GROUP-THEORETICAL APPROACH TO EXTENDED CONFORMAL SUPERSYMMETRY:
FUNCTION SPACE REALIZATIONS AND INVARIANT DIFFERENTIAL OPERATORS

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We give function space realizations of all representations of the
conformal superalgebra, $\mathfrak{su}(2,2/N)$ and of the corresponding supergroup
induced from irreducible finite-dimensional Lorentz and $\text{SU}(N)$ representations
realized without spin and isospin indices. We use the lowest weight
module structure of our $\mathfrak{su}(2,2/N)$ representations to present a general
procedure (adapted from the semisimple Lie algebra case) for the canonical
construction of invariant differential operators closely related to the
reducible (indecomposable) structure of some representations. All conformal
supercovariant derivatives are obtained in this way. Examples of higher
order invariant differential operators are given. The massless UIR of
$\mathfrak{su}(2,2/N)$ are realised as invariant function subspaces.

SUPPLEMENTARY

Supersymmetry, the most appealing new symmetry principle of the
last decade, was immediately combined with conformal symmetry [1]. Apart
from providing examples of models which seem to survive the quantization
[2] the superconformal symmetry has been used as a convenient tool in non-
superconformal invariant theories as well (for reviews see [3]).

Unlike its less stringent in the physical applications (extended)
super-Poincaré subalgebra, the superconformal algebra $\mathfrak{O}_\mathfrak{f} = \mathfrak{su}(2,2/N)$ is
a semisimple (reductive for $N=4$) Lie superalgebra. This allows the
application of powerful mathematical results in the representation theory
of such algebras. However, although representations of $\mathfrak{O}_\mathfrak{f}$ were considered
and employed in supersymmetric theories there is by now no systematic group-
theoretical approach (at least in our understanding) to the study and
construction of these representations. In [4] we announced the development
of such an approach and here we give its detailed exposition.

Our approach is a natural development of earlier work on the
Euclidean conformal group $\text{SO}(5,1)$ (and $\text{SO}(n,1)$) [5], the Minkowski
conformal group $\text{SU}(2,2)$ [6,7] and semisimple Lie groups in general [8].
The present paper has several aims.

We construct and study all representations of $\mathfrak{su}(2,2/N)$ and of
the corresponding supergroup induced from irreducible finite-dimensional
Lorentz and $\text{SU}(N/0)$ representations. These are called (as in the semisimple
conformal group $\text{S}_{5} = \text{SU}(5,1)$ and $\text{S}_{n} = \text{SU}(n,1)$)
the structure of lowest weight modules (LWM) over the complexification
$\mathfrak{O}_\mathfrak{f} = \mathfrak{sl}(4/N; \mathbb{C})$ of $\mathfrak{O}_\mathfrak{f}$. Whenever such a
reducibility condition for the LWM or ER $\chi$ is satisfied there arises an
invariant map from the representation space $C_\chi$ to the space $C_{\chi'}$, where
$\chi'$ is obtained by the action of a definite Weyl reflection on the lowest
weight corresponding to $\chi$. The existence of such a map is equivalent to
the partial equivalence of the LWM (or ER) $\chi$ and $\chi'$. These results were
used in the purely algebraic setting in [4] for the classification of the physically important multiplets\(^\dagger\) of LWM.

The algebraic approach exploited in [4] is more economical and rather powerful (see also [8]). In the function space realizations of the ER, which look more appealing from the physical point of view, the invariant maps discussed above become invariant differential operators. These differential operators correspond to the standard supercovariant derivatives but acting irreducibly in spin and isospin. To provide a compact form for the invariant differential operators as earlier [5-7], we employ realizations of the (finite-dimensional) inducing representations without spin and isospin indices. The advantages of Poincaré superalgebra representations without explicit SO(N) - indices (N= 2,3) were recently noted in another context in [11,12].

The paper is organized as follows.

We start with the structural analysis of the superconformal algebra \(\mathfrak{of}_\mathfrak{f} = \mathfrak{su}(2,2|N)\) relevant for the induction from the so called maximal parabolic superalgebra \(\mathfrak{p}_\mathfrak{m}\) comprised of the Lorentz, \(\mathfrak{u}(1)\), \(\mathfrak{su}(N)\), dilation and special superconformal subalgebras. In section 2 we record some known facts about the complexified Lie superalgebra \(\mathfrak{of}_\mathfrak{f} = \mathfrak{sl}(4|N;\mathbb{C})\) in a basis most suitable for our purposes. In Section 3 we introduce the supergroup \(G = \text{SU}(2,2|N)\) as a matrix group with entries taking values in a Grassmann algebra with countably many (odd) generators.

In Section 4 we give the inducing representations of \(\text{SL}(2,\mathbb{C})\) and \(\text{SU}(N)\) in indexless realizations. In Section 5 we give three realizations of the elementary representations. The first involves functions on \(G\) and is particularly useful in the derivation of the integral intertwining operators in Section 6. As for \(\text{SU}(2,2)\) these operators may serve as 2-point functions of superconformal invariant \(\mathfrak{of}_\mathfrak{f}\)'s while those among them which reduce to differential operators provide (as kernels of invariant forms) \(G\)-invariant actions. The second realization involves functions on the superspace and being most natural from the physical point of view, all results are translated to it. The third realization is used first in section 7 to identify the \(\mathfrak{of}_\mathfrak{f}^{\mathbb{C}}\) lowest weight module (LWM) structure of the ER. That identification enables us to exploit the theory of Kac of LWM (Kac actually uses equivalently highest weight modules) of the basic classical (complex) Lie superalgebras. Here we show explicitly the adoption of a criterion of Kac to obtain the necessary conditions for reducibility of the LWM (and consequently of the ER). In Section 8 a general procedure (adapted from [8]) for the canonical construction of invariant differential operators is presented. Besides the construction of the operators corresponding to the supercovariant derivatives the construction of other higher-order differential conformal operators is discussed. An outline of the proof that the conditions for reducibility are also sufficient is given. In section 9 the massless UIR of \(\mathfrak{su}(2,2|N)\) are realized on invariant subspaces of certain elementary representation spaces. These subspaces are essentially defined as the solution spaces of a set of equations provided by some of the invariant operators constructed in the previous sections.

We work for arbitrary \(N\) noting all peculiarities of the case \(N = 4\).

\[1. \quad \text{STRUCTURE OF THE LIE SUPERALGEBRA } \mathfrak{su}(2,2|N)\]

The Lie superalgebra \(\mathfrak{of}_\mathfrak{f} = \mathfrak{sl}(4|N;\mathbb{C})\) shall be realized by \((4+N) \times (4+N)\) matrices

\[
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} , \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \mathfrak{of}_\mathfrak{f}^{\mathbb{C}} , \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \in \mathfrak{of}_\mathfrak{f}^{\mathbb{C}} , \quad (1.1)
\]

where \(a, b, c, d\) are \(4 \times 4\), \(4 \times N\), \(N \times 4\), \(N \times N\) matrices, respectively. \(\mathfrak{of}_\mathfrak{f}^{\mathbb{C}}(0), \mathfrak{of}_\mathfrak{f}^{\mathbb{C}}(1)\) are the even and odd parts of \(\mathfrak{of}_\mathfrak{f}^{\mathbb{C}}\) and

\(\dagger\) Each multiplet is an (infinite) collection of partially equivalent LWM, or ER, or superfields (supermultiplets) in the usual sense.
str \ Y = \text{tr} a - \text{tr} d = 0 \quad (1.2)

holds. The Lie superalgebra \(\mathfrak{su}(2,2|N)\) is the following \((N^2 + 8N + 15)\)-dimensional real noncompact form of \(\mathfrak{sl}(4/N)\):

\[
\mathfrak{g} = \mathfrak{su}(2,2|N) \cong \left\{ Y \in M(4|N) \left|\begin{array}{c}
\Lambda Y \omega + \alpha \wedge \omega Y \omega = 0, \\
Y \in \mathfrak{g}_{(4)}, \quad \w \equiv \left( \begin{array}{c}
1, 0 \\
0, 1
\end{array} \right)
\end{array} \right. \right\},
\]

where \(Y^+\) is the Hermitian conjugate of the matrix \(Y\).
(This differs from the usual choice of \(u = \text{diag}(1, -1, 1, 1)\) by a real orthogonal transformation.) The even part of \(\mathfrak{g}\)

\[
\mathfrak{g}_{(4)} = \mathfrak{su}(2,2) \oplus \mathfrak{u}(4) \oplus \mathfrak{su}(N)
\]

is \((N^2 + 15)\)-dimensional, and \(\dim \mathfrak{g}_{(4)} = 8N\).

Next we define the Cartan involution \(\hat{\omega}\)

\[
\hat{\omega} Y_{\mu} = \omega^{-1} Y_{\mu} \omega = -i^a Y_{a}^+, \quad Y \in \mathfrak{g}_{(4)}, \quad a = 0, 1,
\]

and obtain the Cartan decomposition of \(\mathfrak{g}\)

\[
\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},
\]

where \(\mathfrak{k}\) is \((N^2 + 8N + 7)\)-dimensional subalgebra (the maximal compact subalgebra in the SSLA case) and \(\mathfrak{p}\) is a \((4N + 8)\)-dimensional vector space such that

\[
Y \in \mathfrak{k} \Rightarrow \mathfrak{c} Y = Y, \quad Y \in \mathfrak{p} \Rightarrow \mathfrak{c} Y = -Y
\]

Explicitly

\[
k = \{ Y = (\begin{array}{c}
0 0 0 0 \\
0 0 0 0 \\
0 0 0 0 \\
0 0 0 0
\end{array}) | \alpha^a = x^a, \beta^a = x^a, \mathfrak{d} = 0, \mathfrak{d} = 0 \}
\]

\[
p = \{ Y = (\begin{array}{c}
0 0 0 0 \\
0 0 0 0 \\
0 0 0 0 \\
0 0 0 0
\end{array}) | \alpha^a = x^a, \beta^a = x^a \}
\]

where \(\alpha, \beta\) are \(2 \times 2\) matrices, \(p\) is \(2 \times N\) matrix, \(\mathfrak{c}(0) \cong \mathfrak{u}(2) \oplus \mathfrak{u}(2) \oplus \mathfrak{su}(N)\),

\[
\dim \mathfrak{c}(1) = \dim \mathfrak{c}(1) = 4N.
\]

We shall consider representations induced from the so-called maximal parabolic subalgebra \(\mathfrak{p}\)

\[
\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n},
\]

In (1.8) \(\mathfrak{m}\) is a 1-dimensional (even) subalgebra contained in \(\mathfrak{p}\)

\[
\alpha = \epsilon_1 [\epsilon_1], \quad \epsilon_1 = \frac{1}{2} \left( \begin{array}{c}
1 0 0 0 \\
0 -1 0 0 \\
0 0 0 1
\end{array} \right)
\]

\(\mathfrak{m}\) is called the dilatation subalgebra (also of the conformal algebra \(\mathfrak{su}(2,2)\)). Further, in (1.8) \(\mathfrak{m} = \mathfrak{m}(0)\) is the centralizer of \(\mathfrak{a}\) in \(\mathfrak{p}\); it is isomorphic to \(\mathfrak{su}(2,1) \oplus \mathfrak{su}(N)\) and is given by

\[
\mathfrak{m} = \left\{ Y = \left( \begin{array}{c}
0 0 0 0 \\
0 0 0 0 \\
0 0 0 0 \\
0 0 0 0
\end{array} \right) | \frac{1}{2} \mathfrak{d} = 0 \right\}
\]

Thus \(\mathfrak{m}\) contains the Lorentz subalgebra of the conformal algebra. For later reference we fix a basis of a Cartan subalgebra \(\mathfrak{d}\) of \(\mathfrak{m}\):

\(\text{**) The parabolic subalgebras of } \mathfrak{su}(2,2|N) \text{ are in 1-to-1 correspondence with}\)

the parabolic subalgebras of \(\mathfrak{su}(2,2)\) which are described in [12,7].
Finally, in (1.9) \( \mathfrak{g} \) is the subalgebra comprised of the negative restricted root spaces with respect to \( \mathfrak{g} \), i.e.

\[
\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+ ; \quad [\epsilon, \gamma] = -\lambda_\epsilon(\gamma) \gamma, \quad \gamma \in \mathfrak{g}_+ \; ; \quad \lambda_\epsilon(\epsilon_j) = \frac{1}{2}, \quad \lambda_2 = 2 \lambda_1,
\]

\[
\mathfrak{g}_- = \mathfrak{l} \{ \left[ \begin{array}{cc} \epsilon_0 & O_n \\ O_n & \epsilon_0 \end{array} \right] \} \quad \text{and} \quad \mathfrak{g}_+ = \mathfrak{l} \{ \left[ \begin{array}{cc} \epsilon_0 & 0 \\ 0 & \epsilon_0 \end{array} \right] \}.
\]

In (1.9d) \( [\ , \ ] \) is the standard Lie superalgebra bracket.

The even part of \( \mathfrak{g} \) (that is \( \mathfrak{g}_2^{-} \)) is the subalgebra of special conformal transformations of the conformal algebra. Analogously, we introduce the subalgebra composed of the positive restricted root spaces:

\[
\mathfrak{h}_+ = \mathfrak{g}_+ \otimes \mathfrak{g}_+^* ; \quad [\gamma, \chi] = -\lambda_\gamma(\chi) \chi, \quad \gamma \in \mathfrak{g}_+ \; ; \quad \lambda_\gamma(\gamma_j) = \frac{1}{2}, \quad \lambda_2 = 2 \lambda_1,
\]

\[
\mathfrak{h}_+ = \mathfrak{g}_+ \mathfrak{g}_+^* = \mathfrak{h}_+ \mathfrak{g}_+^{-} ; \quad \mathfrak{g}_+ = \mathfrak{g}_+ \mathfrak{g}_+^*.
\]

In (1.9e) \( \mathfrak{g} \) is the subalgebra of translations of the conformal algebra. Finally, we write a decomposition of the conformal algebra.

\[
\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \oplus \mathfrak{c} \oplus \mathfrak{m}
\]

and we note that

\[
\mathfrak{l} = \mathfrak{c} \oplus \mathfrak{h}
\]

is a Cartan subalgebra of \( \mathfrak{g} \).

2. THE COMPLEXIFIED LIE SUPERALGEBRA \( \mathfrak{g}_c^\mathbb{C} = \mathbf{sl}(4|n; \epsilon) \)

In the approach of [8] which we adopt here (cf. also [15]) to the superalgebra case there is an important interplay between the complex algebra \( \mathfrak{g}_c^\mathbb{C} \) and the real form \( \mathfrak{g}_c \) under consideration. Matters are arranged so that the roots of the pair (\( \mathfrak{g}_c^\mathbb{C}, \mathfrak{h}_c^\mathbb{C} \)) are the Cartan subalgebra of \( \mathfrak{g}_c^\mathbb{C} \), where \( \mathfrak{h}_c^\mathbb{C} \) is the Cartan subalgebra of \( \mathfrak{g}_c^\mathbb{C} \), take real values on the basis of \( \mathfrak{h}_c^\mathbb{C} \). Also, ordering of roots must be compatible with their restriction to \( \mathfrak{c} \) and positive (resp. negative) restricted root spaces should be subspaces of the positive (resp. negative) root spaces. Bearing this in mind we fix the following basis of \( \mathfrak{h}_c^\mathbb{C} \):

\[
\lambda_i, \lambda_\pm, \lambda_{i+1}, \lambda_{i-1}, \quad \lambda_i = \epsilon_{n} \epsilon_{n+1}, \lambda_{i+1} = \epsilon_{n+1} \epsilon_{n}, \lambda_{i-1} = -\lambda_{i+1}, \lambda_{i} = -\lambda_{i-1}, \quad 0 \leq i \leq n+1,
\]

where \( \epsilon_{ij} \) are the Cartan matrices with only one non-zero element common to the \( i \)-th row and \( j \)-th column. Note that (cf. (1.9a,c))

\[
\mathfrak{l}_i = \left[ \begin{array}{cc} \epsilon_{n} & \epsilon_{n+1} \\ \epsilon_{n+1} & \epsilon_{n} \end{array} \right], \quad \mathfrak{l}_2 = \frac{1}{2} \left[ \begin{array}{cc} \epsilon_{n} & \epsilon_{n+1} \\ \epsilon_{n+1} & \epsilon_{n} \end{array} \right] + \frac{1}{2} \left[ \begin{array}{cc} \epsilon_{n} & -\epsilon_{n+1} \\ -\epsilon_{n+1} & \epsilon_{n} \end{array} \right],
\]

\[
\mathfrak{h}_+ = \left[ \begin{array}{cc} \epsilon_{n} & \epsilon_{n+1} \\ \epsilon_{n+1} & \epsilon_{n} \end{array} \right], \quad \mathfrak{h}_- = \left[ \begin{array}{cc} \epsilon_{n} & -\epsilon_{n+1} \\ -\epsilon_{n+1} & \epsilon_{n} \end{array} \right], \quad 0 \leq i \leq n+1.
\]

Let \( \mathfrak{h}_c^\mathbb{C} \) be the space of linear functionals over \( \mathfrak{g}_c^\mathbb{C} \); let \( \alpha \in \mathfrak{h}_c^\mathbb{C}, \alpha \neq 0 \), belong to the root system of \( \mathfrak{g}_c^\mathbb{C} \); the root spaces and the root system (\( \mathfrak{g}_c^\mathbb{C}, \mathfrak{h}_c^\mathbb{C} \)) are

\[
\mathfrak{g}_c^\mathbb{C} = \{ \gamma \in \mathfrak{g}_c^\mathbb{C} | \ [\gamma, \chi] = \alpha(\chi) \chi, \quad \chi \in \mathfrak{h}_c^\mathbb{C}, \} ;
\]

\[
\Delta = \{ \lambda_\epsilon(\alpha) = \Delta \} = \{ \lambda_\epsilon(\alpha) | \lambda_\epsilon(\alpha) \in \mathfrak{h}_c^\mathbb{C}, \}.
\]

We shall present the set of simple roots (cf. Remark 2.1 below for \( n=4 \))

\[
\Delta_{\epsilon} = \{ \lambda_\epsilon(\alpha) = \Delta_\epsilon | \lambda_\epsilon(\alpha) \in \mathfrak{h}_c^\mathbb{C}, \} = \Delta_\epsilon \Delta_\epsilon.
\]

in terms of their values on \( \mathfrak{h}_c^\mathbb{C} \).
The Cartan killing form \((Y, Y')\) is also defined by the supertrace for arbitrary \(Y, Y' \in \mathfrak{g}^+\). Next we give the non-simple roots through the simple ones:

\[
\begin{align*}
\delta_{ij} &= \delta_i - \delta_j, \\
\delta_{ij} &= \delta_i + \delta_j,
\end{align*}
\]

The corresponding root spaces are complexly spanned by \((i < j)\):

\[
X_{ij}^s = \begin{cases} 
\mathbb{C} \delta_i + \{ (i, j) = (i, j), (j, i) \}, & \text{if } i < j \\
\mathbb{C} \delta_j, & \text{otherwise}
\end{cases}
\]

Thus we can write explicitly the usual decomposition

\[
\mathfrak{g}^+ = \mathfrak{g}^+_{\text{even}} \oplus \mathfrak{g}^+_{\text{odd}}, \\
\mathfrak{g}^+_{\text{even}} = \mathbb{C} \{ X_{ij}^s, \text{ if } i < j \leq N+4 \}
\]

where \(\text{c.l.s.}\) stands for complex linear span. The somewhat peculiar transposal in (2.7a) for \((i,j)=(3,j),(4,j)\) comes from the requirement we mentioned above that the positive (negative) restricted root space \((\text{resp.})\) is contained in \(\overline{\mathfrak{g}}^+\) \((\text{resp. } \mathfrak{g}^-)\); and for \((i,j)=(3,4)\) see formula 7.2 below. Note \(X^+_{ij} \in \mathfrak{g}^+_{\mathfrak{g}^+} \) for \(1 \leq k < 4, 5 \leq i < j < N+4\); \(X^+_{ij} \in \mathfrak{g}^+_{\mathfrak{g}^+} \) for \(1 \leq k < 4, 4 \leq i < j < N+4\). A root \(\alpha_{ij}\) shall be called even (resp. odd) if \(X^+_{ij} \in \mathfrak{g}^+_{\mathfrak{g}^+}\) (resp. \(X^+_{ij} \in \mathfrak{g}^+_{\mathfrak{g}^+}\)) and we shall write \(\Delta = \Delta^{(0)} \cup \Delta^{(1)}\).

Roots are also divided into compact roots with zero values on \(\mathfrak{g}^0\) and non-compact roots (the rest\(^*)\). The compact roots are \(\gamma_1, \gamma_2\) and those in (2.6a); all are even.

In \(\mathfrak{g}^+\) there is an important (for the representation theory) element \(\rho\) defined in [13]

\[
\rho = \rho_{\text{c}} - \rho_{\text{n}}, \quad \rho_{\text{c}} = \frac{1}{2} \sum_{\alpha \in \Delta^+_{\text{c}}} \alpha, \quad \alpha \neq 0, e.
\]

Explicitly we have:

\[
\begin{align*}
J_0 &= \frac{N}{2} \left[ I_{\alpha} + I_{\bar{\alpha}} + \frac{1}{2} I_{\gamma} + \frac{1}{2} I_{\bar{\gamma}} \right] + \frac{1}{2} \left[ \sum_{k \neq s} X_k \left( \frac{1}{2} \sum_{k \neq s} X_k \right) \right], \\
J_\alpha &= \frac{N}{2} \left[ I_{\alpha} + I_{\bar{\alpha}} + \frac{1}{2} I_{\gamma} + \frac{1}{2} I_{\bar{\gamma}} \right] + \frac{1}{2} \left[ \sum_{k \neq s} X_k \left( \frac{1}{2} \sum_{k \neq s} X_k \right) \right], \\
J &= \frac{1}{2} \left[ (\gamma + \gamma) I_{\gamma} + 2 (\gamma + \gamma) I_{\bar{\gamma}} + \frac{1}{2} \sum_{k \neq s} X_k \left( \frac{1}{2} \sum_{k \neq s} X_k \right) \right] + \frac{1}{2} \left[ \sum_{k \neq s} X_k \left( \frac{1}{2} \sum_{k \neq s} X_k \right) \right].
\end{align*}
\]

Note that every non-compact root restricted to \(\mathfrak{g}^0\) coincides with some restricted root (hence the terminology); the even roots coincide with \(2 \lambda_{1}\) and the odd with \(\lambda_{1}\). Thus the restricted counterpart of \(\rho\) is

\[
J^s = J_{|\mathfrak{g}^0} = 2(1-\bar{w}) \lambda_{1}.
\]

\(^*)\) Note that division into compact and noncompact roots depends (via \(\mathfrak{g}^0\)) on the parabolic subalgebra used (cf. [7]).
Further we extend the Cartan-Killing form to \( \mathfrak{e}^{\mathfrak{e}} \) by the standard formula

\[
(i; j, k) \equiv (h_i, h_j).
\]

(2.10)

Next we introduce the Weyl reflections in \( \mathfrak{e}^{\mathfrak{e}} \) corresponding to even roots by the usual formula

\[
W_\alpha : x \equiv x - 2\langle \alpha, x \rangle \beta, \quad \alpha \in \Delta, \beta \in \Delta(\omega).
\]

(2.11)

It is clear that (2.11) cannot be used for odd roots for which \( (\beta, \beta) = 0 \).

However we shall define odd reflections by the formula (which seems like application of l'Hopital's rule to (2.11))

\[
W_\alpha : x \equiv x + \beta, \quad \alpha \in \Delta, \beta \in \Delta(\omega), \quad \alpha + \beta \notin \Delta, \quad (\beta, \beta) = 0.
\]

(2.12)

This definition may seem strange since it takes any root out of \( \alpha \), however it emerges naturally in the context of reducible highest (lowest) weight modules and the action of \( \Delta \) on the corresponding weights. This is explained in detail in Sec. 7 below.

**Remark 2.1.** For \( N=4 \) both \( \mathfrak{g}_4 = \mathfrak{su}(2,2|4) \) and \( \mathfrak{g}_4^\mathfrak{e} = \mathfrak{so}(4|4;\mathbb{C}) \) become reductive since \( \mathfrak{b}_4 \) and \( \mathfrak{b}_4^\mathfrak{e} \) contain the unit matrix (cf. 1.9c).

Thus the rank of the semisimple part is equal to six (and not \( 7 = N+3 \)). However the peculiarity of superalgebras is that one cannot find a basis of the root system \( \Delta \) with six members. Thus the only constraint which concerns \( \mathfrak{b}_4 \) (and not \( \Delta \)) is the following relation between the simple roots \( \gamma \):

\[
\gamma_4 + 2\gamma_3 + \gamma_2 = \gamma_1 + 2\gamma_3 + \gamma_3, \quad (N=4).
\]

(2.14)

(We remind that all roots should be expressed through the simple ones with integer coefficients - all positive or all negative.)

3. **The Lie Supergroup**

There exist by now a number of (equivalent) definitions of Lie supergroups [16-20]. It is most convenient for us to consider the Lie supergroup as an ordinary group (though not a Lie group!) with some additional structure (see Appendix A.1.). In this approach the Lie supergroups arising in our paper such as \( \text{SL}(4|N;\mathbb{C}) \) and \( G = \text{SU}(2,2|N) \) are groups of matrices with entries in a Grassmann algebra \( \Lambda \) with countably many odd generators (cf. [19]). The exponentiation [21] of the Grassmann envelope \( \mathfrak{g}(\Lambda) \) of \( \mathfrak{g} = \text{su}(2,2|N) \) is a Lie subsupergroup of \( G \).

Let \( \Lambda_m = \Lambda_{m-1} \otimes \Lambda_1 \) be a sequence of \( 2^m \)-dimensional complex Grassmann algebras with involution \( ** = \text{id} \). We define \( \Lambda = \{ a \in \Lambda_m, m=0,1, \ldots \} \). Denote by \( \mathfrak{g}^\mathfrak{e}(\Lambda) \) the Grassmann envelope of any subspace \( \mathfrak{g}_\mathfrak{e} = \mathfrak{g}_0(\Lambda) \) of \( \mathfrak{g}_\mathfrak{e} \) with respect to \( \Lambda = \Lambda_0 \Lambda_1 \), i.e. \( \mathfrak{g}^\mathfrak{e}(\Lambda) = \{ \text{finite sums } \sum \zeta_\mathfrak{e} \alpha, \alpha \in \Lambda \} \).

We shall use the following matrix realization of \( \mathfrak{g}^\mathfrak{e}(\Lambda) \):

\[
\mathfrak{g}^\mathfrak{e}(\Lambda) \ni \gamma \rightarrow \mathfrak{g}(\Lambda) \ni \gamma \rightarrow \rho(\gamma) \equiv \sum \zeta_i \alpha_i, \quad (3.1)
\]

We shall identify \( \mathfrak{g}^\mathfrak{e}(\Lambda) \) with the matrix Lie algebra

\[
\mathfrak{g}^\mathfrak{e}(\Lambda) = \left\{ A = \begin{pmatrix} A & \mathbf{E} \\ C & D \end{pmatrix}; A, D \in \Lambda(\omega); \ z_i, z_i \in \Lambda(\omega), \right. \left. (i, \bar{i}) \right\}.
\]

(3.2a)

(We remind that all roots should be expressed through the simple ones with integer coefficients - all positive or all negative.)
The relations (3.1), (3.2a) imply that
\[(P(x)) = P(x'), \quad \text{where} \]
\[(A^\dagger)_{a} = (A_{a})^* \quad a, \ldots, \gamma. \tag{3.2c}\]

The real form \(u(2,2/N)(A)\) of \(Q^c(A)\) is defined as
\[u(2,2/N)(A) = \{X \in q^c(A) \mid X^+ + \omega X = 0\} \tag{3.3}\]
and it can be realized as the Grassmann envelope \(Q(A(R))\) of the
dreal Lie superalgebra \(Q = su(2,2/N)\) (see Table 3). Here \(A(R) = \{x \in A \mid (\tilde{C}_{a}) - (-i)^{a} \tilde{C}_{a} = 0\}\) is a real form of \(A\); we shall
denote for brevity \(Q^c(A)\) instead of \(Q(A(R))\) for \(Q\) a subspace of \(Q\).

We shall first introduce some supergroups which will be sub-

groups of \(G\) and which correspond to the subalgebras of \(Q\)
singled out in Section 1. We start with the even supergroup
\[X_{(0)} \equiv \exp \tilde{C}_{0}(A) = \left\{\begin{bmatrix} 0 & \xi & 0 \\ 0 & 0 & 0 \\ \xi & 0 & 0 \end{bmatrix} \mid \xi \in \Lambda_{0}, \quad X^* = X_{0} \right\} \tag{3.4a}\]
where \(\tilde{C}_{0}(A)\) is defined in (1.9e). The group of supertranslations
\(\mathbb{Z}\) the superspace) is defined as
\[X = X_{(0)} \cdot \exp \tilde{C}_{0}(A) = \left\{\begin{bmatrix} \xi_{e} & 0 & 0 \\ 0 & 0 & 0 \\ \xi_{o} \end{bmatrix} \mid \xi_{e} \in \Lambda_{0}, \quad \xi_{o} \in \Lambda_{0}, \quad \xi_{o}^* = \xi_{o} \right\} \tag{3.4b}\]

\[X = X_{(0)} \cdot \exp \tilde{C}_{0}(A) = \left\{\begin{bmatrix} 0 & \xi & 0 \\ 0 & 0 & 0 \\ \xi & 0 & 0 \end{bmatrix} \mid \xi \in \Lambda_{0}, \quad X^* = X \right\} \tag{3.4c}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5a}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5b}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5c}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5d}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5e}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5f}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5g}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5h}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5i}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5j}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5k}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5l}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5m}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5n}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5o}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5p}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5q}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5r}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5s}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5t}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5u}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5v}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5w}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5x}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5y}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5z}\]

\[X = e^{-i\theta} e^{-i\theta} = \left\{\begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} \mid \theta \in \Lambda_{0} \right\} \tag{3.5} \]
Naturally $\mathfrak{m}$ is the Lie algebra of $M$.

Denote by $X^C$ the submanifold of $X$ described by $(x^2)_{0}=0$.

Then we define the supergroup $G = SU(2,2;\mathbb{N})$ as the union

$$G = X^{C} \cup X^{C} \cup X^{C}$$

(\omega as in (1.3)). The even subgroup of $G$

$$G_{\text{ev}} = X^{C} \cup X^{C} \cup X^{C} \cup X^{C} = \left\{ x_{\alpha}, \eta_{\alpha}, A, L \right\} Z U = S(U(2,2) \times U(\mathbb{N}))$$

is the superanalogue of $G = SU(2,2\times U(N))$ and (3.13), (3.14) may be viewed as generalizations of the (maximal parabolic subgroup related) Bruhat decomposition for $SU(2,2)$ [22].

For $N=4$ $Z$ is a centre of $G$. In that case we can consider the factorgroup $G^Z = G/Z$ along with $G$.

For any $g \in G$ the relation

$$g^\omega g = \omega$$

(3.15)

corresponding to (3.3) holds. Given the supergroup $SL(4;\mathbb{C})$ (e.g. as the matrix group $\left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$ with $A_{\alpha \beta}, D_{\alpha \beta} \in L(0)$, $B_{\alpha \beta}, C_{\alpha \beta} \in \mathfrak{a} L(1)$, $\det(A-BD^{-1}C)/\det(0) = 1$) (3.15) can be taken as a defining relation for $SU(2,2;\mathbb{N})$.

The subgroup $P = NAM = HAN$ is the maximal parabolic subgroup of $G$. We shall consider representations of $G$ induced from representations of $P$ and then it is natural to consider representation spaces of functions on $X \neq G/P = X \cup X^C$, or equivalently on $X$ with some asymptotic conditions as in one of our realizations in Sec.5.

Analogously, we introduce the "maximal compact" subgroup $K$ of $G$ (with Lie superalgebra $\mathfrak{k}$) starting from $K(0)$ (cf.(1.7a,9d,e)):

Note that the Weyl matrix $\omega$ (1.3) belongs to $K(0)$.

4. THE INDUCING REPRESENTATIONS OF $M$

We introduce the representations of $M=\mathbb{L}^{2,0}$ which will be used in the next section for the inducing of the representations of $G$. We shall first describe the representations of the ordinary group $M$ and then we shall extend them to representations of its superanalogue $M$.

The finite-dimensional representations of $SL(2,\mathbb{C})$ are characterized by two nonnegative half-integers $j = (j_1,j_2)$. Following [23] we realize these representations in the space $V_j$ of homogeneous polynomials of a complex two vector $z = (z^1, z^2)$ and its conjugate $\bar{z} = (\bar{z}_1, \bar{z}_2)$

$$V_j \cong \left\{ \psi : \mathbb{C}^+ \rightarrow \mathbb{C} \right\} \left( \psi(z,\bar{z}) = \lambda^{j+1/2} \psi(z,\bar{z}), \lambda \in \mathbb{C} \right)$$

(4.1)

The representation $D^j$ acts in $V_j$ by (cf.(3.9))

$$\left(D^j (\hat{\xi}) \psi \right)(z,\bar{z}) = \psi \left( \hat{\xi} z, \bar{\hat{\xi}} \bar{z} \right)$$

(4.2)

The sesquilinear form

$$\langle \psi, \psi \rangle_j \equiv \frac{\psi^\dagger (2,2) \psi}{(2,2) \psi^\dagger (2,2) \psi}$$

(4.3)

is $SL(2,\mathbb{C})$ invariant. (For the representations of $SL(2,\mathbb{C})$ one uses also the signature $\{j_1,j_2\} \cong \left\{ j_2, j_1, j_2+j_1 \right\}$.)
In our considerations a very important role will be played by the Weyl conjugated representation $D_i^+(\ell)$ acting in the same space $V_j$ by
\[ D_i^+(\ell) \equiv D_i^+(\ell^\omega) \] (4.4a)
where $\ell$ is the Weyl conjugated matrix of $\ell$
\[ \ell^\omega = \ell^T \prod \ell = \left( \begin{array}{cc} \ell^{\tau} & 0 \\ 0 & \ell \end{array} \right) \] (4.4b)

The Weyl image $D_i^+$ is equivalent to the representation $D_j^+$ where
\[ J = \{ j_1, j_2, \ldots \} \text{ and } j = \{ j_1, j_2, \ldots \} \text{ for } J = \{ j_1, j_2 \}. \]

The intertwining operator realizing this equivalence is given by
\[ B_j : V_j \rightarrow V_j, \quad B_j D_i^+ = D_i^+ B_j \] (4.5a)
and its inverse $B_j^{-1} = B_j$ by the same formula for $\Psi \in V_j$.

The representations of $\text{SU}(2; \mathbb{C})$ (cf. (3.9)) will be indexed by the signature $\xi = 0, 1$. Their action in $V_j$ and the action of $L \equiv \text{SL}(2; \mathbb{C})$ in $V_j$ is given respectively by:
\[ (D_i^+ D_j^+)(\ell \psi) = D_{ij} \psi \] (4.6a)

The representations of $\text{Z}^*$ may be parametrized by a complex number $\lambda$ (as of $\text{Z}$ and its Lie algebra). However (cf. (4.21,22) below) defining consistently the representation of the whole group $\mathcal{M}$ will restrict the values of $\lambda$ and fix the value of $\xi$.

\[ B_k : V_k \rightarrow V_k, \quad B_k D_i^+ = D_i^+ B_k \] (4.5b)

The UIR of $\text{SU}(N)$ are characterized by $N-1$ nonnegative integers $r = r_1, \ldots, r_{N-1}$. They are standardly realized [15] in the space of polynomial functions over $\text{SL}(N, \mathbb{C})$
\[ \mathcal{V}_r \equiv \left\{ P : \text{SL}(N, \mathbb{C}) \rightarrow \mathbb{C}; P(\lambda L) = \chi_r(\lambda) P(\lambda) \right\} \] (4.7)
where $\lambda, \mu, \nu \in \text{SL}(N, \mathbb{C})$, $\mathbf{g}_\lambda = (g_{i,k})^0$, $h \in H$ - the Cartan subgroup of $\text{SL}(N, \mathbb{C})$
\[ \mathbf{g}_\lambda = \exp \left( \sum_{j=1}^{N-1} c_j \mathbf{e}_{j,j} \right), \chi_r = \mathbf{i}^{\mu_j + \nu_j}, \xi_j = \mathbf{i}^{\mu_j - \nu_j}, \zeta_j = 0, m_j = 0 \] (4.8)

The representation $D_i^+$ in $\mathcal{V}_r$ is defined by
\[ (D_i^+ P)(\lambda) = P((\lambda^*)^{-1}), \quad \lambda \in \text{SU}(N), \quad \zeta \in \text{SL}(N, \mathbb{C}) \] (4.9)

By the covariance property in (4.8a) $\mathcal{V}_r$ can be also realized [15] as the space of polynomial functions over the upper triangular matrices $U = (u_{i,j})$ of $\text{SU}(N, \mathbb{C})$ exploiting Gauss decomposition:
\[ U^{-1} = U^{-1}(U^*)^{-1} \]

Another identification for $\mathcal{V}_r$ is with the space of homogeneous polynomials of $N-1$ $N$-dimensional complex variables $V^*$
\[ P(\lambda, \nu_1, \ldots, \nu_N) = \left( \prod_{j=1}^{N-1} \frac{\lambda_j^{\nu_j}}{\gamma_j} \right)^{-1} P((\lambda^*)^{-1}) \] (4.10a)
where
\[ \nu_j \in \mathbb{N} \quad \text{and} \quad \mathbb{N} \text{ is otherwise arbitrary subject to the condition} \]
The definition (1.10a) makes sense since by the covariance property in (4.7) \( P(f(y)) = P(f(y')) \) if \( y^k, k = 1, \ldots, N-1 \).

These functions satisfy \( \langle v \cdot v \rangle = \langle k \cdot v \rangle \).

Eqs. (4.11a) are the infinitesimal version of the covariance property in (4.7).

Another useful realization is provided by the restriction of \( P \in \pi \) to \( SU(N) \), i.e.,

\[
\mathcal{V}_f = \{ \rho : SU(N) \to \mathbb{C} ; \rho(u) = P(u) \} \tag{4.12a}
\]

with the property (following from (4)

\[
\Phi_k \rho(u) = \chi_{k^T}^{-1}(u) \Phi(u), \quad \rho_n \in H \cong H / S(U(N)) \tag{4.12b}
\]

where \( H = S(U(1) \times \cdots \times U(1))), (N \text{ factors}) \), is the Cartan subgroup of \( SU(N) \).

The representation formula (4.9) remains the same with the obvious replacements. The functions \( \Phi \) can be continued uniquely to functions of \( \chi_{k^T}^{-1} \) using the (unique) Iwasawa decomposition of \( SL(N, \mathbb{C}) \)

\[
P(f) = \chi_{k^T}^{-1}(u) \Phi(u), \quad f = u^T A \alpha A \tag{4.13}
\]

Similarly to (4.10) we define

\[
\langle \nu \cdot \nu \rangle = \det \langle \nu \rangle = \langle 1 \rangle (\varepsilon_1, \ldots, \varepsilon) = 1. \tag{4.10b}
\]

Remark 4.1. The less restrictive covariance property

\[
P(f) = \chi_{k^T}^{-1}(u) P(s) \tag{4.14a}
\]

instead of that in (4.7) defines a reducible representation of \( SU(N) \).

Similarly abandoning the identification \( \Phi(u) = P(u) \) but keep it \( \chi_{k^T}^{-1} \) the property (4.12b) we get a reducible representation of \( SU(N) \) in terms of functions \( \Phi(u^1, \ldots, u^N; \tilde{u}^1, \ldots, \tilde{u}^N) \). Essentially these reducible representations have been used in [12] as inducing representations for the \( N=2, \tilde{3} \) Poincaré superalgebra. The right action \( u^k \rho(u) \) in (4.12b) of the \( SU(N) \) Cartan subgroup \( H \) on the group \( SU(N) \) defines the "\( H \)-charges" [12] of the \( u^k \)-variables \( (k=1, \ldots, N) \):

\[
\nu^k = (u \cdot \exp \gamma \chi)^{k}\nu_{u^{k}} = u^k \exp \gamma \chi \chi_k^T \nu_k, \quad \chi_k = \delta_{ij} - \delta_{jk} \text{e}_{ij}^T.
\]

The equations

\[
D_{ik} \Phi(u^1, \ldots, u^N; \tilde{u}^1, \ldots, \tilde{u}^N) = 0, \quad 1 \leq i < N, \tag{4.15a}
\]

\[
(D_{ik})_{\tilde{u}^k} \Phi = 0, \quad 1 \leq i \leq N-1 \tag{4.15b}
\]

are the (interior derivative) operators corresponding to \( d_{ik} \) in (4.11), single out an irreducible subrepresentation characterized by the set \( \Gamma \) in (4.12b) which can be realized (up to equivalence) in terms of functions
\( f(u^1, \ldots, u^{N-1}) \) of the first \( N-1 \) arguments. Accordingly, appropriate degrees of \( D_{ik} \) (\( i < k \)) can be used to project out any of the irreducible components entering the (infinite) direct sum described by \( \rho \).

The SU(N)-invariant scalar product in \( V_\chi \) is defined by

\[
\langle \psi, \psi' \rangle_r = \sum_{m=1}^{\infty} \frac{1}{m!} \langle \partial_m^0 \psi \rangle \langle \partial_m^0 \psi' \rangle
\]

(4.16)

The standard SU(N) tensor described by the Young tableaux with \( m_i \) cells in the \( i \)-th row are recovered by straightforward differentiation of the functions of \( V_\chi \) (or \( V_\chi^* \)).

\[
\psi(u^1, \ldots, u^{N-1}) = \sum_{m=1}^{\infty} \frac{1}{m!} \partial_m^0 \psi(u^1, \ldots, u^{N-1})
\]

(4.17a)

(4.17b)

The complex conjugates of the functions of \( V_\chi \) transform according to the representation \( D^r_\chi \) defined by

\[
D^r_\chi(\bar{u}^i) = \bar{D}^r(u^i)
\]

(4.18)

The representations \( D^r \) and \( D^r_\chi \) where \( r = (m_{N-1}, \ldots, m_1) \) are equivalent and the corresponding intertwining operator \( B_\chi \) acting in \( V_\chi \) by

\[
B_\chi : V_\chi \rightarrow V_\chi^*, \quad D^r_\chi B_\chi = B_r D^r
\]

(4.19a)

is defined by

\[
B_r \psi(u^1, \ldots, u^{N-1}) = \psi(-u^1, -u^{N-1}, \ldots, -u^{N-1})
\]

(4.19b)

\[
\sum_{\kappa=1}^{\infty} ( u^\kappa, \ldots, u^{\kappa}, \partial_{\kappa} u^1, \partial_{\kappa} u^2, \ldots, \partial_{\kappa} u^{N-1}) \psi(u^1, \ldots, u^{N-1})
\]

(4.19c)

\[
\left( u^N - 1 \right, \ldots, u^{N-1}, u^1 \right) \text{ from (4.14b)}.
\]

The operator \( B \) corresponds in the tensor language to the raising of all indices of the tensor \( T \) by the Levi-Civita symbol and obtaining a tensor described by the dual Young tableaux. Similarly, the raising of only part of the indices can be described here by an operator (analogous to \( B \)) acting from \( V_\chi \) to the appropriate function space. All resulting realizations are equivalent and are omitted since we shall not use them below.

The operator \( B^*_\chi \), acting in \( V_\chi^* \) is defined by (4.19c). For \( r = r \) the representations \( D^r \) and \( D^r_\chi \) are equivalent and a reality condition

\[
\langle B\psi(u^1, \ldots, u^{N-1}) \rangle = \bar{\psi}(-u^{N-1}, -u^{N-1}, \ldots, -u^1)
\]

(4.20)

can be imposed (non-trivially in some cases) on \( \psi \in V_\chi \).

Let \( V = V_\chi \otimes V^*_\chi \) be the direct product of the two finite-dimensional spaces \( V_\chi \) (4.1) and \( V_\chi^* \) (4.12) realized, say, on \( \mathbb{Z} = \mathbb{R}^d \). The representation of the group \( \mu \) acting in \( V \) will be given by \((\bar{g}(t))\mathbb{Z}^*\) from (3.11):

\[
\begin{align*}
D^r_{\mu}(\bar{g}(t)) & = \exp(\bar{g}(t))D^r \exp(-\bar{g}(t)) & D^r_{\mu}(\bar{g}(t)) & = \exp(\bar{g}(t))D^r \exp(-\bar{g}(t)) & D^r_{\mu}(\bar{g}(t)) & = \exp(\bar{g}(t))D^r \exp(-\bar{g}(t))
\end{align*}
\]

(4.21)

Noting that \( \bar{g}(t) \in \mathbb{Z}^* \) is extended uniquely to a \( \Lambda \)-valued function on \( \mathbb{Z} = \Lambda^* \mathbb{R} \cdot \mathbb{Z} \). The representation \( M \rightarrow V \) is extended to \( V \) by

\[
\gamma_i + \bar{\gamma}_j \in \mathbb{Z}, \quad E \rightarrow (\gamma_i + \bar{\gamma}_j - 2\lambda)_{\mathbb{Z}} = (\lambda_j - \lambda_i)_{\mathbb{Z}}
\]

(4.22)

We note also that for \( j_i - j_j = 0 \) and \( r = r \) the reality condition (4.20) can be extended to \( V \).

Further by Berezin's Grassmann analytic continuation any \( f \in V \) is extended uniquely to a \( \Lambda(0) \)-valued function on \( \mathbb{Z} \). The resulting space \( V_B \) (isomorphic to \( V(0) \)) is further extended to the space \( V \) obtained by taking the linear span of all left actions \( f(g^{-1}) \), \( g \in M \), \( e \in \mathbb{Z} \), \( \lambda = 0 \), \( V(0) \) can be viewed as a Grassmann envelope of \( V_B \) with respect to \( \Lambda(0) \) (cf. App.A.1.). The representations of \( M \) are defined in \( V \) by the same formulae as for \( M \) in \( V \). Another representation space \( V_{(1)} \) of the Lie supergroup \( M \) is provided by the \( \Lambda(1) \)-envelope of \( V_B \). The inner products in \( V_{(a)} \) are \( \Lambda(0) \)-valued. The sign of complex conjugation denotes from now on Grassmann involution.
5. THE ELEMENTARY REPRESENTATIONS OF G

We shall use three equivalent realizations of the elementary representations of $G$ induced by the parabolic subgroup $P$. Let $\chi \in \mathcal{C}$, fix a (non-unitary) character of $A$ and let

$$\chi = [\lambda_1, \lambda_2; \gamma_1, \ldots, \gamma_n] = \chi [\lambda_1, \lambda_2; \gamma_1]$$

(5.1)

denote the signature of the finite-dimensional representation $D^\chi$ of $\mathcal{M}_A$ acting in $V$ by the formula (cf. (4.21,22))

$$D^\chi_{(m,a)} = D^{c-2N} D^{\chi}(m) \chi [\lambda_1, \lambda_2; \gamma_1]$$

(5.2)

($d = c+2-N$ is the scale dimension of conformal superfields in QFT.)

Then we define the representation space

$$\mathcal{F}_\chi \equiv \{ f : G \times \mathbb{C} \rightarrow \mathbb{C}((g)) \mid f \in \mathcal{C}^\infty(G \times \mathbb{C}) (\Lambda, 0), \forall (g) \in U_0 \}$$

$$\mathcal{F}_\chi = C^\infty(G \times \mathbb{C}) (\Lambda, 0)$$

(5.3)

where $\mathcal{C}^\infty(G \times \mathbb{C}) (\Lambda, 0)$ is the Grassmann envelope of $\mathcal{C}^\infty(G \times \mathbb{C}, \Lambda) \equiv \mathcal{C}^\infty(G \times \mathbb{C}, \Lambda)$ both defined in App. A.1. (Analogously, a space of $\Lambda$ valued functions can be defined by imbedding in $\mathcal{C}^\infty(G \times \mathbb{C}) (\Lambda, 1)$.)

The action of the elementary representation $T^\chi$ in $\mathcal{F}_\chi$ is given by the left regular action

$$(T^\chi (g) f)(g') = f(g g'), \qquad g, g' \in G, \quad f \in \mathcal{F}_\chi$$

(5.4)

The corresponding ER of the superalgebra $\mathcal{G}$ is defined in the space $\mathcal{F}_\chi^{alg}$ obtained as in (5.3) with $\mathcal{C}^\infty(G \times \mathbb{C}) (\Lambda, 0)$ replaced by $\mathcal{C}^\infty(G \times \mathbb{C}) (\Lambda) (0)$ in other words $\mathcal{F}_\chi$ can be identified with $\mathcal{F}_\chi^{alg}(\Lambda, 0)$.

We give two more realizations of the elementary representations.

First is the so-called non-compact picture in which the representation space is obtained by restricting the functions of $\mathcal{F}_\chi$ to $X \times V_0$.

The ER $T^\chi$ in $\mathcal{F}_\chi$ is defined by

$$(T^\chi (g) f)(x) = D^\chi_{(n_x g)} f(x)$$

(5.6a)

or using $f(x) \in \mathcal{F}_0$ and dropping the third argument in $f(x, z, \bar{z}, u)$

$$(T^\chi (g) f)(x, z, \bar{z}) = D^\chi_{(n_x g)} f(x, z, \bar{z})$$

(5.6b)

where the quantities on the RHS are defined through the Bruchat decomposition

$$2^{-N} x = X_0 T^{+\chi} z + \bar{z}$$

(5.7)

For the various subgroups of $G$ we obtain from (5.7)

a) $g = x^1, \quad \lambda^1 x = x^1, \quad x^1 = x - x^1 + 2i (\theta^1 - \theta^1 \bar{\theta}^1), \quad \theta^1 = \theta^1 - \theta^1 \bar{\theta}^1$;

b) $g = \exp(z \theta^1 \bar{\theta}^1), \quad \lambda^1 x = x g^{-1}$;

c) $g = \omega = \omega^0, \quad X = X_0 - \eta(x), \quad x_0 = 0, \quad x = x_0 + x, \quad x = x_0 + \sum_{n=1}^N \delta_n \bar{\theta}^1, \quad x^1 = (x_0 + \sum_{n=1}^N \delta_n \bar{\theta}^1)^{\chi}$.
In $C_X$ we can introduce a $K$-invariant ($\Lambda_{(0)}$-valued) "scalar product"
\[
\langle \ell, \ell \rangle_X \equiv \int_{\chi} \frac{d\mu(x)}{\det(\mathcal{A} + i \mathcal{L} - 20\mathbf{e})} \langle \hat{\phi}(\mathcal{A} + i \mathcal{L} - 20\mathbf{e}) \hat{\phi}(\mathcal{A} + i \mathcal{L} - 20\mathbf{e}) \rangle \tag{5.12a}
\]
where $d\mu(x) = d\mu(x, \theta, \overline{\theta})$ is the standard measure on $X$ (see Appendix A.2.);
\[
\chi \equiv \text{SU}(\text{II}/0)-\text{invariant scalar product generalizing (4.16)}
\]
\[
\langle \ell, \ell \rangle_X \equiv \int_{\chi} \frac{d\mu(x)}{\det(\mathcal{A} + i \mathcal{L} - 20\mathbf{e})} \langle \hat{\phi}(\mathcal{A} + i \mathcal{L} - 20\mathbf{e}) \rangle \tag{5.12b}
\]

Note that the analogue of (5.12a) for the Lie group $G$ or equivalently for its noncompact subgroup $\text{SU}(2,2)$ provides a topology with respect to which the corresponding elementary representation of $\text{SU}(2,2)$ is continuous.

Another realization (reminiscent of the induction from the minimal parabolic subgroup) is taken in the space $C_X$ of functions on $X$ which are obtained from the functions of $C_X$ ($\chi' = \text{SL}(2/0;\mathbb{C})$):
\[
\tilde{\varphi}(x) = \mathcal{H} C_X \chi, \quad \mathcal{F}(\tilde{\varphi}(x)) = \chi' \mathcal{F}(\varphi) \chi = \varphi(x), \quad \varphi(x) \equiv \varphi(x), \quad \pi(u)\varphi(x) = \pi(u) \varphi(x),
\]
\[
\pi(u) \equiv \pi \left( \left( \begin{array}{cc} u & 0 \\ 0 & \overline{u} \end{array} \right) \right), \quad \varphi(x) = \varphi(x) \tag{5.13a}
\]
\[
\pi(u) \varphi(x) = \varphi(u \varphi(x) \varphi(u)^{-1}) = \varphi(x) \tag{5.13b}
\]

The functions from $C_X$ have a covariance property following from (4.1) and (4.7, 4.12a):
\[
\mathcal{F}(x \ell \hat{\phi}(x) \ell) = \chi_i' \mathcal{F}(x \ell \hat{\phi}(x) \ell) = \chi_i' \mathcal{F}(x \ell \hat{\phi}(x) \ell), \quad \chi_i' \mathcal{F}(x \ell \hat{\phi}(x) \ell), \quad \chi_i' \mathcal{F}(x \ell \hat{\phi}(x) \ell)
\]
\[
\chi_i' \mathcal{F}(x \ell \hat{\phi}(x) \ell) = \exp \left( i \left( \ell \mathcal{A} + \mathcal{L} \ell \mathbf{e} \right) \right) \mathcal{F}(\chi_i' \mathcal{F}(x \ell \hat{\phi}(x) \ell)), \quad \ell_i = \exp \left( i \left( \ell \mathcal{A} + \mathcal{L} \ell \mathbf{e} \right) \right)
\]
\[
\chi_i' \mathcal{F}(x \ell \hat{\phi}(x) \ell) = \exp \left( i \left( \ell \mathcal{A} + \mathcal{L} \ell \mathbf{e} \right) \right) \mathcal{F}(\chi_i' \mathcal{F}(x \ell \hat{\phi}(x) \ell)), \quad \ell_i = \exp \left( i \left( \ell \mathcal{A} + \mathcal{L} \ell \mathbf{e} \right) \right)
\]
\[
\chi_i' \mathcal{F}(x \ell \hat{\phi}(x) \ell) = \exp \left( i \left( \ell \mathcal{A} + \mathcal{L} \ell \mathbf{e} \right) \right) \mathcal{F}(\chi_i' \mathcal{F}(x \ell \hat{\phi}(x) \ell)), \quad \ell_i = \exp \left( i \left( \ell \mathcal{A} + \mathcal{L} \ell \mathbf{e} \right) \right)
\]
\[
\chi_i' \mathcal{F}(x \ell \hat{\phi}(x) \ell) = \exp \left( i \left( \ell \mathcal{A} + \mathcal{L} \ell \mathbf{e} \right) \right) \mathcal{F}(\chi_i' \mathcal{F}(x \ell \hat{\phi}(x) \ell)), \quad \ell_i = \exp \left( i \left( \ell \mathcal{A} + \mathcal{L} \ell \mathbf{e} \right) \right)
\]
The intertwining operator $\mathcal{A}^\dagger_1$ acts in $\mathcal{G}_x$ by:

\[
(\mathcal{A}^\dagger_1 f)(x, z, u) = \int_{\mathfrak{g}} e^{-C z \nu} (-1)^{\nu} \exp(\chi x \nu) F(x, z, u) \, d\nu(\gamma)
\]

where we have again used the Bruhat decomposition in the form (5.7).

The inverse operator $A^{-1}_1: \mathcal{G}_x \rightarrow \mathcal{G}_x$ is defined by

\[
(\mathcal{A}^{-1}_1 f)(x, z, u) = \mathcal{F}(x, z, u) \mathcal{F}^{-1}(x, z, u)
\]

The intertwining properties of $\mathcal{A}^\dagger_1$ and $A^{-1}_1$ are easily checked.

Along with $\mathcal{A}_1$ we shall use its conjugate representation $\mathcal{A}_1^\dagger$, its Weyl conjugate $\mathcal{A}_1^\dagger$ and its mirror image $\mathcal{A}_1^\dagger$ where

\[
\mathcal{A}_1 = \left[ j, \lambda; r, r' \right], \quad \mathcal{A}_1^\dagger = \left[ j, -\lambda; r, r' \right], \quad \mathcal{A}_1^\dagger = \left[ j, \lambda; r, -r' \right]
\]

We note that $\mathcal{A}_1$ and $\mathcal{A}_1^\dagger$ (and $\mathcal{A}_1$ and $\mathcal{A}_1^\dagger$) are equivalent by

\[
\mathcal{B}_j : \mathcal{G}_x \rightarrow \mathcal{G}_x, \quad \mathcal{A}_1 \mathcal{B}_j = \mathcal{B}_j \mathcal{A}_1^\dagger
\]

(6.1a)

where $\mathcal{B}_j$ is from (4.5a).

Also $\mathcal{A}_1$ and $\mathcal{A}_1^\dagger$ are equivalent:

\[
\mathcal{B}_j : \mathcal{G}_x \rightarrow \mathcal{G}_x, \quad \mathcal{A}_1 \mathcal{B}_j = \mathcal{B}_j \mathcal{A}_1^\dagger
\]

(6.1b)

where $\mathcal{B}_j$ is from (4.19).

6. INTEGRAL INTERTWIXING OPERATORS

Analogously to the SSLG case we first introduce an intertwining operator between the representations $\mathcal{X}$ and $\mathcal{X}^\dagger_1$ which is given by

\[
\mathcal{A}_1 \mathcal{G}_x \rightarrow \mathcal{G}_x
\]

(6.1a)

\[
\langle \mathcal{A}_1 f(x), \mathcal{G}_x \rangle = \gamma(x) \int X(f(x, z, u)) \, d\mu(X)
\]

(6.1b)

where $\gamma(x)$ is a constant, $d\mu(X)$ is defined in Appendix A.2 (This operator is an exact analogue of the Knapp-Stein operator [24]). One checks similarly to the SSLG case that

\[
\mathcal{A}_1 \mathcal{G}_x \rightarrow \mathcal{G}_x
\]

(6.2)

The intertwining operator in the noncomponent picture is

\[
\mathcal{A}_1 f(x) = (\mathcal{A}_1 f)(x) = \gamma(x) \int X(f(x, z, u)) \, d\mu(X)
\]

(6.3a)

\[
\gamma(x) \int X(f(x, z, u)) \, d\mu(X) = X(f(x, z, u)) \, d\mu(X)
\]

(6.3b)

where we have used $X = x, \, \nu, \, \omega, \, \nu, \, \omega$ (cf. (5.8c)). A slightly more explicit formula is written down in Appendix A.2.

Combining (6.1a) and (5.18a) a G-invariant "sesquilinear form" on $\mathcal{G}_x \times \mathcal{G}_x$ can be defined for $c$ real:

\[
\langle f, g \rangle_\mathcal{G}_x = \int X(f(x, z, u)) \, d\mu(X) \langle f, g \rangle_\mathcal{G}_x
\]

(5.4)

where $\mathcal{B}_j$ is from (4.19).
where
\[ \mathcal{W}_n = A_n \circ \mathcal{B}_n \quad \text{and} \quad \mathcal{T}_n \mathcal{W}_n = \mathcal{W}_n \mathcal{T}_n \quad . \]

For \( N=1 \) the integral operators \( \mathcal{W}_n \) were constructed in [25,26].

7. **Lowest Weight Module Structure of the Elementary Representations, Their Reducibility and Related Intertwining Operators**

We start by introducing the right action of \( \mathcal{A}_N \) or \( \mathcal{A}_N^G \) in \( \mathfrak{h}^{\text{alg}} \), \( \mathfrak{h}_N^G \), respectively, by an obvious generalization of a standard formula [27]:
\[ (\mathcal{Y} f)(x) = \frac{1}{i} \sum_{\xi \in i \mathfrak{h}_N} f(\xi) \exp \{ i\xi(x) \}, \quad \xi \in i \mathfrak{h}_N^G, \quad \xi \mathfrak{g} = 0 . \]

Further, in a slight abuse of notation, only the \( G \)-representation spaces \( \mathfrak{h}_N^G \) shall be employed. We immediately notice that (cf. (1.9a,c)(2.9)(5.13))
\[ \mathcal{Y} f = 0 \quad \text{if} \quad \mathfrak{g} \in \mathfrak{h}_N^G . \]

Thus we see that every element \( \gamma \in \mathfrak{h}_N^G \) may play the role of a lowest weight vector of a lowest weight module of \( \mathcal{A}_N \). Indeed \( \mathfrak{g} \) is annihilated by \( \mathfrak{h}_N^G \) and the Cartan subalgebra \( \mathfrak{h}_N \) (of which \( e_j \) form a basis) acts on \( \mathfrak{g} \) by scalars. (In the mathematical literature [13] highest weight modules (annihilation by \( \mathfrak{h}_N^G \)) are usually used; however we follow our traditional approach to conformal theories [5] where lowest weight modules (albeit unnoticed) were used. To make this structure more apparent we first calculate the values of \( \mathfrak{g} \) on \( e_j \) using (2.1b).

\[ \beta (e_1, e_2, e_3, e_4, e_5, \ldots, e_{4a+1}) = (0, -1, 2, -1, 0, -1, \ldots, -1) \quad . \]

Thus (7.3) acquires the standard form:
\[ \mathfrak{g} f = (\Lambda + \gamma) f, \quad \gamma \in \mathfrak{g} . \]

where \( \Lambda = \lambda \mathfrak{h}^G \) is completely determined from the conditions (resulting from the comparison of (7.3) and (7.5); for \( N=4 \) see also Remark 7.1):
\[ \Lambda(e_1) = \eta_1 + \eta_1, \quad \Lambda(e_2) = \eta_2 + \eta_2, \quad \Lambda(e_3) = \lambda, \quad \Lambda(e_4) = \lambda - \frac{1}{2} N - \lambda, \quad \Lambda(e_5) = -\frac{1}{2} N - \lambda \quad (\text{for } 1 \leq k \leq N-1) . \]

It is easy to express \( \gamma \) explicitly in terms of the simple roots \( \gamma \), to reproduce (7.6) but below we shall need only its values on the \( h_j \) basis. For that we use the inverse of (2.1b)
\[ \gamma_1 = e_1 + e_2, \quad \gamma_2 = -e_1 + e_2, \quad \gamma_3 = \frac{1}{2} (1 + \sqrt{5}) e_2 + \frac{1}{2} (1 - \sqrt{5}) e_3, \quad \gamma_4 = \frac{\sqrt{5}}{2} (e_2 + e_3), \quad \gamma_5 = \epsilon e_3 + \epsilon e_4 + \epsilon e_5 \quad (\text{for } 1 \leq k \leq N-1) . \]

Remark 7.1 For \( N=4 \) \( e_1 = \frac{1}{2} H_0 \) and the superalgebras \( \mathcal{A}_N^G \) and the supergroup \( G \) become reductive rather than semisimple. Consequently the highest weight of \( \mathcal{A}_N^G \) splits into the sum \( \Lambda + \lambda \) where the weight \( \lambda \) is such that \( \lambda(e_k) = 0 \) for \( h_0 \lambda \). Because of this our considerations below remain unchanged. One should only keep in mind that for \( N=4 \) \( \lambda \) in the signature \( x \) comes not from the LWM \( \Lambda \) but from \( \lambda \).

*) Note that this form is standard in SSLA representation theory [28,15]; Kac uses equivalently \( \mathcal{Y} f = \lambda(x) f \) (cf. [13]).
Now we are able to exploit the theory of Kac of LWM. From this theory it follows that the LWM or BR are generically irreducible. However we are interested mostly in the reducible LWM. In [4] we used the criterion of Kac [13] adapted to our situation obtaining that the LWM x is reducible only if at least one of the following $4 + 4N$ conditions is true

$$2(\lambda, \beta) = -m(\beta, \beta), \quad \beta \in \Delta', \quad \text{noncompact}, \quad m \in Z^+, \quad (7.8)$$

Remark 7.2 Note that (7.8) is automatically fulfilled for all compact roots, in particular for $\beta = \delta_1', \delta_2', \ldots, \delta_{N-1}'$ with $m = 1 + 2j_1, 1 + 2j_2, 1 + 2j_3, \ldots, 1 + 2j_{N-1}$. However we work with finite dimensional representation of $SL(2/0;F)$ and $SU(N/0)$ and thus our BR live in the invariant subspaces which are present in more general representations of $O_J$. These invariant subspaces are actually annihilated by the corresponding operators - cf.

Conditions (7.8) explicitly are (in (7.9a) an additional $E-$ condition not contained in (7.8) holds (derivation as in [29,7]) for $N=1$ cf. [9,10] for (7.10)):

$$-m = c \in \{1 + j_1, \ldots, j_1\} \in Z^+, \quad E = (c + j_1, j_2)(\text{mod} 2), \quad (7.9a)$$

$$-m = c \in \{1 + j_1, \ldots, j_1\} \in Z^+, \quad C - 1 - 2j_1 + z - R_{\kappa k} = 0, \quad \kappa = 1, \ldots, N, \quad (7.9b)$$

$$C + 1 + 2j_1 + z - R_{\kappa k} = 0, \quad (7.10.1)$$

$$C - 1 + 2j_1 - 2 + R_{\kappa k} = 0, \quad (7.10.2)$$

$$C - 1 - 2j_1 - 2 + R_{\kappa k} = 0, \quad (7.10.3)$$

$$C + 1 + 2j_1 - 2 + R_{\kappa k} = 0, \quad (7.10.4)$$

where

$$Z = \frac{1}{2}(j_1, j_1), \quad R_{\kappa k} = \frac{1}{2}\lambda(\delta_{1'}, \delta_{2'}) \cdot \lambda = (Z, \beta), \quad \beta \in \Delta^+, \quad m \in Z^+, \quad \beta \in \Delta^+, \quad \text{noncompact}, \quad (7.11)$$

In the derivation of (7.9-10) we have used (2.7), (7.7) and

$$\langle \Lambda, \beta \rangle = \sum_{k=1}^{N+1} k \langle \Lambda_k, \beta \rangle = \sum_{k=1}^{N+1} k \langle \Lambda_k, \beta \rangle$$

Conditions (7.9) are obtained for $\beta = \delta_1, \delta_2, \delta_3, \delta_4$ and (7.10a) are obtained for $\beta = \delta_{a_{1}, a_{2}, a_{3}, a_{4}}, \quad a = 1, 2, 3, 4$ respectively. (Note that for a given LWM and fixed $a = 1, 2, 3, 4$ only one of the $N$ conditions in (7.10a) may hold. Thus no more than four of the conditions (7.10) may hold for a given LWM. Compatibility between even and odd conditions should also be checked bearing in mind (4.22). For the representations of $O_J$ (and $E$ and the conditions containing it should be dropped as well as the restriction on $\lambda$ in (4.22)) such compatibility was essential for the classification in [4].)

Whenever (7.8) is satisfied for some $B$ and same $m$, there arises an intertwining (differential) operator defined in the space $\tilde{X}_\chi$ (or $\tilde{X}_\xi$, or $\tilde{X}_\eta$) with the image in a partially equivalent representation space $\tilde{X}_\chi'$ where $\chi'$ is determined easily (as in the SSLG case [28]) as follows. We start with the even case. We use (2.12) with $\Lambda$ replaced by $\Lambda'$ and obtain

$$\Lambda' = \Lambda + \Delta, \quad \Lambda = \Lambda + \frac{2(\lambda, \beta)}{(\beta, \beta)}, \quad m \in Z^+, \quad \beta \in \Delta^+, \quad \text{noncompact}, \quad j = (7.13)$$

where we have used (7.8). Thus the signature $\chi'$ of the BR shall be determined using (7.6) with $\Lambda'$ replaced by $\Lambda' + \Delta$. Explicitly for $\beta = \delta_{a_1}, \delta_{a_2}, \delta_{a_3}, \delta_{a_4} = 23$ we obtain ($\chi = [j_1, j_2, c, \xi_1, \xi_2, \ldots, \xi_{N-1}] = [j_1, j_2, c, k, \xi]$)

$$\chi' = \left[ j_1 + \frac{c}{2}, j_2 + \frac{c}{2}, c + m, \lambda / \gamma \right], \quad m = C - 1 + j_1 + j_2 = 0, \quad (7.14a)$$

$$\chi' = \left[ j_1 + \frac{c}{2}, j_2 + \frac{c}{2}, c + m, \lambda / \gamma \right], \quad m = C - 1 - j_1 - j_2 = 0, \quad (7.14b)$$

$$\chi' = \left[ j_1 + \frac{c}{2}, j_2 + \frac{c}{2}, c + m, \lambda / \gamma \right], \quad m = C + j_1 - j_2 = 0, \quad j_2 = \frac{\alpha}{2}, \quad (7.14c)$$

$$\chi' = \left[ j_1 + \frac{c}{2}, j_2 + \frac{c}{2}, c + m, \lambda / \gamma \right], \quad m = C - j_1 + j_2 = 0, \quad j_2 = \frac{\alpha}{2}, \quad (7.14d)$$
Note that (7.8) for \( B \) even are also the reducibility conditions for the SU(2,2) LWM [7] and (7.14) were obtained as above in [7].

Formulas (7.14) were earlier established directly for the elementary representations of the Euclidean conformal group [5] and of SU(2,2) [6].

We want to stress an important difference between the reducibility conditions of LWM and ER which was discussed in [8]. For LWM (7.8) means only that \( A \) is reducible and contains as a submodule the LWM \( A' = A + m \).

In the language of ER that would mean that the map from \( A \) to \( A' \) is onto (although it has a kernel). However typically in the function space realizations these maps are not onto and thus \( A' \) is also reducible. To account for this the reducibility conditions (7.9) for ER are replaced [6,7] by

\[
\mathcal{C} + j_1 - j_2 \in \mathbb{Z} \quad , \quad \mathcal{E} = \left( \mathcal{C} + j_1 - j_2 \right)_{(mod \, 2)} ,
\]

while (7.10) remain unchanged. However note that (7.9) are the conditions for existence of an intertwining map between the LWM \( \mathcal{A} \sim \mathcal{A}' \) and (equivalently) of an invariant differential operator between the ER \( \mathcal{X} \sim \mathcal{X}' \).

Remark 7.1 For \( N = 4 \) \( \sigma = 0 \) (cf. (7.11)) and the value of \( \lambda \) is not relevant for the reducibility of LWM or ER (cf. Remark 7.1).

We now turn to the odd case. We use (2.12) with \( a \) replaced by \( \mathcal{A} \) and obtain instead of (7.13).

\[
\mathcal{A}' \equiv \mathcal{A}' \sim \mathcal{A}' = \mathcal{A} \sim \mathcal{A}' , \quad \mathcal{A} \sim \mathcal{A}' = \mathcal{A} \sim \mathcal{A}' , \quad \mathcal{A} \sim \mathcal{A}' = \mathcal{A} \sim \mathcal{A}' , \quad \mathcal{A} \sim \mathcal{A}' = \mathcal{A} \sim \mathcal{A}' , \quad \mathcal{A} \sim \mathcal{A}' = \mathcal{A} \sim \mathcal{A}' .
\]

A different vector from the lowest weight vector (LWV) \( v \) of \( \mathcal{A} \) and having the characteristic of the LWV \( v' \) of \( \mathcal{A}' \). (cf. [13] and in the SSLA case [28]).

In the next section we give the construction of the operators which realize the partial equivalences between \( \mathcal{C} \) and \( \mathcal{C}' \) in (7.14) and (7.17) and also of some higher-order invariant differential operator.

8. CONSTRUCTION OF INTERTWINING DIFFERENTIAL OPERATORS

We shall present a very general method for the construction of intertwining differential operators. It is an adaptation to the superalgebra case of a method developed in the semisimple Lie algebra (or group) context and which exhausts all possible intertwining operators there [8].

This method is based on several facts most of which are known.

1. An intertwining map between two LWM \( \mathcal{A} \) and \( \mathcal{A}' \) is equivalent to the existence in \( \mathcal{A} \) of a vector \( v \), called singular, different from the lowest weight vector (LWV) \( v \) of \( \mathcal{A} \) and having the characteristic of the LWV \( v' \) of \( \mathcal{A}' \). (cf. [13] and in the SSLA case [28]).

2. In the Verma module realization of the LWM \( \mathcal{A} \) (that is \( \mathcal{A} \) is identified with the universal enveloping algebra of \( \mathfrak{g} \) acting on \( v \)) any singular vector is a homogeneous polynomial of the root space vectors \( x_k \) corresponding to the simple roots \( \chi_k \) of the root system \( \mathfrak{g} \) [28].
3. In the SSLA case all intertwining maps are compositions of those determined by the positive roots \([2,30]\) and if we work with a real form \(G^0\) of \(G^0\) by the positive noncompact roots \([8,15]\). The intertwining map corresponding to the positive (noncompact) root \(\beta\) and acting from the LWMA when \((7.8)\) holds is realized by the singular vectors \(v_\beta\)

\[
V_\beta = \mathcal{P}_\beta(\lambda_1', \ldots, \lambda_\ell') \nu, \quad (\ell = r_m \nu \mathcal{O}_\ell),
\]

where \(\mathcal{P}_\beta\) is a homogeneous polynomial (determined up to the nonzero multiple) in \(\lambda_1', \ldots, \lambda_\ell'\) of degrees \([2\Sigma]\) respectively:

\[
\mathcal{P}_\beta = \sum_{k_1, \ldots, k_\ell} \mathfrak{m}_{k_1, \ldots, k_\ell} \mathfrak{r}^{i \lambda_{k_1} \ldots \lambda_{k_\ell}}, \quad x \in \mathbb{Z}, \quad \lambda_1 \gt \lambda_2 \gt \cdots \gt \lambda_\ell, \quad \mathfrak{m} = \sum_{k=1}^\ell k r_{k},
\]

where \(m = \frac{-2(\mathfrak{h}, \mathfrak{h})}{(\mathfrak{h}, \mathfrak{h})} \in \mathbb{Z}\) is obtained from \((7.8)\) and \(\nu\) is the lowest weight vector of \(\Lambda\) ; \(\mathfrak{m}_{k_1} \ldots \mathfrak{m}_{k_\ell}\) may be identified with the LWV of \(\Lambda' = \Lambda + m \mathfrak{h}\).

This holds also in the superalgebra case for the even roots \(\beta\) \((k = N + 3)\) without changes. (Naturally in our case \(k_1 = 0\) or \(1; (8.6) = \mathfrak{m}\) for all even noncompact \(\beta\).)

More remarkable is that it holds also for the odd roots if we replace the formula for \(m\) in \((8.2)\) with \(m = 1\). However, unlike the SSLA case, the singular vectors are not exhausted by the ones corresponding to the positive noncompact roots. We discuss this at the end of the section.

4. Finally the intertwining differential operator \(A_\beta\) corresponding to the singular vector \(v_\beta\) \((\text{and the root } \beta)\) is given by \([8]\)

\[
A_\beta = \mathcal{P}_\beta(\lambda_1', \ldots, \lambda_\ell'),
\]

where \(\mathfrak{k}_\ell'\) are defined in \((7.1)\).

Remark 8.1: It is important for the intertwining properties of \(A_\beta\) that \(\mathfrak{k}_\ell'\) act from the right while the \(\mathfrak{h}\) act from the left but that obviously is not sufficient. We stress that the action of a root space vector \(\mathfrak{k}_\ell'\)

The derivation of \((8.5)\) and of the odd operators below is done in the space \(\mathfrak{g}_x\) and then is translated by \((5.13a, 5.15)\) to the space \(\mathfrak{c}_x\) corresponding to a nonsimple root does not produce an intertwining map.

Remark 8.2: The above procedure can be also applied to the simple compact roots \((\text{cf. Remark 7.2})\). The resulting invariant operators annihilate the whole spaces \(V(x)\) or \(\mathfrak{g}_x\). Their explicit expressions for \(\chi = a - x\),

\[
\mathfrak{g}_x = \mathfrak{g}_x \times \mathfrak{g}_a \times \mathfrak{g}_y \times \mathfrak{g}_z \times \mathfrak{g}_u,
\]

are given in \((4.15)\).

Now we proceed directly to the construction of the differential operators in the case of the positive noncompact roots.

The differential operators corresponding to the even roots are known from our earlier work on the conformal groups \([5-7]\). Thus the operators acting from \(x\) to \(x'\) given in \((7.14)\) are (respectively)

\[
\begin{align*}
(d_{x}^{m} f)(x, z, \bar{z}) &= (2 \bar{z} \bar{z})^{m} f(x, z, \bar{z}), \quad \bar{z} \bar{z} = (\mathfrak{g}_x, \mathfrak{z}) \mathfrak{g}^{*}(\mathfrak{z}), \\
(d_{x}^{m} f)(x, z, \bar{z}) &= (\mathfrak{g}_x, z) \mathfrak{g}^{*}(z),
\end{align*}
\]

and we have suppressed the \(u\) dependence in \(f(x, z, \bar{z}, y)\).

The derivation of \((8.5)\) and of the odd operators below is done in the space \(\mathfrak{g}_x\) and then is translated by \((5.13a, 5.15)\) to the space \(\mathfrak{c}_x\).
which is most natural.

We start the derivation of the differential operator corresponding to the odd roots with $N=1$. For the two simple roots $\alpha \epsilon, \alpha \epsilon$ (cf. (2.4)) we immediately obtain by direct application of (7.1):

\[ \mathcal{D}'(x, z, \bar{z}) = \mathcal{X}(x, z, \bar{z}) - (Dz) \mathcal{X}(x, z, \bar{z}), \quad (Dz) = D_\alpha x, \alpha \epsilon, \alpha \epsilon \quad \text{(8.6a)} \]

\[ \mathcal{Z}'(x, z, \bar{z}) = \mathcal{X}_\alpha(x, z, \bar{z}) - (Dz) \mathcal{Y}(x, z, \bar{z}), \quad (Dz) = D_\alpha x, \alpha \epsilon, \alpha \epsilon \quad \text{(8.6b)} \]

where

\[ D_\alpha \equiv \frac{1}{2} D_{\alpha \epsilon} - i D_{\beta \epsilon}, \quad \overline{D_\alpha} \equiv \frac{1}{2} D_{\alpha \epsilon} + i D_{\beta \epsilon} , \quad \text{(8.7)} \]

and $f \in \mathfrak{g}'$, $x$ satisfies the corresponding constraints in (7.10).

For the root $\alpha_1 = \alpha_1 + \alpha_3$ the polynomial (8.1) takes the form (up to nonzero multiple)

\[ \mathcal{D}'(x_1, x_3) = 2j_1 x_1^1 x_3^3 - (2j_1 + 1) x_1^1 x_3^1 \quad \text{(8.8a)} \]

and the operator is:

\[ \mathcal{D}'(x, z, \bar{z}) = \mathcal{D}'(x_1, x_3) f(x_1, x_3) - (Dz) f(x_2, x_3) . \quad \text{(8.8b)} \]

Analogously for the root $\alpha_{35} = \alpha_2 + \alpha_4$

\[ \mathcal{D}'(x_2, x_4) = -2j_2 x_2^1 x_4^4 + (2j_2 + 1) x_2^1 x_4^1 \quad \text{(8.9a)} \]

\[ \mathcal{D}'(x, z, \bar{z}) = \mathcal{D}'(x_2, x_4) f(x_2, x_4) - (Dz) f(x_3, x_4) . \quad \text{(8.9b)} \]

In (8.8), (8.9) we have used (8.4a) and (cf. (4.1)).

\[ z_\alpha \frac{D}{Dz} f(x, z, \bar{z}) = 2j_1 f(x, z, \bar{z}) \quad \text{and} \quad \overline{z_\alpha} \frac{D}{Dz} f(x, z, \bar{z}) = 2j_1 f(x, z, \bar{z}) . \quad \text{(8.10)} \]

Now we present the operators for arbitrary $N$. These shall be denoted by

\[ \mathcal{D}_k', \mathcal{D}_k'^* , \mathcal{D}_k' , \mathcal{D}_k'^*, \quad k = 1, \ldots, N \] (8.11)

corresponding to the roots $\alpha_{2N-5+4k}, \alpha_{2N-5+4k}'$ respectively (and to the roots $\alpha_{4k-4}, \alpha_{4k-4}'$ in the notation of [4]).

The operators $\mathcal{D}_N'$, $\mathcal{D}_1'$ corresponding to the simple roots $\alpha_2$, $\alpha_4$ resp. are derived as in (8.7) by direct application of (7.1)

\[ \mathcal{D}'(x, z, \bar{z}, y) = \mathcal{X}_\alpha(x, z, \bar{z}, y) - (Dz) f(x, z, \bar{z}, y) = \frac{1}{2} (D_{\alpha \epsilon} x_\alpha f(x, z, \bar{z}, y)) \quad \text{(8.12a)} \]

\[ \mathcal{D}'(x, z, \bar{z}, y) = \mathcal{X}_\alpha(x, z, \bar{z}, y) = \frac{1}{2} (D_{\beta \epsilon} x_\alpha f(x, z, \bar{z}, y)) \quad \text{(8.12b)} \]

\[ D_{ka} \equiv \frac{1}{2} D_{ka} \quad \text{and} \quad \overline{D_{ka}} \equiv \frac{1}{2} D_{ka} + i \frac{D_{ka}}{\alpha \beta} , \quad \text{(8.12c)} \]

The operators $\mathcal{D}_k'$ and $\mathcal{D}_k'^*$ are derived in Appendix C. The result being (action of $f(x, z, \bar{z}, y)$ is assumed)

\[ \mathcal{D}' = \sum_{\alpha \beta=1}^{N-1} \mathcal{D}'(x, z, \bar{z}, y) \quad \text{(8.13a)} \]

\[ \mathcal{D}' = \sum_{\alpha \beta=1}^{N-1} \mathcal{D}'(x, z, \bar{z}, y) \quad \text{(8.13b)} \]

\[ \mathcal{D}' = \sum_{\alpha \beta=1}^{N-1} \mathcal{D}'(x, z, \bar{z}, y) \quad \text{(8.13c)} \]

\[ \mathcal{D}' = \sum_{\alpha \beta=1}^{N-1} \mathcal{D}'(x, z, \bar{z}, y) \quad \text{(8.13d)} \]

\[ \mathcal{D}' = \sum_{\alpha \beta=1}^{N-1} \mathcal{D}'(x, z, \bar{z}, y) \quad \text{(8.13e)} \]

\[ \mathcal{D}' = \sum_{\alpha \beta=1}^{N-1} \mathcal{D}'(x, z, \bar{z}, y) \quad \text{(8.13f)} \]

\[ \mathcal{D}' = \sum_{\alpha \beta=1}^{N-1} \mathcal{D}'(x, z, \bar{z}, y) \quad \text{(8.13g)} \]

\[ \mathcal{D}' = \sum_{\alpha \beta=1}^{N-1} \mathcal{D}'(x, z, \bar{z}, y) \quad \text{(8.13h)} \]

\[ \mathcal{D}' = \sum_{\alpha \beta=1}^{N-1} \mathcal{D}'(x, z, \bar{z}, y) \quad \text{(8.13i)} \]

\[ \mathcal{D}' = \sum_{\alpha \beta=1}^{N-1} \mathcal{D}'(x, z, \bar{z}, y) \quad \text{(8.13j)} \]
where in (8.13a), (8.14a) the terms with * are omitted, m \neq 0.

\[ (u^I, u^J, u^K) = (u^I_J, u^J_I, u^K), \quad j = 1, \ldots, N \]

D_{ik} are given in (4.15). The operator \( H^N \) has also a very simple expression (cf.(4.10b)) which is obvious in the form (8.14a)

\[ H^N = \left( B^2 \frac{\partial}{\partial u^I}, \ldots, \frac{\partial}{\partial u^J} \right). \]  

(8.16)

The operators \( D^I, D^J, D^K \) are given (cf. Appendix C) by the same formulae (8.13) and (8.14), resp. with the replacement (respectively)

\[ D^I \longrightarrow \frac{\partial}{\partial z^I}, \quad D^J \longrightarrow \frac{\partial}{\partial z^J}. \]  

(8.17)

Remark 8.3 We stress again that all expressions are first obtained in the space \( \mathbb{C}_X \) and then translated to \( \mathbb{C}_X \) via (5.13a, 15). If we apply the procedure to the functions of \( \mathbb{C}_X \) we shall obtain equivalent operators with spin and isospin indices which are much more complicated than (8.13, 14) but are again SL(2,C) and SU(N) irreducible. The usually used in the super-Poincaré context invariant reducible operators with spin and isospin indices corresponding to (8.12c) and (8.14b) can be extracted from any of the p=0 terms in (C.6) and (8.14b).

\[ \mathbb{C}_X \] 

Remark 8.4 Let x and \( x' \) be such that (7.10.4) with \( k=1 \) and (7.10.2) with \( k=N \) hold respectively. Then the conditions

\[ B^r = 0, \quad f \in \mathbb{C}_X \]  

(8.18a)

\[ D_{ij} = 0, \quad g \in \mathbb{C}_X \]  

(8.18b)

single invariant subspaces (cf. also the end of this Section) of \( \mathbb{C}_X \) and \( \mathbb{C}_X \), resp. if \( x' = (\ldots, -a, a), \quad x' \) for \( x = (j, l, o, a, l, r) \), \( o, a \) real, then \( g \) can be identified with \( D^r \) i.e. (cf.(4.19))

\[ g(y) = \left( \frac{\partial}{\partial z} \right)(y) = \left( \frac{\partial}{\partial z^I}, \ldots, \frac{\partial}{\partial z^J} \right) \equiv f(y); \]  

(8.19)

(we omit the \( x, z, \bar{z} \) dependence of \( f, g, \)). For \( j_r = 0 = r \) (\( j_r = 0 = r \)) Eq. (8.18a); (8.18b) resp. expresses the chirality (antichirality resp.) condition. If both (7.10.4) (\( k=1 \)) and (7.10.2) (\( k=N \)) hold for \( x \) (and hence for \( x' \) as well) the conditions

\[ B^r = 0, \quad f \in \mathbb{C}_X, \quad g \in \mathbb{C}_X \]  

(8.20)

are the superconformal invariant analogues of the harmonicity conditions introduced and explored [for \( N = 2,3 \) in [12]].
Next we discuss the intertwining maps (singular vectors) corresponding to elements $\alpha$ of the positive root lattice $Q^+$

$$Q^+ \equiv \left\{ \alpha = \sum_{i=1}^{N+1} k_i a_i, \mid k_i \in \mathbb{Z}, k_i > 0 \right\} \supset \Delta^*$$

(8.21)

which are not positive roots, $\beta \in Q^+ \setminus \Delta^*$. (As before we are interested in operators which are not compositions of other operators.) At the moment we can state only some necessary conditions in this respect. Namely if $\beta \in Q^+ \setminus \Delta^*$ gives rise to an intertwining map between the spaces $C_{\alpha}$ and $C_{\alpha'}$, or equivalently between the LWR $\Lambda = \Lambda^\mathbf{k}$ and $\Lambda' = \Lambda^\mathbf{k'}$ the following conditions on the coefficients $k_i$ in (8.18) certainly hold:

$$k_i^2 + k_i' = 0, \quad (k_j - k_j') \leq 2N$$

(8.22a)

$$k_i, k_i' \leq k_j, k_j' \leq k_j', k_i + k_i' \leq k_j$$

(8.22b)

The singular vector is built analogously to the case of the odd roots namely we use (8.1,2) with $m = 1$. The difference with that case is that we do not know in advance the conditions replacing $(\Lambda, \beta) = 0$.

These conditions are obtained from the requirement that $\nu_s$ is a singular vector. (Referring to the derivation of the (odd root) operators in Appendix C we recall that one of the conditions $\nu_s - \nu_s' = 0$ just recovered $(\Lambda, \beta) = 0$.)

There are cases here when $-2(\Lambda, \beta) = (\beta, \beta) = 0$ and then this gives also the conditions on the signature.

Next we give a conjecture (which is stronger than (8.2)): if $\beta \in Q^+ \setminus \Delta^*$ gives rise to an intertwining map as above then

$$\beta = \sum_{i=1}^{N+1} \left( k_i a_i + k_i' a_i' \right), \quad k_i, k_i' > 0$$

(8.23a)

or

$$\beta = \sum_{i=1}^{N+1} \left( k_i a_i - k_i' a_i' \right), \quad k_i, k_i' > 0$$

(8.23b)

In other words we conjecture that $\beta$ is built from the subset $\left( a_{11}, a_{21} \right)$ or $\left( a_{31}, a_{41} \right)$ of the odd roots in a way similar (to some extent) to the way roots are built from the simple roots. For $N = 1$ there are only two possible operators given by $k_5 = k_5 - 1$ (otherwise $\beta$ is just a root).

$$\beta = \sum_{i=1}^{N+1} \left( a_{11} + a_{21} \right), \quad (8.24a)$$

$$\beta = \sum_{i=1}^{N+1} \left( a_{31} + a_{41} \right), \quad (8.24b)$$

In Appendix D we prove that (8.24) indeed give rise to invariant differential operators. The operators are derived for arbitrary $N$ (use only $\beta = a_2 + a_4$ in (8.24b)) and are given in (8.31, 8.5) respectively. Their action on the signatures (for which we take into account (8.31, 8.4) resp.) is

$$f(\theta, \phi) : X = \left( \theta, \phi, \gamma, \delta, \xi, \eta, \zeta \right) \mapsto X' = \left( \theta', \phi', \gamma', \delta', \xi', \eta', \zeta' \right)$$

(8.25a)

$$f(\theta, \phi) : Y = \left( \theta, \phi, \gamma, \delta, \xi, \eta, \zeta \right) \mapsto Y' = \left( \theta', \phi', \gamma', \delta', \xi', \eta', \zeta' \right)$$

(8.25b)

(8 = \frac{\lambda (z, z)}{2N} ). Elsewhere [31] we shall prove that (8.25) are really the only possible higher order invariant differential operators for $N = 1$.

Investigation of these operators for $N > 1$ is also in progress and in Appendix D we give some examples.

Finally we shall indicate how to show the actual reducibility of the ER for which at least one of the conditions (7.10) holds (referring also to (7.6) we recall that only their necessity is proven in general [13]).

The crucial step in the proof is that every ER, for which one of the condition (7.10) holds, appears in an infinite sequence of ER connected by
The operators $D_1^N$, $D_1^N$ and one operator $B_{(1,3)}$ (or $D_1^1$, $B_1^{(1,3)}$ and $B_{(2,4)}$), for $N=1$ (plus higher order operators for $N>1$).

The composition of any two consequent operators is zero: $A \circ B = 0$ because they appear only in the following combinations $A = B = 0$, or $A = D_1^N$, $B = B_{(1,3)}$ or $A = B_{(1,3)}$, $B = D_1^N$ (for the analogues with $A_1^1$, $D_1^1$, $B_{(2,4)}$). (Note that $A \circ A = 0$ is equivalent to their anticommutativity in tensor language.) The complete proof shall appear elsewhere [32].

9. THE MASSLESS UIR OF EXTENDED CONFORMAL SUPERSYMMETRY

The massless UIR of extended conformal supersymmetry are representations which remain irreducible when $\mathfrak{g}$ is restricted to the extended super-Poincaré (SP) subalgebra. They have the same field content as the resulting massless UIR of SP. We realize the massless representations of $\mathfrak{g}$ as invariant subspaces of the ER spaces characterized by the signatures (here $\chi = \{j_1, j_2; d = c_2 = N, z; \chi_1, \ldots, \chi_{N-1}\}$):

\[
\chi_+ \oplus \chi_- \equiv \left[ 1, 0; \frac{4s-1}{2}, N; 0, \ldots, 0 \right] \oplus \left[ 0, 1; \frac{4s-1}{2}, 1; 0, \ldots, 0 \right],
\]

\[
\chi'_+ \oplus \chi'_- \equiv \left[ 0, 0; \frac{4s-1}{2}, 0; 0, \ldots, 0 \right] \oplus \left[ 0, 1; \frac{4s-1}{2}, -1; 0, \ldots, 0 \right].
\]  

Here $s = 0, \frac{1}{2}, 1, \ldots; n$ is an integer, $N/2 < n < N$, $\chi_+ = \chi$ for $s=0$ and $\chi'_+ = \chi'_-$ for $n = N/2$. $C_{\chi_+}$ can be identified with the complex conjugation of $C_{\chi_+}$, i.e., if $g \in C_{\chi_+}$, $g(x, \bar{z}) = r(x, \bar{z})$, $r \in C_{\chi_+}$ while any $g \in C_{\chi'_+}$, $n \neq N/2$ can be identified with some $r'$, $r' \in C_{\chi'_+}$ as in (8.19). For $n = N/2$, even, such identification provides a reality condition which in our realization reads (cf.(4.20) and (8.19))

\[
(\mathcal{D}_l^T)(u) = \overline{f(u)}, \quad r(u) = \overline{f(u)}.
\]

The ER in (9.1) provide examples of unitarizable representations of $\mathfrak{g}$ [32] (see [33] for $N=1$).

For $\mathfrak{e} \in \mathfrak{e}_{\chi}$, $\mathfrak{e} \in \mathfrak{e}'_{\chi}$ the differential eqs. (cf.(8.12), (8.17), (D.2-5)

\[
\mathcal{D}_l^T = 0 = \mathcal{D}_l^S, \quad t \neq 0;
\]

\[
\mathcal{D}_l^T = 0 = \mathcal{D}_l^S, \quad t \neq 0,
\]

(9.3a)

(9.3b)

single out an invariant subspaces for any $C_{\chi_+}$ or $C_{\chi'_+}$ which is represented by a supermultiplet containing maximal spin $s_{min} = \frac{N}{2}$ (spin refers here to $t_2 = t_1 + t_2$; the helicity 1, $t_2 = t_1 - t_2$). When reduced to the even subalgebra $\mathfrak{g}_{(0)} = \mathfrak{su}(2) \oplus \mathfrak{su}(1) \oplus \mathfrak{su}(N)$, the corresponding subrepresentation split into a direct sum of irreducible representations of $\mathfrak{g}_{(0)}$ given by

\[
\chi_+ = \bigoplus_{m=0}^{N/2} \left[ 1, 0; \frac{4s-1}{2}, \frac{N}{2}; 0, \ldots, 0 \right] \oplus \left[ 0, 1; \frac{4s-1}{2}, 1; 0, \ldots, 0 \right],
\]

\[
\chi'_+ = \bigoplus_{m=0}^{N/2} \left[ 0, 1; \frac{4s-1}{2}, 0; 0, \ldots, 0 \right] \oplus \left[ 0, 1; \frac{4s-1}{2}, -1; 0, \ldots, 0 \right].
\]  

As mentioned above (9.3a) are the chirality (or antichirality) conditions. The eqns. (9.3b) are an index-less version of the eqns. of motion [31] satisfied by the corresponding massless on-shell fields in Poincaré (linear) supersymmetry. Their conformal invariance is assured here since the signatures $\chi_+$ and $\chi'_+$ satisfy all the relevant conditions in (7.10). For $n=0$ the operators $\mathcal{D}_l^T$ and $\mathcal{D}_l^S$ act trivially while the second order operators in (9.2b) are needed to extract the massless subrepresentations.
Similarly the signatures in (9.1b) satisfy both (7.10.4), k=1 and
(7.10.2), k=N. The corresponding differential operators define invariant
subspaces of $\chi^L_n$ and $\chi^R_n$.

\[
\mathcal{D}^L f = 0 = \mathcal{D}^R g, \quad f \in \chi^L_n, \quad g \in \chi^R_n
\]  

represents a supermultiplet containing max spin $\ell_{\text{max}} = \frac{n}{2}$ for $n = \frac{N}{2}$

and the maximally extended theories are recovered if the reality condition
(9.2) is improved.

The $\mathfrak{su}(2,2) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(N)$ content in this case is

\[
\chi^L_n = \bigoplus_{l=0}^{\ell_{\text{max}}} \left[ l \frac{1}{2}, 0 \right] \mathcal{O} \left[ l - l \frac{1}{2}, -l - l \frac{1}{2} \right] \oplus \left[ 0, 0, 0, 0, 0 \right] \oplus (9.6)
\]

and the corresponding formula for $\chi^R_n$ obtained by replacing
\[(J_1, J_2, \ldots, J_N) \rightarrow (J_1, J_2, \ldots, J_N) \rightarrow (J_1, J_2, \ldots, J_N) . \]

The massless representations of $\mathfrak{g}$ have been realized earlier [34]
directly as irreducible reps. The particular multiplet (9.6) for

\(n=4, N=8\) appeared also in [35].

As is clear from (9.1) the case $N=4$ is exceptional since only
the representation $\chi^L_n$, $n=2$ survives if the (as has been stressed in [34])

factorized algebra $\mathfrak{g}/\mathfrak{u}(1)$ is considered. Accordingly, the only massless
unitary subrepresentation of $\mathfrak{g}/\mathfrak{u}(1)$ present is that (if (9.6)) describing
the on-shell YM field strength (n=2). In particular, the maximal spin
\(\ell_{\text{max}} = 2 \ (n=4)\) multiplet which has the same field content as the on-shell
multiplet relevant for the Poincaré supergravity requires the full $\mathfrak{g}$.

However the use of this representation as well as of the rest $n=4$

massless representations in a conformally invariant framework is rather
doubtful. Note that the on-shell subrepresentations describing maximal
spin 2 and relevant for the (linearized) Weyl conformal supergravity

\[\text{(see [3]) are not unitary reps. of } \mathfrak{g} \text{ and the corresponding imbedding}
\]

\[\mathcal{M}_{\text{max}} \text{ characterized (for } N=4 \text{) by the signatures}
\]

\[
\left[ \frac{q_k}{2}, 0; \frac{q_k}{2} \right] + \left[ \frac{q_k}{2}, 1, 1, \cdots, 1 \right] \oplus \left[ 0, \frac{q_k}{2}, \frac{q_k}{2}, 1, 1, \cdots, 1 \right] \ (9.7)
\]

are beyond the series (9.1). (Clearly (9.7) allows a factorization of
$\mathfrak{u}(1)$ for $N=4$).

Finally we note that the massless unitary representations
realized here on invariant subspaces of elementary representations
actually split into a direct sum of positive and negative
energy UIR.

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APPENDIX A

A.1. LIE SUPERGROUPS AND REPRESENTATION SPACES

We sketch here the basic notions involved in the definition of Lie supergroups and their representations in a form most convenient to us (for various equivalent approaches to supermanifolds see [16-20]).

Let \( \{ q_j, j = 1, \ldots, q \} \) be the set of generators of a (complex) Grassmann algebra \( A_q \) and \( U \) an open region in \( \mathbb{R}^P \). Denote by \( C(U, A_q) \) the commutative superalgebra of functions (pointwise multiplication assumed)

\[
\{ f : U \to A_q, \; f = \sum_{i_0} f_{i_0} \cdots i_{q_j} q_j \cdots i_{q_j-1} q_j \cdots i_{i_0} \in C(U, A_q) \} \tag{A.1}
\]

isomorphic to \( C^*(U, A_q) \). Let \( \Lambda \) be the \( \infty \)-dimensional Grassmann algebra considered in Section 3. and \( U \) a superregion in \( \Lambda(P, q) \) over \( U \), i.e. \( U = \{ (x, \xi) \in \Lambda(P, q) \; | \; x_0 = x - n(x) \notin U \} \); here \( n_1(x) \) is the nilpotent part of \( x \) in \( \Lambda_q(0) \). By Berezin’s Grassmann analytic continuation any \( f_{i_0} \cdots i_{q_j} \in C(U, \Lambda) \) in (A.1) can be extended uniquely to a \( \Lambda(0) \)-valued function \( \hat{f}_{i_0} \cdots i_{q_j} \) on \( U \) defined by the (finite) Taylor expansion

\[
\hat{f}(x_0, \xi(x, \xi)) = \sum_n (x)^k (k_1)_{k_1} \cdots (k_p)_{k_p} f(x_0) : k = (k_1, \ldots, k_p) \text{ is a multiindex}. \tag{A.2}
\]

analogously, any \( f_{i_0} \) can be extended to a function \( \hat{f}_{i_0} : U \to (x, \xi) \to \hat{f}(x_0, \xi(x, \xi)) = x_{i_0} \in \Lambda(1) : (x_{i_0} \hat{f}_{i_0})(x, \xi) = x_{i_0} \hat{f}(x_0, \xi(x, \xi)) \). Then to any \( f = C^*(U, A_q) \) there corresponds

\[
\hat{f} : U \to \Lambda, \quad \hat{f}(x, \xi) = \sum_{i_0} f_{i_0} \cdots i_{q_j} q_j \cdots i_{q_j-1} q_j \cdots i_{i_0} \in \Lambda \tag{A.3}
\]

The resulting superalgebra (to be denoted by \( C^*(M, \Lambda) \) or \( C^*(U, A_q) \)) isomorphic to \( C^*(U, A_q) \) is a subsuperalgebra of the commutative superalgebra of all maps \( U \to \Lambda \). The superregion \( U \) is the simplest example of a (p,q) - dimensional supermanifold; \( U \) is called the underlying region of the superregion. In particular \( X = (4,4) \) in (3.5) is a (4,4) - dimensional supermanifold.

In general a supermanifold \( M \) and the superalgebra of smooth superfunctions \( C^*(M, \Lambda) \) can be defined by a standard procedure [18-20]. The essential point is that the spaces \( C^*(U) \) provide a definition of a smooth map of superregions in \( \Lambda(P, q) \) : \( \varphi : U \to U' \) is smooth if for \( \forall \psi \in C^*(U') \) \( \varphi \psi \in C^*(U) \). (The topology on superregions is defined by introducing open superregions \( U' \subset U \) as an open subsuperregion iff it is a superregion itself.) The isomorphism \( C^*(M) = C^*(U, A_q) \) assures the equivalence of the resulting theory of supermanifolds with that of Berezin-Leites [16] (cf. [18, 20]).

With the help of \( C^*(M) \) supersmooth maps of supermanifolds are defined as above. The requirement of supersmoothness of the "multiplication map" and the "inverse element map" turns a supermanifold which is at the same time a group into a Lie supergroup. An important theorem [16, 20] states that if \( G \) is a (p,q) - dimensional Lie supergroup with an underlying p-dimensional Lie group \( G \), then \( C^*(G) = C^*(G) \otimes A_q \) thus extending the isomorphism \( C^*(U) = C^*(U, A_q) \). The latter implies also that all operations on \( f \) in (A.1) extend to the corresponding \( \hat{f} \) in (A.2), e.g.,

\[
\begin{align*}
\hat{f} \cdot \hat{g} & = \sum_{i_0} f_{i_0} \cdots i_{q_j} q_j \cdots i_{q_j-1} q_j \cdots i_{i_0} \cdot \sum_{j_0} g_{j_0} \cdots j_{q_j} q_j \cdots j_{q_j-1} q_j \cdots j_{j_0} \\
\hat{f}^{-1} & = \sum_{i_0} f_{i_0}^{*} \cdots i_{q_j} q_j \cdots i_{q_j-1} q_j \cdots i_{i_0} \in \Lambda
\end{align*}
\]

The representations of a superalgebra \( \varphi \) can be defined in \( C^*(G) \).
(or in our case on $C^\infty(G \times \mathcal{L})$, where $\mathcal{L} = \mathbb{L}(2,0)_{\mathbb{R}}$, cf. Sec. 4.).

To define a representation of the Lie supergroup $G$ however we shall need the Grassmann envelope [17, 20]

$$C^\infty(\mathcal{L})(\Lambda)_0 \equiv \left\{ \mathcal{F} = \sum_{i=1}^m \lambda^i \cdot \mathcal{F}^i \left| \lambda^i \in \Lambda_0, \mathcal{F}^i \in C^\infty(G_0) \right. \right\} \quad (A.4)$$

(the usual multiplication in $\Lambda$ is meant). Formally any $\mathcal{F}$ of the type (A.4) admit the same expansion as the elements of (A.2), e.g. for $\mathcal{F} \in C^\infty(\mathbb{A}^{n/2} \otimes \mathbb{R})(\Lambda)_0$  

$$F(x, \mathcal{L}) = \sum_{i=1}^n F_{x_1 \ldots x_n}(x) \zeta_{x_1} \ldots \zeta_{x_n} \in \Lambda_0$$

however the coefficients of $F_{x_1 \ldots x_n}(x)$ around $x_0 \in \mathbb{R}^n$ are no more $\mathcal{C}$-valued but $\Lambda_{(a)}$-valued, (a) depending on i according to (A.4). We shall refer to both (A.2) and (A.5) as superfields although one should keep in mind the entirely different nature of the two spaces.

The left action of $G$ in $C^\infty(G)(\Lambda)_0$ is defined according to

$$T(g)F(g') = F(g^{-1}g') = \sum_{i=1}^n \lambda^i \hat{F}(g^{-1}g') \left( = \sum_{i=1}^n \lambda^i \hat{F}(g') \right)$$

(A.6)

A generalization of (A.4) is provided by the space $C^\infty(G_1)(\Lambda)$ obtained as in (A.4) with (a) omitted. The action of the Lie superalgebra $\mathfrak{g}_{1/2}$ corresponding to (A.6) leaves $C^\infty(G)(\Lambda)$ (or $C^\infty(G) \subset C^\infty(G_1)(\Lambda)$) invariant.
APPENDIX B

EVALUATION OF THE SECOND ORDER CASIMIR OPERATOR

We start with the expression for \( C_2 \) in the Cartan-Weyl basis adapted from the semisimple case

\[
C_2 = \sum_{\ell \neq j} \frac{1}{k(\ell^*)} \frac{1}{k(j^*)} + \sum_{\ell \neq j} \left[ (\varepsilon_{\ell^*}, \varepsilon_{j^*}) \varepsilon_{\ell^*} + (\varepsilon_{\ell^*}, \varepsilon_{j^*}) \varepsilon_{\ell^*} \right], \quad N \neq 0
\]

(3.1)

where \( A_{ij} = (h_i, h_j) \) (cf. (2.5)), \( e \) are the root space vectors \( X^i \) in (2.8).

We introduce the canonical generators \( h_0 \) also for the nonsimple roots by \( h_0 (e, e) = [e, e] \); note \( |(e, e)| = 1 \) in our basis.

Computed on the lowest weight vector \( \psi, e^N \psi = 0 \) (B.1) gives (the basis (2.1c) is convenient).

\[
C_2 \psi = \left\{ \left( \frac{1}{k(\ell^*)} \frac{1}{k(j^*)} - 2 \left( \frac{1}{k(\ell^*)} \frac{1}{k(j^*)} - \frac{1}{2} \right) \sum_{\ell \neq j} \varepsilon_{\ell^*} - \frac{1}{2} \sum_{\ell \neq j} \varepsilon_{\ell^*} \right) \right\} \psi = \psi
\]

(8.2a)

\[
= \left[ \frac{1}{k(\ell^*)} \frac{1}{k(j^*)} - 2 \left( \frac{1}{k(\ell^*)} \frac{1}{k(j^*)} - \frac{1}{2} \right) \sum_{\ell \neq j} \varepsilon_{\ell^*} - \frac{1}{2} \sum_{\ell \neq j} \varepsilon_{\ell^*} \right] \psi
\]

(8.2b)

where \( \psi \) is the lowest weight vector of the LWM \( \Lambda \) in consideration we obtain that \( \psi^l = -\psi^l \) and using \( h_0 \psi = 0 \) (cf. (7.3), (7.7))

\[
\psi = \left( \frac{1}{k(\ell^*)} \frac{1}{k(j^*)} \right) \psi = \psi
\]

(8.3)

\[
\psi = \left( \frac{1}{k(\ell^*)} \frac{1}{k(j^*)} \right) \psi = \psi
\]

(8.4)

is the SU(N) Casimir operator eigenvalue. Recalling from (7.11) that \( \psi = \frac{1}{2} \frac{z}{N} \psi \)

we see that the final expression is valid also for \( N=4 \) (the term with \( z^2 \) disappears).

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APPENDIX C

DERIVATION OF THE COVARIANT DERIVATIVES CORRESPONDING TO NONSIMPLE ODD ROOTS

We start with the derivation of the operators \( \mathcal{D}_j \) corresponding to the roots \( \alpha, \beta \in N+4, N+5-\frac{j}{N+1} \), \( j = 1, ..., N-1 \). We denote for fixed \( k \), \( i \in k N+1 \) the homogeneous polynomial (8.1) of degree one in each of \( X^i_X^i, X^i_X^i, ..., X^i_X^i \) by \( \gamma_k \). Recalling commutation relations between the \( X^i_X^i \) we immediately can write

\[
\gamma_k = \frac{1}{k} \frac{1}{k} \gamma_k^i \gamma_k^i + \frac{1}{k} X^i \gamma_k^i \gamma_k^i
\]

(8.1)

where \( \gamma_k^i, \gamma_k^i \) are(a priori) different polynomials in \( X^i_X^i, X^i_X^i, ..., X^i_X^i \)

Finally using (7.3) we obtain

\[
X^i \gamma_k^i \gamma_k^i = \left( \frac{1}{k} \right) \gamma_k^i \gamma_k^i + \frac{1}{k} X^i \gamma_k^i \gamma_k^i
\]

(8.2)

\[
X^i \gamma_k^i \gamma_k^i = \left( \frac{1}{k} \right) \gamma_k^i \gamma_k^i + \frac{1}{k} X^i \gamma_k^i \gamma_k^i
\]

(8.3)

\[
X^i \gamma_k^i \gamma_k^i = \left( \frac{1}{k} \right) \gamma_k^i \gamma_k^i + \frac{1}{k} X^i \gamma_k^i \gamma_k^i
\]

(8.4)

we obtain

\[
\gamma_k^i \gamma_k^i X^i \gamma_k^i \gamma_k^i = \gamma_k^i \gamma_k^i X^i \gamma_k^i \gamma_k^i
\]

(8.5)

Analogously applying

\[
X^i \gamma_k^i \gamma_k^i = \left( \frac{1}{k} \right) \gamma_k^i \gamma_k^i + \frac{1}{k} X^i \gamma_k^i \gamma_k^i
\]

(8.6)

\[
X^i \gamma_k^i \gamma_k^i = \left( \frac{1}{k} \right) \gamma_k^i \gamma_k^i + \frac{1}{k} X^i \gamma_k^i \gamma_k^i
\]

(8.7)

we obtain

\[
\gamma_k^i \gamma_k^i X^i \gamma_k^i \gamma_k^i = \gamma_k^i \gamma_k^i X^i \gamma_k^i \gamma_k^i
\]

(8.8)

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where $\mathcal{G}_k^1$ is a homogeneous polynomial in $X_3, X_5^*, \ldots, X_4^{*k}$ ($\mathcal{G}_k^0 = X_3^*$).

The only remaining equation to be satisfied is $X_3^*T^{-k}v_0 = 0$ which reproduces condition (7.10.2) for our fixed $L$. It is easy to solve relations (C.5) and recalling that $\mathcal{G}_k^k$ corresponds to the operator $\mathcal{A}^{N-k}$ and using the explicit expressions for $X_3 = x_3^{*25}$ in (8.1a) and for $X_4^{*k}$ in (8.4b) we obtain (reproducing also (8.12a))

$$D_k = \sum_{j=1}^{N-k} \sum_{i=1}^{p} \sum_{j_1, j_2, \ldots, j_p} \frac{N!}{(j_1-1)!(j_2-1)\cdots!(j_p-1)!} \left( L_{i_1-1} \Delta_{i_2-1} \cdots \Delta_{i_p-1} + \tau_{i_1} \Delta_{i_2} \cdots \Delta_{i_p} \right).$$

(C.6)

These operators contain $2^{N-k}$ terms for $k \neq N$ each term with fixed $p_1, j_1 \neq N, j_2, \ldots, j_p$ is combined with the term $p_1 = p_1, j_1 = N, \ldots, j_p = j_p$ and after some play with the $\psi$ symbols is brought to the form (8.13b) or equivalently (8.13a) (cf. (4.14b)).

For the derivation of the operators $\mathcal{D}^j$ (corresponding to the roots $\lambda, N+5-j = N+4+1 \ldots, N+1$, $j = 1, \ldots, N$) we denote for fixed $K, l, K \neq N$, the homogeneous polynomial (8.1) of degree one in each of $X_1^*, X_3^*, X_5^*, \ldots, X_4^{*k}$ by $\mathcal{G}_k^k$ and we immediately see that

$$\mathcal{G}_k^k = 2^{j_1} X_1^* \mathcal{G}_k^k - (2^{j_1+1}) \mathcal{G}_k^k X_4^*.$$  

(C.7)

where $\mathcal{G}_k^k$ is the polynomial constructed above. Proceeding as in (8.8) we obtain that the operators $\mathcal{D}^j$ are obtained from $\mathcal{D}_j$ by the replacement $Dz = D_e$.

The derivation of the operators $\mathcal{D}^j$ and $\mathcal{D}^j$ proceeds in complete analogy to the derivation of $\mathcal{D}^j$ and $\mathcal{D}^j$. It is even simpler since (8.14b) is obtained directly being analogous to the intermediate form (C.6) for $\mathcal{D}^j$.

Remark C.1 Relations (C.5) were derived using only the generators $X_3^{*k}$ of (the complexification of) the SU(N) subalgebra which is contained also in the extended Poincaré superalgebra. (For (C.7) $X_3^*$ from the $\mathfrak{sl}(2,1)$ subalgebra is also used.) Thus formulae (C.6), (8.13) and (8.14) are valid also for Poincaré supersymmetry as they should. Naturally constraints (7.10) are not meaningful in the Poincare framework as are the odd root space condition $X_3^*T^{-k}v_0 = 0$ (or its equivalent with $X_4^*$ for $\mathcal{D}^k, 2^{k}K$ from which they are derived.

Since formulae (8.13) and (8.14) may look complicated we write down explicitly the simplest ones including all those occurring for $N \leq 4$:
We remind that \((C.9b,c,d)\) for \(N=2,3,4\) respectively sum up to the simpler expression (8.16).

\[
\text{APPENDIX D}
\]

\text{HIGHER ORDER INVARIANT DIFFERENTIAL OPERATOR CORRESPONDING TO NON-ROOT ELEMENTS OF THE POSITIVE ROOT LATTICE}

We start with the operators corresponding to \(B = \chi_4 + 2 \chi_3\) in (8.21a). Obviously the only possible polynomial in \(X_1, X_3^*\) of degree 1 and 2 respectively is

\[
\mathcal{D}_{(1,1)}(X_1^*, X_3^*) = X_1^* X_3^* X_4^* .
\]

It is easy to see (using (7.10.2) with \(k=N\) and (8.4)) that \(T_0\) (where \(v_0\) is the LWV of the LWL \(A\) in consideration) is a singular vector iff

\[
\hat{d}_4 = 0, \quad \Lambda(k_3) - 1 = (\Lambda, \omega_{\lambda(k)}) - 1 = 0 \implies \hat{c} - 1 + 2 - R_{NN} = 0 .
\]

The explicit form of the corresponding invariant operator (acting on \(f(x, z, \bar{z}, u)\)) is

\[
D^1 = \mathcal{D}_{(1,1)}(X_1^*, X_3^*) = (u^D \delta z) (u^\lambda \delta z) (u^\kappa \delta z) = (u^D \delta z) (u^\lambda \delta z) .
\]

Analogously the operator corresponding to \(B = \chi_2^* + 2 \chi_4\) with \(\mathcal{D}_{(2,4)} = X_4^* x_2^* y_4^*\) is invariant iff

\[
\hat{d}_4 = 0, \quad \Lambda(\kappa_4) - 1 = (\Lambda, \omega_{\lambda(\kappa)}) - 1 = 0 \implies c - 1 + 2 - R_{NN} = 0 .
\]

the explicit expression being

\[
\mathcal{D}^1 = \mathcal{D}_{(2,4)}(X_1^*, X_3^*) = (u^D \delta z) (u^\lambda \delta z) (u^\kappa \delta z) = -(u^D \delta z) (u^\lambda \delta z) = \frac{1}{2} u^D \delta z .
\]
Further we give two operators for \( N \geq 2 \) presenting the elements \( \beta \), the corresponding polynomial in \( \chi_{k+} \), the conditions on the signature, the changes of the signature and the function space realization:

\[
\beta = k_1 + k_2 = 2 \ell_1 + \ell_2, \quad \mathcal{D}(\chi_{k+}^*, \chi_{k-}^*) = \chi_{k+}^* \chi_{k-}^*, \quad (D.6a)
\]

\[
\beta_{N+} = 0, \quad \Lambda(\beta) = -1 \Rightarrow \ell_1 + \ell_2 + 2 - K_{WW} = 0. \quad (D.6b)
\]

\[
\mathcal{D}(\chi_{k+}^*, \chi_{k-}^*) = (\tilde{\alpha} \tilde{\alpha} D\tilde{\alpha}^*) D_{N-1}^{[N]}(\tilde{\alpha} \tilde{\alpha} D\tilde{\alpha}^*) - (\tilde{\alpha} \tilde{\alpha} D\tilde{\alpha}^*) (\tilde{\alpha} \tilde{\alpha} D\tilde{\alpha}^*) = (N \geq 2) \quad (D.6c)
\]

\[
\mathcal{D}(\chi_{k+}^*, \chi_{k-}^*) = \left\{ \begin{array}{l}
(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_8, \ell_9, \ell_{10}) \\
(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_8, \ell_9, \ell_{10}) \\
(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_8, \ell_9, \ell_{10})
\end{array} \right. \quad (N \geq 2). \quad (D.6d)
\]

where we have used (4.10b).

\[
\beta^\mu = \sum_{k=1}^{K^+} k_1 + k_2 = N \chi_{k+} + 2 \chi_{k-} + 2 \sum_{k=1}^{K^+} k_1 (N + 1 - k) \quad (D.7a)
\]

\[
\mathcal{D}^\mu = \prod_{k=1}^{K^+} (\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7 \ell_8 \ell_9 \ell_{10}) \quad (D.7b)
\]

\[
\Lambda(\beta) = 2 - N, \quad \ell_1 + \ell_2 = - \sum_{k=1}^{K^+} k_1 = 0 \Rightarrow \ell_1 + N - 2 = 0. \quad (D.7c)
\]

\[
\beta^\mu : \gamma = (0, 0, 0, 1, 1, 1) \rightarrow \chi_k \quad (D.7d)
\]

\[
\mathcal{D}^\mu = \sum_{k=1}^{K^+} (\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7 \ell_8 \ell_9 \ell_{10})(\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7 \ell_8 \ell_9 \ell_{10})(\ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7 \ell_8 \ell_9 \ell_{10}) \quad (D.7e)
\]

where \( (D.7e) = (D.7e) \). The above operator is maximal in the sense that it is obtained from (8.20a) with \( K_1 = K_1' = 1 \) for all \( \ell_1 \).

The operators (D.6,7) come from (8.20a). The reader can easily obtain their counterparts coming from (8.20b) with the obvious substitutions.

REFERENCES

A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, 

(1978) 507.


in Russian).


[17] F. Berezin, Lie supergroups, Moscow preprint ITEP-78 (1977);
Introduction to the Algebra and Analysis with Anticommuting 
Variables (Moscow Univ. Press, 1983, in Russian).


[20] V.V. Molotkov, Talk at the XIII International Conference on 
Differential-Geometric Methods in Theoretical Physics, Shumen 
(1984), to be published in the Proceedings, Eds. H.D. Doebner 
and T.D. Palev; ICTP Trieste preprint IC/84/183 (1984) and in 
preparation;

V. Rittenberg, in Lecture Notes in Physics, Vol. 79 (Springer Verlag, 
Berlin, 1978) 3;


[23] I.M. Gel'fand and N.Ya. Vilenkin, Generalized Functions, 


(1976) 188.


(Nauka, Moscow, 1974, in Russian).