}{2k}\right)} \sum_{a=1}^{d_G} \sum_{n=-\infty}^{\infty} : \beta^a_{-n} \beta^a_n : \tag{5.3}$$

N_d and N_G are the occupation number operators for the left movers satisfying $[N_d,\alpha^i_n] = -n\,\alpha^i_n$, $[N_G,\beta^a_n] = -n\,\beta^a_n$. Analogously one defines the occupation number operators for the right movers, $\tilde{N}_d$ and $\tilde{N}_G$ with α and β replaced by $\tilde{\alpha}$ and $\tilde{\beta}$, respectively. These number operators must satisfy the following closed string constraint:

$$N_d + N_G = \tilde{N}_d + \tilde{N}_G \quad . \tag{5.4}$$

To identify the mass spectrum we need to study the representations of the Kac-Moody algebra [20] of the operators β^a_n . (For a review see Refs.[21].) From now on, we shall specialize to the case of simply laced G. (The more complicated case of non-simply laced algebras has been treated in Ref.[22].) Furthermore we restrict ourselves to $k = 1$. As for the ordinary Lie algebras every representation is characterized by a highest weight vector. The basis of the representation space is obtained by successive application of the step operators on it. A weight vector obtained in this way is denoted by [21]

$$\ell = (\Lambda^I, \kappa, \delta) \qquad I = 1, \ldots, \mathrm{rank}\, G \quad , \tag{5.5}$$

where Λ^I are the components of a vector on the weight lattice of G, κ is the eigenvalue of k (see (4.4b)).and δ is the eigenvalue of the derivation operator d [21] which, in our case, is nothing but $(1-N_G)$.

We shall consider the basic representation [20] which is characterized by the highest weight $\ell_0 = (0,1,1)$. This weight corresponds to the ground state $|0\rangle$. Given the highest weight ℓ_0, the remaining bases of the representation space are obtained by successive application of the appropriate step operators, $\beta^a_{-n}\ldots\beta^b_{-m}|0\rangle$ with positive $n,\ldots,m$. Using the infinite discrete Weyl group, $\hat{W}$, Frenkel and Kac [20] have shown that the weight vectors of the states obtained in this way must have the form (Proposition (2.1) in Ref.[20])

$$\left(\alpha^I,\ 1,\ 1-p-\tfrac{1}{2}\alpha^I\alpha^I\right) \qquad p = 0,1,2,\ldots \tag{5.6}$$

with degeneracy $M_p(r)$ given by

$$\sum_{p=0}^{\infty} M_p(r)\, x^p = \prod_{p=1}^{\infty} (1-x^p)^{-r} = 1 + r x + \frac{r(r+3)}{2} x^2 + \cdots \tag{5.7}$$

Here r is the rank of G, and α^I is a vector on the root lattice. From (5.6) it follows that $N_G = p + (\alpha^I\alpha^I/2)$. Thus

$$\tfrac{1}{4}\alpha' M^2 = p + \tfrac{1}{2}\alpha^I\alpha^I - 1 \quad . \tag{5.8}$$

This formula can be compared with the mass formula for the closed bosonic string compactified on an r-dimensional torus [12,23] which is given by

$$\tfrac{1}{4}\alpha' M^2 = \bar{n} + \tfrac{1}{2} p^I p^I - 1 \quad , \qquad \bar{n} = 0,1,2,\ldots \tag{5.9}$$

Here $\bar{n}$ is an occupation number operator [23] and p^I are the internal momentum components. Since one identifies p^I with points on the root lattice of a group G as [23]:

$$\sqrt{2\alpha'}\, p^I = \alpha^I \quad , \tag{5.10}$$

it follows that the two mass formulae, (5.8) and (5.9), are identical. In other words, the mass spectrum of a bosonic string moving on $M_d \times G$ is the same as that of a bosonic string moving on $M_d \times$ r-dimensional torus, where r is the rank of G. It should be emphasized, however, that the states are obtained in the torus case by the Cartan subalgebra oscillators and the vertex operators [20,24] while in the group manifold case all states are obtained from the β^a_{-n} oscillator operators alone.

To illustrate the details of the spectrum let us consider the case of $M_{24} \times SU(3)$ as an example. The states $\beta^a_{-n_1}\beta^b_{-n_2}\ldots\beta^c_{-n_i}|0\rangle$ are eigenstates of $d = 1 - N_G$ with eigenvalues $1 - (n_1 + n_2 + \ldots + n_i)$. Hence the states $\beta^a_{-1}|0\rangle$ have $d = 0$. These have the weights $(\alpha, 1,0)$ and $(0,1,0)$ where α is a root of SU(3). The six weights $(\alpha,1,0)$ have unit multiplicity each, while the multiplicity of $(0,1,0)$ is two. Therefore they form an SU(3) octet.

The next set of states (corresponding to $d = -1$) are $\beta^a_{-1}\beta^b_{-1}|0\rangle$ and $\beta^a_{-2}|0\rangle$. By virtue of the commutation relations satisfied by the β's, the state $\beta^a_{-2}|0\rangle$ is the antisymmetric part of $\beta^a_{-1}\beta^b_{-1}|0\rangle$. Thus this state is not independent.

The states $\beta^a_{-1}\beta^b_{-1}|0\rangle$ have the weights $(\alpha,1,-1)$ and $(0,1,-1)$ with multiplicities 2 and 5, respectively. They therefore form the representation $1 \oplus 8 \oplus 8$ of SU(3). Note that the states 10, $\bar{10}$ and 27 in the product 8×8 are not present because their weights are not of the form given in (5.6). One can show that they have vanishing norm.

To apply this construction to the case of a closed bosonic string propagating on $M_d \times G$ we need to include $\tilde{\beta}^a_n$ as well as α^i_n and $\tilde{\alpha}^i_n$ in the operator algebra. Since β and $\tilde{\beta}$ generate two commuting Kac-Moody algebras the spectrum has $G \times G$ symmetry.

The ground state $|0\rangle$ here is a singlet of the Poincaré group of M_d as well as of $G \times G$. It is annihilated by all α^i_n, $\tilde{\alpha}^i_n$, β^a_n and $\tilde{\beta}^a_n$ for which $n \geq 1$. Hence its mass is given by $\frac{1}{4}\alpha' M^2 = -1$ and therefore it is a tachyonic state.

The first excited level is massless and consists of the following states: $\alpha^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle$ containing a graviton, a second rank antisymmetric tensor and a dilaton; the states $\alpha^i_{-1}\tilde{\beta}^a_{-1}|0\rangle$ and $\tilde{\alpha}^i_{-1}\beta^a_{-1}|0\rangle$ which have spin one and transform in the adjoint representation of $G \times G$; and finally there are the scalars $\beta^a_{-1}\tilde{\beta}^a_{-1}|0\rangle$ which transform as (adj, adj) of $G \times G$. For $G = SU(3)$ in Table I we have given the spin-zero states obtained by the operation of β and $\tilde{\beta}$ on $|0\rangle$ for the first five levels.

For the closed spinning string the mass formula reads:

$$\frac{1}{4}\alpha' M^2 = N^d_B + N^G_B + N^d_F + N^G_F + \Delta \qquad ,$$

where

$$N^d_F = \frac{1}{2}\sum_{n=-\infty}^{\infty} n : d^i_{-n} d^i_n :$$

$$N^G_F = \frac{1}{2}\sum_{n=-\infty}^{\infty} n : S^a_{-n} S^a_n :$$

and N^d_B and N^G_B are given as before with the constraint

$$N^d_B + N^G_B + N^d_F + N^G_F = \tilde{N}^d_B + \tilde{N}^G_B + \tilde{N}^d_F + \tilde{N}^G_F .$$

Corresponding to periodic (P) and antiperiodic (AP) boundary conditions, there are nine possible sectors. However to simplify the discussion we consider only four of these sectors corresponding to the cases where $\lambda^{(1)}$ and $\lambda^{(2)}$ (idem $\chi^{(1)}$ and $\chi^{(2)}$) obey the same boundary conditions. This will not change our results. In Table II we give Δ as well as the transformation properties of the ground state and the masses of the first few levels. We can infer from this table that if we desire to have a non-Abelian group G we cannot have massless fermions. To clarify this, note

that physical fermions must have periodic boundary conditions in Minkowski space-time and therefore are contained in the last two rows of Table II. Thus, supersymmetry in space-time (as contrasted to supersymmetry in d = 2) is unlikely to arise in such theories. The only possibility for massless fermions is when $d + d_G = 10$, which when combined with Eq.(1.2) gives the unique solution $c_A = 0$. This would permit only a product of U(1)'s (e.g. $U(1)^4$ for d = 6). *)

VI. CONCLUDING COMMENTS

One of our motivations in undertaking this study was to investigate the possibility of constructing a string theory in d = 6 which in the low energy limit could give rise to the anomaly free matter coupled N = 2 d = 6 supergravity with $E_6 \times E_7 \times U(1)$ symmetry [18]. In that theory the anomaly cancellations are highly non-trivial, while the theory admits a (phenomenologically interesting) chiral monopole compactification down to d = 4. From (1.4) and (1.5) we see that there may exist the possibility of constructing a string model in d = 6 which would consist of both the bosonic and the fermionic sectors. Evidently, the rank 20 group quoted in (1.4) as an example is large enough to contain $E_6 \times E_7 \times U(1)$. For example, we could construct a heterotic string where the left moving sector is realized with a spinning string in d = 6 with $G_L = SU(3)$ (k = 1) while the right moving sector is realized with $G_R = SU(3) \times U(1)^{18}$. This $U(1)^{18}$ can give rise (through Frenkel-Kac construction, Refs.[12,20]) to a rank 18 group, e.g. $E_8 \times E_8 \times SU(3)$.

It is perhaps of interest to note that $(E_6 \times SU(3) \times (E_7 \times U(1))$ is contained in $E_8 \times E_8$. Note also that one may consider the possibility of a heterotic string with spinning left movers on $U(1)^4$ while the right movers are realized through Frenkel-Kac construction yielding $E_8 \times E_8 \times U(1)^4$ although such a model may perhaps be looked upon more simply as arising from the conventional heterotic string picture in d = 10 where the ten space-time dimensions have been compactified to six space-time dimensions. These constructions have no d = 2 gravitational anomalies. The consistency of these models must, of course, be investigated with interactions taken into account.

---

*) Note that there are two possible G-parity [15] operators in the theories of this type, $(-1)^{2N^F_d - 1}$ or $(-1)^{2N^F_G - 1}$. These may be useful for suppressing the tachyons.

The results of this paper can be extended in the following directions. (a) Generalization of our model to the case of coset spaces; (b) generalization to extended (N = 2,4,8,16) supersymmetries in d = 2; (c) study of interacting strings (d) and finally and most importantly the construction of a model in which the spinors χ^i are replaced by objects which are spinors of both d = 2 and the higher dimensional space-time, as in the superstring model of Green and Schwarz [18]. This may affect formula (1.2), as well as restore supersymmetry which as noted in Sec.VI, is lost for non-Abelian G for the fermionic string treated in this paper.

ACKNOWLEDGMENTS

We appreciate discussions with L. Alvarez-Gaume, M. Green, D. Olive, A. Neveu, A. Schwimmer, J. Strathdee and E. Witten.

REFERENCES

[1] S. Deser and B. Zumino, Phys. Lett. 65B, 369 (1976).

[2] L. Brink, P. di Vecchia and P. Howe, Phys. Lett. 65B, 471 (1976).

[3] A. Neveu and J.H. Schwarz, Nucl. Phys. B31, 86 (1971); Phys. Rev. D4, 1109 (1971);
P. Ramond, Phys. Rev. D3, 2415 (1971).

[4] J. Wess and B. Zumino, Phys. Lett. 37B, 95 (1971);
E. Witten, Nucl. Phys. B223, 422, 433 (1983).

[5] S.J. Gates, Jr., C.M. Hull and M. Roček, Nucl. Phys. B248, 157 (1984).

[6] T. Curtright and C. Zachos, Phys. Rev. Lett. 53, 1799 (1984);
R. Rohm, "Anomalous interactions for the supersymmetric nonlinear σ-model in two dimensions", Princeton preprint (October 1984);
E. Braaten, T.L. Curtright and C.K. Zachos, "Torsion and geometrostasis in non-linear sigma models", Florida preprint (1985)

[7] D. Nemeschansky and S. Yankielowicz, "Critical dimension of string theories in curved space", SLAC, preprint SLAC-PUB-3513 (November 1984).

[8a] P.G. Freund, Santa Barbara, preprint (1984);
V.G. Krizhnik and A.B. Zamolodchikov, Nucl. Phys. B247, 83 (1984);
V.G. Kac, "Infinite dimensional Lie algebras" in Prog. in Maths. Vol.44, (Birkhäuser, Boston, 1983) p.245;
P. Goddard, D. Olive and A. Schwimmer, Imperial College, preprint TP/84-85/23;
P. Goddard, W. Nahm and D. Olive, "Symmetric spaces and chiral field theories in two dimensions", Imperial College, preprint (1985);

[8b] S. Jain, R. Shankar and S.R. Wadia, "Conformal invariance and string theories in compact space: Bosons", Tata Inst., Bombay, preprint TIFR/TH/85-3 (1985).

[9] E. Witten, Comm. Math. Phys. 92, 455 (1984).

[10] I. Antoniadis and C. Bachas, "Conformal invariance and parastatistics in two dimensions", SLAC, preprint SLAC-PUB-3625 (April 1985).

[11] F. Ardalan and F. Mansouri, Phys. Rev. D9, 3341 (1974);
J.F.L. Hopkinson and R.W. Tucker, Phys. Rev. D10, 558 (1974).

[12] D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, "Heterotic string theory - I. The free heterotic string", Princeton preprint (January 1985).

[13] L. Alvarez-Gaumé and D.Z. Freedman, Comm. Math. Phys. 80, 443 (1981).

[14] P. Forgács and N.S. Manton, Comm. Math. Phys. 72, 15 (1980).

[15] See for example,
J.H. Schwarz, Phys. Rep. Vol. 89, 223 (1982).

[16] P. Goddard and D. Olive, "Kac-Moody algebras, conformal symmetry and critical exponents", Imperial College, preprint (1984).

[17] S. Randjbar-Daemi, Abdus Salam, E. Sezgin and J. Strathdee, Phys. Lett. 151B, 351 (1985);
Abdus Salam and E. Sezgin, ICTP, Trieste, preprint IC/85/19 (1985).

[18] M.B. Green and J.H. Schwarz, Nucl. Phys. B181, 502 (1981).

[19] J. Paton and Chan Hong-Mo, Nucl. Phys. B10, 519 (1969).

[20] I. Frenkel and V.G. Kac, Inventiones Math. 62, 23 (1980).

[21] D. Olive, "Kac-Moody algebras: An introduction for physicists", Imperial College, London, preprint (March 1985);
P. Goddard, "Infinite dimensional Lie algebras: Representations and applications", Cambridge, preprint (1985).

[22] A.J. Feingold and I. Frenkel, "Classical affine algebras", preprint.

[23] L. Dolan and R. Slansky, "The physical spectrum of compactified strings", Los Alamos, preprint (January 1985).

[24] N.S. Craigie, W. Nahm and K.S. Narain, "Realization of the Kac-Moody algebras of 2-D QFT's through soliton operators", Brookhaven preprint (1985).

Table I

$\frac{1}{4}\alpha' M^2$	States	SU(3) x SU(3) representation
-1	$\|0\rangle$	(1,1)
0	$\beta^a_{-1}\tilde{\beta}^b_{-1}\|0\rangle$	(8,8)
+1	$\beta^a_{-1}\ \beta^b_{-1}\ \tilde{\beta}^c_{-1}\ \tilde{\beta}^d_{-1}\|0\rangle$	(1,1) + 2(1,8) + 2(8,1) + 4(8,8)
+2	$\beta^{a_1}_{-1}\ \beta^{\cdot}_{-1}\ \beta^{\cdot}_{-1}\ \tilde{\beta}^{\cdot}_{-1}\ \tilde{\beta}^{\cdot}_{-1}\ \tilde{\beta}^{a_6}_{-1}\|0\rangle$	4(1,1) + 6(1,8) + 6(8,1) + 2(1,10) + 2(10,1) + 2(1,$\overline{10}$) + 2($\overline{10}$,1) + 9(8,8) + 3(8,10) + 3(10,8) + 3(8,$\overline{10}$) + 3($\overline{10}$,8) + (10,10) + (10,$\overline{10}$) + ($\overline{10}$,10) + ($\overline{10}$,$\overline{10}$)
+3	$\beta^{a_1}_{-1}\ \beta^{\cdot}_{-1}\ \beta^{\cdot}_{-1}\ \beta^{\cdot}_{-1}$ x x $\tilde{\beta}^{\cdot}_{-1}\ \tilde{\beta}^{\cdot}_{-1}\ \tilde{\beta}^{\cdot}_{-1}\ \tilde{\beta}^{a_8}_{-1}\|0$	(1,1) + 6(1,8) + 6(8,1) + (1,10) + (10,1) +(1,$\overline{10}$) + ($\overline{10}$,1) + (1,27) + (27,1) + 36(8,8) + 6(8,10) + 6(10,8) + 6(8,$\overline{10}$) + 6($\overline{10}$,8) + 6(8,27) + 6(27,8) + (10,10) + (10,$\overline{10}$) + ($\overline{10}$,10) + ($\overline{10}$,$\overline{10}$) + (10,27) + (27,10) + ($\overline{10}$,27) + (27,$\overline{10}$) + (27,27)

The scalars obtained by the operation of β^a_{-n} and $\tilde{\beta}^a_n$ on $|0\rangle$ for the first five levels for G = SU(3).

Table II

λ^α	χ^a	Δ	ground state	$\frac{1}{4}\alpha' M^2$
A.P.	A.P.	$-\frac{1}{2}$	scalar of M_d and G	$-\frac{1}{2}$, 0, $\frac{1}{2}$,...
A.P.	P	$\frac{d_G-8}{16}$	scalar of M_d and spinor of $SO(d_G)$	$\frac{d_G-8}{16}$, $\frac{d_G}{16}$, $\frac{d_G+8}{16}$,...
P	A.P.	$\frac{d-10}{16}$	spinor of M_d scalar of G	$\frac{d-10}{16}$, $\frac{d-2}{16}$, $\frac{d+6}{16}$,...
P	P	$\frac{d+d_G-10}{16}$	spinor of M_d and $SO(d_G)$	$\frac{d+d_G-10}{16}$, $\frac{d+d_G+6}{16}$, $\frac{d+d_G+22}{16}$,...

The values of Δ , the transformation properties of the ground state and the mass levels for different sectors, corresponding to periodic (P) or antiperiodic (AP) boundary conditions or the fermions λ^α and χ^a.