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ON SU(3) x SU(2) x U(1) INVARIANT COMPACTIFYING SOLUTIONS TO 11-DIMENSIONAL SUPERGRAVITY *

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ABSTRACT

Harmonic expansions on $\frac{SU(3)xSU(2)xU(1)}{SU(2)x\ U(1)xU(1)}$ quotient spaces are described and it is shown that none of these spaces can account for the $SU(3)xSU(2)xU(1)$ quantum numbers of even a single generation of quarks and leptons. This is in addition to the well-known chirality difficulty .

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I. INTRODUCTION

It was observed by Witten[1] that the minimum dimensionality for a manifold which admits SU(3)xSU(2)xU(1) as a group of motions is seven. Such a manifold might arise by virtue of spontaneous compactification[2] in an eleven-dimensional theory of gravity. It therefore seems appropriate to search for such compactifying solutions in the most well known theory, N = 1 supergravity in eleven dimensions[3]. The Lagrangian of this system is completely specified in terms of one coupling parameter: the 11-dimensional analogue of Newton's constant.

An interesting class of solutions was discovered by Freund and Rubin[4] in which the vacuum geometry factors into the product of 4-dimensional *anti-de Sitte* spacetime with a 7-dimensional Einstein space of Euclidean signature. The most well-studied examples from this class are the cases where the 7-dimensional Einstein space is a seven-sphere, either "round" or "squashed"[5]. Both of these are examples of quotient spaces, O(8)/O(7) and $\frac{O(5)xO(3)}{O(3)xO(3)}$ respectively[6].

It was Witten's idea to examine all possible quotient spaces of the form SU(3)xSU(2)xU(1)/SU(2)xU(1)xU(1) with a view to finding a possibly realistic explanation for the symmetries of the strong and electroweak interactions, based in eleven dimensional supergravity. There are serious obstacles to the realization of this proposal. In particular, it does not seem possible to get light fermions with realistic quantum numbers. However, our understanding of this theory is, at best, sketchy and we believe that, in spite of the negative outlook, a careful analysis of the Witten proposal is worth undertaking.

Witten has arrived at a 3-parameter set of manifold M^{pqr}, where the integers p,q and r characterize the embedding of SU(2)xU(1)xU(1) into SU(3)xSU(2)xU(1). However, in assigning a metric tensor to these spaces, three additional parameters become available. The 6-parameter family of Riemannian space $M^{pqr}(\alpha,\beta,\gamma)$ – with α,β and γ as the three metrical scales – all have SU(3)xSU(2)xU(1) as isometries. We shall redesignate these parameters and denote the spaces as $M^{st}(\alpha,\beta,\gamma)$. This new characterization has been done in Sec.II and has the advantage that the components of the curvature appear more naturally in terms of them. The relation of s and t to the parameters p, q and r is discussed in Sec.III (cf. Eq.(3.17)).

In Sec.III we examine the SU(3)xSU(2)xU(1) content of harmonic expansions and show that it is sensitive to the parameters s and t, only. One conclusion which emerges from this analysis is that no choice of s and t will yield fermions with realistic quantum numbers. It was pointed out by Witten that chiral fermions are not to be expected: the 4-dimensional effective theory is necessarily vectorlike. Our observation is that the theory is not only vectorlike but also, that not even one full generation of quark and leptons can be realized, so far as their SU(2)xU(1) quantum numbers are concerned.

In Sec.IV we consider the vacuum solutions of the Freund- Rubin class. It turns out that the Riemannian spaces $M^{st}(\alpha,\beta,\gamma)$ are restricted to be Einstein spaces with curvature scalar

$$R = \frac{7}{2}\Lambda \quad ,$$

where Λ denotes the (negative) cosmological constant of the 4-dimensional anti-de Sitter spacetime. The scale parameters α, β and γ are specified in terms of s,t and Λ which are themselves unrestricted.

II. THE RIEMANNIAN STRUCTURE

Witten compiled a list of manifolds, M^{pqr}, labelled by triplets of integers p,q and r. These manifolds are quotient spaces G/H where G=SU(3)xSU(2)xU(1) and H = SU(2)xU(1)xU(1). The labels p,q and r correspond to different embeddings of H in G.

Following Witten we shall take the SU(2) factor in H to be embedded as the isospin subgroup of SU(3). let $\lambda_1,\ldots,\lambda_8$ be the antihermitian generators of SU(3), T_1,T_2, T_3 those of SU(2), and Y the generator of U(1). Then the Lie algebra of G is as follows,

$$[\lambda_\alpha,\lambda_\beta] = f_{\alpha\beta\gamma}\,\lambda_\gamma$$

$$[T_k,T_\ell] = \epsilon_{k\ell m}\,T_m$$

$$[\lambda_\alpha,T_k] = [\lambda_\alpha,Y] = [T_k, Y] = 0 \quad , \qquad (2.1)$$

where $\epsilon_{123} = f_{123} = 1$, $f_{458} = f_{678} = \sqrt{3}/2$ and $f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = 1/2$.

Now we identify the subgroup H. The SU(2) factor in H is generated by $\lambda_1,\lambda_2,\lambda_3$. Among the generators of G there are three linearly independent combinations which commute with each other and with $\lambda_1,\lambda_2,\lambda_3$. These are linear combinations of λ_8, T_3 and Y. Two such combinations can be adopted as generators of the U(1)xU(1) factor in H. The third can then be identified with a vector in the tangent space of G/H.

The group G is not semi-simple and there is therefore no natural definition of orthogonality in the Lie algebra. Because of this it is not sufficient to specify merely the combination which goes into the tangent space of G/H. The U(1)xU(1) generators must be linearly independent of it but they cannot be said to be orthogonal. For this reason we choose to specify the generators, $Q_{\bar{a}}$, of H = SU(2)xU(1)xU(1) as follows ,

$$Q_{\bar{k}} = \lambda_k \ , \ k = 1,\ 2,\ 3$$

$$Q_{\bar{4}} = \tilde{\lambda}_8 + s\,Y$$

$$Q_{\bar{5}} = T_3 + tY \quad , \qquad (2.2)$$

where $\tilde{\lambda}_8 = (2/\sqrt{3})\,\lambda_8$ is the SU(3) hypercharge. The parameters s and t are free. The remaining seven generators, Q_a , are given by

$$Q_0 = Y$$

$$Q_1 = \lambda_6 + i\,\lambda_7 \quad , \quad Q_{\dot 1} = \lambda_6 - i\,\lambda_7$$

$$Q_2 = \lambda_4 + i\,\lambda_5 \quad , \quad Q_{\dot 2} = \lambda_4 - i\,\lambda_5$$

$$Q_3 = T_1 + i\,T_2 \quad , \quad Q_{\dot 3} = T_1 - i\,T_2 \tag{2.3}$$

and they are to be associated with the tangent space of G/H. We must make the association more precise.

A convenient device[8)] for discussing the geometry of the factor space G/H is the mapping into G described by the "boosts", L_y,

$$L : \quad G/H \to G$$
$$y \to L_y \quad . \tag{2.4}$$

Here y indicates a point in G/H while L_y is itself an element of G. It can be parametrized in many ways. For example, in an open set containing the identity element we could take

$$L_y = \exp\,(y_\alpha\, Q_\alpha) \tag{2.5}$$

and regard y^α as coordinates on a patch of G/H [*)].

Under the action of left translations by G the elements L_y transform according to

$$g : L_y \to L_{y'} = g\, L_y\, h^{-1} \quad , \tag{2.6}$$

where $g \in G$ and $h = h(y,g) \in H$. The explicit form of h(y,g) depends upon the particular choice of the boost elements L_y.

A set of frames for the tangent space of G/H is obtained from the 1-form $L_y^{-1}\, dL_y$ which belongs to the Lie algebra of G.

$$L_y^{-1} dL_y = e^\alpha\, Q_\alpha + e^{\bar\alpha}\, Q_{\bar\alpha} \quad . \tag{2.7}$$

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*)Regarding G as a principal H-bundle on the base space G/H, we see that L_y represents a local section. Under the projection $\pi : G \to G/H$ we have $\pi\, L_y = y$. In general, more then one patch will be needed to cover G/H.

The required frames are given by

$$e^\alpha = dy^\mu\, e^\alpha_\mu\,(y) \quad . \tag{2.8}$$

These expressions generally depend upon parameters, s,t. Notice in particular that the choice of $Q_{\dot 0}$, $Q_{\bar 4}$ and $Q_{\bar 5}$ governs the definition e^0 as a particular linear combination of the 1-forms associated with λ^8, T^3 and Y.

Under the action of G the frames transform according to

$$e^\alpha\,(y') = e^\beta\,(y)\; D^\alpha_\beta\,(h^{-1}) \quad , \tag{2.9}$$

where $D_\beta{}^\alpha\,(h^{-1})$ is an orthogonal matrix. It is defined by

$$h\, Q_\alpha\, h^{-1} = D_\alpha{}^\beta(h^{-1})\; Q_\beta \tag{2.10}$$

and serves to specify the embedding of H in the tangent space group of G/H. In our problem this group is O(7). Under the action of SU(2)xU(1)xU(1) the tangent space 7-vector branches as follows,

$$7 = 2_{1,0} + 2_{-1,0} + 1_{0,1} + 1_{0,-1} + 1_{0,0} \tag{2.11}$$

i.e. two doublets and three singlets of SU(2) with U(1)xU(1) quantum numbers indicated by subscripts. The corresponding pieces of the 7-vector Q_α are exhibited in (2.3) where the doublet $2_{1,0}$ is represented by the pair Q_1, Q_2, the singlet $1_{0,1}$ by Q_3 , their conjugates $2_{-1,0}$ and $1_{0,-1}$ by $Q_{\dot 1}$, $Q_{\dot 2}$ and $Q_{\dot 3}$, respectively. The neutral singlet $1_{0,0}$ corresponds to Q_0.

We now introduce a G-invariant metric on G/H. The action of G upon G/H is defined by the formula (2.6) which fixes the mapping $y \to y' = gy$ and the tangent space rotation h(y,g) corresponding to each element $g \in G$. The most general symmetric tensor $g_{\alpha\beta}\,(y)$ with the invariance

$$g_{\alpha\beta}\,(y') = D_\alpha{}^\gamma(h)\; D_\beta{}^\delta(h)\; g_{\gamma\delta}(y)$$

reduces, in the basis (2.8) to three parameters

$$g_{1\dot 1} = g_{2\dot 2} = \alpha^2$$
$$g_{3\dot 3} = \beta^2$$
$$g_{00} = \gamma^2 \quad , \tag{2.12}$$

where α, β and γ are independent of y^{μ}. This results from the tangent space decomposition (2.11).

We therefore conclude that the Riemannian manifolds G/H with frames $e^{\alpha}(y)$ given by (2.8) and metric $g_{\alpha\beta}$ by (2.12), depend on five independent parameters s, t and α, β, γ. To clarify their geometrical role we have computed the components of the Riemann tensor. Details are contained in the Appendix. The results are,

$$R_{ab\dot{c}\dot{d}} = (\tfrac{3}{4}\, s\gamma)^2 (\delta_{a\dot{d}}\,\delta_{b\dot{c}} - \delta_{a\dot{c}}\,\delta_{b\dot{d}}), \quad a,b,.. = 1,2$$
$$R_{30\dot{3}0} = -(t\,\gamma^2/\beta)^2$$
$$R_{a3\dot{c}\dot{3}} = -\tfrac{3}{4}\, s\, t\, \gamma^2\, \delta_{a\dot{c}}$$
$$R_{a\dot{3}\dot{c}3} = \tfrac{3}{4}\, s\, t\, \gamma^2\, \delta_{a\dot{c}}$$
$$R_{a0\dot{c}0} = -(\tfrac{3}{4}\, s\,\gamma^2/\alpha)^2\, \delta_{a\dot{c}}$$
$$R_{3\dot{3}3\dot{3}} = 2\,\beta^2 - 3(t\,\gamma)^2$$
$$R_{3\dot{3}c\dot{d}} = -\tfrac{3}{2}\, s\, t\, \gamma^2\, \delta_{c\dot{d}}$$
$$R_{a\dot{b}c\dot{d}} = \tfrac{3}{2}(\alpha^2 - \tfrac{15}{16}(s\gamma)^2)\,\delta_{a\dot{b}}\,\delta_{c\dot{d}} + \tfrac{1}{2}(\alpha^2 - \tfrac{9}{16}(s\gamma)^2)(\tau^k)_{a\dot{b}}(\tau^k)_{c\dot{d}} \quad (2.13)$$

where the components refer to the basis e^{α} with metric given by (2.12).

To obtain quantities with an absolute significance we shall extract from (2.13) the eigenvalues of the Riemann tensor considered as a 21x21 symmetric matrix acting on the space of rank 2 antisymmetric 7-tensors. We shall also rescale the components (2.13) so that they refer to an orthonormal basis. The distinct eigenvalues are

$$(\tfrac{3}{4}\, s\gamma/\alpha^2)^2,\ -(t\gamma/\beta^2)^2,\ -\tfrac{3}{4}\, s\, t\, (\gamma/\alpha\beta)^2,$$

$$\tfrac{3}{4}\, s\, t\, (\gamma/\alpha\beta)^2,\ -(\tfrac{3}{4}\, s\,\gamma/\alpha^2)^2$$

corresponding to the first five lines of (2.13), and

$$(1/\alpha)^2 - (\tfrac{3}{4}\, s\,\gamma/\alpha^2)^2$$

corresponding to the SU(2) triplet form in the last line of (2.13). Two more eigenvalues could be obtained by diagonalizing the 2x2 matrix

$$\begin{bmatrix} \frac{2}{\beta^2} - 3\,(\frac{t\gamma}{\beta^2})^2 & -\frac{3}{4}\, st\,(\frac{\gamma}{\alpha\beta})^2 \\ -\frac{3}{4}\, st\,(\frac{\gamma}{\alpha\beta})^2 & \frac{3}{2\alpha^2} - \frac{5}{2}(\frac{3}{4}\, s\, \frac{\gamma}{\alpha^2})^2 \end{bmatrix} \quad (2.14)$$

but we shall not pursue this. Suffice it to say that all eigenvalues of the Riemann tensor are constructed from four algebraically independent scales: $1/\alpha^2$, $1/\beta^2$, $s\gamma/\alpha^2$ and $t\gamma/\beta^2$. Out of our five parameters, only four combinations are significant for the local geometry.

To clarify the global significance of our parameters we shall examine the SU(3)xSU(2)xU(1) content of harmonic expansions.

III. HARMONIC EXPANSIONS AND SPECTRA

Elsewhere [7] a general scheme for developing harmonic expansions on quotient spaces was discussed. If the functions $\phi_i(y)$ belong to an irreducible representation $\mathbb{D}$ of the stability group H then

$$\phi_i(y) = \sum_{n,p} \sqrt{\frac{d_n}{d_{\mathbb{D}}}}\; D^n_{i,p}(L_y^{-1})\,\phi^n_p \quad , \qquad (3.1)$$

where D^n denotes an irreducible unitary representation of G. It has dimension d_n and the columns of this matrix are labelled by the index,p. The rows, labelled by the index i, correspond to a sector of dimension $d_{\mathbb{D}}$ which is invariant with respect to the subgroup H. This sector supports the irreducible representation $\mathbb{D}$ to which ϕ belongs. Naturally, the sum in (3.1) must include only those D^n which contain $\mathbb{D}$ on restriction to H. The coefficients ϕ^n_p are given by

$$\phi^n_p = \frac{1}{V} \sum_i \sqrt{\frac{d_n}{d_{\mathbb{D}}}} \int_{G/H} d\mu\; D^n_{p,i}(L_y)\;\phi_i(y) \quad , \qquad (3.2)$$

where $d\mu$ denotes the invariant measure on G/H normalized to volume V.

The harmonic content is controlled by the embedding of H, both in G and in the tangent space group of G/H. In our case the embedding of SU(2)xU(1)xU(1) in SU(3)xSU(2)xU(1) is given by the formulae (2.2) and thereby depends on the two parameters s and t. On the other hand, the embedding in the tangent space SO(7) is given by (2.10) which does not involve any parameters. In particular, the SU(2)xU(1)xU(1) content of the 7-vector is given by

$$7 = 2_{1,0} + 2_{-1,0} + 1_{0,1} + 1_{0,-1} + 1_{0,0} \quad . \tag{3.3}$$

From this it follows that the 8-spinor decomposes as follows

$$8 = 2_{0,1/2} + 2_{0,-1/2} + 1_{1,1/2} + 1_{-1,-1/2} + 1_{1,-1/2} + 1_{-1,1/2} \tag{3.4}$$

other multiplets of SO(7) can be similarly reduced.

If the manifold admits a spinor structure then it should be possible to expand, say, the 8-spinor in irreducible representations of SU(3)xSU(2)xU(1). But from the general discussion above we know that this requires one or more of the SU(2)xU(1)xU(1) pieces listed in (3.4) to be contained in representations of SU(3)xSU(2)xU(1). According to (2.2), the simplest representations of the latter group decompose as follows,

$$\begin{aligned} (3,1)_0 &= 2_{1/3,0} + 1_{-2/3,0} \\ (1,2)_0 &= 1_{0,1/2} + 1_{0,-1/2} \\ (1,1)_1 &= 1_{s,t} \quad , \end{aligned} \tag{3.5}$$

(where the subscript on the left corresponds to the U(1) quantum number, Y). In order to find, say the lepton doublet in the harmonic expansion of the 8-spinor (3.4), since

$$(1,2)_{-1} = 1_{-s,-t+1/2} + 1_{-s,-t-1/2} \tag{3.6}$$

we must have

$$-s = 1 \text{ or } -1 \text{ and } -t \pm \frac{1}{2} = \frac{1}{2} \text{ or } -\frac{1}{2} ,$$

i.e. the pair (s,t) must be one of eight possibilities

$$\begin{aligned} &(1,1),\ (1,0),\ (-1,0),\ (-1,1) \\ &(-1,-1),\ (-1,0),\ (1,0),\ (1,1) \ . \end{aligned} \tag{3.7}$$

On the other hand, in order to find the lepton singlet

$$(1,1)_{-2} = 1_{-2s,-2t} \tag{3.8}$$

we must have

$$-2s = 1 \text{ or } -1 \text{ and } -2t = \frac{1}{2} \text{ or } -\frac{1}{2} ,$$

and this is clearly not compatible with (3.7). This means that fermions with the SU(2)x U(1) quantum numbers of left and right handed leptons, i.e. doublets and singlets of the standard kind, cannot both exist in any one of these manifolds. We have examined here only the 8-spinor of SO(7) but it is not difficult to persuade oneself the argument extends to all spinors.

We must conclude that the seven dimensional SU(3)xSU(2)xU(1) invariant internal spaces cannot give rise to quarks and leptons with appropriate quantum numbers. This problem is in addition to the chirality difficulty.

To discuss the harmonic expansions in general we shall make use of Gel'fand Tsevtlin[11] patterns to label the basis vectors of the irreducible representations of SU(3). In this notation the vectors are characterized by arrays of integers, positive and negative,

$$\left| \begin{matrix} n_1 & & n_2 & & n_3 \\ & m_1 & & m_2 & \\ & & k_3 & & \end{matrix} \right\rangle , \tag{3.9}$$

where

$$\begin{aligned} &n_1 \geq m_1 \geq n_2 \geq m_2 \geq n_3 \\ &m_1 \geq k_3 \geq m_2 \quad . \end{aligned} \tag{3.10}$$

These labels are appropriate for representations of U(3) and therefore contain some redundancy in so far as SU(3) is concerned. To remove this we shall suppose that a uniform shift of all six numbers by the same integer, $n_1 \to n_1 + r$, $n_2 + r, \ldots,$

$k_3 + k_3 + r$, leaves the state invariant. The numbers in the first row, n_1, n_2, n_3 which characterize the irreducible representation may therefore be restricted such that

$$n_1 + n_2 + n_3 = 0, 1 \text{ or } 2 \tag{3.11}$$

and the labelling is thereby made unique. The numbers in the second row label the irreducible representations of SU(2)xU(1) which are contained,

$$\begin{aligned} 2k &= m_1 - m_2 \\ \lambda_8 &= m_1 + m_2 - \frac{2}{3}(n_1 + n_2 + n_3) \ , \end{aligned} \tag{3.12}$$

where k denotes the SU(2) spin and λ_8 the U(1) charge. Finally, in the third row, k_3 labels the U(1) representations contained in SU(2). The value of this spin component is

$$k_3 = k_3 - \frac{1}{2}(m_1 + m_2) \ . \tag{3.13}$$

Basis vectors for irreducible representations of $SU(3)xSU(2)_T xU(1)_Y$ may be taken in the form

$$\left| \begin{matrix} n_1 & n_2 & n_3 & ; T & ; Y \\ m_1 & m_2 & & T_3 & \\ & k_3 & & & \end{matrix} \right\rangle , \tag{3.14}$$

where 2T are integers.

Now consider the harmonic expansion of an irreducible multiplet of the isotropy group SU(2)xU(1)xU(1) with SU(2) spin k and U(1)xU(1) charges $Q_{\bar 4}$ and $Q_{\bar 5}$. According to the general rule[9] we must include all those representations of SU(3)xSU(2)xU(1) which contain states for which

$$\begin{aligned} m_1 - m_2 &= 2k, \\ \lambda_8 + sY &= Q_{\bar 4} \\ T_3 + tY &= Q_{\bar 5} \ , \end{aligned} \tag{3.15}$$

where λ_8 is given by (3.12). Since, according to (3.3) and (3.4), $Q_{\bar 4}$ is necessarily an integer while $Q_{\bar 5}$ may be an integer or half-integer, it follows from (3.15) that 3sY and 2tY must be integers,

$$3sY = N_1 \quad \text{and } 2tY = N_2 \ . \tag{3.16}$$

These equations imply

$$3s = \nu a_1 \quad \text{and} \quad 2t = \nu a_2 \ , \tag{3.17}$$

where a_1 and a_2 are relatively prime integers and ν is rational. No generality is lost by taking ν to be an integer since this can always be achieved by an appropriate rescaling of the spectrum of Y values. The integers a_1, a_2 and ν presumably correspond to Witten's p, q and r, respectively.

We can solve Eq.(3.15) for m_1, m_2 and T_3. Making the substitutions from (3.12) and (3.17) into (3.15),

$$\begin{aligned} m_1 + m_2 &= Q_{\bar 4} - \frac{\nu a_1 Y}{3} + \frac{2}{3}(n_1 + n_2 + n_3) \\ m_1 - m_2 &= 2k \\ T_3 &= Q_{\bar 5} - \frac{\nu a_2 Y}{2} \ . \end{aligned} \tag{3.19}$$

Since a_1 and a_2 are integers, it follows that T_3 takes integer or half-integer values, as it should, but m_1 and m_2 can be integers only if the triality, $n_1+n_2+n_3$, is chosen appropriately. In other words, these equations serve to determine the triality as well as m_1, m_2 and T_3. One finds,

$$n_1 + n_2 + n_3 = \begin{cases} \dfrac{\nu n_1 Y}{2} \bmod 3, & Q_{\bar 4} + 2k \ \text{even} \\[2ex] \dfrac{\nu a_1 Y - 3}{2} \bmod 3, & Q_{\bar 4} + 2k \ \text{odd} \end{cases} \tag{3.20}$$

Notice, in particular, that if a_1 is even it will not be possible to satisfy (3.19) for odd values of $Q_4 + 2k$. Inspection of (3.3) and (3.4) shows that spinors all carry odd values of this quantum number. This means that manifolds with a_1 even do not admit a spinor structure.

Finally, we observe that the harmonic expansions generally take the form

$$\sum_{=-\infty}^{\infty} \sum_{T \geq |T_3|} \sum_{n_1 \geq m_1 \geq n_2 \geq m_2 \geq n_3} \qquad , \tag{3.20}$$

where T_3, m_1, m_2 and $n_1 + n_2 + n_3$ are fixed in terms of N.

With the harmonic expansion problem solved we now have available some useful indicators of global features of the spaces $M^{st}(\alpha,\beta,\gamma)$. In particular, spectra of invariant operators such as the Laplacian and Dirac operators, which are sensitive to global aspects of the space, can be extracted by algebraic methods. We can show that the eigenvalues of such operators depend upon the pairs s and t which were shown above to govern the local geometry.

The simplest operator to evaluate is the Laplacian,

$$\tilde{\Box} = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \tag{3.21}$$

constructed from the canonical connection,

$$\Gamma_\alpha{}^\beta = e^{\tilde{\gamma}} C_{\alpha\tilde{\gamma}}{}^\beta \tag{3.22}$$

which differs from the Riemannian connection by an invariant tensor (see Appendix for details). The spectrum of (3.21) is given by

$$\begin{aligned}\tilde{\Box} &= g^{\alpha\beta} Q_\alpha Q_\beta \\ &= \frac{1}{\alpha^2}\{Q_a, Q_{\bar{a}}\} + \frac{1}{\beta^2}\{Q_3, Q_{\bar{3}}\} + \frac{1}{\gamma^2} Q_0^2 \\ &= -\frac{2}{\alpha^2}(C^3 - C^2 - C^1) \\ &\quad -\frac{2}{\beta^2}(C_T^2 - C_T^1) - \frac{1}{\gamma^2} Q_0^2 \qquad ,\end{aligned} \tag{3.23}$$

where the various Casimir operators are given by

$$\begin{aligned} Q_0 &= Y = \frac{N}{\nu} \\ C_T^2 - C_T^1 &= T(T+1) - T_3^2 \\ &= T(T+1) - (Q_{\bar{5}} - \frac{a_2}{2} N)^2 \\ C^3 - C^2 - C^1 &= \frac{1}{6}\left[(n_1-n_2+3)^2 + (n_2-n_3+3)^2 + (n_1-n_3)^2 - 18\right] - \\ &\quad - k(k+1) - \frac{3}{4}(Q_{\bar{4}} - \frac{a_1}{3} N)^2 \end{aligned} \tag{3.24}$$

and the ranges of N, n_1, n_2, n_3 and T are as determined above.

Evaluation of the Laplacian $\Box$ associated with the Riemannian connection is considerably more laborious and we shall not attempt it here. The magnitude of the general problem can be guessed from the expressions we have constructed for the case of a 7-vector A_γ,

$$\begin{aligned} \Box A_c &= \left[\tilde{\Box} - \frac{9}{8}\left(\frac{s\gamma}{\alpha^2}\right)^2\right] A_c \\ &\quad + i\,\frac{3}{2}\frac{\tilde{s}}{\alpha^2}(\tilde{\nabla}_c A_0 + \tilde{\nabla}_0 A_c) \\ \Box A_3 &= \left[\tilde{\Box} - 2\left(\frac{t\gamma}{\beta^2}\right)^2\right] A_3 \\ &\quad + i\,2\frac{t}{\beta^2}(\tilde{\nabla}_3 A_0 + \tilde{\nabla}_0 A_3) \\ \Box A_0 &= \left[\tilde{\Box} - \frac{9}{4}\frac{s\gamma^2}{\alpha^4} - 2\frac{t\gamma^2}{\beta^4}\right] A_0 \\ &\quad + i\,\frac{3}{2}\frac{s\gamma^2}{\alpha^4}(\tilde{\nabla}_{\bar{c}} A_c - \tilde{\nabla}_c A_{\bar{c}}) + 2i\,\frac{t\gamma^2}{\beta^4}(\tilde{\nabla}_{\bar{3}} A_3 - \tilde{\nabla}_3 A_{\bar{3}}) \end{aligned}$$

It would be straightforward, though tedious, to use the harmonic expansions of A_c, A_3 and A_0 to reduce this problem to the diagonalization of finite dimensional matrices.

We conclude that the eigenvalues of $\Box$ are sensitive to all parameters characterizing $M^{st}(\alpha,\beta,\gamma)$.

IV. SOLUTIONS

The Freud-Rubin class of solutions describes vacuum geometries in which the 11-dimensional spacetime factorizes into the product of 4-dimensional anti-de Sitter spacetime and a compact 7-dimensional Einstein space. In a suitable basis the Ricci tensor *) R_{AB} reduces to the form

$$R_{ab} = -\Lambda g_{ab} \qquad , \; a,b = 0,1,2,3$$

$$R_{a\beta} = 0$$

$$R_{\alpha\beta} = \frac{\Lambda}{2} g_{\alpha\beta} \qquad \alpha,\beta = 4,5,\dots,10 \tag{4.1}$$

where the (negative) cosmological parameter Λ is related to the vacuum value of the Cremmer-Julia gauge field

$$F_{abcd} = \sqrt{\frac{-3\Lambda}{\kappa^2}}\; \epsilon_{abcd} \frac{1}{\sqrt{-g}} \qquad \kappa = \text{11-dimensional Newtonian constant} \; . \tag{4.2}$$

A subset of the Witten-type spaces, $M^{st}(\alpha,\beta,\gamma)$ can be Einstein spaces and therefore fall into the Freund-Rubin class. From the expressions (2.13) for the Riemann tensor one easily extracts the Ricci tensor. In the notation of Sec. II

$$R_{a\dot{b}} = (-3 + \frac{9}{8}(s\gamma/\alpha)^2)\,\delta_{a\dot{b}}$$

$$R_{3\dot{3}} = -2 + 2\,(t\gamma/\beta)^2$$

$$R_{00} = -(\frac{3}{2} s\,\gamma^2/\alpha^2)^2 - 2(t\gamma^2/\beta^2)^2 \; . \tag{4.3}$$

On setting $R_{\alpha\beta} = \frac{\Lambda}{2} g_{\alpha\beta}$ we obtain three equations

*) The Ricci tensor used here is defined by $R_{ABC}{}^{B} = R_{AC}$ where the Riemann tensor is defined in the Appendix.

$$3 - \frac{9}{8}(s\gamma/\alpha)^2 = -\alpha^2\frac{\Lambda}{2}$$

$$2 - 2(t\gamma/\beta)^2 = -\beta^2\frac{\Lambda}{2}$$

$$(\frac{3}{2} s\gamma^2/\alpha^2)^2 + 2(t\gamma^2/\beta^2)^2 = -\gamma^2\frac{\Lambda}{2} \tag{4.4}$$

which can be solved for the three scale factors α, β and γ. It is convenient to replace these variables by three new ones,

$$x = -\alpha^2\frac{\Lambda}{2} \; , \; y = -\beta^2\frac{\Lambda}{2} \; , \; z = -\gamma^2\frac{\Lambda}{2} \; . \tag{4.5}$$

The equations for x,y and z are then

$$3 - \frac{9}{8} s^2 \frac{z}{x} = x$$

$$2 - 2t^2\frac{z}{y} = y$$

$$(\frac{3}{2}\frac{s}{x})^2 + 2(\frac{t}{y})^2 = \frac{1}{z} \; . \tag{4.6}$$

Of these the first two are solved by

$$x = \frac{3}{2}\left[1 \pm\sqrt{1 - \frac{1}{2}s^2 z}\right]$$

$$y = 1 \pm\sqrt{1 - 2t^2 z} \tag{4.7}$$

Eliminating x and y from the third equation we obtain

$$\left(\frac{s}{1 \pm\sqrt{1-\frac{1}{2}s^2 z}}\right)^2 + 2\left(\frac{t}{1 \pm\sqrt{1-2t^2 z}}\right)^2 = \frac{1}{z} \tag{4.8}$$

It is not difficult to prove that this equation has a positive root if and only if the positive roots for x and y are chosen in (4.7). Suppose $4t^2 \leqslant s^2$ and write

$$\frac{1}{ch^2\theta} = \frac{1}{2} s^2 z \quad . \tag{4.9}$$

Then (4.8) becomes, if positive roots are chosen

$$2\, e^{-2\theta} + \left(\frac{2t/s}{ch\theta + \sqrt{ch^2\theta - 4t^2/s^2}} \right)^2 = 1 \tag{4.10}$$

and, since the left hand side decreases monotonically with θ from a value > 1 at $\theta = 0$ to 0 at $\theta = \infty$, this equation clearly has one real solution. A similar result obtains when $4t^2 \gg s^2$.

We may therefore conclude that the equations of motion serve to fix the scale factors α, β, and γ of the internal space in terms of the anti-de Sitter scale, Λ.

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APPENDIX

RIEMANN TENSOR FOR WITTEN SPACES

To compute the Riemann tensor for the 7-dimensional spaces SU(3)xSU(2)xU(SU(2)xU(1)xU(1), it is necessary to specify the metric and work out the component of the Riemannian connection in a suitable basis. We shall employ the basis defined by the 1-forms, e^α, discussed in Sec.II. In this basis

$$R_{\alpha\beta\gamma}{}^{\delta} = e_\alpha \Gamma_{\beta\gamma}{}^{\delta} - e_\beta \Gamma_{\alpha\gamma}{}^{\delta} - \Gamma_{\alpha\gamma}{}^{\epsilon} \Gamma_{\beta\epsilon}{}^{\delta} + \Gamma_{\beta\gamma}{}^{\epsilon} \Gamma_{\alpha\epsilon}{}^{\delta} - \Omega_{\alpha\beta}{}^{\epsilon} \Gamma_{\epsilon\gamma}{}^{\delta} \quad , \tag{A.1}$$

where e_α denotes the differential operator

$$e_\alpha = e_\alpha{}^{\mu}(y) \frac{\partial}{\partial y^\mu} \tag{A.2}$$

The metric components $g_{\alpha\beta}$ are specified in terms of three independent constants,

$$g_{\dot{1}\dot{1}} = g_{\dot{2}\dot{2}} = \alpha^2, \quad g_{\dot{3}\dot{3}} = \beta^2, \quad g_{00} = \gamma^2 \tag{A.3}$$

The connection is given by

$$\Gamma_{\alpha[\beta\gamma]} = \frac{1}{2}(\Omega_{[\alpha\beta]\gamma} - \Omega_{[\alpha\gamma]\beta} - \Omega_{[\beta\gamma]\alpha}) \quad , \tag{A.4}$$

where

$$\Omega_{[\alpha\beta]\gamma} = e_\alpha{}^{\mu} e_\beta{}^{\nu} (\partial_\mu e_\nu{}^{\delta} - \partial_\nu e_\mu{}^{\delta}) g_{\delta\gamma} \quad . \tag{A.5}$$

The metric (A.3) is invariant and we expect the components of the Riemann tensor in this basis to be likewise invariant. It will therefore be adequate to compute them at a fixed point in G/H. We shall adapt the coordinate system accordingly. Thus, we shall adopt the parametrization.

$$L_y = e^{y^\alpha Q_\alpha} \tag{A.6}$$

and attempt to compute the curvature at the point $y^\alpha = 0$. Near $y = 0$ we have

$$L_y^{-1}\, dL_y = dy^{\alpha} Q_{\alpha} + \frac{1}{2}\,[dy^{\beta} Q_{\beta}\,,\, y^{\gamma} Q_{\gamma}] + O(y^2)$$

so that, according to the definitions (2.7),

$$e^{\alpha} = dy^{\alpha} + \frac{1}{2}\, dy^{\beta}\, y^{\gamma}\, C_{\beta\gamma}^{\ \ \alpha} + O(y^2) \qquad \text{(A.7a)}$$

$$e^{\bar\alpha} = \frac{1}{2}\, dy^{\beta}\, y^{\gamma}\, C_{\beta\gamma}^{\ \ \bar\alpha} + O(y^2)\,, \qquad \text{(A.7b)}$$

where the structure constants of SU(3)x SU(2)xU(1) appear,

$$[Q_{\alpha}\,,\, Q_{\beta}] = C_{\alpha\beta}^{\ \ \gamma} Q_{\gamma} + C_{\alpha\beta}^{\ \ \bar\gamma} Q_{\bar\gamma}\,. \qquad \text{(A.8)}$$

The generators Q_{α} and $Q_{\bar\alpha}$ are listed in Eqs. (2.2) and (2.3).

Using (A.7) in (A.5) and (A.4) gives, at y = 0,

$$\Omega_{[\alpha\beta]\gamma} = C_{\alpha\beta}^{\ \ \delta}\, g_{\delta\gamma} \equiv C_{[\alpha\beta]\gamma} \qquad \text{(A.9)}$$

$$\Gamma_{\alpha[\beta\gamma]} = \frac{1}{2}\,(C_{[\alpha\beta]\gamma} - C_{[\alpha\gamma]\beta} - C_{[\beta\gamma]\alpha}) \qquad \text{(A.10)}$$

$$\begin{aligned} e_{\alpha}\,\Gamma_{\beta[\gamma\delta]} = & -\frac{1}{4}\, C_{\alpha\beta}^{\ \ \bar\epsilon}\,(C_{[\bar\epsilon\gamma]\delta} - C_{[\bar\epsilon\delta]\gamma}) \\ & +\frac{1}{4}\, C_{\alpha\gamma}^{\ \ \bar\epsilon}\,(C_{[\bar\epsilon\beta]\delta} + C_{[\bar\epsilon\delta]\beta}) \\ & -\frac{1}{4}\, C_{\alpha\delta}^{\ \ \bar\epsilon}\,(C_{[\bar\epsilon\beta]\gamma} + C_{[\bar\epsilon\gamma]\beta}) \end{aligned} \qquad \text{(A.11)}$$

The non-vanishing components of $C_{[\alpha\beta]\gamma}$, in the basis (2.2),(2.3) are given by

$$C_{[a\dot b]0} = i\, \delta_{a\dot b}\, \frac{3}{2}\, s\, \gamma^2$$

$$C_{[3\dot 3]0} = i\, 2t\, \gamma^2 \qquad \text{(A.12)}$$

The non-vanishing components of $C_{[\alpha\bar\beta]\gamma}$ and $C_{[\alpha\beta]}^{\ \ \bar\gamma}$ are

$$\begin{aligned} C_{[a\bar k]\dot b} &= -C_{[\dot b\bar k]a} = -\frac{i}{2}\,(\tau_k)_{a\dot b}\,\alpha^2 \\ C_{[a\bar 4]\dot b} &= -C_{[\dot b\bar 4]a} = i\,\delta_{a\dot b}\,\alpha^2 \\ C_{[a\dot b]}^{\ \ \bar k} &= i(\tau_k)_{a\dot b} \\ C_{[a\dot b]}^{\ \ \bar 4} &= -i\,\delta_{a\dot b}\,\frac{3}{2} \\ C_{[3\bar 5]\dot 3} &= -C_{[\dot 3\bar 5]3} = i\,\beta^2 \\ C_{[3\dot 3]}^{\ \ \bar 5} &= -2i\,, \end{aligned} \qquad \text{(A.13)}$$

where $(\tau_k)_{a\dot b}$ denotes the Pauli spin matrices. The non-vanishing components of the connection at y = 0 are obtained by substituting from (A.12) into (A.10),

$$\begin{aligned} \Gamma_{a[\dot b o]} &= -\Gamma_{\dot b[ao]} = i\,\delta_{a\dot b}\,\frac{3}{4}\, s\,\gamma^2 \\ \Gamma_{o[a\dot b]} &= -i\,\delta_{a\dot b}\,\frac{3}{4}\, s\,\gamma^2 \\ \Gamma_{o[3\dot 3]} &= -i\, t\,\gamma^2 \\ \Gamma_{3[\dot 3 0]} &= -\Gamma_{\dot 3[30]} = i\, t\,\gamma^2\,; \end{aligned} \qquad \text{(A.14)}$$

From the list (A.13) it appears that $C_{[\alpha\bar\beta]\gamma} = -C_{[\gamma\bar\beta]\alpha}$. This implies some simplifications in (A.11) and one finds

$$e_{\alpha}\Gamma_{\beta[\gamma\delta]} - e_{\beta}\Gamma_{\alpha[\gamma\delta]} = C_{\alpha\beta}^{\ \ \bar\epsilon}\; C_{[\gamma\bar\epsilon]\delta}\,. \qquad \text{(A.15)}$$

It is now straigthforward to compute the components $R_{\alpha\beta\gamma\delta}$ by substituting into (A.1). The result is contained in Eq. (2.13).

In Sec.III we made reference to the "canonical" connection, $\tilde{\Gamma}$, defined by Eq. (3.23).

$$\tilde{\Gamma}_{\alpha}{}^{\beta} = e^{\bar{\gamma}} \, C_{\alpha\bar{\gamma}}{}^{\beta} \quad ,$$

where the 1-forms $e^{\bar{\gamma}}$ correspond to the part of $L^{-1}dL$ which lies in the algebra of SU(2)xU(1)xU(1) contained in O(7). The Riemannian connection, $\Gamma_{\alpha}{}^{\beta}$, takes its values in the algebra of O(7). The difference between these two connections,

$$\Gamma_{\alpha}{}^{\beta} - \tilde{\Gamma}_{\alpha}{}^{\beta} = e^{\gamma} \, t_{\gamma\alpha}{}^{\beta} \quad , \tag{A.16}$$

defines a tensor, $t_{\gamma\alpha}{}^{\beta}$, which is invariant with respect to SU(3)xSU(2)xU(1). In the basis employed above, its components are independent of y. Then, in view of (A.7b) that $t_{\gamma\alpha}{}^{\beta}$ coincides with $\Gamma_{\gamma\alpha}{}^{\beta}$ at y = 0: the components $t_{\gamma\alpha\beta} = t_{\gamma\alpha}{}^{\delta} g_{\delta\beta}$ are given directly by the expressions $\Gamma_{\gamma[\alpha\beta]}(y = 0)$ in Eq. (A.14).

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