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ON KALUZA KLEIN COSMOLOGY *

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ABSTRACT

A generalization of Friedmann-Robertson-Walker cosmology to 4+K dimensions is considered. The space-time manifold $R^1 \times S^3 \times S^K$ is characterized by two time-dependent scales, $R(t)$ and $a(t)$. The equations of motion for R and a are derived from the 4+K-dimensional Einstein action supplemented by a one-loop thermal term, corresponding to a gas of non-interacting scalar particles. It is shown that in the approximation when $T < 1/a$ the equations of motion admit a solution in which the internal space has a constant radius $\underline{a}$ while the external $R(t)$ evolves in the usual manner.

[illegible]

The idea of an internal space whose symmetries correspond to the observed internal symmetries of low energy physics has received much attention, recently. It is assumed in this picture that space-time has more than four dimensions and that all but four of them are highly curved and compactified so as to be unresolvable with the energies available at present. The relevant distance (measured in units of Planck length) would be inversely correlated with the gauge coupling constants in the theory.

As with other questions involving inaccessibly small distance scales it seems appropriate to look for cosmological implications of the Kaluza-Klein hypothesis. Were the compact dimensions always compact and always so much more curved than the others? If there is a temporal evolution of the distance scales, is it compatible with the observed constancy of gauge coupling parameters like the fine structure constant? We shall demonstrate that it is indeed possible to maintain a fixed internal scale while the external radius evolves in the conventional manner.

To study the cosmological aspects [1] of the Kaluza-Klein hypothesis we shall begin by setting up a straightforward generalization of the homogeneous Friedmann, Robertson, Walker cosmologies for space-times with extra dimensions. To be specific we shall assume the geometry $R^1 \times S^3 \times S^K$ where R^1 corresponds to the time dimension, S^3 to the "large" space dimensions and S^K to the "small" or internal space dimensions. Instead of the sphere S^3 one could choose an open space of constant curvature and, instead of the sphere S^K one could choose a compact quotient space G/H of K dimensions. In any case the metric tensor could be presented in the form

$$g_{MN} = \begin{pmatrix} -1 & & \\ & R(t)^2\, \tilde{g}_{mn}(x) & \\ & & a(t)^2\, \tilde{g}_{\mu\nu}(y) \end{pmatrix} , \qquad (1.1)$$

where $\tilde{g}_{mn}(x)$ and $g_{\mu\nu}(y)$ represent metrics on the three- and K-dimensional spaces, respectively. The sizes of these two spaces are characterized by time-dependent scale factors, $R(t)$ and $a(t)$.

The metric (1.1) admits a number of isometries. For the case $R^1 \times S^3 \times S^K$, the symmetry group of the spatial section is $O(4) \times O(K+1)$. The stress tensor, whose form would be dictated by Einstein's equations, must have the same invariances. In the coordinate system where the metric is given by (1.1), the stress tensor is

$$\mathbb{T}_{MN} = \begin{pmatrix} \rho & & \\ & p\, g_{mn} & \\ & & p'\, g_{\mu\nu} \end{pmatrix} , \qquad (1.2)$$

where the energy density ρ and pressures p, p' may depend on time, but not on the space coordinates x^m, y^μ.

The Einstein equations for the scale factors R and $\underline{a}$ in terms of ρ, p and p' can be derived from the action functional,

$$\Gamma = \int d^{4+K}x\, (-g)^{1/2} \left[-\frac{1}{\bar{G}} (\mathbb{R} + \bar{\Lambda}) \right] + \Gamma_1(g) \quad , \qquad (1.3)$$

where $\mathbb{R}$ denotes the 4+K-dimensional Ricci scalar, $\bar{G}$ is the coupling parameter and $\bar{\Lambda}$ is a cosmological constant. The source term, Γ_1, represents the contributions of matter fields, vacuum fluctuations, etc. Its variation defines the stress tensor,

$$\delta\Gamma_1 = \int d^{4+K}x\, (-g)^{1/2} \frac{1}{2}\, \delta g^{MN} \mathbb{T}_{MN} \quad . \qquad (1.4)$$

By substituting the $O(4) \times O(K+1)$ invariant ansatz (1.1) into (1.3), one can reduce this action to the form

$$\Gamma = \int dt\, \mathcal{L}(R,a) \quad ,$$

where the Lagrangian, $\mathcal{L}$, is given by

$$\mathcal{L} = -\frac{\Omega_3\, \Omega_K}{\bar{G}} \left[6 \frac{\dot{R}^2}{R^2} + 6K \frac{\dot{R}}{R} \frac{\dot{a}}{a} + K(K-1) \frac{\dot{a}^2}{a^2} \right] - U \quad . \qquad (1.5)$$

Here Ω_3 and Ω_K denote the spatial volumes,

$$\Omega_3 = 2\pi^2 R^3 \quad , \qquad \Omega_K = (2\pi)^{\frac{K+1}{2}} a^K \Big/ \Gamma\left(\frac{K+1}{2}\right) \qquad (1.6)$$

and U is an effective "potential". The construction of this potential will be discussed more fully in the following section. It incorporates a classical gravity term arising from $\mathbb{R} + \Lambda$ (and displayed in the first

[illegible] of (1.7)) plus terms in the second bracket which arise from Γ_1. The latter has a (one-loop) quantum part plus a thermal part depending on the entropy, S

$$U = \frac{\Omega_3\, \Omega_K}{\bar{G}} \left[-\frac{6}{R^2} - \frac{K(K-1)}{a^2} + \bar{\Lambda} \right] + \left[\sigma_K \frac{\Omega_3}{a^4} + \tau \frac{S^{4/3}}{R} + \zeta\, m \right] \quad . \qquad (1.7)$$

Here σ_K, τ and ζ are numerical parameters which can be computed. The approximations which go into the form of the entropy (S)-dependent term in (1.7) will be discussed in the next section, but we believe that the given expression should be valid in the regime

$$R \gg 1/T \gg a \quad , \qquad (1.8)$$

where the temperature T is given by

$$T = \frac{\partial U}{\partial S} \sim \frac{S^{1/3}}{R} \quad . \qquad (1.9)$$

The time evolution of the Kaluza-Klein universe is governed by the Euler Lagrange equations derived from (1.5). We now show that for an appropriate choice of the parameter $\bar{\Lambda}$ in U, it is possible to obtain a solution where the internal radius is constant ($\dot{a} = 0$) while the large radius evolves as in standard model cosmology.

The two equations of motion simplify, on setting $\dot{a} = 0$, to the following:

$$\partial_t(R\dot{R}) - \frac{1}{2}\dot{R}^2 = \frac{\bar{G}}{12} \frac{R^2}{\Omega_3\, \Omega_K} R \frac{\partial U}{\partial R} \quad , \qquad (1.10a)$$

$$\frac{1}{R}\partial_t(R^2\dot{R}) - \dot{R}^2 = \frac{\bar{G}}{6K} \frac{R^2}{\Omega_3\, \Omega_K} a \frac{\partial U}{\partial a} \quad . \qquad (1.10b)$$

The "energy" integral corresponding to these equations reduces, when $\dot{a} = 0$, to

$$E = -\frac{\Omega_3\, \Omega_K}{\bar{G}} \left[6 \frac{\dot{R}^2}{R^2} \right] + U \quad . \qquad (1.11)$$

The compatibility of (1.10a), (1.10b) and (1.11) requires the algebraic relation,

$$R\frac{\partial U}{\partial R} - \frac{2}{K}a\frac{\partial U}{\partial a} = E - U \quad . \tag{1.12}$$

On taking $E = \zeta m$, this condition reduces to

$$f(a) = \frac{df}{da} = 0 \quad , \tag{1.13a}$$

where *)

$$f(a) = \frac{\Omega_K}{\bar{G}}\left(-\frac{K(K-1)}{a^2} + \bar{\Lambda}\right) + \frac{\sigma}{a^4} \quad . \tag{1.13b}$$

Determining $\bar{\Lambda}$ and a in terms of $\bar{G}$ from (1.13a) and substituting the results into (1.10b), we obtain,

$$\partial_t(R\dot{R}) = -1 \quad , \tag{1.14}$$

which is solved by

$$R^2 = t(t_0 - t) \quad , \tag{1.15}$$

where t_0, the time of collapse, is given by (1.11), i.e. $t_0^2 = \frac{\tau}{3\pi^2}\frac{\bar{G}}{\Omega_K}S^{4/3}$.

The expression (1.15) for $R^2(t)$ gives the conventional Friedmann-Robertson-Walker cosmology for the case when the standard 3—curvature parameter k equals unity. In the general case, the $6/R^2$ term in the first bracket in (1.7) reads $6k/R^2$ and the right-hand side of (1.14) equals $(-k)$, with appropriate changes in (1.15).

To summarize, if $R \gg 1/T > a$, and if $\bar{\Lambda}$ and a are determined in terms of $\bar{G}$ from (1.13a), we find $\dot{a}(t) = 0$, while $R(t)$ evolves in the conventional manner.

* Eqs.(13a) are the same as those of Candelas and Weinberg [2] for the static case considered by them ($R = \infty$).

II. THE FREE ENERGY

In order to have a meaningful expression for the components of the stress tensor, it is necessary to derive them from a free energy function, $F(R,a,T)$. From F one obtains the energy as a function of the entropy, $S = -\partial F/\partial T$, by the usual Legendre transform

$$U = F + TS = \Omega_3\,\Omega_K\,\rho(R,a,S) \quad . \tag{2.1}$$

If the pressures p and p' of Eq.(1.2) are defined by the thermodynamic identity,

$$d(\Omega_3\,\Omega_K\,\rho) = TdS - \Omega_K p\, d\Omega_3 - \Omega_3 p'\, d\Omega_K \tag{2.2}$$

then as is well-known the conservation of energy, $\nabla_M T^M{}_N = 0$, implies conservation of entropy. It follows that the Einstein equations deriving from (1.5) require that S be independent of time. This was tacitly assumed in Sec.I.

Since we have no a priori idea of what equations of state would be appropriate for Kaluza-Klein cosmology, we shall proceed on the basis of a simple model. We shall suppose that the density and pressures are due to a gas of non-interacting, spinless particles in thermal equilibrium. The free energy of such a system can be expressed by a one-loop integral,

$$\beta F = \frac{1}{2}\ln \mathrm{Det}(-\Box + \mu^2) \quad , \tag{2.3}$$

where μ is a mass parameter and $\Box$ denotes the Laplacian on the compact manifold $S^1 \times S^3 \times S^K$. This manifold is characterized by three radii, $\beta/2\pi = 1/2\pi T$, R and a. The expression (2.3) is given formally by a triple sum over the eigenvalues of $-\Box + \mu^2$,

$$\beta F = \frac{1}{2}\sum_{r=-\infty}^{\infty}\sum_{m,n=0}^{\infty} D_{mn}\,\ln\left[\left(\frac{2\pi r}{\beta}\right)^2 + \frac{m(m+2)}{R^2} + \frac{n(n+K-1)}{a^2} + \mu^2\right] , \tag{2.4}$$

where D_{mn} is a multiplicity factor. It is equal to the dimension of the $O(4) \times O(K+1)$ representation $(m,0) \times (n,0,\ldots,0)$ in Gel'fand Zetlin notation,

$$D_{mn} = (m+1)^2\,\frac{(2n+K-1)(n+K-2)!}{(K-1)!\,n!} \tag{2.5}$$

The sum (2.4) is of course divergent. But it is easy to regularize [3]. For this purpose write

$$\ell n\ X = \frac{d}{ds} X^s \Big|_{s=0} = \frac{d}{ds}\left[\frac{1}{\Gamma(-s)}\int_0^\infty dt\ t^{-s-1}\ e^{-tX}\right]_{s=0}$$

for each term in the sum. After some rearrangements one obtains an expression for the finite part

$$\beta F = \frac{d}{ds}\left[\frac{1}{2\Gamma(-s)}\int_0^\infty dt\ t^{-s-1}\ e^{-t\mu^2}\ \sigma_1\left(\frac{4\pi^2 t}{\beta^2}\right)\ \sigma_3\left(\frac{t}{R^2}\right)\ \sigma_K\left(\frac{t}{a^2}\right)\right]_{s=0}, \quad (2.6)$$

where the functions σ_1, σ_3 and σ_K are defined, respectively, by

$$\sigma_1\left(\frac{4\pi^2 t}{\beta^2}\right) = \sum_{-\infty}^{\infty} e^{-t(2\pi r/\beta)^2}$$

$$\sigma_3\left(\frac{t}{R^2}\right) = \sum_{0}^{\infty} (m+1)^2\ e^{-m(m+2)t/R^2}$$

$$\sigma_K\left(\frac{t}{a^2}\right) = \sum_{0}^{\infty} \frac{(2n+K-1)(n+K-2)!}{(K-1)!\ n!}\ e^{-n(n+K-1)t/a^2}. \quad (2.7)$$

These sums converge and define analytic functions on the half-plane Re $t > 0$. At $t = 0$ they are singular,

$$\sigma_1 \sim t^{-1/2}, \quad \sigma_3 \sim t^{-3/2}, \quad \sigma_K \sim t^{-K/2}.$$

To evaluate the integral (2.6) it is necessary to make approximations to the σ_i. The most important observation in this connection [4] is that these functions are closely related to the Jacobi Theta functions [5]. In particular,

$$\theta_3\left(0\left|\frac{iu}{\pi}\right.\right) = \sum_{-\infty}^{\infty} e^{-r^2 u} = \sqrt{\frac{\pi}{u}}\sum_{-\infty}^{\infty} e^{-(\pi r)^2 u^{-1}}. \quad (2.8)$$

The two series on the right-hand side provide highly convergent expansions for the σ_i around $u = \infty$ and around $u = 0$ *), respectively. In particular, the zero-temperature limit corresponds to the term

$$\sigma_1 \simeq (4\pi t/\beta^2)^{-1/2}, \quad (2.9)$$

and gives

$$F(R,a,T{=}0) = \frac{d}{ds}\left[\frac{1}{4\sqrt{\pi}\Gamma(-s)}\int_0^\infty dt\ t^{-s-3/2}\ e^{-t\mu^2}\sigma_3\left(\frac{t}{R^2}\right)\ \sigma_K\left(\frac{t}{a^2}\right)\right].$$

In a similar fashion, the flat space limit, $R \to \infty$, of the energy density, $F/2\pi^2 R$, corresponds to

$$\sigma_3 \simeq \frac{\sqrt{\pi}}{4} R^3\ t^{-3/2}. \quad (2.10)$$

The effective potential of Sec.I was obtained by assuming $R \gg \beta > a$, while the mass parameter μ for the scalar particles is small (more precisely $\mu < T$). To derive it, use (2.9) and (2.11) to obtain,

$$F \simeq \frac{R^3}{16}\frac{d}{ds}\left[\frac{1}{\Gamma(-s)}\int_0^\infty dt\ t^{-s-3}\left(1 + 2\sum_{1}^{\infty} e^{-r^2\beta^2/4t}\right)\sigma_K\left(\frac{t}{a^2}\right)\right]_{s=0}$$

$$= \frac{R^3}{16}\left[\frac{d}{ds}\left\{\frac{a^{-4-2s}}{\Gamma(-s)}\int_0^\infty du\ u^{-s-3}\ \sigma_K(u)\right\}_{s=0} - 2\int_0^\infty dt\ t^{-3}\sum_{1}^{\infty} e^{-r^2\beta^2/4t}\ \sigma_K\left(\frac{t}{a^2}\right)\right]. \quad (2.11)$$

The "s-regularization" can be safely removed in the temperature dependent terms, which converge at $t = 0$. It can also be removed in the zero-temperature term if K is odd, since the integral involving $\sigma_K(u)$ is analytic at $s = 0$ in that case. (For even K one obtains $\sim a^{-4}\ell na$ in the limit $s = 0$.) Thus

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*) This seems to be true for the odd-dimensional spheres. For the even-dimensional spheres one probably has to employ asymptotic series $\sigma_K(u) \sim u^{-K/2}\ (a + \beta u + \gamma u^2 + \ldots)$, near $u = 0$.

$$F \simeq \Omega_3\left(\sigma_K a^{-4} - \frac{\pi^2}{90}\beta^{-4} + \ldots\right) \quad , \qquad (2.12)$$

where

$$\sigma_K = -\frac{1}{32\pi^2}\int_0^\infty du\, u^{-3}\, \sigma_K(u) \quad . \qquad (2.13)$$

In obtaining the second term of (2.12) we have taken $\sigma_K = 1$, in effect discarding the exponentially small terms $\sim \exp(-\beta/a)$. (Otherwise stated, only the internal space zero modes are retained in the sum over states.) Details of these various approximations will be discussed elsewhere.

The one-loop contribution to the energy, used in Sec.I, is then obtained by taking the Legendre transform of (2.12)

$$U \simeq \sigma_K \frac{\Omega_3}{a^4} + \tau \frac{S^{4/3}}{R} \quad , \qquad (2.14)$$

where

$$\tau = \frac{3}{4}\left(\frac{45}{4\pi^4}\right)^{1/3} \quad . \qquad (2.15)$$

(If there are other species of particles, in addition to scalars which circulate around the loop (for example, gravitons, gauge particles, or spinors) the numerical values of the parameters σ_K and τ will change.)

III. CONCLUSIONS

It is interesting to confirm that the Kaluza-Klein cosmology does admit of a time-independent internal radius a (consistent with lack of variability of gauge couplings with time), in an approximation to the free energy where $R \gg 1/T > a$, while $\bar{\Lambda}$ and a are suitably correlated by Eq.(1.13a). For the regime of temperatures where our precise approximation for the thermal contribution to the free energy, $(S^{4/3}/R)$, may be expected to break down - for example for Planckian temperatures, $T \gtrsim 1/a$ - phase transitions [2] in the behaviour of a (and R) may occur. These will considered elsewhere.

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REFERENCES

[1] P.G.O. Freund, Nucl. Phys. B209, 146 (1982);
T. Appelquist and A. Chodos, Phys. Rev. Lett. 50, 141 (1983);
E. Alvarez and M. Belen Gavela, Phys. Rev. Lett. 51, 931 (1983).
M. Rubin and B.D. Roth, Phys. Lett. 127B, 55 (1983); Nucl. Phys. B226, 44 (1983);
T. Appelquist, A. Chodos and E. Myers, Yale, preprint, YPT 83-04 (1983);
Q. Shafi and C. Wetterich, CERN preprint TH.3613 (1983).

[2] P. Candelas and S. Weinberg, Texas, preprint UTTG-8-83 (1983).

[3] Abdus Salam and J.A. Strathdee, Nucl. Phys. B90, 203 (1975).

[4] C.M.Mukku, J.A. Magpantay and W.A. Sayed, Ann. Phys. 145, 27 (1983).

[5] Higher Transcendental Functions, Eds. A. Erdelyi et al., (McGraw-Hill New York, 1953) Vol.II.