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CHIRAL FAMILIES AND STABLE COMPACTIFICATIONS

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ABSTRACT

In the context of Einstein-Yang-Mills theories, we show that for eight-dimensional $(d = 8)$ space-times, the embedding of an SU(2)-instanton in an external gauge-group K, leads to a stable compactification and the emergence of chiral families of fermions, as well as massless scalars.

These latter are shown to arise for $K = SO(N+4)$ when the SU(2)-instanton is embedded in the 3×3 upper left-corner of the $(4+N) \times (4+N)$ orthogonal matrices for K.

For a class of $d = 4+p$ EYM systems where the spin connection of the internal space M^p is embedded in the gauge group (K) larger than the tangent space group of the internal space and is used to induce compactification we get instability.

I. INTRODUCTION

In this note we would like to report on some results about families of chiral fermions as well as perturbative stability properties of compactification of higher dimensional Einstein Yang-Mills (EYM) systems. More specifically we show:

(1) If for $d = 8$, an $SU(2)$-instanton is embedded in an external gauge group $K \supset SU(2)$, we obtain stable compactification. We show further that such compactifications can lead to chiral families of fermions.

(2) For a class of $d = 4+p$ EYM systems where the spin connection of the internal space M^p is embedded in a gauge group (K) larger than the tangent space group of the internal space and is used to induce compactification we get instability. This instability exists for any internal space which is an Einstein manifold and any number of dimensions. *)

An interesting outcome of our analysis is that if we take $K = SO(N+4)$ and embed the 1-instanton solution as the 3×3 upper left corner of the $(4+N) \times (4+N)$ orthogonal matrices for K, then in the spectrum of the small oscillations we encounter a multiplet of massless spin-0 states. The appearance of such modes in the low energy sector of the theory should be welcome because they can in principle induce a spontaneous symmetry breaking of the final symmetry $SO(N+1) \times SO(5)$, where the $SO(5)$ arises from the Kaluza-Klein side.

II. THE STABILITY OF THE INSTANTON-INDUCED COMPACTIFICATION IN $d = 8$

Consider the d-dimensional action integral **)

$$S = \int d^4z\, (-g)^{1/2} \left[-\frac{1}{\kappa^2} R - \frac{1}{4} \vec{F}^2 - \lambda \right] , \qquad (1)$$

where κ^2 and λ are constants while $\vec{F}$ is the YM field strength of an external gauge group K. If we require the field equations derived from (1) to admit a solution of the form $(\text{Minkowski})_4 \times B_p$ for any B_p, we obtain [3]

$$R_{AB} = -\frac{\kappa^2}{2} \vec{F}_{AC} \cdot \vec{F}_{BC} , \quad \frac{1}{\kappa^2} R + \frac{1}{4} \vec{F}^2 + \lambda = 0, \quad \nabla_A \vec{F}_{AC} = 0 \qquad (2)$$

*) Unfortunately, this class includes some of the solutions (with desirable chiral properties) suggested by Witten [1] for $d = 10$.

**) Our notation and conventions are the same as in Ref.(2). Throughout we use an orthonormal frame relative to which $g_{AB} = \text{diag}(-1,+1,\ldots+1)$.

To check the classical stability of a solution of Eq.(2) we investigate the spectrum of the small perturbations through the substitutions

$$g_{MN} \to g_{MN} + \kappa\, h_{MN} , \quad \vec{A}_M \to \vec{A}_M + \vec{V}_M . \qquad (3)$$

As usual we retain in the action integral bilinears in h and V. In the light cone gauge we impose the conditions

$$h_{-A} = 0 , \quad \vec{V}_- = 0 , \qquad (4)$$

where $(-,+,i,\alpha)$; $i = 1,2, \alpha = 1 \ldots p$, denote the light cone indices [2]. In this gauge the fields with a $+$ index turn out to have algebraic field equations. They can be eliminated in terms of a set of independently propagating fields, after which we obtain the following bilinear action:

$$\begin{aligned}
\mathcal{L}_2 = {} & \frac{1}{4} h^t_{ij} (\partial^2 + \nabla^2) h^t_{ij} + \\
& + \frac{1}{2} \vec{V}_i (\partial^2 + \nabla^2) \vec{V}_i + \frac{1}{2} h_{i\alpha} (\partial^2 + \nabla^2) h_{i\alpha} + \\
& + \frac{1}{2} R_{\alpha\beta} h_{\alpha i} h_{\beta i} - \kappa \vec{F}_{\alpha\beta} \cdot \nabla_\beta \vec{V}_i h_{\alpha i} + \\
& + \frac{1}{2} \vec{V}_\alpha (\partial^2 + \nabla^2) \vec{V}_\alpha + \frac{1}{2} R_{\alpha\beta} \vec{V}_\alpha \cdot \vec{V}_\beta \\
& - \vec{F}_{\alpha\beta} \cdot (\vec{V}_\alpha \times \vec{V}_\beta) - \frac{\kappa^2}{2} (\vec{F}_{\alpha\beta} \cdot \vec{V}_\beta)^2 + \\
& + \frac{1}{4} h^t_{\alpha\beta} (\partial^2 + \nabla^2) h^t_{\alpha\beta} + \frac{2+d}{8d} h_{ii} \left(\partial^2 + \nabla^2 + \frac{4R}{d(2+d)}\right) h_{jj} \\
& + \frac{1}{2} R_{\alpha\beta} h^t_{\alpha\gamma} h^t_{\beta\gamma} - \frac{1}{d} R_{\alpha\beta} h^t_{\alpha\beta} h_{ii} \\
& + \frac{1}{2} \left(R_{\alpha\gamma\beta\delta} - \frac{\kappa^2}{2} \vec{F}_{\alpha\gamma} \cdot \vec{F}_{\beta\delta} \right) h^t_{\alpha\beta} h^t_{\gamma\delta} \\
& - \kappa \nabla_\gamma \vec{F}_{\alpha\beta} \cdot \vec{V}_\beta h^t_{\alpha\gamma} - \kappa \vec{F}_{\alpha\beta} \cdot \nabla_\beta \vec{V}_\gamma h^t_{\alpha\gamma} + \frac{\kappa}{d} \vec{F}_{\alpha\beta} \cdot \nabla_\beta \vec{V}_\alpha h_{ii} ,
\end{aligned} \qquad (5)$$

where ∇_α contains both the Yang-Mills and the Riemannian background connections and $\nabla^2 = \delta^{\alpha\beta} \nabla_\alpha \nabla_\beta$. We also have

$$h^t_{ii} = 0 = h^t_{\alpha\alpha} .$$

The expression (5) is completely general. It is applicable to any background with an arbitrary internal space B_p. Here we shall apply it to two special classes of compactifying solutions, whose physical relevance arises on account of their potential usefulness in inducing chirality of the 4-dimensional fermions. These are;

1) instanton induced compactification [3] in $d = 8$;

2) spin-connection induced compactification in $d = 4+p$ into $M_4 \times B_p$ for any Einstein manifold B_p.

For case (1) we have considered two different embeddings of the SU(2) instanton into the external gauge group K.

To clarify this we shall take K to be $O(4+N)$ for some integer N and denote by $T_{\hat{r}\hat{s}} = -T_{\hat{s}\hat{r}}$, $\hat{r} = 1,\dots 4+N$, a basis of the Lie algebra of K. Divide $\hat{r}$ into two non-overlapping ranges $\bar{r} = 1,\dots,4$ and $r = 5,\dots,N$. Then $T_{\bar{r}\bar{s}}$ will denote the generators of an $O(4)$ subgroup of $O(4+N)$. This $O(4)$ subgroup consists of the 4×4 submatrices in the upper left corner of the $(4+N) \times (4+N)$ orthogonal matrices.

To illustrate the two different embeddings of the instanton in K decompose $T_{\bar{r}\bar{s}}$ into its self-dual and antiself-dual parts. This is done by using 't Hooft's η-symbols

$$S^I = \tfrac{1}{4} \eta^I_{\bar{r}\bar{s}} T_{\bar{r}\bar{s}} , \quad \bar{S}^I = \tfrac{1}{4} \bar{\eta}^I_{\bar{r}\bar{s}} T_{\bar{r}\bar{s}} , \qquad I = 1,2,3 \quad . \qquad (6)$$

S^I and $\bar{S}^I$, $I = 1,2,3$, generate the two commuting SU(2) subgroups of $O(4)$. The $O(3)$ subgroup of $O(4)$ under which $4 \to 3+1$, is generated by

$$M^I = S^I + \bar{S}^I \quad .$$

In this notation the Lie-algebra valued 2-form F may be written as

$$F^{\hat{r}\hat{s}} T_{\hat{r}\hat{s}} = F^I S^I + \bar{F}^I \bar{S}^I + 2F^{\bar{r}s} T_{\bar{r}s} + F^{rs} T_{rs} \quad . \qquad (7)$$

As part of our ansatz we shall assume

$$F^{\bar{r}s} = 0 \; ; \; F^{rs} = 0 \quad .$$

Now we can characterize the non-vanishing components of F on S^4 for the two different embeddings of SU(2) in K:

1a) we set $\bar{F}^I = 0$ and F^I = the SU(2) instanton on S^4,

1b) we set $F^I = \bar{F}^I$ = the SU(2) instanton on S^4.

In case 1a) the instanton lies in the direction of S^I and therefore the unbroken subgroup of $O(4+N)$ is $SU(2) \times O(N)$. Hence the final compact invariance subgroup of the background is $SO(5) \times SU(2) \times O(N)$. In the case 1b) the instanton is effectively embedded as a 3×3 submatrix in the upper left corner of the $(4+N) \times (4+N)$ orthogonal matrices. In this case the unbroken subgroup of $SO(4+N)$ is $SO(N+1)$ and the final compact invariance group of the background is $SO(5) \times SO(N+1)$. In either case Eq.(5) will decompose into a part which couples to $\vec{F}_{\alpha\beta}$ and a part which is orthogonal to $\vec{F}$. We have already shown that the part which mixes with F has no ghosts or tachyons [4]. Here we give the spectrum of the remaining sector.

Spin	$a^2(\text{mass})^2$	SO(5) x SU(2) x SO(N) quantum numbers	
0	$n(n+3) + 2$	$[(n,1) \; ; \; 3 \; ; \; 1]$	$n \geq 1$
0	$n(n+3) + \frac{9}{4}$	$[(n,\frac{3}{2}) \; ; \; 2 \; ; \; N]$	$n \geq \frac{3}{2}$
0	$n(n+3) - \frac{3}{4}$	$[(n,\frac{1}{2}) \; ; \; 2 \; ; \; N]$	$n \geq \frac{3}{2}$
0	$n(n+3) + 2$	$[(n,1) \; ; \; 1 \; ; \; \frac{N(N-1)}{2}]$	$n \geq 1$
1	$n(n+3)$	$[(n,0) \; ; \; 3 \; ; \; 1]$	$n \geq 0$
1	$n(n+3) - \frac{3}{4}$	$[(n,\frac{1}{2}) \; ; \; 2 \; ; \; N]$	$n \geq \frac{1}{2}$
1	$n(n+3)$	$[(n,0) \; ; \; 1 \; ; \; \frac{N(N-1)}{2}]$	$n \geq 0$

Table I(a) for the embedding (1a).

Spin	$a^2(\text{mass})^2$	SO(5) x SO(N+1) quantum numbers	
0	$n(n+3) + 2$	$[(n,2) \; ; \; N+1]$	$n \geq 2$
0	$n(n+3) - 2$	$[(n,1) \; ; \; N+1]$	$n \geq 2$
0	$n(n+3) + 2$	$[(n,1) \; ; \; \frac{N(N+1)}{2}]$	$n \geq 1$
0	$n(n+3) - 4$	$[(n,0) \; ; \; N+1]$	$n \geq 1$
1	$n(n+3) - 2$	$[(n,1) \; ; \; N+1]$	$n \geq 1$
1	$n(n+3)$	$[n,0) \; ; \; \frac{N(N+1)}{2}]$	$n \geq 0$

Table I(b) for the embedding (1b).

In both Tables (Ia) and (Ib) the SO(5) representations have been characterized by the Gel'fand-Zetlin numbers (n_1,n_2), while the SU(2) and SO(N) representations have been indicated by their dimensionalities.

In addition to the non-negativeness of all $(\text{mass})^2$ in Tables (Ia) and (Ib) it is interesting to notice the appearance of a massless spin zero multiplet. This occurs in the 4^{th} row of Table (Ib) for n = 1. Thus it is a vector of O(5) x O(N+1). The presence of such massless modes can be potentially useful in that they can induce a spontaneous symmetry breaking of both the O(5) and O(N+1) symmetries through the Coleman-Weinberg mechanism. Unfortunately these scalars do not have a Yukawa coupling to the massless fermions.

III. THE INSTABILITY OF THE SPIN-CONNECTION INDUCED COMPACTIFICATION

If B_p is an Einstein manifold such that $SO(p) \subseteq K$ then one can identify the spin connection of B_p with an SO(p) gauge field. The components of the gauge potential outside the image of SO(P) in K are set to zero.

With this ansatz, Eq.(2) reduces to $\nabla_\alpha R_{\alpha\beta\gamma\delta} = 0$. This is identically satisfied for any Einstein manifold B_p.

To exhibit the tachyonic mode we consider as an example the case K = SO(p+N) and embed the SO(p) in the upper left corner of the (p+N) x (p+N) orthogonal matrices. Hence the generators $T_{\hat{r}\hat{s}}$ of K decompose into three subsets $T_{\alpha\beta}$, $T_{\alpha r}$ and T_{rs} where $\alpha,\beta = 1,\ldots,p$ and $r = p,\ldots,p+N$. T_{rs} generate the unbroken SO(N). Next we concentrate on those V's in Eq(5) which lie in the direction of $T_{\alpha r}$. After some straightforward manipulation we observe that the relevant part of the Lagrangian is given by

$$\mathcal{L}' = V_\alpha^{[\beta s]} (\partial^2+\nabla^2+\Lambda) V_\alpha^{[\beta s]} + 2V_\alpha^{[\gamma s]} [\nabla_\alpha,\nabla_\beta] V_\beta^{[\gamma s]} \quad , \tag{8}$$

where Λ is defined by

$$R_{\alpha\beta} = -\Lambda\, \delta_{\alpha\beta} \quad . \tag{9}$$

The field $V_{\alpha[\beta s]}$ in (8) may be decomposed into irreducible representations of SO(p), i.e. (for $p \geq 3$).

$$V_{\alpha\beta}^{\ s} = V_{(\alpha\beta)}^{\ s} + V_{[\alpha\beta]}^{\ s} + \frac{1}{p} \delta_{\alpha\beta} V^s \tag{10a}$$

where $V_{(\alpha\beta)}^{\ s}$ is traceless and symmetric in $\alpha\beta$ and $V_{[\alpha\beta]}^{\ s}$ is antisymmetric. We are interested only in the singlet of O(6), i.e. V^s. Its Lagrangian will be given by

$$\frac{1}{p} V^s (\partial^2 + \nabla^2 + \Lambda) V^s \quad , \tag{10b}$$

where ∇^2 is the Laplace-Beltrami operator on B_p. In a compact manifold ∇^2 always has a normalizable zero mode. Along this mode the $(\text{mass})^2$ of V^s is just $-\Lambda$. Hence it will be tachyonic for a positive Λ. On the other hand, it follows from Eq.(2) and Eq.(9) that $\Lambda = \frac{\kappa^2}{2d} \vec{F}^2$, i.e. $\Lambda > 0$. Hence we conclude that these classes of solutions are all unstable.

IV. CHIRAL FERMIONS

The physical motivation for considering the above-mentioned classes of solutions lies in their usefulness in providing topologically non-trivial backgrounds for generating 4-dimensional massless chiral fermions [5].

If we follow a line of argument already given in Ref.(3) we can show that starting from an irreducible spinor of SO(1,7) x SO(4+N) in d = 8 we obtain the following families of chiral fermions in four dimensions (N even):

Case 1a)

In this case the $SL(2,\mathbb{C})$ x SO(5) x SU(2) x SO(N) quantum numbers of the chiral fermionic family are $\left((1,2);\ 1;\ 1;\ 2^{N/2-1}\right)$.

Case 1b)

In this case the $SL(2,\mathbb{C})$ x SO(5) x SO(N+1) quantum numbers of chiral fermionic families are $\left((1,2),\ 1;\ 2^{N/2}\right)$.

Notice that in both cases we get families and never antifamilies of chiral fermions for the relevant groups SO(N) and SO(N+1) respectively. However there is a fundamental difference between cases (1a) and (1b). The case (1a) includes all those unbroken SO(N) groups which have a complex spinor representation. Thus if we start from K = SO(4+10) or K = SO(4+14) - which are anomaly free in 8-dimensions [1] - then we get one left-handed family of SO(10) or SO(14) respectively. In case (1b), on the other hand, we always obtain real spinor representations of SO(N+1).

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5) See 1 and 3 above, P.H. Frampton, P.J. Moxhay and K. amamoto, IFP/229/UNC (1984); and A.N. Schellenkens, IFP/SB/84/29 (1984), where also the stability of the instanton-compactification is demonstrated. These papers arrived after our investigation had been completed. These authors do not consider issues of chirality or masslessness of scalars.