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STABILITY OF INSTANTON INDUCED COMPACTIFICATION IN 8-DIMENSIONS *

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Abstract

We consider the 8-dimensional Einstein-SU(2) Yang-Mills system when the Yang-Mills system assumes a 1-instanton configuration on an internal S_4 with the vacuum solution possessing the geometry of $M_4 \times S_4$ and the invariance group P_4(Poincaré) x SO(5). We demonstrate the classical stability of the solution by computing the spectrum of the physical states and showing the absence of ghosts and tachyons amongst them. We also discuss the possibility of obtaining SO(5) multiplets of massless fermions.

I. INTRODUCTION

The concept of spontaneous compactification is illustrated in a number of models involving the dynamics of a metric tensor in more than four dimensions [1]. In typical cases the field equations are solved by a metric which describes a vacuum geometry of the form $M^4 \times B^K$ where M^4 denotes four-dimensional Minkowski space and B^K denotes a K-dimensional compact manifold. More generally, there may arise four-dimensional anti-de Sitter space instead of M^4. Since observationally the cosmological constant is nearly zero, to be at all realistic the four-dimensional factor M^4 would have to be flat when compared to the K-dimensional "internal" space.

Among the cases where compactification can occur, little is known about their stability. To our knowledge only three cases have been analyzed: pure gravity in 5 dimensions [2], Einstein-Maxwell theory in 6 dimensions [3] and N = 1 supergravity in 11 dimensions *) [4]. We are therefore encouraged to examine one more case: gravity coupled to SU(2) Yang-Mills fields in 8-dimensions.

It has long been realized that topological considerations are central to the question of stability. For this reason models which couple gravity to gauge fields are particularly favoured. If the vacuum gauge field assumes a topologically non-trivial configuration with respect to the internal space then one might expect this vacuum to be stable. The solution by Horvath et al. to the six-dimensional Einstein-Maxwell system was prompted by this idea [5]. The Maxwell field in this case takes the form of a magnetic monopole on the internal 2-sphere. More recently it was verified that the solution is indeed stable - against small fluctuations at least [3]. To prove its stability against vacuum tunneling phenomena one would need something akin to a "positive energy theorem" but this remains to be discovered **).

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*) A supersymmetric ground state is necessarily stable owing to the structure of the superalgebra which implies non-negative energies. Unfortunately, in all known cases the ground state geometry takes the form $AdS^4 \times B^K$ where the anti-de Sitter and internal curvatures are of comparable magnitude. This precludes the identification in any meaningful way of a low energy sector.

**) The importance of the topological criterion is fortified by the examples of instability which arise when it is not met. The five-dimensional pure gravity solution of Kaluza, although classicaly stable, succumbs to vacuum tunneling [2]. The six-dimensional Einstein-Yang-Mills compactification on $M^4 \times S^2$ is not even classically stable since the possible Yang-Mills configurations on S^2 are necessarily trivial [6].

In this paper we consider the eight-dimensional Einstein-SU(2) Yang-Mills system. The vacuum solution to be analyzed here has the geometry of $M^4 \times S^4$ and the Yang-Mills field assumes the 1-instanton configuration on the internal 4-sphere. The invariance group of this vacuum is $P(4) \times SO(5)$ where P(4), the four-dimensional Poincaré group acts on M^4 and SO(5), the five-dimensional orthogonal group acts on S^4.

The non-vanishing second Chern class of the instanton ensures the non-triviality of the vector bundles over S^4 on which the instanton acts as a connection form. We therefore expect the proposed vacuum solution to be at least classically stable. In the following sections this will be established.

Our method for proving the classical stability of the vacuum solution is to compute the spectrum of physical states - those which couple to appropriately conserved (on-shell) currents - and show that this spectrum contains no tachyons or negative-metric states. We do this by linearizing the field equations on the vacuum solution and evaluating the propagator in a chosen gauge. Each pole which persists when the propagator is sandwiched between conserved currents corresponds to a physical state and its metric is determined by the sign of the residue. In this way it is found that the spectrum includes the helicity ±2 graviton, ten helicity ±1 massless states corresponding to the SO(5) Yang-Mill- symmetry of the vacuum, and several towers of massive states with spins 0, 1 and 2.

In Sec.II we discuss the vacuum solution and its symmetries. We then derive the linear equations for fluctuations on this background. Sec.III is devoted to the solution of the linear problem. This involves defining harmonic expansions for the fluctuations on S^4 and, at this point, a good deal of kinematic discussion is unavoidable. The solutions are classified into irreducible pieces with respect to the vacuum symmetry $P(4) \times O(5)$ and, in each piece a number of poles are identified. To distinguish those poles which correspond to physical states we sandwich the propagator between conserved source currents. This is done in Sec.IV. Finally, in Sec.V the results are discussed and the outlook considered.

[illegible]

Consider the action integral

$$S \sim -\int d^8z\, g^{1/2} \left\{ \frac{1}{\kappa^2} R + \frac{1}{4} \vec{F}_{MN} \cdot \vec{F}^{MN} + \lambda \right\} \quad . \tag{1}$$

where κ^2 is the four-dimensional Newtonian coupling constant. S describes the interaction of gravity with an SU(2) gauge field $\vec{F}_{MN}$ in eight-dimensions. λ is an eight-dimensional cosmological constant *).

The field equations following from (1) are

$$R_{AB} - \frac{1}{2} \eta_{AB} R = \frac{-\kappa^2}{2} \left(T_{AB} - \lambda \eta_{AB} \right) \quad , \tag{2a}$$

$$\vec{F}^{AB}{}_{\cdot A} = 0 \quad , \tag{2b}$$

$$T_{AB} = \vec{F}_{AC} \cdot \vec{F}_B{}^C - \frac{1}{4} \eta_{AB} \vec{F}_{CD} \cdot \vec{F}^{CD} \quad . \tag{2c}$$

The latin indices A,B,C,.... refer to the orthonormal frames in the eight-dimensional manifold and $\cdot A$ signifies a covariant derivative with respect to both Riemannian as well as the Yang-Mills connections.

We shall look for solutions of (2) such that

$$\left. \begin{array}{l} R_{aA} = 0 \\ \vec{F}_{aA} = 0 \end{array} \right\} \quad a = 0,1,2,3, \quad A = 0\ldots3,5\ldots8 \quad . \tag{3}$$

and

By substituting this into Eq.(2a) we get

$$\frac{1}{\kappa^2} R + \frac{1}{4} \vec{F}^2 + \lambda = 0 \quad . \tag{4a}$$

$$R_{AB} = -\frac{\kappa^2}{2} \vec{F}_{AC} \cdot \vec{F}_B{}^C \quad . \tag{4b}$$

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*) Sign $\eta_{AB} = (-1,+1,\ldots,+1)$. The Riemann tensor is defined such that $R^K_{LMN} = \partial_M \Gamma^K_{LN} - \partial_N \Gamma^K_{LM} \cdots$ and the Ricci tensor is $R_{LM} = R^K{}_{LMK}$.

[illegible] eight-dimensional manifold is a product $M_4 \times S_4$, where M_4 denotes the four-dimensional Minkowski space and S^4 is the four-dimensional sphere. Thus the isometry group of our ansatz for the metric is $P_4 \times SO(5)$. If we demand that the ansatz for the SU(2) gauge potential $A = A_M(x,y)\, dz^M$ is also $P_4 \times SO(5)$ invariant then our problem has a <u>unique</u> solution. This is given by $A_a = 0$, $a = 0,1,2,3$, with $A_\mu(x,y)$, $\mu = 5,\ldots,8$ describing a single instanton on S_4.

To exhibit the SO(5) invariance (up to a local SU(2) gauge transformation) as well as the uniqueness let us consider the intrinsic (projective) co-ordinates y^μ, on S_4 defined by

$$u^\mu(y) = a \frac{2by^\mu}{b^2 + y^2} \qquad \mu = 5,\ldots,8 \quad . \tag{5a}$$

$$u^5(y) = a \frac{b^2 - y^2}{b^2 + y^2} \quad . \tag{5b}$$

where $y^2 = (y^1)^2 + \cdots + (y^4)^2$ and $u^\mu u^\mu + u^5 u^5 = a^2$. Here $\infty < y^\mu < \infty$ and b is an arbitrary const. The co-ordinate functions y^μ on S_4 associate with any point of S_4 a <u>unique</u> point in $\mathbb{R}^4$. However they do not cover S_4 fully. The south pole, i.e. ($u^\mu = 0$, $u^5 = -a$) on S_4 should be contained in a different co-ordinate patch. (This point would have corresponded to $y^2 \to \infty$ irrespective of the direction of approach to ∞ in $\mathbb{R}^4$. Therefore it cannot be put in correspondence with a <u>unique</u> point of $\mathbb{R}^4$.) Let U^+ denote the domain of definition of the co-ordinate functions y^μ , and consider the local section $L : U^+ \to SO(5)$ defined by

$$L(y) = \begin{pmatrix} \delta_{\mu\nu} - \dfrac{2y^\mu y^\nu}{b^2 + y^2} & \dfrac{2by^\mu}{b^2 + y^2} \\[2ex] \dfrac{-2by^\nu}{b^2 + y^2} & \dfrac{b^2 - y^2}{b^2 + y^2} \end{pmatrix} \quad , \tag{6}$$

$L(y)$ is clearly a 5×5 orthogonal matrix. We may construct the matrix of the 1-forms $L^{-1}(y)\, dL(y)$. This is a 5×5 antisymmetric matrix which may be decomposed into any convenient basis of the Lie algebra of SO(5). We choose a basis $\{Q_{\alpha 5}, Q_{\alpha\beta}\}$ $\alpha,\beta,\ldots,1,\ldots,4$, such that $\{Q_{\alpha\beta}\}$ constitute a basis of the O(4) algebra generating the isotropy subgroup SO(4) in $S_4 = \frac{SO(5)}{SO(4)}$. Then we may write

$$L^{-1}(y)\, dL_y = \frac{1}{b} e^\alpha Q_{\alpha 5} + \frac{1}{2} A^{\alpha\beta} Q_{\alpha\beta} \quad , \tag{7a}$$

where

$$e^{\alpha} = \frac{2ab}{b^2 + y^2} \; dy^{\alpha} \tag{7b}$$

$$A^{\alpha\beta} = \frac{1}{b^2} \; (y^{\alpha} e^{\beta} - y^{\beta} e^{\alpha}) \quad . \tag{7c}$$

The $[\alpha\beta]$ are antisymmetric $O(4)$ indices. Projecting out the self dual part of $A^{\alpha\beta}$ yields the instanton, i.e.

$$-g\, A^{i} = \frac{1}{2} \; \eta^{i}_{\alpha\beta} \; A^{\alpha\beta}$$

$$= \frac{2\, \eta^{i}_{\alpha\mu}\, y^{\alpha}}{b^2 + y^2} \; dy^{\mu} \qquad i = 1,2,3 \quad , \tag{8}$$

where g is the SU(2) coupling constant and $\eta^{i}_{\alpha\beta}$ are the 't Hooft symbols [7].

The A^{i} defined by Eq.(8) is an instanton of "size" b on an S_4 of radius a. The SO(5) invariant metric on this sphere can itself be constructed from (7b). As is clear from (7b), the coefficients $e_{\mu}^{\ \alpha}$ of dy^{μ} given by

$$e^{\alpha}_{\ \mu} = \frac{2ab}{b^2 + y^2} \; \delta_{\mu}^{\ \alpha} \tag{9a}$$

are non-singular 4×4 matrices everywhere on U^{+} . Therefore $e^{\alpha}(y)$ can be regarded as a basis of the cotangent space of U^{+} at the point y. Now the SO(5) invariant metric of S_4 is clearly the one relative to which e^{α} are orthonormal, i.e.

$$g_{\mu\nu}(y) = e^{\alpha}_{\mu} \; e^{\beta}_{\nu} \; \delta_{\alpha\beta} = \frac{4a^2 b^2}{(b^2 + y^2)^2} \; \delta_{\mu\nu} \quad . \tag{9b}$$

This way of constructing the instanton links it intimately with the Riemannian structure of S_4. It makes its SO(5) invariance [8] manifest and consequently simplifies the problem of mode decomposition (cf. Sec.III). To summarize our ansatz for the solution of Eqs.(2a) and (2b) consists of

$$dS^2 = -{dx^0}^2 + {dx^1}^2 + {dx^2}^2 + {dx^3}^2 + \frac{4a^2 b^2}{(b^2 + y^2)^2} \; ({dy^1}^2 + \dots + {dy^4}^2) \tag{10a}$$

$$A^{i} = \frac{-2}{g} \; \frac{\eta^{i}_{\alpha\beta}\, y^{\alpha}}{b^2 + y^2} \; dy^{\beta} \qquad i = 1,2,3 \quad . \tag{10b}$$

The substitution of this ansatz into Eqs.(2a) and (2b) – or equivalently into Eq.(4) reduces these equations into the following algebraic relations between κ, a, g and λ :

$$\frac{\kappa}{ga} = \sqrt{2} \quad , \tag{11a}$$

$$\frac{6}{a^2 \kappa^2} = \lambda \quad . \tag{11b}$$

Notice that this solution is independent of the instanton scale parameter b.

III. FLUCTUATIONS AND GAUGE FIXING

To define the low-energy four-dimensional theory we have to study the spectrum of the small perturbations around the configuration (10). For this, we write an arbitrary metric $g_{MN}(x,y)$ of the eight-dimensional space-time as

$$g_{MN}(x,y) = \langle g_{MN} \rangle + \kappa h_{MN}(x,y) \qquad M,N = 0\ldots 3,\ 5\ldots 8 \tag{12a}$$

where $\langle g_{MN} \rangle$ are the metrical components defined by (10a) and $h_{MN}(x,y)$ are the small perturbations which depend – in a completely arbitrary but continuous way – on the all eight co-ordinates (x^{m}, y^{μ}). We similarly write

$$A^{i}_{M} = \langle A^{i}_{M} \rangle + V^{i}_{M}(x,y) \qquad M = 0,\ldots,8,\ i = 1,2,3 \quad , \tag{12b}$$

where $\langle A^{i}_{M} \rangle$ should be read off Eq.(10b) and V^{i}_{M} are the perturbations. The next step is to substitute (12a) and (12b) into Eq.(1) and expand it in a power series of h and V. This yields

$$\begin{aligned} S_2(h,V^{i}) = \frac{1}{\kappa^4} \int d^8z\, g^{1/2} \Big\{ & \frac{1}{4} h_{AB} \nabla^2_C h_{AB} - \frac{1}{8} h_{AA} \nabla^2_C h_{BB} + \\ & \frac{1}{2} R_{BC}\, h_{AB}\, h_{AC} + \\ & \frac{1}{2} \vec{V}_A \nabla^2_B \vec{V}_A - g \langle \vec{F}_{AB} \rangle \cdot \vec{V}_A \times \vec{V}_B + \\ & + \frac{1}{2} R_{AB}\, \vec{V}_A \cdot \vec{V}_B \\ & - \kappa \Big[h_{AB} \langle \vec{F}_{BC} \rangle - \frac{1}{4} h_{BB} \langle \vec{F}_{AC} \rangle \Big] \cdot \Big[\nabla_C \vec{V}_A - \nabla_A \vec{V}_C \Big] \\ & + \frac{1}{2} \nabla_A \Big[h_{AB} - \frac{1}{2} \eta_{AB} h_{CC} \Big] \nabla_D \Big[h_{BD} - \frac{1}{2} \eta_{BD} h_{EE} \Big] + \frac{1}{2} (D_A \vec{V}_A)^2 \Big\} \end{aligned} \tag{13}$$

All the tensorial indices in Eq.(13) refer to an orthonormal frame of $M_4 \times S_4$, and ∇_A is a covariant derivative containing both the Riemannian connection ω_A of $M_4 \times S_4$ and the Yang-Mill connection given by (10b).
To study the stability of the configuration (10) one must study the response of the system to some external physical disturbances. This is done most conveniently by coupling the perturbations h_{AB} and $\vec{V}_A$ to appropriate sources T_{AB} and J_A, respectively. Thus S_2 should be replaced by

$$S_2'(h,V) = S_2 + \int d^8 z \; g^{1/2} \left\{ \frac{1}{2} T_{AB} h_{AB} + \vec{J}_A \cdot \vec{V}_A \right\} \quad . \tag{14}$$

The source term is required to respect the symmetries of S_2 . Thus, since S_2 is invariant under the following local gauge transformations:

$$\delta h_{AB} = \xi_{A\cdot B} + \xi_{B\cdot A} \tag{15a}$$

$$\delta \vec{V}_A = \xi^B \vec{F}_{AB} + \vec{\Omega}_{\cdot A} \quad , \tag{15b}$$

where $\xi_A(x,y)$ and $\vec{\Omega}(x,y)$ are erbitrary, we must have

$$\begin{cases} T_{AB\cdot B} - \kappa \langle \vec{F}_{AB} \rangle \cdot \vec{J}_B = 0 & (16a) \\ \vec{J}_{A\cdot A} = 0 & (16b) \end{cases}$$

We choose to fix the gauge by imposing the conditions

$$(h_{AB} - \frac{1}{2} \eta_{AB} h_{CC})_{\cdot B} = 0 \quad . \tag{17a}$$

$$\vec{V}_{A\cdot A} = 0 \quad . \tag{17b}$$

The equations of motion then take the form

$$\begin{aligned} \nabla_C^2 h_{AB} - \frac{1}{4} \eta_{AB} \nabla_C^2 h_{DD} + (R_{BC} h_{AC} + R_{AC} h_{BC}) - \\ -\kappa \left(\langle \vec{F}_{BC} \rangle \cdot (\nabla_C \vec{V}_A - \nabla_A \vec{V}_C) + \langle \vec{F}_{AC} \rangle \cdot (\nabla_C \vec{V}_B - \nabla_B \vec{V}_C) \right) \\ + \frac{\kappa}{2} \eta_{AB} \langle \vec{F}_{DC} \rangle \cdot (\nabla_C \vec{V}_D - \nabla_D \vec{V}_C) + T_{AB} = 0 \quad . \end{aligned} \tag{18a}$$

$$\begin{aligned} \nabla_B^2 \vec{V}_A - g \langle \vec{F}_{AB} \rangle \times \vec{V}_B + R_{AB} \vec{V}_B + \kappa \left\{ h_{AB} \langle \vec{F}_{BC} \rangle - \frac{1}{4} h_{DD} \langle \vec{F}_{AC} \rangle \right\}_{\cdot C} \\ - \kappa \left\{ h_{BC} \langle \vec{F}_{BA} \rangle - \frac{1}{4} h_{EE} \langle \vec{F}_{CA} \rangle \right\}_{\cdot C} + \vec{J}_A = 0 \end{aligned} \tag{18b}$$

Eqs.(17) and (18) should be solved for h_{AB} and $\vec{V}_A$ in terms of T_{AB} and $\vec{J}_A$.

IV. SOLUTION OF THE LINEARIZED EQUATIONS

Eq.(18) are a set of linear inhomogeneous partial differential equations. To solve them one has to look for a complete set of basis functions which can be used to expand h_{AB} and $\vec{V}_A$. In our case this set should be chosen to be the eigenfunctions of an appropriately defined Laplacian. Such a Laplacian must be an invariant operator of the symmetry group of the background fields. For our problem this group is $P_4 \times SO(5)$. Since the manifold is a product $M_4 \times S_4$, therefore the Laplace operator ∇_A^2 will also decompose into $\nabla_a^2 + \nabla_\alpha^2$, where $\nabla_a^2 = \partial_a^2$ is the usual P_4 invariant Laplacian and ∇_α^2 is an SO(4) invariant operator on S_4.

In the formalism of Gel'fand and Zetlin an irreducible unitary representation of the group SO(5) is characterized by two numbers (n_1,n_2) both integers or half an odd integer. Then for a given value of (n_1,n_2) the basis vectors of the representation space are written as

$$\left| \begin{matrix} n_1 & n_2 \\ m_1 & m_2 \\ & j \\ & \lambda \end{matrix} \right\rangle \qquad \begin{matrix} n_1 \geq m_1 \geq n_2 \geq m_2 \geq -n_2 \\ m_1 \geq j \geq m_2 \\ j \geq \lambda \geq -j \end{matrix} \quad , \tag{19}$$

where (m_1, m_2) characterize the SO(4) content of the given SO(5) representation. Similarly j and λ specify the SO(3) and the U(1) contents, respectively.

Now we can follow the prescriptions for harmonic expansion discussed in detail in Ref.9. The first step is to decompose the fields in Eq.(18) into their $M_4 \times S_4$ components, i.e.

$$\begin{aligned} h_{AB} &= \{ h_{ab}, h_{a\alpha}, h_{\alpha\beta} \} \\ \vec{V}_A &= \{ \vec{V}_a , \vec{V}_\alpha \} \end{aligned} \qquad \begin{pmatrix} a = 0,1,2,3 \\ \alpha = 5,6,7,8 \end{pmatrix} \quad .$$

The next step is to decompose the (α,i)-pair into the irreducible representations of SO(4) specified by (m_1,m_2). From these values of (m_1,m_2) we can read off the allowed ranges of (n_1,n_2) for each field. We give the results in the following table:

Field	O(4) dimensionality	(m_1,m_2)	(n_1,n_2)	
n_{ab}	1	(0,0)	(n,0)	$n \geq 0$
$h_{a\beta}$	4	(1,0)	(n,1),(n,0)	$n > 1$
$h_{\alpha\alpha}$	1	(0,0)	(n,0)	$n \geq 0$
$h^t_{\alpha\beta}$	9	(2,0)	(n,2),(n,1),(n,0)	$n \geq 2$
$V_{a[\alpha\beta+]}$	3	(1,1)	(n,1)	$n \geq 1$
$V^t_{\alpha[\beta\gamma+]}$	8	(2,1)	(n,2),(n,1)	$n \geq 2$
V_α	4	(1,0)	(n,1),(n,0)	$m \geq 1$

Table I

The list of fields to be expanded into SO(5) harmonics

In this table the following definitions have been used:

$$h^t_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{4}\,\delta_{\alpha\beta}\,h_{\gamma\gamma} \tag{20a}$$

$$V_{a[\alpha\beta+]} = n^i_{\alpha\beta}\; V^i_a = +\frac{1}{2}\,\epsilon_{\alpha\beta\gamma\delta}\,V_{a[\gamma\delta+]} \tag{20b}$$

$$V_\alpha = \frac{1}{3}\;n^i_{\beta\alpha}\;V^i_\beta \tag{20c}$$

$$V^t_{\alpha[\beta\gamma+]} = n^i_{\beta\gamma}\,V^i_\alpha - \left[\delta_{\alpha\beta}V_\gamma - \delta_{\alpha\gamma}V_\beta + \epsilon_{\alpha\beta\gamma\delta}\,V_\delta\right] = +\frac{1}{2}\,\epsilon_{\beta\gamma\epsilon\omega}\,V^t_{\alpha[\epsilon\omega+]} \tag{20d}$$

$$V^t_{\alpha[\alpha\gamma+]} = 0 \;. \tag{20e}$$

The dimensionality of an SO(5) representation defined by (n_1,n_2) is given by

$$d(n_1,n_2) = \frac{1}{6}\,(n_1+n_2+2)\,(n_1-n_2+1)(2n_1+3)(2n_2+1) \;. \tag{21a}$$

Similarly for the SO(4) representation

$$d(m_1,m_2) = (m_1+m_2+1)\,(m_1-m_2+1) \;. \tag{21b}$$

For the mode decomposition on S_4 we use the matrix elements of SO(5) in the basis (19). We denote them by $D^{n_1 n_2}_{p,q}(L_y^{-1})$ and assume the following orthogonality relations:

$$\sum_i \int_{S_4} D^{n_1 n_2}_{p,i}(L_y)\,D^{n_1' n_2'}_{i,p'}(L_y^{-1}) = \frac{d(m_1,m_2)}{d(n_1,n_2)}\;\delta^{n_1 n_1'}\,\delta^{n_2 n_2'}\,\delta^{pp'}\;V_{S_4} \tag{21c}$$

where $d(m_1m_2)$ is the dimensionality of the SO(4) representation denoted by the index i.

It can be shown that [10] $D^{n_1,n_2}_{p,q}(L_y^{-1})$ are the eigenfunctions of the Laplacian ∇^2_α

$$\nabla_\alpha \nabla^\alpha D^{n_1 n_2}_{p,q}(L_y^{-1}) = \frac{-1}{a^2}\left(C_2^{n_1 n_2} - C_2^{m_1 m_2}\right) D^{n_1 n_2}_{p,q}(L_y^{-1}) \tag{21d}$$

Here $C_2^{n_1 n_2}$ and $C_2^{m_1 m_2}$ are the second order Casimirs of SO(5) and SO(4) respectively. They are given by

$$C_2^{n_1 n_2} = n_1(n_1+3) + n_2(n_2+1) \;, \tag{21e}$$

$$C_2^{m_1 m_2} = m_1(m_1+2) + m_2^2 \;. \tag{21f}$$

The representation (m_1,m_2) is associated with the index p of Eq.(21d). With these conventions we can now expand the fields in Table I in the basis of $D^{n_1 n_2}_{p,q}(L_y^{-1})$. For further details see Ref.9.

$$h_{ab}(x,y) = \sum_{\substack{n\geq 0\\ q}} \left(\sqrt{d(n,0)}\;\; h_{ab}(x)^{n0}_q\;\; D^{n,0}_{1,q}(L_y^{-1})\right) \tag{22a}$$

$$h_{a\beta}(x,y) = \sum_{\substack{n\geq 1\\ q}} \left(\sqrt{\frac{d(n,1)}{4}}\; h_a(x)^{n1}_q\, D^{n,1}_{\beta,q}(L_y^{-1}) + \sqrt{\frac{d(n,0)}{4}}\; h_a(x)^{n0}_q\, D^{n,0}_{\beta,q}(L_y^{-1})\right) \tag{22b}$$

$$h_{\alpha\alpha}(x,y) = \sum_{\substack{n\geq 0\\ q}} \left(\sqrt{d(n,0)}\;\; \tilde{h}(x)^{n0}_q\;\; D^{n,0}_{1,q}(L_y^{-1})\right) \tag{22c}$$

$$h^t_{(\alpha\beta)}(x,y) = \sum_{\substack{n\geq 2\\ q}} \left(\sqrt{\frac{d(n,2)}{9}}\; h(x)^{n2}_q\; {}^tD^{n,2}_{(\alpha\beta),q}(L_y^{-1}) + \sqrt{\frac{d(n,1)}{9}}\; h(x)^{n1}_q\; {}^tD^{n1}_{(\alpha\beta),q}(L_y^{-1}) + \sqrt{\frac{d(n,0)}{9}}\; h(x)^{n0}_q\; {}^tD^{n0}_{(\alpha\beta),q}(L_y^{-1})\right) \tag{22d}$$

$$V_{\alpha[\alpha\beta\gamma]} = \sum_{\substack{n\geq 1\\ q}} \left(\sqrt{\frac{d(n,1)}{3}}\; V_{\alpha}(x)^{n1}_{q}\; D_{[\alpha\beta\gamma],q}(L_y^{-1}) \right) \tag{22e}$$

$$V^{t}_{\alpha[\beta\gamma\delta]}(x,y) = \sum_{\substack{n\geq 2\\ q}} \left(\sqrt{\frac{d(n,2)}{6}}\; V(x)^{n2}_{q}\; D^{n,2}_{\alpha[\beta\gamma\delta],q}(L_y^{-1}) + \sqrt{\frac{d(n,1)}{6}}\; \tilde{V}(x)^{n1}_{q}\; D^{n1}_{\alpha[\beta\gamma\delta],q}(L_y^{-1}) \right) \tag{22f}$$

$$V_{\alpha}(x,y) = \sum_{\substack{n\geq 1\\ q}} \left(\sqrt{\frac{d(n,1)}{4}}\; V(x)^{n1}_{q}\; D^{n,1}_{\alpha,q}(L_y^{-1}) + \sqrt{\frac{d(n,0)}{4}}\; V(x)^{n0}_{q}\; D^{n,0}_{\alpha,q}(L_y^{-1}) \right) \tag{22g}$$

As is easily seen only three series of SO(5) modes enter these expansions. They are characterized by (n,0), (n,1) and (n,2). The mode decomposition of the linearized equations which are obtained from the substitution of Eqs.(22) into Eqs.(18) will also be classified into three separate classes. In substituting the above expansions into Eqs.(18) one has to employ all the relations (20), (21). Here we shall omit the details and give the result after Fourier transformations to the p-space. We shall also suppress the label q, as it plays no role in the spectrum. We also separate the trace part of h_{ab}, i.e.

$$h^{t}_{ab} = h_{ab} - \tfrac{1}{4}\, \eta_{ab}\, h_{cc} \;;\; h^{t}_{aa} = 0 \;.$$

Similarly for T^{t}_{ab}.

<u>(n,0) sector</u>

$$\left(p^2 + \frac{n(n+3)}{a^2}\right) h^{t\,n0}_{ab} = T^{t\,n0}_{ab} \tag{23a}$$

$$\left(p^2 + \frac{n(n+3)}{a^2}\right) h^{n0}_{aa} + 2\left(p^2 + \frac{n(n+3)}{a^2}\right)\tilde{h}^{n0} + \frac{12}{a^2}\sqrt{2n(n+3)}\; V^{n0} = -T^{n0}_{aa} \tag{23b}$$

$$\left(p^2 + \frac{n(n+3)-6}{a^2}\right)\tilde{h}^{n0} + 2\left(p^2 + \frac{n(n+3)}{a^2}\right) h^{n0}_{aa} = -\tilde{T}^{n0} \tag{23c}$$

[illegible] (23d)

$$\left(p^2 + \frac{n(n+3)-4}{a^2}\right) V^{n0} + \frac{1}{a^2}\sqrt{\tfrac{1}{3}(n-1)(n+4)}\; h^{n0} + \frac{1}{4a^2}\sqrt{2n(n+3)}\; \tilde{h}^{n0} = J^{n0} \tag{23e}$$

$$\left(p^2 + \frac{n(n+3)-2}{a^2}\right) h^{n0} - \frac{2}{a^2}\sqrt{3(n-1)(n+4)}\; V^{n0} = T^{n0} \tag{23f}$$

<u>(n-1) sector</u>

$$\left(p^2 + \frac{n(n+3)+2}{a^2}\right) h^{n1}_{a} + \frac{1}{a^2}\sqrt{2(n+1)(n+2)}\; V^{n1}_{a} - \frac{3\sqrt{2}}{a}\, i p_a V^{n1} = T^{n1}_{a} \tag{24a}$$

$$\left(p^2 + \frac{n(n+3)-2}{a^2}\right) V^{n1}_{a} + \frac{1}{a^2}\sqrt{2(n+1)(n+2)}\; h^{n1}_{a} = J^{n1}_{a} \tag{24b}$$

$$\left(p^2 + \frac{n(n+3)-2}{a^2}\right) V^{n1} + \frac{\sqrt{2}}{a}\, i p_a h_a - \frac{2}{3a^2}\sqrt{2(n-1)(n+4)}\; h^{n1} = J^{n1} \tag{24c}$$

$$\left(p^2 + \frac{n(n+3)-2}{a^2}\right) h^{n1} + \frac{2}{a^2}\sqrt{\tfrac{1}{3}(n+1)(n+2)}\; \tilde{V}^{n1} - \frac{2}{a^2}\sqrt{2(n-1)(n+4)}\; V^{n1} = T^{n1} \tag{24d}$$

$$\left(p^2 + \frac{n(n+3)-2}{a^2}\right) h^{n1} + \frac{2}{a^2}\sqrt{\tfrac{1}{3}(n+1)(n+2)}\; h^{n1} = \tilde{J}^{n1} \tag{24e}$$

<u>(n,2) sector</u>

$$\left(p^2 + \frac{n(n+3)+4}{a^2}\right) h^{n2} + \frac{2}{a^2}\sqrt{(n+1)(n+2)}\; V^{n2} = T^{n2} \tag{25a}$$

$$\left(p^2 + \frac{n(n+3)+2}{a^2}\right) V^{n2} + \frac{2}{a^2}\sqrt{(n+1)(n+2)}\; h^{n2} = J^{n2} \tag{25b}$$

In writing these equations we have used the harmonic expansion of the gauge conditions (17) to simplify some of the expressions. If we expand Eq.(17) we get

$$i p_b\, {}^{t}h_{ab}^{n0} - \frac{1}{4} i p_a\, h_{bb}^{n0} - \frac{1}{2} i p_a\, \tilde{h}^{n0} - \frac{1}{a}\sqrt{n(n+3)}\; h_a^{n0} = 0 \tag{26a}$$

$$i p_a\, h_a^{n0} - \frac{1}{a}\sqrt{\tfrac{3}{2}(n-1)(n+4)}\; h^{n0} - \frac{1}{2}\frac{1}{a}\sqrt{n(n+3)}\left(\frac{1}{2}\tilde{h}^{n0} + h_{aa}^{n0}\right) = 0 \tag{26b}$$

$$i p_a\, h_a^{n1} - \frac{1}{a}\sqrt{(n-1)(n+4)}\; h^{n1} = 0 \tag{26c}$$

$$i p_a\, V_a^{n1} - \frac{1}{a}\sqrt{\tfrac{2}{3}(n-1)(n+4)}\; \tilde{V}^{n1} - \frac{1}{a}\sqrt{(n+1)(n+2)}\; V^{n1} = 0 \tag{26d}$$

As a check on our calculations we have substituted from Eqs.(23)-(25) into the harmonic expansion of Eqs.(16) and shown that they are satisfied by virtue of Eqs.(26). The harmonic expansion of Eq.(16) is given by

$$i p_b\, {}^{t}T_{ab}^{n0} + \frac{1}{4} i p_a\, T_{bb}^{n0} - \frac{1}{a}\sqrt{n(n+3)}\; T_a^{n0} = 0 \tag{27a}$$

$$i p_a\, T_a^{n0} - \frac{1}{a}\sqrt{\tfrac{3}{2}(n-1)(n+4)}\; T^{n0} + \frac{1}{4a}\sqrt{n(n+3)}\; \tilde{T}^{n0} - \frac{3\sqrt{2}}{a} J^{n0} = 0 \tag{27b}$$

$$i p_a\, T_a^{n1} - \frac{1}{a}\sqrt{(n-1)(n+4)}\; T^{n1} - 3\frac{\sqrt{2}}{a} J^{n1} = 0 \tag{27c}$$

$$i p_a\, J_a^{n1} - \frac{1}{a}\sqrt{\tfrac{2}{3}(n-1)(n+4)}\; \tilde{J}^{n1} - \frac{1}{a}\sqrt{(n+1)(n+2)}\; J^{n1} = 0 \tag{27d}$$

Now we can solve Eqs.(23)-(25). The results are as follows:

(n,0) sector

$${}^{t}h_{ab}^{n0} = \frac{{}^{t}T_{ab}^{n0}}{p^2 + {}^{n0}M_0^2} \tag{28a}$$

$$V^{n0} = \frac{\left(p^2 + {}^{n0}M_4^2\right)\left[\left(p^2 + {}^{n0}M_5^2\right) J^{n0} - \frac{1}{a^2}\sqrt{\tfrac{1}{3}(n-1)(n+4)}\; T^{n0}\right]}{\left(p^2 + {}^{n0}M_1^2\right)\left(p^2 + {}^{n0}M_2^2\right)\left(p^2 + {}^{n0}M_3^2\right)} + \frac{\frac{1}{6a^2}\sqrt{2n(n+3)}\left(p^2 + {}^{n0}M_5^2\right)\left(T_{aa}^{n0} - \frac{1}{2}\tilde{T}^{n0}\right)}{\left(p^2 + {}^{n0}M_1^2\right)\left(p^2 + {}^{n0}M_2^2\right)\left(p^2 + {}^{n0}M_3^2\right)} \tag{28b}$$

$$h_a^{n0} = \frac{T_a^{n0} + \frac{3\sqrt{2}}{a^2}\, i a p_a V^{n0}}{\left(p^2 + {}^{n0}M_0^2\right)} \tag{28c}$$

$$\tilde{h}^{n0} = \frac{\frac{1}{3}\tilde{T}^{n0} - \frac{2}{3}T_{aa}^{n0} - \frac{8}{a^2}\sqrt{2n(n+3)}\; V^{n0}}{\left(p^2 + {}^{n0}M_0^2\right)} \tag{28d}$$

$$h^{n0} = \frac{T^{n0} + \frac{2}{a^2}\sqrt{3(n-1)(n+4)}\; V^{n0}}{\left(p^2 + {}^{n0}M_5^2\right)} \tag{28e}$$

$$h_{aa}^{n0} = \frac{\left(p^2 + {}^{n0}M_1^2\right)\left[\frac{1}{3}T_{aa}^{n0} + \frac{4}{a^2}\sqrt{2n(n+3)}\; V^{n0}\right] - \frac{2}{3}\left(p^2 + \frac{1}{4}{}^{n0}M_1^2 + \frac{3}{4}{}^{n0}M_4^2\right)\tilde{T}^{n0}}{\left(p^2 + {}^{n0}M_1^2\right)\left(p^2 + {}^{n0}M_4^2\right)} \tag{28f}$$

Here the tower of masses is given by

$$ {}^{n0}M_0^2 = \frac{n(n+3)}{a^2} \tag{29a} $$

$$ {}^{n0}M_1^2 = \frac{n(n+3)-6}{a^2} \tag{29b} $$

$$ {}^{n0}M_2^2 = \frac{n(n+3)+1-\sqrt{2n(n+3)+1}}{a^2} \tag{29c} $$

$$ {}^{n0}M_3^2 = \frac{n(n+3)+1+\sqrt{2n(n+3)+1}}{a^2} \tag{29d} $$

$$ {}^{n0}M_4^2 = \frac{n(n+3)+2}{a^2} \tag{29e} $$

$$ {}^{n0}M_5^2 = \frac{n(n+3)-2}{a^2} \tag{29f} $$

We notice that for n = 1, $({}^{no}M_1)^2 = \frac{-2}{a^2}$ corresponds to a tachyonic mass. This could be a sign of instability. We shall show however that the residue at the simple pole $p^2 = {}^{no}M_1^2$ vanishes for all n. Thus this pole is a gauge artifact and has no physical significance.

We shall show that there are also several massless poles corresponding to n = 0. However except for graviton the rest are illusory and non-physical.

<u>(n,1) sector</u>

In this sector there is a massless spin-1 pole at n = 1. It is convenient to handle these modes separately. We choose a Lorentz frame in which

$$ p_a = (p_0, 0, 0, p_3) $$

Then the solutions are given by

$$ h^{11} = 0 \tag{30a} $$

$$ V^{11} = \frac{J^{11}}{p^2 + \frac{2}{a^2}} \tag{30b} $$

$$ \tilde{V}^{11} = \frac{\tilde{J}^{11}}{p^2 + \frac{2}{a^2}} \tag{30c} $$

$$ h_a^{11} = \frac{\left(p^2 + \frac{2}{a^2}\right) T_a^{11} + \frac{3\sqrt{2}}{a}\, i p_a J^{11} - 2\frac{\sqrt{3}}{a^2} J_a^{11}}{p^2\left(p^2 + \frac{2}{a^2}\right)} \tag{30d} $$

$$ V_a^{11} \quad \frac{\left(p^2 + \frac{6}{a^2}\right) J_a^{11} - \frac{2\sqrt{3}}{a^2} T_a^{11} - 6\frac{\sqrt{6}}{a^3} \frac{i p_a J^{11}}{p^2 + \frac{2}{a^2}}}{p^2\left(p^2 + \frac{8}{a^2}\right)} \tag{30e} $$

For n > 1 we only have massive modes. They are described most conveniently in the rest frame

$$ p_a = (p_0, \underset{\sim}{0}) $$

$$ h^{n1} = \frac{\frac{2}{a^2}\sqrt{2(n-1)(n+4)}\; J^{n1} + \left(p^2 + {}^{n}M_3^2\right) T^{n1} - \frac{2}{a^2}\sqrt{\frac{1}{3}(n+1)(n+2)}\; \tilde{J}^{n1}}{\left(p^2 + {}^{n}M_1^2\right)\left(p^2 + {}^{n1}M_2^2\right)} \tag{31a} $$

$$ V^{n1} = \frac{\left[\left(p^2 + {}^{n}M_1^2\right)\left(p^2 + {}^{n}M_2^2\right) - \frac{4}{3a^4}(n-1)(n+4)\right] J^{n1} - \frac{1}{3a^2}\sqrt{2(n-1)(n+4)}\; \left(p^2 + {}^{n}M_3^2\right) T^{n1}}{\left(p^2 + {}^{n}M_1^2\right)\left(p^2 + {}^{n}M_2^2\right)\left(p^2 + {}^{n}M_3^2\right)} + $$

$$ + \frac{\frac{2}{3a^4}\sqrt{\frac{2}{3}(n+1)(n+2)(n-1)(n+4)}\; \tilde{J}^{n1}}{\left(p^2 + {}^{n}M_1^2\right)\left(p^2 + {}^{n}M_2^2\right)\left(p^2 + {}^{n1}M_3^2\right)} \tag{31b} $$

$$V^{n1} = \frac{[\text{illegible}]}{(p^2+{}^{n}M_1^2)(p^2+{}^{n}M_2^2)(p^2+{}^{n}M_3^2)} +$$

$$+ \frac{-\frac{2}{a^2}\sqrt{\frac{1}{3}(n+1)(n+2)}\,(p^2+{}^{m}M_3^2)T^{n1} - \frac{4}{a^4}\sqrt{\frac{2}{3}(n+1)(n+2)(n-1)(n+4)}\;J^{n1}}{(p^2+{}^{m}M_1^2)(p^2+{}^{m}M_2^2)(p^2+{}^{n}M_3^2)} \tag{31c}$$

$$h_a^{n1} = \frac{(p^2+{}^{m}M_3^2)T_a^{n1} - \frac{1}{a^2}\sqrt{2(n+1)(n+2)}\;T_a^{n1} + \frac{3\sqrt{2}}{a}\,i p_a (p^2+{}^{m}M_3^2)V^{n1}}{(p^2+{}^{m}M_4^2)(p^2+{}^{m}M_5^2)} \tag{31d}$$

$$V_a^{n1} = \frac{(p^2+{}^{m}M_2^2)J_a^{n1} - \frac{1}{a^2}\sqrt{2(n+1)(n+2)}\,(p^2+{}^{m}M_3^2)T_a^{m} - \frac{6}{a^3}\sqrt{(n+1)(n+2)}\,i p_a V^{n1}}{(p^2+{}^{m}M_4^2)(p^2+{}^{m}M_5^2)} \tag{31e}$$

The spectrum of masses is given by

$${}^{m}M_1^2 = \frac{n(n+3)-4}{a^2} \tag{32a}$$

$${}^{m}M_2^2 = \frac{n(n+3)+2}{a^2} \tag{32b}$$

$${}^{m}M_3^2 = \frac{n(n+3)-2}{a^2} \tag{32c}$$

$${}^{n1}M_4^2 = \frac{n(n+3)+\sqrt{2(n^2+3n+4)}}{a^2} \tag{32d}$$

$${}^{n1}M_5^2 = \frac{n(n+3)-\sqrt{2(n^2+3n+4)}}{a^2} \tag{32e}$$

In this sector there are no tachyonic poles.

Finally we give the solutions in the (n,2) sector

$$V^{n2} = \frac{\left(p^2+\frac{n(n+3)+4}{a^2}\right)J^{n2} - \frac{2}{a^2}\sqrt{(n+1)(n+2)}\;T^{n2}}{(p^2+{}^{n2}M_1^2)(p^2+{}^{n2}M_2^2)} \tag{33a}$$

$$h^{n2} = \frac{\left(p^2+\frac{n(n+3)+2}{a^2}\right)T^{n2} - \frac{2}{a^2}\sqrt{(n+1)(n+2)}\;J^{n2}}{(p^2+{}^{n2}M_1^2)(p^2+{}^{n2}M_2^2)} \tag{33b}$$

The masses are given by

$${}^{n2}M_1^2 = \frac{n(n+1)}{a^2} \tag{34a}$$

$${}^{n2}M_2^2 = \frac{n(n+5)+6}{a^2} \; . \tag{34b}$$

For all $n \geq 2$ the masses are non tachyonic.

Thus except for the n = 1 tachyonic pole in the (n,0) sector given by Eq.(38b) all of the M^2 are positive. In order to complete the proof of stability we have to compute the residues at $p^2 = -M^2$ for all M^2 and show that all these residues are non-negative. The residue at the tachyonic pole should vanish in order for the theory to be stable.

V. COMPUTATION OF THE RESIDUES AT $p^2 = -M^2$

In this section we shall present the result of the computation of the residues of the poles of the propagators sandwiched between conserved sources. Let us start with (n,0) sector. In this sector there is a massless pole at n = 0. It is convenient to start from Eqs.(23) for n = 0 and solve the The result is given by

$$h_{ab}^{t\,00} = \frac{[illegible]}{p^2} \qquad (35a)$$

$$\tilde{h} = \frac{-\frac{2}{3} T_{aa}^{00} + \frac{1}{3} \tilde{T}^{00}}{p^2 + \frac{2}{a^2}} \qquad (35b)$$

$$h_{aa}^{00} = \frac{-T_{aa}^{00}}{p^2} - 2\,\tilde{h}^{00} \quad . \qquad (35c)$$

For the massles pole at $p^2 = 0$

$$R_0^{00} = \lim_{p^2 \to 0} p^2 \left\{ \frac{1}{2} h_{ab}^{t00}\, T_{ab}^{00^*} + \frac{1}{8} h_{aa}^{00}\, T_{ac}^{00^*} \right\}. \qquad (36)$$

If we choose a Lorentz frame in which

$$p_a = (p_0, 0, 0, p_3) \qquad p_0 = p_3 \quad . \qquad (37)$$

Then the conservation laws (27a) yield

$$T_{00}{}^t = T_{30}{}^t + \frac{1}{4} T_{aa} \qquad (38a)$$

$$T_{03}{}^t = T_{33}{}^t + \frac{1}{4} T_{aa} \qquad (38b)$$

$$T_{0r}{}^t = T_{3r}{}^t \quad . \qquad (38c)$$

With the help of these identities one can compute the right-hand side of (36). This result is

$$R_0^{00} = \frac{1}{2} \left| T_{11}^{00} - T_{22}^{00} \right|^2 + \left| T_{12}^{00} \right|^2 . \qquad (39)$$

This is clearly positive and it corresponds to the two degrees of freedom of a graviton which is a singlet of SO(5).

One can similarly show that the residue at $p^2 = -\frac{2}{a^2}$ is given by

$$R = \lim_{(p^2 + \frac{2}{a^2}) \to 0} (p^2 + \frac{2}{a^2}) \left\{ \frac{1}{8} h_{aa} T_{bb}^* + \frac{1}{8} h^* T^* \right\}$$

$$= \frac{1}{6} \left| T_{aa} - \frac{1}{2} \tilde{T} \right|^2 \quad . \qquad (40)$$

[illegible] is a singlet of SO(5).

Now let us consider the rest of the modes in the (n,0) sector. We can assume $n \neq 0$. There are six simple poles as listed in Eq.(29). We must compute the following six residues:

$$R_i = \lim_{(p^2 + {}^{no}M_i^2) \to 0} (p^2 + {}^{no}M_i^2) \left\{ \frac{1}{2} h_{ab}^{t\,no} T_{ab}^{t\,no^*} + \frac{1}{8} h_{aa}^{no} T_{aa}^{no^*} + \frac{1}{8} \tilde{h}^{no} \tilde{T}^{no^*} + h^{no} T^{no^*} + h_a^{no} T_a^{no^*} + 3 V^{no} J^{no^*} \right\} \qquad (41)$$

Here $i = 0,1,\ldots,5$ and ${}^{no}M_i^2$ are given by Eqs.(29). After a tedious calculation and numerous applications of the conservation laws (27) we obtain the following result:

$$R_0 = \frac{1}{2} \left| T_{ij} - \frac{1}{3} \delta_{ij} T_{kk} \right|^2 \qquad i,j = 1,2,3 \qquad (42a)$$

$$R_1 = R_4 = R_5 = 0 \qquad (42b)$$

$$R_2 = \left[\prod_{\substack{i=0 \\ i \neq 1 \neq 2}}^{5} (-{}^{no}M_2^2 + {}^{no}M_i^2) \right]^{-1} \Big| -\sqrt{3} (-{}^{no}M_2^2 + {}^{no}M_5^2)(-{}^{no}M_2^2 + {}^{no}M_4^2) J^{no} + \frac{1}{a^2} \sqrt{(n-1)(n+4)}\, (-{}^{no}M_2^2 + {}^{no}M_4^2)\, T^{no} + \frac{1}{a^2} \sqrt{\frac{n}{24}(n+3)}\, (-{}^{no}M_2^2 + {}^{no}M_5^2) (\tilde{T}^{no} - 2\, T_{aa}^{no}) \Big|^2 \qquad (42c)$$

where

$$\prod_{\substack{i=0 \\ i \neq 1 \neq 2}}^{5} (-{}^{no}M_2^2 + {}^{no}M_i^2) = \frac{2}{a^8} \left(-1 + \sqrt{2n(n+3)+1}\right) \sqrt{2n(n+3)+1} \left(1 + \sqrt{2n(n+3)+1}\right)\left(-3 + \sqrt{2n(n+3)+1}\right) \qquad (42d)$$

Similarly

$$R_3 = \left[\prod_{\substack{i=0 \\ i\neq 1,3}}^{5} \left(-{}^{no}M_3^2 + {}^{no}M_i^2 \right) \right]^{-1} \left| \left(-{}^{no}M_3^2 + {}^{no}M_4^2 \right) \left[-\sqrt{3} \left(-{}^{no}M_3^2 + {}^{no}M_5^2 \right) J^{no} + \frac{1}{a^2} \sqrt{(n-1)(n+4)}\; T^{no} \right] + \frac{1}{a^2} \sqrt{\frac{n(n+3)}{24}} \left(-{}^{no}M_3^2 + {}^{no}M_5^2 \right) \left(\tilde{T}^{no} - 2 T_{ac}^{no} \right) \right|^2 \tag{42e}$$

where

$$\prod_{\substack{i=0 \\ i\neq 1,3}}^{5} \left(-{}^{no}M_3^2 + {}^{no}M_i^2 \right) = \frac{2}{a^8} \left(1 + \sqrt{2n(n+3)+1}\right)\left(\sqrt{2n(n+3)+1}\right)\left(-1 + \sqrt{2n(n+3)+1}\right)\left(3 + \sqrt{2n(n+3)+1}\right) \tag{42f}$$

It is interesting to note that the only propagating massive modes with $(\text{mass})^2 = {}^{no}M_0^2$ are spin-2. Although the vector field h_a^{no} exhibits this pole it does not contribute to the residue. Similarly the contributions of ${}^{no}M_1^2$, ${}^{no}M_4^2$ and ${}^{no}M_5^2$ are all zero. We recall that the tachyonic pole was at ${}^{no}M_1^2 = -\frac{2}{a^2}$. Therefore its residue is also zero and it is not relevant for the stability of our background solution.

<u>(n,1) sector</u>

Now we consider the (n,1) sector. In this sector we have to compute the following residues:

$$R_i = \lim_{p^2 \to 0} p^2 \left\{ h_a^{10}\, T_a^{10^*} + V_a^{10}\, J_a^{10^*} \right\} \tag{43}$$

Here also it is convenient to analyze the massless states corresponding to n = 1 separately. For these states we have to consider

$$R = \lim_{p^2 \to 0} p^2 \left\{ h_a^{10}\, T_a^{10^*} + V_a^{10}\, J_a^{10^*} \right\} . \tag{44}$$

It is again convenient to choose a frame such that

$$p_a = (p_0, 0, 0, p_3) \qquad p_0 = p_3 .$$

Then the conservation laws (27c) and (27d) become

$$T_0 - T_3 = \sqrt{3}\;(J_0 - J_3) . \tag{45}$$

With the help of (45) and (30) one can show that the value of (44) is given by

$$R = \frac{1}{4} \left| T_1^{10} - \sqrt{3}\, J_1^{10} \right|^2 + \frac{1}{4} \left| T_2^{10} - \sqrt{3}\, J_2^{no} \right|^2 . \tag{46}$$

This residue corresponds to the two degrees of freedom of a massless spin-1 field belonging to the 10-dimensional (adjoint) representation of SO(5). These are the SO(5) gauge fields resulting from compactification. As is seen clearly from Eq.(46) they consist of well defined linear mixtures of the perturbations of the metric as well as the instanton. This confirms our previous result in 6 dimensions.

Now we consider the massive modes for which we choose the rest frame

$$p_a = (p_0, \underset{\sim}{0}) .$$

Again with the help of the conservation laws (27c), (27d) and (31) and (32) we can evaluate the value of (43). After a very long and a very tedious calculation one obtains the following gratifying result:

$$R_1 = 0 = R_3 \tag{47a}$$

$$R_2 = \frac{1}{(n+1)(n+2)} \left| 2 T^{n1} - \sqrt{2(n-1)(n+4)}\; J^{n1} + \sqrt{\tfrac{1}{3}(n+1)(n+2)}\; \tilde{J}^{n1} \right|^2 \tag{47b}$$

$$R_4 = \frac{1}{2\sqrt{2(n^2+3n+4)}} \left| \sqrt{2 + \sqrt{2(n^2+3n+4)}}\; T_i^{n1} + \sqrt{-2 + \sqrt{2(n^2+3n+4)}}\; J_i^{n1} \right|^2 \tag{47c}$$

$$R_5 = \frac{1}{2\sqrt{2(n^2+3n+4)}} \left| \sqrt{-2 + \sqrt{2(n^2+3n+4)}}\; T_i^{n1} - \sqrt{-2 + \sqrt{2(n^2+3n+4)}}\; J_i^{n1} \right|^2 \tag{47d}$$

Thus all the non-vanishing residues are manifestly positive. Note that the only physically propagating spin-zero mass is ${}^{n1}M_2$. We also note that R_4 and R_5 indicate the propagation of massive spin-1 particles belonging to the representation (n,1) of SO(5).

(n,2) sector

Finally we compute the residues for the (n,2) sector. Here we have to substitute from Eq.(33) into

$$R_i^{n2} = \lim_{(P^2+M_i^{n2^2})} (P^2 + M_1^{n2^2})(h^{n2}\, T^{n2^*} + V^{n2}\, J^{n2^*}) \quad i = 1,2 \quad , \tag{48}$$

where $({}^{no}M_i)^2$, i = 1,2 are given by (34). The results are

$$R_1^{n2} = \frac{1}{2n+3} \left| \sqrt{n+2}\; J^{n2} - \sqrt{n+1}\; T^{n2} \right|^2 \tag{49a}$$

$$R_2^{n2} = \frac{1}{2n+3} \left| \sqrt{n+1}\; J^{n2} + \sqrt{n+2}\; T^{n2} \right|^2 \tag{49b}$$

We conclude that there is no ghost in the (n,2) sector.

Thus our vacuum solution is stable under small perturbations. As the result of these computations we can list the masses of the physical states in the following table:

O(5) - Rep.	MASS	SPIN
(n,0) n > 0	${}^{n0}M_0{}^2 = \frac{n(n+3)}{a^2}$	2
	${}^{n0}M_2{}^2 = \frac{n(n+3)+1-\sqrt{2n(n+3)+1}}{a^2}$	0
	${}^{n0}M_3{}^2 = \frac{n(n+3)+1+\sqrt{2n(n+3)+1}}{a^2}$	0
(n,1) n > 1	${}^{n1}M_2{}^2 = \frac{n(n+3)+2}{a^2}$	0
	${}^{n1}M_4{}^2 = \frac{n(n+3)+\sqrt{2(n^2+3n+4)}}{a^2}$	1
	${}^{n1}M_5{}^2 = \frac{n(n+3)-\sqrt{2(n^2+3n+4)}}{a^2}$	1
(n,2) n > 2	${}^{n2}M_1{}^2 = \frac{n(n+1)}{a^2}$	0
	${}^{n2}M_2{}^2 = \frac{n(n+5)+6}{a^2}$	0

Table II

The list of all propagating massive modes. The massless graviton in the (n,0)-sector with n = 0 and the ten massless SO(5) Yang-Mills spin-1 particles in the (n,1)-sector with n = 1 are not contained in this table.

VI. [illegible]

In this paper we have proved that the solution (10,11) of the coupled Einstein-Yang-Mills equations in 8 dimensions is classically stable. Therefore fore this solution is a reasonable candidate for the ground state of the eight-dimensional theory. it describes the spontaneous compactification of the eight-dimensional manifold into $M^4 \times S^4$.

Due to the non-trivial topology of the various vector bundles associated with the instanton bundle over S^4 we had anticipated the stability of our solution earlier [11] and analyzed the zero mode patterns of an eight-dimensional Dirac operator acting on an 8 component spinor of SO(1,7). It was shown that if this spinor belongs to a 2t+1 dimensional representation of SU(2) then in M^4 there will be an SO(5) multiplet of left-handed (Weyl) fermions. These massless (Weyl) fermions transform according to a real representation of Kaluza SO(5) gauge group.

Now the fermionic Lagrangian possesses an additional global U(1) symmetry relative to which these massless fermions transform in a complex representation. We could have gauged this global U(1) by introducing the appropriate Maxwell field. Our four-dimensional massless fermions would then have transformed according to a complex representation of such a local gauge symmetry. We have chosen not to gauge this U(1), *) since the resulting theory would exhibit axial anomalies. [12]

It is worth mentioning that the spectrum of small oscillations around the vacuum solution exhibits a number of massive modes - among them one spin-zero tachyon. The tachyonic mode, as well as a number of other massive modes, do not appear in the on-shell Born amplitude and we therefore conclude that these are gauge artifacts.

As far as the stability test is concerned there is still the outstanding problem of finding a more efficient technique - something analogous to the positive energy theorem of the four-dimensional general relativity - which would avoid the tedious process of harmonic expansion and guarantee positivity, not just for small amplitude oscillations. Gauged supergravity may provide such positivity, but apparently not when the cosmological constant is zero.

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*) The analogy of this global U(1) is with the (B-L) symmetry which is present in a grand unified minimal SU(5) model and which within that model is not gauged.

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A D D E N D U M

STABILITY OF INSTANTON INDUCED COMPACTIFICATION IN 8-DIMENSIONS

S. Randjbar-Daemi, Abdus Salam and J. Strathdee

We have investigated the stability of small oscillations around the vacuum solutions $M^4 \times S^7$ and $M^4 \times S^3 \times S^4$ for the case of $d = 11$ supergravity theory where M^4 is Minkowski space. These solutions apparently do exist when supersymmetry is explicitly broken by the addition of a cosmological term to the $d = 11$ supergravity Lagrangian. To our surprise, for both the cases we have investigated, there exist tachyons in the spectrum, signalling instability.

The bosonic field equations of 11-dimensional supergravity, modified by the addition of a (supersymmetry-breaking) cosmological term are given in an orthonormal basis by

$$R_{AB} - \frac{1}{2}\eta_{AB}(R + \kappa^2\lambda) = -\frac{\kappa^2}{2} T_{AB} \tag{1}$$

$$\nabla_A F_{AB_1B_2B_3} + \kappa\, \varepsilon_{B_1\ldots B_{11}} F_{B_4\ldots B_7} F_{B_8\ldots B_{11}} = 0 ,$$

where the stress energy tensor is given by

$$T_{AB} = \frac{1}{6}\left(F_{AC_1C_2C_3} F_{BC_1C_2C_3} - \frac{1}{8}\eta_{AB} F^2\right) . \tag{2}$$

In these equations the indices $A,B,\ldots$ take the values $0,1,\ldots,11$. The cosmological constant, λ , breaks supersymmetry explicitly.

Assuming that the vacuum geometry is $M^4 \times S^7$, to be obtained by solving (1), (2) with

$$R_{aB} = 0 \qquad a = 0,1,2,3$$
$$R_{\alpha\beta} = -\frac{6}{a^2}\delta_{\alpha\beta} \qquad \alpha,\beta = 5,\ldots,11 \tag{3}$$

and

$$F_{\alpha BCD} = 0 \qquad \alpha = 5,\ldots,11$$
$$F_{abcd} = f\,\varepsilon_{abcd} \qquad a,b, = 0,1,2,3 . \tag{4}$$

The equations of motion reduce to the relations

$$\lambda = 3 f^2 = \frac{36}{\kappa^2 a^2} , \tag{5}$$

where a denotes the radius of the 7-sphere. Observe that the field strength parameter, f, which appeared in the Freund-Rubin solution as an undetermined constant of integration, is now related to the cosmological parameter, λ (> 0).

To see the tachyons, consider the bilinear part of the bosonic sector (where h_{AB} and V_{ABC} indicate the perturbations of g_{AB} and A_{ABC} respectively, R_{ABCD} and F_{ABCD} are given by the background solutions Eqs.(3.5)):

$$\begin{aligned} S_2(h,V) = \int d^{11}x \Big\{ & \frac{\kappa^2}{4}\left(\frac{1}{\kappa^2}R + \lambda + \frac{1}{48}F^2\right)\left(h_{AB}h_{AB} - \frac{1}{2}h^2\right) + \\ & + \frac{1}{2}\left(-h_{AC}h_{BC}R_{AB} + hh_{AB}R_{AB} + h_{AB}h_{CD}R_{ACBD}\right) \\ & - \frac{1}{4}h_{AB\cdot C}h_{AB\cdot C} + \frac{1}{8}h_{\cdot C}h_{\cdot C} + \frac{1}{2}\left(h_{AB} - \frac{1}{2}\eta_{AB}h\right)_{\cdot B}\left(h_{AC} - \frac{1}{2}\eta_{AC}h\right)_{\cdot C} \\ & - \frac{\kappa^2}{12}F_{AA_1A_2A_3}F_{BA_1A_2A_3}\left(h_{AC}h_{BC} - \frac{1}{2}hh_{AB}\right) - \frac{\kappa^2}{8}h_{AB}h_{CD}F_{ACA_1A_2}F_{BDA_1A_2} \\ & + \frac{2}{3}\kappa\, h_{AB}F_{AA_1A_2A_3}\nabla_{[B}V_{A_1A_2A_3]} - \frac{\kappa}{12}h\,F_{A_1\ldots A_4}\nabla_{A_1}V_{A_2A_3A_4} \\ & - \frac{1}{3}\nabla_{[A_1}V_{A_2A_3A_4]}\nabla_{[A_1}V_{A_2A_3A_4]} \\ & + \frac{\kappa}{(12)^3}\epsilon_{A_1\ldots A_{11}}\nabla_{A_1}V_{A_2\ldots A_4}F_{A_5\ldots A_8}V_{A_9\ldots A_{11}} \\ & + \frac{1}{2}T_{AB}h_{AB} + \frac{1}{3!}J_{ABC}V_{ABC}\Big\} , \end{aligned} \tag{6}$$

where $h = \eta^{AB}h_{AB}$ and $F_{A_1\ldots A_4} = 4\nabla_{[A_1}A_{A_2\ldots A_4]} = \nabla_{A_1}A_{A_2\ldots A_4}$ + cycl. permutations. All tensors are referred to an orthonormal basis of $M_4 \times B_7$, relative to which $\eta_{AB} = \mathrm{diag}(-1,+1,\ldots,+1)$. Also $h_{AB\cdot C} = \nabla_C h_{AB}$, etc.

In order for $S_2(h,V)$ to be invariant under the linearized gauge transformations

$$\left.\begin{aligned} \delta h_{AB} &= \frac{1}{\kappa}\left(\zeta_{A\cdot B} + \zeta_{B\cdot A}\right) \\ \delta V_{ABC} &= \nabla_{[A}\Lambda_{BC]} - F_{ABCD}\zeta_D \end{aligned}\right\} \qquad \zeta_A \;\&\; \Lambda_{BC} = -\Lambda_{CB} \text{ arbitrary}$$

the sources must be constrained to satisfy

$$T_{AB \cdot B} = \frac{\kappa}{6} F_{ABCD} J_{BCD}$$

$$\nabla_C J_{ABC} = 0 \quad .$$

Given a solution of the form $M_4 \times G|H$ to the background field equations we can expand the fields h_{AB} and V_{ABC} into normal modes on $G|H$ and evaluate the spectrum of the physical masses M^2. For the two cases of $G|H = S^7$ and $G|H = S^4 \times S^3$, we exhibit below only those equations which give rise to tachyonic $(\text{masses})^2$. First consider the solution $G|H = S^7$. In this case the fields have to be expanded into harmonics of O(8). Each component field will carry four O(8) labels $(n_1,\ldots,n_4)$ and the sector in question is characterized by $(n_1,\ldots,n_4) = (n,0,0,0)$. The spin zero parts of the field equations in this sector are given by

$$\left(p^2 + \frac{n(n+6)-12}{a^2}\right)(h_{00} + h_{\gamma\gamma}) + \left(p^2 + \frac{n(n+6)+12}{a^2}\right) h_{ii} + 2\kappa f i p_a V_a - 2T_{00} = 0$$

$$3\left(p^2 + \frac{n(n+b)+12}{a^2}\right) h_{00} - \left(p^2 + \frac{n(n+b)+36}{a^2}\right) h_{ii} - 3\left(p^2 + \frac{n(n+6)-12}{a^2}\right) h_{\gamma\gamma} - 6\kappa f i p_a V_a - 2\, T_{ii} = 0$$

$$\left(p^2 + \frac{n(n+6)-12}{a^2}\right)(-7h_{00} - 7h_{ii} + 5h_{\gamma\gamma}) - 14\kappa f i p_a V_a + 2\, T_{\gamma\gamma} = 0$$

$$\left(p^2 + \frac{n(n+6)-12}{a^2}\right) h_0 - \frac{\kappa f}{a} \nu_1(n)\, V_0 - T_0 = 0$$

$$p_0 p_a V_a + \frac{n(n+6)}{a^2} V_0 - \kappa f i p_0 h_{ii} - J_0 = 0$$

$$\left(p^2 + \frac{n(n+b)-12}{a^2}\right) h = T \quad . \tag{7}$$

In writing the above we have chosen a Lorentz frame $p_a = (p_0, \underset{\sim}{0})$ and the gauge conditions $\nabla_A(h_{AB} - \frac{1}{2}\eta_{AB}h) = 0$, $\nabla_\alpha V_{\alpha AB} = 0$, where $a = 0,1,2,3$ and $\alpha = 5,\ldots,11$. The field V_a is $\frac{1}{6}\varepsilon_{abcd}V_{bcd}$ while h_{ii} is the three-dimensional (spatial) trace of h_{ab}. ($\nu_1(n)$ are constants which we shall not write out in detail.)

It can be shown that the solution of these equations exhibitsthe following <u>propagating</u> poles in the p^2 plane:

$$-p^2 = M_\pm^2 = \frac{n(n+6) + 8 \pm 4\sqrt{3n(n+6)+4}}{a^2} \tag{8}$$

$$-p^2 = \frac{n(n+6)}{a^2} \quad . \tag{9}$$

As is easily checked, for the spin-zero poles exhibited above, M^2 is tachyonic for $n = 1,2,3$ and its contribution to the interaction energy $\int \frac{1}{2} h_{AB} T_{AB}$ is non-vanishing.

The analogous equations for $S^4 \times S^3$ are considerably more cumbersome. Here again we give only the relevant sector of the spin-0 equations in the representation of the $O(5) \times O(4)$ characterized by $(n_1,n_2,m_1,m_2) = (n_1,0,m_1,0)$, where (n_1,n_2) and (m_1,m_2) indicate the irreducible O(5) and O(4) representations, respectively. Now we adopt the gauge condition $\nabla_A V_{ABC} = 0$ instead of $\nabla_\alpha V_{\alpha AB} = 0$. The relevant equations are:

$$(p^2 + M^2 - \kappa^2 f^2)(h_{00} + h_{\alpha\alpha} + h_{\alpha'\alpha'}) + (p^2 + M^2 + \kappa^2 f^2) h_{ii} - 2\kappa k i p_0 V_0 - 2T_{00} = 0$$

$$3(p^2 + M^2 + \kappa^2 f^2) h_{00} - (p^2 + M^2 + 3\kappa^2 f^2) h_{ii} - 3(p^2 + M^2 - \kappa^2 f^2)(h_{\alpha\alpha} + h_{\alpha'\alpha'}) + 6\kappa f i p_0 V_0 - 6\, T_{ii} = 0$$

$$(p^2 + M^2 - \kappa^2 f^2)(-2h_{00} + h_{\alpha\alpha} + 2h_{\alpha'\alpha'} + 2h_{ii}) + 4\kappa k i p_0 V_0 + T_{\alpha\alpha} = 0$$

$$(p^2 + M^2 - \kappa^2 f^2)(-3h_{00} + 3h_{\alpha\alpha} + h_{\alpha'\alpha'} + 3h_{ii}) + 6\kappa f i p_0 V_0 - 2T_{\alpha'\alpha'} = 0$$

$$(p^2 + M^2) V_0 - \kappa f i p_0 h_{ii} - J_0 = 0 \; . \tag{10}$$

The notation is as before except that $\alpha = 1,\ldots,4$ on S^4 and $\alpha' = 1,\ldots,3$ on S^3. M^2 is an abbreviation for

$$M^2 = \frac{m(m+2)}{a'^2} + \frac{n(n+3)}{a^2} \quad , \tag{11}$$

where a and a' are the radii of S^4 and S^3 respectively, which by virtue of the background equations are now related to $\kappa^2 f^2$

$$\frac{1}{a^2} = \frac{\kappa^2 f^2}{6} \; , \quad \frac{1}{a'^2} = \frac{\kappa^2 f^2}{4} \quad . \tag{12}$$

Propagating tachyons with $O(5) \times O(4)$ quantum numbers $(n,m) = (1,0)$, $(0,1)$ and $(1,1)$ are found. Their masses are given by

$$M_{\mp}^2 = \frac{m(m+2)}{a'^2} + \frac{n(n+3)}{a^2} + \frac{2}{3}\kappa^2 f^2 \mp 2\kappa f \sqrt{\frac{m(m+2)}{a'^2} + \frac{n(n+3)}{a^2} + \frac{\kappa^2 f^2}{9}} \,, \tag{13}$$

where the radii are given by (12).

The existence of tachyons in the S^7 solution was confirmed by E. Sezgin for which we express appreciation.