ON THE PROPAGATION AND STABILITY OF WAVE MOTIONS
IN RAPIDLY ROTATING SPHERICAL SHELLS:
II. HYDROMAGNETIC TWO-DIMENSIONAL MOTIONS

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ABSTRACT

The linear propagation properties and stability of wave motions in spherical shells examined in paper I (Eltayeb, 1980) are here extended to the case of a toroidal magnetic field together with an associated shear flow. The analysis is restricted to moderate values of the magnetic field's amplitude in which case the ensuing motions are two-dimensional. They occur in thin cylindrical cells coaxial with the axis of rotation.

For every set of the relevant parameters an infinity of modes exists and is divided into two uncoupled categories. One category is associated with a temperature perturbation even in the axial coordinate $z$ and the other category odd in $z$. In the presence of an inner solid core the even set persists only outside the cylindrical surface, $C_r$, whose generators touch the inner core at its equator while the odd set persists everywhere. This property is used to clarify the relationship between motions in thin and thick spherical shells.

The direction of propagation of these waves depends on the ratio, $q$, of thermal to magnetic diffusivities and on the modified Chandrasekhar number $Q$ (which is the ratio of Lorentz to Coriolis forces). For small values of $q$ relevant to geophysical applications both eastward and westward propagation is possible if $Q$ is small but as $Q$ increases beyond a certain value only eastward propagation is possible. For the case of large $q$ applicable to astrophysical situations both eastward and westward propagation is possible.

All these results apply for a variety of temperature gradients, in which both internal and differential forms of heating are invoked, and various forms of toroidal magnetic fields.

The stability of these wave motions is examined and the most preferred mode of convection is identified in each case. The unstable cell always lies on $C_r$ or outside it. Its precise location depends on the types of magnetic field and temperature gradient. The sloping boundary of the spherical shell tends to stabilize westward propagating waves.

Two-dimensional wave motions in spherical shells are found to provide the preferred mode of convection for $Q < 0$ (max $(1, q^{1/2})$) for all toroidal fields which depend only on the distance, $s$, from the rotation axis. For fields which depend both on $s$ and on the latitude two-dimensional motions are preferred only if $Q < q^{1/2}$. For larger values of $Q$ they are stabilized by the presence of the shear and instability occurs in the form of three-dimensional motions.
I. INTRODUCTION

The propagation properties and stability of wave motions in rotating magnetic systems bear on a number of observed geomagnetic and astrophysical phenomena (see, e.g., Moffatt 1972, Acheson 1978). The relevance of hydromagnetic-inertial waves to the geomagnetic secular variations was initiated by Hide (1966) who studied the free oscillations of a diffusionless, unstratified, perfectly conducting, incompressible fluid sphere rotating uniformly in a uniform magnetic field that is predominantly toroidal, using a beta-plane approximation. By appealing to the axial vorticity in thin coaxial cylindrical filaments, he found that hydromagnetic-inertial waves, which have no preferred zonal direction of phase propagation in an infinite medium (Lehnert 1954), are constrained by the sloping boundary of the spherical surface to propagate eastwards if the spherical shell is thin. Hide also argued that the same waves would propagate westwards if the shell was thick. Subsequent studies by Stewartson (1967) and Hide and Stewartson (1972) showed that westward propagation of the slow waves is unlikely except possibly for modes associated with large radial wavenumbers. On the other hand Malkus (1967) studied the same problem with the uniform toroidal field replaced by a non-uniform one corresponding to a uniform axial electric current to find that no preference for westward propagation is evident. However, in contrast with the uniform field, the Malkus model can exhibit instability if the amplitude of the magnetic field is large enough. This field gradient instability was later studied by Acheson (see Acheson and Hide (1973)) for various profiles of magnetic fields varying with the distance, s, from the rotation axis. Acheson found that provided the magnetic field varies faster than $s^{3/2}$, magnetic instabilities occur in the form of westward propagating slow hydromagnetic-inertial waves. Braginskii (1967) also studied slow hydromagnetic-inertial modes in the presence of stratification and pointed out their relevance to the geodynamo, where he showed that a hierarchy of modes of these waves can produce a mean poloidal field from a toroidal one.

Now all the above studies deal with diffusionless fluids. The studies on diffusive rotating magnetic systems usually deal with the stability aspects i.e. the identification of the most unstable mode in the linear theory.

The first such study was made by Chandrasekhar (1961) for a Benard layer in the presence of vertical rotation and magnetic field. Subsequently a number of publications on the stability of the hydromagnetic rotating Benard layer have appeared (Eltayeb 1972, 1975; Soward 1979). The preferred mode of convection has no preferred direction if the field is uniform but a non-uniform field can provide a selection mechanism for the direction of phase propagation of the unstable mode. Motivated by the application of convective motions to planetary interiors Eltayeb and Kumar (1974, 1977) extended the study of the instabilities of diffusive rotating magnetic systems to a full sphere (also, see Eltayeb 1972b) using the Malkus field. In contrast with the plane layer model instability in a full sphere is necessarily oscillatory. The unstable waves propagate eastward or westward depending on the relative strength of the Lorentz and Coriolis forces as well as on the relative magnitudes of magnetic and thermal diffusivities. Another important result of these studies is that provided the order of magnitude of the Lorentz forces does not exceed that of the Coriolis forces, convection occurs in the form of coaxial cylindrical filaments of the type found by Roberts (1968) for the same problem in the absence of the magnetic field. The thickness of the filaments depends on the ratio, $Q$, of magnetic to Coriolis forces. The stability of the magnetic rotating system was also examined by Busse (1976) using a sliced cylindrical shell. That model made a reasonable representation of the spherical problem when $Q<1$ and has been used by Busse (1977) and Soward (1979) to construct dynamo models because of its simplicity.

Two-dimensional motions in the magnetic rotating sphere has recently been studied by Fearn (1979) who isolated some of the properties of the unstable waves. However, this study is also limited to the preferred mode of convection.

In paper I (Eltayeb 1980) the propagation properties and stability of rotating spherical shells were investigated in the absence of a magnetic field. The role played by the slope of the boundary in tending to promote (oppose) eastward (westward) phase propagation was clarified. Although the preferred mode of convection is a wave always propagating eastward it was found that two infinite sets of waves one of which propagates eastward and the other westward exist. Instability for every mode was examined to find that it becomes unstable in the form of a coaxial filament whose distance from the
rotation axis depends on the temperature gradient driving the instability. Moreover, every infinite set of waves is divided into two groups one of which has a temperature perturbation even in the axial coordinate z and the other odd in z. The presence of an inner core suppresses the even set within the cylindrical surface, \( C \), whose generators touch the inner core at its equator (see figure 1).

In this paper we extend the study made in I to include magnetic effects. Unlike previous studies which used magnetic fields varying only as the distance from the rotation axis, we here examine the influence of three toroidal magnetic fields. In addition to the Malkus field two other profiles of magnetic field are used. Both of them vanish on the outer surface of the shell and hence depend both on the distance from the axis of rotation and latitude. The only difference between them being that one is even in \( z \) and the other odd in \( z \). Because of the dependence on latitude the basic state magnetic field is associated with a zonal flow (Taylor, 1963). The amplitude of the zonal flow is linked to that of the magnetic field, with the consequence that as the amplitude of the magnetic field increases, the shear flow becomes more important until \( q \) tends to \( 1/2 \) when the shear flow stabilizes the two dimensional instabilities.

For small values of the amplitude of the magnetic field \( B_0 \), the wave motions discussed in I are present and slightly modified by its presence. As the amplitude of \( B_0 \) is increased other modes appear. These modes have the lowest member providing the critical mode of instability of the sphere (Eltayeb 1972a; Eltayeb and Kumar 1977). However, higher modes exist and both westward and eastward phase propagation are possible. When the ratio, \( q \), of thermal (\( \kappa \)) to magnetic (\( \eta \)) diffusivities is small the family of waves propagates by the inertial-buoyancy mechanism (Fearn 1979) but for large \( q \) the waves propagate with the hydromagnetic-inertial mechanism (Hide 1966). In both cases, however, instability is strongly dependent on both magnetic and thermal diffusivities.

Both zonal phase speed and temperature gradient of the critical mode depend on the product \( kr_1 \).

In § 2 the perturbation equations are derived; in § 3 the propagation and stability of waves are studied in the case \( q \leq 0(1) \) while § 4 is devoted to the case \( q > 1 \). § 5 contains some concluding remarks. In appendix A, an approximate solution is used to provide insight into the problem. Appendix B contains the complicated mathematical expression used in the analysis of § 3 and § 4.

II. THE PERTURBATION EQUATIONS

A Boussinesq fluid is contained between two spherical shells \( S_1 \) and \( S_2 \) of radii \( r_1 \) and \( r_2 \) respectively. The fluid is rotating uniformly, with angular velocity \( \Omega \), in the presence of a magnetic field \( \mathbf{B} \) and flow \( \mathbf{U} \) (see figure 1). The fluid has a viscosity \( \nu \), thermal diffusivity \( \kappa \), magnetic diffusivity \( \eta \), magnetic permeability \( \mu \) and coefficient of thermal expansion \( \alpha \).

The system is governed by the equations of motion, induction, heat, state and continuity together with Gauss' law:

\[
\rho \left( \frac{D\mathbf{U}}{Dt} + \mathbf{U} \cdot \nabla \mathbf{U} + 2 \Omega \times \mathbf{U} \right) = -\nabla p + \rho \mathbf{U} \times \mathbf{E} + \frac{\rho}{c_p} \mathbf{F} + \rho \mathbf{U} \times \mathbf{B} + \rho \mathbf{U} \cdot \nabla \mathbf{U}, \quad (2.1)
\]

\[
\frac{D\mathbf{E}}{Dt} = \mathbf{E} \times \mathbf{B} + \frac{1}{c_p} \nabla \mathbf{T}, \quad (2.2)
\]

\[
\frac{D\mathbf{T}}{Dt} = \kappa \nabla^2 \mathbf{T} + \mathbf{e}, \quad (2.3)
\]

\[
\mathbf{U} \cdot \nabla \rho = 0. \quad (2.4)
\]

Here \( \rho \) is the density, \( p \) the fluid pressure, \( \mathbf{T} \) the temperature, \( \mathbf{g} \) the gravitational acceleration, \( \rho_0 \) the density where \( T = T_0 \) and \( \mathbf{F} \) a uniform heat source.

We take a cylindrical coordinate system \((s, \varphi, z)\) with origin, \( O \), at the centre of the spherical shell and \( O\varphi \) along \( \mathbf{B} \). Consider a basic state

\[
\mathbf{T}_c = \mathbf{T}_c(s, z); \quad (\mathbf{B}, \mathbf{U}) = \left[ B_0(s, z), U(s, z) \right] \hat{\mathbf{e}}; \quad \rho = \rho_0(s, z); \quad \mathbf{F} = \mathbf{0}, \quad (2.6)
\]

where \( \rho_0 \) is a constant, and a unit vector in the \( \varphi \)-direction will always be denoted by \( \hat{\mathbf{e}} \).
We shall assume that the fluid shell is self-gravitating so that
\[
\frac{g}{c_s^2} - \frac{3}{r} \frac{\partial \rho}{\partial r} = \frac{\partial}{\partial r} \left[ \frac{\rho}{r^2} \frac{\partial \eta}{\partial \theta} \right], \tag{2.7}
\]
The basic state (2.6) is then governed by the equations
\[
(2 \Omega u - \mathbf{g} + \mathbf{B}_o + \mathbf{U}^2) \mathbf{U}^2 = -N \left( \frac{\rho}{r^2} + \frac{\rho}{r^2} \mathbf{e} \cdot \mathbf{U}^2 + \nabla \mathbf{U}^2 \right) + \mathbf{U}^2 \nabla \mathbf{U}, \tag{2.9}
\]
\[
\mathbf{U} = \mathbf{U} \cdot \mathbf{r}, \quad \mathbf{U} = (u, v, w), \tag{2.3}
\]
\[
\mathbf{U} = \mathbf{U} \cdot \mathbf{r}, \quad \mathbf{U} = (u, v, w), \tag{2.10}
\]
Since we are interested in rapidly rotating systems we shall assume that the shear flow \( \mathbf{U} \) measured in the rotating frame is much smaller than the zonal speed due to the rotation. Thus
\[
\left| \frac{\mathbf{U}}{2 \Omega r} \right| \ll 1. \tag{2.11}
\]
Assuming viscosity is small it can be shown that its effects are concentrated in Ekman boundary layers on the surface of the shell and the contribution of these layers can be neglected in the leading order. Thus
\[
\mathbf{U} = \mathbf{B}_o / 2 \Omega r \mathbf{e}, \tag{2.12}
\]
and another expression for \( \mathbf{B}_o \).
Equation (2.10) can be solved for spherically symmetric temperature \( T \) to obtain
\[
- \frac{\partial T}{\partial r} = \frac{\varepsilon}{3} \left[ 1 - \frac{\eta}{\Delta T} \right] - \frac{\varepsilon}{\Delta T} \frac{T}{r^2} \left[ 1 - \eta \right] \left[ \frac{\Delta T}{r^2} \right], \tag{2.13}
\]
where \( \Delta T \) is the temperature difference across the shell (i.e. \( T_1 - T_0 \)), \( r \) the radial distance and
\[
\eta = \frac{\varepsilon}{\Delta T} \tag{2.14}
\]
If we take \( (r, \theta, \phi) \) as spherical polar coordinates with the axis of rotation \( \phi = 0 \), we can solve (2.9) to get
\[
\mathbf{B}_o = \sum_{n=1}^{\infty} \left[ A_n \mathbf{r}^n + C_n \mathbf{r}^{-n} \right] P_n^1(\cos \theta), \tag{2.15}
\]
in which \( P_n^1(x) \) are the associated Legendre polynomials and \( A_n, C_n \) are constants.
For a full sphere (i.e. \( r = 0 \)) \( C_n = 0 \) for all \( n \) and the mode \( n = 1 \) corresponds to the Malkus field
\[
\mathbf{B}_o = A_1 \mathbf{r} \sin \theta = \mathbf{A}_1 \mathbf{r}. \tag{2.16}
\]
Our interest here is to study the stability of the basic state (2.12) and (2.13) when \( \mathbf{B}_o \) vanishes on the outer surface of the shell. Such fields are more likely to mimic the field of the Earth's outer core since the conductivity is too weak to allow the field to penetrate it. We then take \( C_n = A_n \) for all \( n \) and obtain
\[
\mathbf{B}_o = \sum_{n=1}^{\infty} A_n \left( r^n - r^{-n} \right) P_n^1(\cos \theta). \tag{2.17}
\]
The values of \( A_n \) can only be determined by specifying the field on the surface \( S_1 \). Now the various terms (corresponding to different values of \( n \)) here can be categorized as either symmetric or antisymmetric about the equatorial plane \( \phi = 0 \). We shall therefore investigate the stability of the two fields \( n = 1 \) and \( n = 2 \). The field \( n = 1 \) is symmetric about the equatorial plane. Indeed this is the modification of the Malkus field to vanish on the outer surface of the shell. The field \( n = 2 \) is anti-symmetric about the equatorial plane, a property believed to apply to the toroidal field of the Earth.

If we take \( p, u, b, \theta \) as the dimensionless perturbations in pressure, velocity, magnetic induction and temperature respectively, the dimensionless equations of the linear problem are
Here we have taken $r_0$ as a unit of length, $\delta/r_0$ as a unit of perturbation velocity and $r_0^2/k$ as a unit of time. $\mathcal{B}$ and $\mathcal{U}$ are dimensionless forms of $\mathcal{B}_0$ and $\mathcal{U}_0$, and

\[
K(r) = \begin{cases} 
1 - \alpha r^{-3} & \beta = \epsilon/3k \quad \epsilon \neq 0 \\
\eta r^{-3} & \beta = \Delta T/r_0^2(1-\eta) \quad \epsilon = 0
\end{cases}
\]

(2.22)

in which

\[
\alpha = \frac{1}{2} \eta (1+\eta) - 3 \eta K \Delta T / \epsilon r_0^2 (1-\eta).
\]

(2.23)

The dimensionless parameters $\beta$, $\eta$, $Q$, $E$ and $q$ are the Prandtl number, the modified Rayleigh number, the modified Chandrasekhar number, the Ekman number and a magnetic number. They are defined by

\[
\beta = \frac{\alpha k}{h}, \quad R = \frac{\alpha k r_0^2}{2 \xi h}, \quad Q = \frac{\mathcal{B}}{2 \xi h}, \quad E = \frac{\mathcal{E}}{2 \xi h}, \quad q = \frac{\mathcal{H}}{h}.
\]

(2.24)

We now express $u$ and $b$ in their toroidal and poloidal parts

\[
\begin{align*}
u &= \text{curl}(W \xi) + \text{curl}^2(V \xi) \\
b &= \text{curl}(\mathcal{B} \xi) + \text{curl}^2(\mathcal{E} \xi)
\end{align*}
\]

(2.25)

so that equations (2.21) are automatically satisfied, and assume that $W$, $V$, $\mathcal{V}$, $\mathcal{S}$, $\mathcal{F}$ have the dependence

\[
F(\xi, \phi, z, t) = F(\xi) J_n(kz) \exp i(m\phi + \gamma t).
\]

(2.26)

Guided by previous investigations of the spherical problem (El-tayeb 1972b), we assume that

\[
\frac{\partial}{\partial \phi} \gg \frac{\partial}{\partial \xi} \gg \frac{\partial}{\partial z} = O(1),
\]

(2.27)

and reduce equations (2.18)-(2.20) to (see 1)

\[
\begin{bmatrix}
\Psi \\
\frac{\partial}{\partial \xi} \\
T
\end{bmatrix}
= \begin{bmatrix}
Y_1 \\
Y_2 \\
Y_3
\end{bmatrix}
= 0,
\]

(2.28a)

in which

\[
Y_1 = ik W, \quad Y_2 = -k^2 V, \quad Y_3 = (i\omega + k^2) \mathcal{H}/hk, \quad T = m/kh = t a \ell, \quad \mathcal{R} = R h^2 k/(i k^2 - \omega), \quad \Psi = ik h \left[ E(i\omega r^2 + k^2) + m^2 Q f^2/(i\omega r^2 + k^2) \right], \quad z = h \xi, \quad \ell = \mathcal{B}/s.
\]

(2.28b)

In (2.28) we have defined the intrinsic (Doppler-shifted) frequency $\omega$ by

\[
\omega = \sigma + m \mathcal{S}/s = \sigma + m^2 Q f^2.
\]

(2.29)

The magnetic field $b$ is given by
\[
\tau = \text{Im} \left( \frac{W}{i\omega + k^2} \right), \quad \tilde{V} = \text{Im} \left( \frac{V}{i\omega + k^2} \right). \tag{2.30}
\]

The system (2.28) is of second order in \(d/d\xi\) and must be solved subject to the condition that the normal component of velocity on the boundary vanishes. The remaining boundary conditions (see Eltayeb 1975) are adjusted by boundary layers on the surface. Thus

\[
Y_3'(\pm i) = 0, \tag{5.7.10}
\]

for filaments outside \(C_0\) (see figure 1 above), and

\[
Y_3(i) = Y_3(h_i) = 0, \tag{2.311}
\]

for filaments inside \(C_0\), where

\[
h_i = \left\{ \frac{(1 - \xi_i^2)/(1 - \xi_i^2)}{2} \right\}^{1/2}. \tag{2.32}
\]

The system (2.28), (2.31) possesses two uncoupled sets of solutions one of which has \(Y_1\) and \(Y_3\) even in \(\xi\) while \(Y_2\) is odd and the other set has \(Y_1\) and \(Y_3\) odd while \(Y_2\) even in \(\xi\). Since \(Y_3\) is proportional to the temperature perturbation and \(Y_1\) is a measure of the axial vorticity we see that the sign of temperature perturbation and axial vorticity are always the same. We shall see later that the preferred mode of convection has an even \(Y_3\) with no zeros inside the shell and consequently the axial vorticity has the same sign along the filament. In the treatment below we shall refer to the solution in which \(Y_3\) is even by the even solution and the other by the odd solution.

For every set of the parameters \(p, q, E, Q, R, k, s, a\) the system (2.28), (2.31) possesses an infinite set of solutions each one of them is characterized by an index \(n\), which is the scaled axial wavenumber. For each \(n\) the quantities \(p, q, E, Q, R, k, s, a\) are related by a dispersion relation. Our interest is in neutrally stable modes for which \(a\) is real. Separation of real and imaginary parts then leads to two relations

\[
\sigma = \sigma(k, n, s_n), \quad R = (k, n, s_n), \tag{2.33}
\]

for every set of values of \(p, q, E\) and \(Q\). The expression for \(\sigma\) gives the propagation properties while that for \(R\) governs the stability properties. The stability is studied by first minimizing \(R\) over \(k\) to get

\[
R = R_m = \tilde{R}_n(n, s_n). \tag{2.34}
\]

\(R_m\) is then examined as a function of \(n\) and \(s\) to determine the preferred mode i.e. the mode associated with the smallest value \(R_m(s, n)\), for which \(R\), \(\sigma\), \(k\), \(s\), \(n\) take the values \(R_m, \sigma, k_m, s_m, n_m\). In the analysis below \(k_m\) may be replaced by \(k\) since they are related as given in (2.28b).

Before we conclude this section we found it convenient to express \(K(r)\) in the form

\[
K(r) = \alpha_1 - \alpha_2 r^{-3}, \tag{2.35}
\]

for easy reference to the various forms of temperature gradient.

We may also point out here that the three magnetic fields studied are such that

\[
\int_M = 1, \quad \int_S = 1 - r^{-3}, \quad \int_A = 2(1 - r^{-5}) \tag{2.36}
\]

corresponding respectively to the Malkus field, the symmetric (with respect to the equatorial plane) field \(n = 1\) and the anti-symmetric field \(n = -2\).

III. PROPAGATION AND STABILITY WHEN \(q << 1\).

When \(q << 1\), the nature of the wave motions depend on the quantity \(\gamma\) defined by

\[
\gamma = Q p^\frac{3}{2} E, \tag{3.1}
\]

as has been shown by Eltayeb and Kumar (1974, 1977). See also appendix B. For \(\gamma << 0(1)\) the magnetic field serves to modify the waves of the non-magnetic case (see I) but as \(\gamma\) increases to higher values another type of wave is present. It is then convenient to consider the two cases (i) \(\gamma << 0(1)\) and (ii) \(\gamma >> 1\) separately.
3.1. The case $\gamma \in O(1)$

We make the transformation

\begin{equation}
(m, k, \sigma, R) = p^{\frac{1}{3}} E^{-\frac{1}{3}} \left( M, q, p^{\frac{1}{3}} E^{-\frac{1}{3}} \sigma, p R \right),
\end{equation}

and expand in powers of $p$ (see I)

\begin{equation}
Y = Y^{(0)} + p Y^{(1)} + \cdots, \\
\sigma = \sigma^{(0)} + p \sigma^{(1)} + \cdots, \\
R = R^{(0)} + p R^{(1)} + \cdots,
\end{equation}

so that

\begin{equation}
\Psi = \Psi^{(0)} + p \Psi^{(1)} + \cdots, \\
\bar{R} = p \bar{R}^{(0)} + \cdots.
\end{equation}

where

\begin{equation}
\Psi^{(0)} = -h \sigma^{(0)} a, \quad \Psi^{(1)} = h a \left( -\sigma^{(0)} + \frac{1}{2} a^2 + \frac{1}{6} M^2 + \frac{1}{3} \sigma^{(0)} \right), \\
\bar{R}^{(0)} = \frac{a h^3 \bar{R}^{(0)}}{a^2 - \sigma^{(0)}},
\end{equation}

Note here that

\begin{equation}
\omega = p^{\frac{1}{3}} E^{-\frac{1}{3}} \left[ \sigma^2 + Y p \left( \frac{E^2}{p^2 q^2} \right) \right]^2,
\end{equation}

and the influence of the shear is potent only if $p^{2/3} - \frac{3}{2} q \ll 1$. However, for the geophysical case this inequality is far from being satisfied and we have therefore neglected this term so that $\omega = \sigma$.

The expressions (3.3) and (3.4) are substituted into (2.28) and the coefficients of different powers of $p$ are equated to zero to obtain a hierarchy of equations which can be solved seriatim. For our purposes here it suffices to consider the first two such problems. These are

\begin{equation}
\begin{aligned}
\sigma^{(0)} &= 0, \\
\Psi^{(0)} &= 0,
\end{aligned}
\end{equation}

which must be solved subject to the conditions that both $Y^{(0)}$ and $Y^{(1)}$ satisfy (2.31). The operators $L^{(0)}$ and $L^{(1)}$ are given by

\begin{equation}
\begin{bmatrix}
\Psi^{(0)} \\
\frac{\partial \Psi^{(0)}}{\partial x} \\
-\bar{R}^{(0)}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
-1
\end{bmatrix}
\begin{bmatrix}
\Psi^{(0)} \\
\frac{\partial \Psi^{(0)}}{\partial x} \\
-\bar{R}^{(0)}
\end{bmatrix}.
\end{equation}

The process of obtaining expressions for $\Psi^{(0)}$ and $\sigma^{(0)}$ is identical to that in I and we may therefore omit the details and state the results.

**Convection outside $C$**

The solutions here fall into two categories of even and odd solutions. The even solution is

\begin{equation}
\begin{aligned}
Y_{1e}^{(e)} &= \cos \Psi_e S, \\
Y_{2e}^{(e)} &= \sin \Psi_e S, \\
Y_{3e}^{(e)} &= T \cos \Psi_e S - S \sin \Psi_e S,
\end{aligned}
\end{equation}

where

\begin{equation}
\Psi_e^{(n)} = \Psi_e(n) = Q_e + n \Pi, \quad n = 0, \pm 1, \pm 2, \ldots.
\end{equation}

Hence

\begin{equation}
\begin{aligned}
\sigma^{(e)} &= \sigma^{(0)} = -\Psi_e^{(n)}/h a, \\
\sigma^{(1)} &= \sigma^{(0)} = -\sigma^{(n)} \left\{ 1 + \frac{(n+1)^2}{a^2} \right\} \sigma^{(n)} \left\{ [\sigma^{(n)}]^2 \right\},
\end{aligned}
\end{equation}

in which the integrals $I_n$ and $H_n$ are defined by

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\[
I_n(l, o) = \int_0^1 K(s) [\gamma_{(s)}^o] dS, \quad H_n(I, o) = \int_0^1 [f(s)]^2 dS, \quad (3.12)
\]

and are evaluated in appendix B for the three expressions (3.1) for \( f \).

The odd solution is given by

\[
Y_{l, 0}^{(o)} = \sin \Psi_0^{(o)} S, \quad Y_{l, 0}^{(b)} = - \cos \Psi_0^{(o)} S, \quad Y_{l, 0}^{(e)} = T \sin \Psi_0^{(o)} S + \cos \Psi_0^{(o)} S, \quad (3.13)
\]

where

\[
\Psi_0^{(o)} = \Psi_0^{(o)}(n) = (\theta_n - \frac{\pi}{2}) + n \pi, \quad n = 0, 1, 2, \ldots, \quad (3.14)
\]

Thus

\[
\sigma^{(o)} = \sigma^{(o)} = - \Psi_0^{(o)}(n)/h, \quad (3.15)
\]

and \( \sigma^{(1)} \) and \( \sigma^{(0)} \) are given by (3.11) provided we replace \( I_n \) by

\[
\int_0^1 K(s) [\gamma_{(s)}^o] dS, \quad (3.16)
\]

which is evaluated in appendix B.

Here the solution cannot be divided into even and odd sets. If we define

\[
\psi = \Psi^{(o)}(s-1) + \theta, \quad (3.17)
\]

the solution here takes the form

\[
Y_{l, 1}^{(o)} = C \sin \psi, \quad Y_{l, 2}^{(o)} = \sin \psi, \quad Y_{l, 3}^{(o)} = T \psi - \xi \cos \psi, \quad (3.18)
\]

Application of the boundary conditions leads to

\[
\Psi^{(o)} = \Psi^{(o)}(n) = (\bar{\bar{\theta}} + n \pi)/(1 - h), \quad n = 0, 1, 2, \ldots, \quad (3.19)
\]

in which \( \bar{\bar{\theta}} \) is the principal value of \( \bar{\bar{\theta}} \) of

\[
\tan \left[ (1 - h) \Psi^{(o)}(n) \right] = \frac{h}{1 - h}, \quad (3.20)
\]

Consequently we obtain

\[
\sigma^{(o)} = \sigma^{(o)} = - \Psi^{(o)}(n)/h, \quad (3.21)
\]

\[
\sigma^{(o)} = \frac{\sigma^{(o)}}{\sigma^{(o)}} \left\{ [1 - h + (i - q)] A^2 \right\} H_n(l, h) / \left\{ (a^2 + \frac{q^2}{4} \sigma^{(o)})^2 \right\}, \quad (3.22)
\]

\[
H_n(l, h) = \left\{ (a^2 + \frac{q^2}{4})^2 \right\} \left\{ [1 - h + (i - q)] A^2 \right\} H_n(l, h) / \left\{ (a^2 + \frac{q^2}{4} \sigma^{(o)})^2 \right\}, \quad (3.23)
\]

The expressions \( H_n(l, h) \) and \( J(l, h) \) are given in appendix B.

Having derived the expressions for the frequency \( \sigma^{(o)} \) and the Rayleigh number \( R^{(o)} \) both inside and outside \( C \), we now proceed to discuss their dependence on the magnetic field (as measured by \( \gamma \)) and on the ratio \( \gamma \), of thermal to magnetic diffusivities. In general the presence of any toroidal magnetic field results in an increase in the Rayleigh number whatever the value of \( \gamma \). However, the dependence of the phase speed \( \sigma_p = \sigma / \sigma \) on the magnetic field is related to \( \gamma \). If we note that the expression (3.19) for \( H^{(o)}(n) \) applies for all values of \( \gamma \), we see that \( \sigma_p \) is decreased (increased) in magnitude for \( \gamma < (>) 1 \).

To isolate the influence of the form of magnetic field on the phase propagation and stability of the waves we find it convenient to study the cases \( a_1 = 1, a_2 = 0 \); \( b_1 = 0, a_2 = - a_1 \); \( c_1 = 1, a_2 < 0 \); \( d \) \( a_1 = 1, a_2 > 0 \); discussed in 1 in order to facilitate comparison between the corresponding cases. The Kennedy-Higgins model will also be discussed. For each case results in the three fields (2.36) are presented.

\[
\sigma^{(o)} = \Psi^{(o)}(n) = (\bar{\bar{\theta}} + n \pi) / (1 - h), \quad n = 0, 1, 2, \ldots, \quad (3.24)
\]

The temperature gradient here corresponding to the model appropriate to the full sphere and has been the subject of study by many authors (see e.g., Eltayeb and Kumar 1977, Fearn 1979). It is, however, possible to introduce an inner solid core provided that it has the same temperature on the surface as that possessed by the same fluid surface in a full sphere i.e. by taking the temperature difference.
\[ \Delta T = \epsilon \gamma^2 (1 - \eta^2) / 6 \kappa. \tag{3.20} \]

The functions (3.11) and (3.19) for \( R(n) \) have been evaluated numerically subject to the condition \( \delta R(\eta)/\delta \eta = 0 \) to find \( R(n) \) as a function of \( n \) and \( \eta \). Such a value of \( R(n) \) will be denoted by \( R_n \). In figure 2, \( R \) is drawn as a function of \( \eta \) for the first few modes. The location of the preferred cell for all values of \( n \) depends on \( \eta \) as well as on the form of the ambient magnetic field. For the Malkus field the distance \( \eta \) for the preferred cell for each \( n \) moves towards the axis as \( \eta \) increases. The critical mode is provided by the fundamental even model (FEM) as given by \( n = 0 \) in (3.11) if \( \eta \) is not too large. As \( \eta \) increases \( R_n \) for this mode increases and the associated \( \eta \) moves towards the axis of rotation. If an inner core of radius, \( \eta \), exceeding 0.03 exists then this mode will always be preferred. However, if \( \eta < 0.03 \) then large values of \( \eta \) make \( R_n \) for the FEM large compared with that associated with the mode \( n = 1 \) of the odd set (3.13) which then becomes preferred.

For the symmetric and anti-symmetric fields on the other hand the preferred cell moves away from the axis as \( \eta \) is increased from zero. The critical mode is provided by the FEM for all values of \( \eta \).

Within \( C \) convection is similar to that described in I when \( \eta \) is small. For the \( M \)-field the curves for \( R_m \) versus \( \eta \) flatten as \( \eta \) increases until they become flat and for yet larger values of \( \eta \) they change curvature and possess minima on the axis. For the \( S \) and \( A \) fields, however, the curves steepen with the increase in \( \eta \) so that their minima always lie on \( C \). For all fields the smallest \( R \) is always provided by the mode \( n = 1 \) in (3.19).

The dependence of the critical mode on \( \eta \) is similar to that found by Eltayeb and Kumar (1977) and is not shown here. In figure 3 the dependence of the critical mode on \( \eta \) is illustrated. Although \( R \) decreases with the increase of \( \eta \) for all fields \( M, S \) and \( A \), the decrease of \( R \) in the case of the \( M \) field is sharper. For the \( M \) field \( s \) and \( a \) increase with \( \eta \) but they decrease for both \( S \) and \( A \) fields. \( a \) decreases steadily with \( \eta \) for both \( S \) and \( A \) fields but for the \( M \) field it initially decreases, provided \( n > 0.23 \), before it starts to increase. Moreover, all the quantities \( R, s, a \) tend to values independent of \( \eta \) as \( \eta \) increases beyond about 10. We shall see later that when \( \eta \) is large, the dependence of \( R \) on \( \gamma \) is much less pronounced.

3.2. \( \eta = 0, \gamma = 0 \).

This is the case of a differentially heated spherical shell. In the absence of the magnetic field the critical mode lies on \( C \) for all values of \( n \).

The influence of the \( M \) field here only increases the value of \( R \) but the unstable cell rests on \( C \) for all \( n \). For the \( S \) and \( A \) fields \( s \) always exceeds the value 0.52 predicted by the non-magnetic case because the effect of either of these fields is to push the unstable cell towards the equator and away from the axis (see figure 4).

3.3. \( \eta > 0, \gamma = 0 \).

Except for very small values of \( \gamma \) (see I, figure 5) this case is similar to the preceding one. This is clearly brought out in figure 5 where the critical mode for cases (1) - (3) are compared.

3.4. \( \eta = 1, \gamma = 0 \).

In this case the magnetic field has no influence on the region of convection (i.e. where \( R > 0 \)) but \( R \) is again increased by the presence of the field. For the \( M \) field the behaviour of \( R \) versus \( \eta \) is similar to that obtained for the non-magnetic case of paper I. For the \( S \) and \( A \) fields, however the behaviour is slightly different.

3.5. The \( K - H \) model

The influence of the magnetic field on the model of a spherical shell whose lower part is unstably stratified while the upper part is stably stratified will now be discussed. As we showed in I this situation can be obtained from (2.22), (2.23) by replacing \( \gamma \) by \( -\gamma \) so that in (2.35) \( a_1 = 1 \), \( a_2 > 0 \) and \( R \) is replaced by \( -R \). To avoid confusion \( R \) here will be referred to as \( R^* \).

In the absence of the magnetic field the minimum value of \( R^* \) for all modes occurs on \( C \); a situation which persists even when the \( M \)-field is present. For the \( S \) and \( A \) fields, however, \( R^* \) decreases from its value on \( C \) to a minimum at \( s = s \) provided \( n \) is large before it increases to infinity.
at some value of $s$, depending on $n$. For larger values of $|n|$, $R^*$ behaves in a way similar to that for the $M$-field. For all fields the region of convection within the unstable cell is unaffected by the presence of the field. However, this result will not survive when the amplitude the field is large (see §3.2 below).

The analysis of this (3.1) section showed that the location of the preferred cell of convection according to the linear theory depends on the form of the magnetic field. Whereas the cell moves towards the axis in the case of the $M$-field it is pushed away from the axis in the case of the other two fields $A$ and $S$. A detailed examination of the expression for $R(0)$ showed that this is due to the dependence of the integral $H_n$ on $s_0$. For the $M$-field $f(=1)$ is a constant and $H_n$ is proportional to the normalized length of the cell ($=2$) and consequently the contribution of the magnetic field to $R(0)$ is proportional to $s_0^2$, so that this contribution is smallest where $s_0$ is smallest.

For the $A$ and $S$ fields $H_n$ decreases sharply with $s_0$ and consequently the critical cell of the non-magnetic case tends to move to where the field is weakest. The result that the cell lies at a finite distance from both axis and equator is due to the relative strength of the component of buoyancy normal to the axis which promotes instability and the slope of the boundary which tends to inhibit it.

The location of the preferred cell depends also on the magnitude of the amplitude of the field. For fields which decrease with $s$, as in the case of the $A$ and $S$ fields, the critical cell stays almost at a constant distance from the axis as $\gamma$ increases because the steep slope of the boundary near the equator provides a very strong inhibiting influence. In the case of the $M$ field the critical cell moves towards the axis, as $\gamma$ increases, until it coincides with $C_0$ (if present). The critical mode in all cases being the FEM, the only exception occurring when $n<0(p)$ and $p<<1$. In this situation the critical cell for fields of the $M$ type can belong to the first eastward propagating odd mode. As is clearly shown in figure 2(a) and (b) above, $R_c$ for the FEM increases sharply within a region of order $p$ around the axis of rotation due to the strong influence of viscous diffusion there (see 1) and hence $R_c$ for the FEM is too large to be preferred. This situation was overlooked by Fearn (1979 §4c).

### §3.2 The case $\gamma >> 1$, $q << 1$

As $\gamma$ increases while $q$ remains small, two more waves appear (Eltayeb and Kumar 1974). The stability of these two modes for values of $q = O(1)$ was studied by Eltayeb and Kumar (1977). For $q<<1$, these waves have frequencies whose ratio is $q$ and it is found that the wave with the smaller frequency (the 'slow' wave) is associated with a Rayleigh number which is smaller than that associated with the 'fast' wave by a factor of $q$. The mechanism responsible for their propagation is the same and we will therefore discuss the slow wave here.

If we let

$$ R = R_0 Q^{-1}, \quad \sigma = \frac{1}{2} Q \omega^2, \quad m = \frac{1}{2} Q \gamma, \quad k = a \frac{Q}{Q} \gamma^{-1} $$

and substitute in (2.28) adopting an expansion in powers of $q^{1/2}$, we find that, to leading order,

$$ d^2 Y_3^{(0)}/ds^2 + R^{(0)} T Y_3^{(0)} = 0 $$

$$ Y_2^{(0)} = -d Y_3^{(0)}/d s, \quad Y_2^{(0)} = T \left[ Y_3^{(0)} - 5 d Y_3^{(0)}/d s \right], $$

where

$$ R^{(0)} = R_0^{(0)} \frac{3}{2} k \frac{h}{a} \frac{1}{(-\sigma^{(0)})}. $$

The first of (3.22) subject to the condition that $Y_3^{(0)}$ vanishes at the ends of the interval $[-1, 1]$ or $[h_1, 1]$, depending on whether cells outside or inside $C_0$ are under consideration, possesses a discrete set of solutions and every solution is a diffusionless wave whose properties are determined by Coriolis and buoyancy forces. The direction of phase propagation of the waves depends on the nature of the stratification. Assuming, without loss of generality, that $R^{(0)}$ is positive and realizing that a solution for $Y_3^{(0)}$ vanishing at the ends of a finite interval exists only if $R^{(0)}$ is positive, the phase speed...
\( a = -\sigma^{0}/a \) is positive, corresponding to eastward propagation if \( K(c) \) is positive everywhere (and the shell is unstably stratified). If, on the other hand, \( K(c) \) is negative everywhere and the fluid is subject to bottom heavy stratification then \( a \) is negative and the wave will drift westward. In the situation in which \( K(c) \) changes sign somewhere along the cell both eastward propagating waves are possible, in general. Thus the direction of propagation of these diffusionless waves depends on the nature of the stratification. Now the stability problem is, to leading order, determined by the next order problem (i.e. the problem for \( Y^{(2)} \)). Here a balance between the Lorentz and thermal diffusion forces is maintained resulting in the instability of some of these waves so that the direction of propagation of the unstable wave depends also on magnetic forces and thermal diffusion.

If we let

\[
P = -R^{(0)} \frac{1}{h} \frac{\partial}{\partial x} \sigma^{0},
\]

so that the first of (3.22) assumes the form

\[
\frac{d^2 Y^{(0)}}{dS^2} + P K Y^{(0)} = 0,
\]

then the problem for \( Y^{(1)} \) yields a solvability condition which leads to

\[
\sigma^{(0)} = 0, \quad \sigma^{(0)} = -\frac{\sigma^{(0)}}{2} \frac{P}{h} \frac{1}{s} \frac{d^2}{dS^2} \frac{1}{s}, \quad \frac{R^{(0)}}{h} = P \frac{2}{s} \frac{d^2}{dS^2} \frac{1}{s}, \quad \frac{(3.26)}{(3.25)}
\]

in which

\[
\frac{S^{(0)}}{S} = \int_{a}^{b} K(3) [Y^{(0)}(S)]^2 dS, \quad \frac{S^{(0)}}{S} = \left[ \int_{a}^{b} S^{(0)}(S) \frac{d^2}{dS^2} \frac{1}{s} \right] dS.
\]

where \( a \) takes the values \( 0, h \), for cells outside and inside \( C \), respectively. Whereas \( S^{(0)} \) is always positive, \( S^{(0)} \) may take positive or negative values depending on the form of \( K(c) \). If \( K(c) \) does not change sign within the cell then \( S^{(0)} \) will have the sign of \( K(c) \) but if \( K(c) \) changes sign somewhere between \( a \) and \( h \) then \( S^{(0)} \) may take positive or negative values depending on the value of \( l \) at which \( K(c) \) changes sign.

It is noteworthy that although the leading order problem makes it possible for westward propagating waves to exist in a shell with bottom-heavy stratification (i.e. \( K < 0 \)), the relationship (3.26) shows that all such waves must decay when Lorentz and thermal diffusion forces are included because if we replace \( P \) and \( K \) by \( -P \) and \( -K \) then \( S^{(0)} \) becomes \( -S^{(0)} \) and consequently \( K^{(0)} \) becomes \( -K^{(0)} \) i.e. \( a \) takes the value \( -a \) and hence the local temperature gradient remains positive. Thus instability can occur only if \( K(c) \) is positive somewhere within the shell.

When \( a = 1, a = 0, K(c) = 1 \) and the problem posed by (3.25) and (3.26) can be solved analytically. Outside \( C \), the even solution is

\[
Y_{2}^{(0)} = \frac{\omega_{n}^{0} \sin \omega_{n}^{0} S}{\omega_{n}^{0} \sin \omega_{n}^{0} S + \omega_{n}^{0} \sin \omega_{n}^{0} S}, \quad Y_{2}^{(0)} = T^{(0)} \left[ \sin \omega_{n}^{0} S + \cos \omega_{n}^{0} S \right],
\]

where

\[
\omega_{n}^{0} = \frac{1}{n + \frac{1}{2}} \pi, \quad Y = 0, \pm 1, \pm 2, \ldots.
\]

Consequently

\[
Y_{2}^{(0)} = \frac{\omega_{n}^{0}}{K^{(0)}}, \quad \frac{S^{(0)}}{S^{(0)}} = 0, \quad \frac{S^{(0)}}{S^{(0)}} = -\frac{1}{2} \frac{\omega_{n}^{0}}{h^{2}},
\]

and \( P(1,0) \), as given by equations (8.8)-(8.10) of appendix B, is defined as

\[
F_{n}(1,0) = \frac{1}{n + \frac{1}{2}} T^{2} \int_{a}^{b} \left( \frac{Y_{n}^{(0)}(S)}{s} \right)^{2} dS.
\]

The odd solution is given by

\[
Y_{0}^{(0)} = \sin \omega_{n}^{0} S, \quad Y_{0}^{(0)} = -\frac{\omega_{n}^{0}}{K^{(0)}}, \quad Y_{0}^{(0)} = T^{(0)} \left[ \sin \omega_{n}^{0} S - \cos \omega_{n}^{0} S \right].
\]
where
\[ \widehat{\alpha}_n = n \pi / (1 - h_1) \quad ; \quad n = \pm 1, \pm 2, \ldots \quad \gamma \]  
(3.33)

and then
\[ \widehat{R}(n) = -\frac{\gamma_n}{h^2} \sigma_p \quad \sigma_p = -\frac{1}{\beta} \frac{\gamma_n}{h^2} \sigma_p \frac{1}{G_n(1,0)} \quad \]  
(3.34)

\( G_n \) being defined as
\[ G_n(1,0) = \int_0^1 \left\{ \left[ Y_{10}(s) \right]^2 + \left[ Y_{11}(s) \right]^2 \right\} ds , \]  
(3.35)

and is also evaluated in appendix B.

Within \( C_c \) the solution takes the form
\[ \gamma_3(0) = \gamma_2 \left( 1 - \gamma_1 \right) ; \quad \gamma_2 = -\frac{\gamma_3}{C_t \gamma_1 (1 - 1)} ; \quad \gamma_1 = T^{-1} \left( \gamma_3 \frac{\gamma_1 (1 - 1)}{\gamma_2 \gamma_1 (1 - 1)} \right) , \]  
(3.36)
in which
\[ \Delta_n = n \pi / (1 - h_1) \quad ; \quad n = \pm 1, \pm 2, \ldots \quad \gamma \]  
(3.37)

and
\[ \widehat{R}(n) = -\frac{\gamma_n}{h^2} \sigma_p \quad \sigma_p = -\frac{1}{\beta} \frac{\gamma_n}{h^2} \sigma_p \frac{1}{G_n(1,0)} . \]  
(3.38)

The expressions (3.30), (3.34), (3.38) have been evaluated for all the fields \( N, S \) and \( A \) for various values of \( a_0, \epsilon \) to find that neutrally stable waves necessarily drift eastward for all values of \( n \) so that westward propagation is not possible. The dependence of \( \widehat{R}(n) \) and \( \sigma_p \) on \( a_0 \) is illustrated in figure 6 for various values of \( n \). It is clear that the unstable cell tends to move to where the field is strongest. For the \( A \) and \( S \) fields the preferred cell is always situated on \( C_c \) but for the Malkus field it is forced by the steep slope of the boundary at the equator to lie away from the region of maximum field amplitude. The expression for \( \sigma_p \) for the \( A \) and \( S \) fields also shows marked difference in its dependence on \( a_0 \) from that predicted by the \( M \)-field.

When \( R(n) \) is evaluated as a function of \( a_0 \) inside \( C_c \) it is found that it increases sharply from its value on \( C_c \) as \( a_0 \) is decreased. Indeed at \( a_0 = 0 \) the value of \( R(n) \) is drastically increased to an order \( \left( \frac{\sqrt{q}}{Eq} \right)^{1/3} \) larger and the possible wave on the axis is associated with a primary balance in which Coriolis, inertial, buoyancy Lorentz and magnetic diffusion forces are all potent.

In figure 7 the critical mode is drawn as a function of \( n \). For the \( M \)-field the critical cell is situated at \( a_0 = 0.68 \) so that it is uninfluenced by the presence of \( n \) for all \( n < 0.68 \). However, when \( n \) increases beyond 0.68 the critical cell is situated on \( C_c \) with \( R \) increasing with \( n \) and the associated phase speed \( \sigma_c \) decreasing with \( n \). For the \( S \) and \( A \) fields the critical mode is situated on \( C_c \) for all \( n \). Both \( R \) and \( \sigma_c \) increase steadily with the increase in \( n \).

To investigate the relative importance of the form of temperature gradient on these waves and their instability we briefly discuss the two cases

(i) a differentially heated shell and (ii) a shell for which \( K(\xi) \) changes sign.

(i) Differentially heated shell

Here \( \alpha_1 = 0, \alpha_2 = -\alpha_3 \). Equation (3.22) was integrated numerically and a sample of the results is shown in figure 8. The surprising result here is that the dependence of the location of the unstable cell depends on the type of node under consideration. For the lowest even mode \( R(n) \) increases steadily form its value on \( C_c \) to infinity at the equator. The influence of differential heating is to promote convection near \( C_c \). However, its effect on the odd modes of the \( M \)-field is not strong enough to make their preferred mode lie on \( C_c \).

(ii) \( K(\xi) \) changes sign within shell

The interest in this situation is motivated by the suggestion of Kennedy and Higgins (1973) that the earth's liquid core is composed of an instably stratified lower part and stably stratified upper part. Such a
situation is obtained by setting \( a_1 = 1 \) and \( a_2 > 0 \). To illustrate the influence of such a model on the propagation properties and stability of inertial-buoyancy waves of this case we shall take \( n = 0.3 \) and \( a_2 = 0.2 \) (so that \( T_0 \approx 0.58 \)) and replace \( \epsilon \) by \( -\epsilon \) (see I). The numerical integrations of (3.22) within and outside \( C_0 \) showed that an infinity of modes can exist and all of them can be made unstable if \( \epsilon \) is large enough. Because of the change of sign of \( K(\epsilon) \) the solution, which is characterized by a primary balance between Coriolis and buoyancy forces, is oscillatory within the lower unstable stratified portion and exponentially decaying elsewhere. The first few modes are illustrated in figure 9. The amplitude of \( Y_3^{(0)} \) adjusts itself so as to make the integral \( S_1 \) positive.

The dependence of \( R(n) \) and \( \sigma_p^{(1)}(n) \) on \( n_0 \) is illustrated in figure 10.

IV. STABILITY AND PROPAGATION FOR \( \epsilon \gg 1 \).

The analysis of § 3.1 above showed that \( R \) tends to a constant as \( \epsilon \) increases beyond about 10. This is because the dependence of \( R_c \) on \( \epsilon \) changes as \( \epsilon \) increases to large values. Indeed if we define

\[
\lambda = \frac{\gamma \beta}{q}
\]

then the critical mode remains the same as that of the nonmagnetic mode, to leading order, until \( \lambda \) takes values of order unity. As a result the primary balance in (2.27) is between inertial, Coriolis and Lorentz forces. The propagation properties of the neutral waves is then strongly influenced by magnetic forces and the resulting waves are closely related to those studied by Hinds (1966) for the homogeneous sphere.

The equations of the problem in this case are (3.2)-(3.4), (3.7) and (3.8) provided we define \( \psi^{(0)} \) and \( \psi^{(1)} \) by

\[
\psi^{(0)} = \frac{he}{-\sigma + M^2 \lambda^2 \sigma^2 / \sigma (a)}
\]

\[
\psi^{(1)} = \frac{he}{\sigma^2 + i \alpha^2 + M^2 \lambda^2 \sigma^2 / \sigma (a)}
\]

while \( R^{(0)} \) is again given by the last of (3.5) and the magnetic Prandtl number

\[
\frac{p_c}{p} = \frac{p}{\gamma} = \frac{\mu}{\rho}
\]

The leading order problem \( L^{(0)} \psi^{(0)} = 0 \) can be written in component form as

\[
\begin{align*}
\psi^{(0)} Y_1^{(0)} - d Y_2^{(0)}/d \xi &= 0 \\
d Y_1^{(0)}/d \xi + \psi^{(0)} Y_2^{(0)} &= 0 \\
Y_3^{(0)} - T Y_1^{(0)} + S Y_2^{(0)} &= 0
\end{align*}
\]

For the M-field \( \psi^{(0)} \) is a constant and the first two equations can be reduced to a second order linear differential equation which can easily be solved. However, in general \( \psi^{(0)} \) depends on \( c \) (as for the \( A \) and \( S \) fields here) and a more general method of solution is required. This can readily be found. We introduce \( Z \) by

\[
Z = Y_1^{(0)} + i Y_2^{(0)}
\]

so that the first two of (4.4) become

\[
d Z_1/d \xi - i \psi^{(0)} Z = 0
\]

Consequently

\[
Z = C \exp i \Phi(\xi), \quad \Phi(\xi) = \int_\alpha^\xi \psi^{(0)}(\xi) d \xi
\]

in which \( \alpha \) takes the values \( a, \beta \) depending on whether the cell is inside or outside \( C_c \), respectively.

Convection outside \( C_c \) occurs in the form of the even modes

\[
\begin{align*}
Y_{1e} &= \cos \Phi, \\
Y_2 &= \sin \Phi, \\
Y_{3e} &= T \cos \Phi - S \sin \Phi
\end{align*}
\]
and the odd modes
\[ y_{0}^{(e)} = - i m \Phi \quad y_{1}^{(e)} = i n \nabla \Phi \quad y_{0}^{(o)} = - T \lambda \Phi - \zeta \cos \Phi. \] (4.9)

The boundary condition \( \Phi \big|_{\delta} (1) = 0 \) demands that
\[ \Phi (1) = \Phi (n) \] for the even modes \( n = 0, 1 \) for the odd modes \( n > 0 \).

It then follows from (4.2), (4.7) and (4.10) that
\[ \left[ \sigma^{(e)} \right]^{2} + \frac{\Phi_{n}}{\beta \alpha} \sigma^{(e)} - \lambda \alpha^{2} H_{n}(1, \beta) = 0. \] (4.11)

The three terms in (4.11) represent, respectively, inertial, Coriolis and Lorentz forces. This relation then represents propagation of waves in a diffusionless incompressible unstratified fluid rotating in the presence of a corotating magnetic field. For each set of values of \( n, \beta, \alpha \) and \( \lambda \), two frequencies are possible
\[ \sigma_{\lambda}^{(e)} = \frac{1}{2} \left[ - \frac{\Phi_{n}}{\beta \alpha} \pm \frac{\Phi_{n}^{2}}{\beta \alpha^{2}} + 4 \lambda \alpha^{2} H_{n}(1, \beta) \right]^{1/2}. \] (4.12)

It is clear from (4.10) that both even and odd families of solutions are composed of two subsets one of which is associated with \( \Phi < 0 \) and the other with \( \Phi > 0 \). Since the dispersion relation (4.11) closely resembles that obtained by Hide (1966) it is of interest to investigate the relationship between the two cases. It turns out that the case studied by Hide is the same as the fundamental even mode with \( n = 0 \). Here \( \Phi_{n}^{(e)} \) is positive and \( \Phi^{(e)} \) correspond, respectively, to Hide's fast and slow waves. If \( \Phi < 0 \) then the roles of fast and slow waves are reversed. Consequently both fast and slow waves can propagate either way.

The relationship between waves in thin and thick spherical shells has been a matter of controversy (see Hide 1966, Stewartson 1967, Hide and Stewartson 1972) and it is in order to use the above results to clarify the situation. It was shown in I that when \( h_{1} \) tends to zero in (4.16), \( \sigma_{\lambda}^{(e)} \) has an axial vorticity even in \( \zeta \) while (4.11) and (4.12) show that the odd modes with odd axial vorticity, also possess fast and slow waves and both fast and slow waves can propagate in either direction.

Within the \( C \) solution vanishing on \( \zeta = 1, h_{1} = 0, \)
\[ \Phi_{1}^{(e)} = - \lambda \Phi \left[ \Phi^{(e)} + \alpha \right], \quad \Phi_{1}^{(o)} = \Phi \left[ \Phi^{(o)} + \alpha \right], \]
\[ \Phi_{2}^{(e)} = - T \lambda \Phi \left[ \Phi^{(e)} + \alpha \right] - \zeta \cos \left[ \Phi^{(e)} + \alpha \right], \] (4.13)
in which
\[ \alpha = \tan^{-1} \left( - h / \gamma \right), \quad \Phi^{(e)} = \int_{h}^{\gamma} \Phi^{(e)} d \xi, \]
\[ \Phi^{(o)} = \Phi_{n} = - \Phi + n \pi; \quad n = 0, 1, 2, \ldots, \] (4.14)
where \( - \Phi \) is the principal value of
\[ \tan^{-1} \left( \frac{\Phi^{(e)}}{h_{1} + \gamma} \right). \] (4.16)

The dispersion relation for \( \phi^{(e)} \) here is obtained from (4.11) by replacing \( H_{1}(1, 0) \) by \( H_{1}(1, \beta) \) and likewise for the two frequencies \( \phi^{(o)} \) in (4.12).

The lowest mode \( n = 0 \) is associated with \( \Phi < 0 \) and hence the slow mode is \( \Phi^{(e)} \) and is eastward propagating while the fast mode is westward propagating. In fact the mode \( n = 0 \) is precisely the one studied by Hide for a thin shell. For values of \( | n | > 0 \) the direction of phase propagation of the waves \( \phi^{(e)} \) depends on the sign of \( n \). If \( n > 0 \) then \( \phi^{(e)} \) correspond, respectively to eastward and westward propagation with \( \phi^{(o)} \) being the fast wave. If \( n < 0 \) the roles of \( \phi^{(e)} \) are reversed. Consequently both fast and slow waves can propagate either way.
to τ/2 - θ_0 so that the modes (4.11) match uniformly to the odd modes (4.10). Thus while the motions outside \( C_C \) can occur in the form of even and odd modes it is only the odd modes which can occur within \( C_C \). Now the stability theory in the full sphere has predicted a westward propagating wave which is the lowest slow mode of the even solution which has no counterpart within \( C_C \) while the lowest odd mode, which exists both within and outside \( C_C \), is associated with a larger Rayleigh number and is therefore not preferred.

Although the propagation properties are governed by a dispersion relation independent of diffusion and stratification, the stability of these modes is strongly dependent on both stratification and diffusion. The important role played by diffusion here is to prevent the wavelength of the most unstable mode from shrinking to zero as occurs in the diffusionless theory of Braginskii (1967). The role of stratification here is to provide the necessary driving force for the instability. In the absence of buoyancy all the modes (4.11) will be damped.

The order \( p \) terms in the expansion (3.3) yield a consistency condition which represents a balance between buoyancy and diffusive processes (thermal and magnetic). We obtain

\[
\sigma^{(1)} = \sigma^{(0)} \left[ 1 + \frac{2 \lambda \lambda H_n(1,0)}{\int_0^\infty \sigma^{(0)} \sigma^{(0)} d\zeta} \right] \left\{ 1 + \frac{2 \lambda \lambda H_n(1,0)}{\int_0^\infty \sigma^{(0)} \sigma^{(0)} d\zeta} \right\},
\]

outside \( C_C \) for both even and odd solutions; and

\[
R^{(1)} = \left\{ \frac{\sigma^{(0)}}{\int_0^\infty \sigma^{(0)} \sigma^{(0)} d\zeta} \right\}^\frac{1}{2} \right\} \left\{ 1 + \frac{2 \lambda \lambda H_n(1,0)}{\int_0^\infty \sigma^{(0)} \sigma^{(0)} d\zeta} \right\}
\]

for even modes

\[
R^{(1)} = \left\{ \frac{\sigma^{(0)}}{\int_0^\infty \sigma^{(0)} \sigma^{(0)} d\zeta} \right\}^\frac{1}{2} \left\{ 1 + \frac{2 \lambda \lambda H_n(1,0)}{\int_0^\infty \sigma^{(0)} \sigma^{(0)} d\zeta} \right\}
\]

for odd modes

where

\[
\bar{I}_n(1,0) = \int_0^1 K(\zeta) \sigma^{(0)}(\zeta)^2 d\zeta, \hspace{1cm} \sigma_0(1,0) = \int_0^1 K(\zeta) \bar{I}_n(1,0) d\zeta.
\]

in which \( \sigma^{(0)} \) takes the appropriate values in (4.10) and \( \sigma^{(0)}_3, \sigma^{(0)}_1, \sigma^{(0)}_0 \) are given by (4.8), (4.9) respectively.

Inside \( C_C \), we get

\[
\sigma^{(0)} = \sigma^{(0)} \left\{ 1 + \frac{2 \lambda \lambda H_n(1,0)}{\int_0^\infty \sigma^{(0)} \sigma^{(0)} d\zeta} \right\}^\frac{1}{2} \left\{ 1 + \frac{2 \lambda \lambda H_n(1,0)}{\int_0^\infty \sigma^{(0)} \sigma^{(0)} d\zeta} \right\},
\]

\[
R^{(0)} = \left\{ \frac{\sigma^{(0)}}{\int_0^\infty \sigma^{(0)} \sigma^{(0)} d\zeta} \right\}^\frac{1}{2} \left\{ 1 + \frac{2 \lambda \lambda H_n(1,0)}{\int_0^\infty \sigma^{(0)} \sigma^{(0)} d\zeta} \right\}
\]

in which \( \sigma^{(0)}_3 \) is given by (4.13).

The expressions (4.18) and (4.20) for \( R^{(0)} \) were evaluated numerically for the three fields A, S and M for various values of \( \lambda, a, n \) and \( s_0 \). It was found that \( R^{(0)} \) for the slow mode tends to zero as a approaches zero for all \( n, s_0 \). This indicates that the most unstable slow mode is associated with an asymptotically smaller \( a \). The fast mode on the other hand possesses a minimum \( R^{(0)} \) for a finite value of \( a \). The critical mode always belongs to the lowest even mode \((n = 0)\) which exists outside \( C_C \). The dependence of the critical mode on \( \lambda \) and \( p \) is illustrated in figure 12 and tables 1 and 2. For both symmetric and anti-symmetric fields the location of the critical cell, \( s_c \), is almost independent of \( a \) because for values of the magnetic field amplitude given by \( \lambda = 0(1) \) the cell has been pushed nearer the equator where the steep slope of the boundary has a strong stabilizing influence. For the M field, however, the critical cell lies on \( C_C \). For a given value of \( p \), the critical Rayleigh number, \( R_c \), increases with \( \lambda \) for all fields. However, for fixed values of \( \lambda \) it decreases with \( p \).

When \( \lambda \) increases beyond 0(1) values the critical Rayleigh number belongs to one of the magnetic modes. Here the appropriate scaling is that of § 3.2 above and the expansion scheme is obtained by replacing \( p \) by \( a^{-1/2} \) in (3.3) above. The slower of the two modes has a Rayleigh number an order of magnitude \( q \) larger than that of the faster. Indeed this wave, to leading order, is stationary and has a large \( z \)-dependence with Coriolis, Lorentz and buoyancy together with magnetic and thermal diffusion forces all sharing in the primary balance. We shall not pursue this wave any further.

The faster wave, to leading order, is given by
in which
\[ \Psi^{(0)} = \frac{h a M^2 T^2}{\sigma^{(0)}} \]
and the next order problem is governed by
\[
\begin{bmatrix}
\Psi^{(0)} & -\frac{d}{d\xi} & 0 \\
\frac{d}{d\xi} & \Psi^{(0)} & 0 \\
T & -\xi & 0
\end{bmatrix}
\begin{bmatrix}
\Psi^{(0)} \\
\Psi^{(0)} \\
Y_2^{(0)}
\end{bmatrix}
= 0
\] (4.21)

The two problems must be solved subject to the conditions
\[
Y_3^{(0)}(\pm 1) = 0, \quad Y_3^{(1)}(\pm 1) = 0
\] (4.25)
for cells outside \( C_c \), and
\[
Y_3^{(0)}(h_1) = Y_3^{(0)}(i) = Y_3^{(1)}(h_1) = Y_3^{(1)}(i) = 0
\] (4.26)
for cells inside \( C_c \).

The leading order problem governing \( Y^{(0)} \) is characterized by a balance being maintained between Coriolis and Lorentz forces resulting in hydromagnetic-inertial waves of the slow type as studied by Hide (1966).

If we define
\[
\chi \equiv \frac{h a}{\sigma^{(0)}} \int_0^s \left[ \frac{\dot{f}(s)}{s} \right]^2 ds
\] (4.27)
then the even solution outside \( C_c \) is
\[
Y_{1,0}^{(0)} = \sin \chi(s); \quad Y_{2,0}^{(0)} = \sin \chi(s); \quad Y_3^{(0)} = T \xi \sin \chi(s) - 5 \xi \eta \chi(s)
\] (4.28)
when the odd solution is given by
\[
Y_{1,0}^{(0)} = \sin \chi(s); \quad Y_{2,0}^{(0)} = -\cos \chi(s); \quad Y_3^{(0)} = T \xi \sin \chi(s) + 5 \xi \eta \chi(s)
\] (4.29)
The boundary conditions (4.25) yield
\[
\chi = \chi_n = \begin{cases} 
\theta_c + n \pi & \text{for even modes} \\
\theta_c - \frac{n}{2} \pi + n \pi & \text{for odd modes}
\end{cases}
\] (4.30)
in which \( n = 0, \pm 1, \pm 2, \ldots \) Thus
\[
\sigma^{(0)} = h a \frac{1}{\chi_n} \frac{H_n(1)}{\gamma_n}
\] (4.31)
so that both even and odd modes possess two infinite subsets of modes one of which propagates westward and the other eastward. The lowest even mode propagates westward and the lowest odd mode propagates eastward.

The problem for \( Y^{(1)} \) provides a solvability condition which yields
\[
\sigma^{(1)} = 0
\] (4.32)
for both even and odd solutions, and
\[
R_n^{(0)} = \frac{\chi_n}{h^2} \pi n(1,h) \eta_n(1,h)
\] (4.33)
for the even solution; and
\[ R^{(0)}(n) = \frac{\xi_n}{\omega} \left\{ T^2 + \frac{3}{2} + \int_0^T \left[ \frac{T^2}{T^2 + 3/2} \right] \frac{d\xi}{\omega} \right\}, \quad (4.34) \]

for the odd solution. Here
\[ I_n(1, 0) = \frac{\xi_n}{\omega} \left\{ T^2 + \frac{3}{2} + \int_0^T \left[ \frac{T^2}{T^2 + 3/2} \right] \frac{d\xi}{\omega} \right\}, \]
\[ - \frac{\xi_n}{\omega} \left\{ T^2 + \frac{3}{2} + \int_0^T \left[ \frac{T^2}{T^2 + 3/2} \right] \frac{d\xi}{\omega} \right\}, \quad (4.35) \]

and \( J_n(1, 0) \) is given by (4.41) below by setting \( h = 0 \).

Numerical computations for the expressions (4.33) and (4.34) for
\( R(n) \) for the three fields \( A, M, S \) for various forms of the temperature gradient
showed that the location of the unstable cell depends on the forms of the field
and temperature gradient in a way similar to that described in § 3 above for
the small \( q \) case and we shall not discuss it in detail here. However it maybe
of interest to note that the solution within \( C_1 \) takes the form
\[ Y_{1}^{(0)} = \sin \left[ \chi(0) + \xi \right], \quad Y_{2}^{(0)} = - \cos \left[ \chi(0) + \xi \right], \]
\[ Y_{3}^{(0)} = T \cos \left[ \chi(0) + \xi \right] + \xi \sin \left[ \chi(0) + \xi \right], \quad (4.36) \]
in which
\[ \tan \xi = h_1/T, \quad \chi(0) = \frac{h_1}{\xi_n^{(0)}} \left\{ \int h_1^{'(0)} \right\}^2, \quad (4.37) \]
The boundary conditions (4.26) then demand that
\[ \tan \left[ \chi(0) \right] = - \frac{h_1^{(0)}}{h_1 + T^2}, \quad (4.38) \]
so that
\[ \chi(0) = \chi_n = - \theta + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots \quad (4.39) \]

and \( a_0 \) is the principal value of (4.38). The expression for \( R(n) \) is again given by
\[ R^{(0)}(n) = \frac{\xi_n}{\omega} \left\{ T^2 + \frac{3}{2} + \int_0^T \left[ \frac{T^2}{T^2 + 3/2} \right] \frac{d\xi}{\omega} \right\}, \quad (4.40) \]

where
\[ \int h_1^{(0)}, h_1 = \frac{\xi_n}{\omega} \left\{ T^2 + \frac{3}{2} + \int_0^T \left[ \frac{T^2}{T^2 + 3/2} \right] \frac{d\xi}{\omega} \right\}, \]
\[ - \frac{\xi_n}{\omega} \left\{ T^2 + \frac{3}{2} + \int_0^T \left[ \frac{T^2}{T^2 + 3/2} \right] \frac{d\xi}{\omega} \right\}, \quad (4.41) \]

The modes (4.39) form the analytic continuation of the odd modes
(4.30) present outside \( C_1 \). The lowest mode \( n = 0 \) here corresponds to Hides
slow magnetohydrodynamic wave in a thin spherical shell and, as predicted by
Hides, it propagates eastward. When the expression (4.40) for \( R(n) \) is evaluated
for various values of \( n \) it is found that the preferred mode of convection within
\( C_1 \) (i.e., the mode possessing the lowest \( R(n) \) there) is the mode \( n = 1 \) which is
eastward propagating. The mode \( n = 1 \) is also the preferred odd mode outside
\( C_1 \). The overall preferred mode, however, is the lowest even mode outside \( C_1 \)
whatever the form of magnetic field and temperature gradient.

V. CONCLUDING REMARKS

The analysis presented above has shown that two-dimensional linear
waves in a rapidly rotating, fully diffusive, stratified spherical shell in
the presence of a toroidal magnetic field are possible provided that
\[ Q \ll \xi_n^{(0)} \frac{1}{2} \quad (5.1) \]

Whatever the form of magnetic field in the presence of any form of temperature
gradient which is adverse in some part of the fluid shell. When \( Q \gg 1 \), (5.1)
applies even if \( Q \gg 1 \). Since this range of \( Q \) is also known to be associated
with motions of three-dimensional character (Eltayeb and Kumar 1977) it would be interesting to know whether three-dimensional or two-dimensional motions are preferred. However, in order to resolve this question a thorough investigation of all the possible wave motions in this range of \( Q \) must be made. Such an analysis has been carried out by Howard (1979) for the plate layer model in the presence of the Malkus field, and a comparison of his results with the above ones in the range \( 1 \leq Q \leq q^{1/2} \) indicates that \( R_c = 0 \) \( (Q^{-1}) \) in both cases, although the coefficient of proportionality for the three-dimensional motions of the layer is small suggesting their preference. However the influence of the spherical boundary may have a slightly stabilizing influence.

For the geophysically relevant case of small \( q \) the condition (5.1) restricts \( Q \) to values well below unity. It transpires that two-dimensional motions can still exist in the range \( 0(1/2) \leq Q \leq 0(1) \) in the case of magnetic fields depending on \( s \) only because the basic field and shear flow can be chosen independently. The expression for (dimensionless) intrinsic frequency \( \omega \) is

\[
\omega = \sigma + m R_B \frac{V}{U},
\]

(5.2)
in which \( R_B \) is the magnetic Reynolds number defined by

\[
R_B = \frac{\mathcal{L} \mathcal{V} \rho_o}{\eta},
\]

(5.3)
where \( \mathcal{L} \mathcal{V} \) is a typical flow amplitude. Thus the analysis of the magnetic modes can be extended to all values of \( Q \ll 1 \) provided

\[
R_B \ll \frac{Q}{Q},
\]

(5.4)
and scaling for \( \sigma, m, k \) is altered to

\[
\sigma = \sigma_0 Q^{1/2}, \quad m = M Q^{1/2}, \quad k = k Q^{1/2}.
\]

(5.5)

However, for spherical fields (i.e. fields depending on both \( s \) and \( \eta \) ) the legitimate basic state \( B_0 \) and \( U_0 \) are necessarily coupled by (2.12) and it is not possible to find any scaling that can neglect the Doppler shift \( Q \frac{V}{U} \) in the expression for \( \omega \). Moreover the search for two-dimensional motions for \( Q = O(1/2) \) using the method of appendix A showed that the influence of the shear is to stabilize these motions in the sense that \( R_c \) increases with \( (Q/q) \). It then appears that the range \( O(q^{-1}) \leq Q \leq 0(1) \) may be associated with three-dimensional motions when \( q \ll 1 \) and the field is spherical.

The preferred mode of convection identified above is always associated with a temperature perturbation which is even in the axial coordinate \( z \).

Diffusive process are instrumental in determining all of the wavenumber, wave frequency and temperature gradient responsible for the onset of instability. For example, take the magnetic modes when \( q \gg 1 \). To leading order, these waves are of the hydromagnetic-inertial type studied by Hide (1966). However, their instability is provided by buoyancy providing the necessary energy. In the absence of diffusion the maximum growth rate is attained for the radial wavenumber approaching infinity. The presence of diffusion (magnetic and thermal) prevents the wavelength from tending to zero with the consequence that the wavelength of the preferred mode is finite in which case the damping influence of diffusion is provided for by the buoyancy energy. The dependence of \( R_c, \sigma_c \) and \( \beta_c \) on diffusion is clearly shown by

\[
\frac{\sigma_c}{\sigma_c} \sim \left( \frac{k}{Q} \right)^{1/2}, \quad \frac{\beta_c}{\beta_c} \sim \left( \frac{k}{Q} \right)^{3/2}, \quad \beta_c \sim \frac{\eta^4}{Q},
\]

(5.6)
and even the phase speed of the preferred mode \( c_p \) is dependent on both magnetic and thermal diffusivities.

The most significant effect introduced by the presence of an inner solid core is that it suppresses a certain class of two-dimensional modes in the region enclosed by the coaxial cylindrical surface, \( C_c \), touching the inner core at its equator. The motions inside and outside \( C_c \) can be explained as follows. If \( \eta \) is the radius of the inner core then for every coaxial cylindrical cell whose distance from the axis \( s \) is greater than \( \eta \) has two types of modes.
One type has an axial vorticity even in $z$ so that $Y(-z)=Y(z)$ and the other type has $Y(-z)=-Y(z)$. For a cell with $s<n$ all modes have $Y(-z)=-Y(z)$ (for $z>s$ because of the 'cut' made by the solid core in the cell). The lowest mode outside $C_c$ belongs to the even vorticity type and the lowest mode within $C_c$ belongs to the odd vorticity type. This is the explanation for Hide's suggestion that cells in thin spherical shells are different in nature from those in thick shells.

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**APPENDIX A**

The application of the procedure described in the appendix to Eltayeb and Kumar (1977) to a cell $C_0$ yields the dispersion relation

$$\frac{\omega}{\omega_0} = K \sin^2 \theta_0 I_1 + E I_3 = R \sin^2 \theta_0 I_2$$

(A.1)

where

$$I_1 = \frac{1}{\omega_0} \int_0^{\omega_0} \frac{\omega^2}{\omega^2 + k^2} d\omega$$,

$$I_2 = \frac{1}{\omega_0} \int_0^{\omega_0} \frac{k^2}{\omega^2 + k^2} d\omega$$,

$$I_3 = \frac{1}{\omega_0} \int_0^{\omega_0} (\omega^2 + k^2) d\omega$$.

(A.2)

Note that a solution of the form

$$F(z) \exp i (\sigma t + z \theta_1 + \xi x)$$

(A.3)

has been assumed and the assumption $|z|<<|\xi|$ is made. Here $\xi$ is the zonal distance and $x$ the radial one. Also we have defined

$$\omega_0 = \sigma + \ln \frac{U}{\sin \theta_0}, \quad k = \xi + z^2$$.

(A.4)

Because $U(\xi, z^2)$ for the $A, S$-fields is a function of $z$ the integrand in $I_3$ depends on $z$. For the $N$-field $U$ is uniform and can be taken as zero. The dispersion relation (A.1) then reduces to

$$K \sin^2 \theta_0 = (\omega_0 - \xi^2) \left[ E(\omega_0^2 + \xi^2) + \frac{\sin \theta_0}{\omega_0} \left( \frac{\omega_0^2 + \xi^2}{\frac{\omega_0^2 + \xi^2}{\omega_0^2 + \xi^2}} \right) \right]$$

(A.5)

when $\omega_0 = 0$. This is, of course, the same equation (A.3) in Eltayeb and Kumar (1977). (Note here that the last term within the square brackets is preceded by a plus sign and not a minus sign as misprinted in Eltayeb and Kumar (1977)).

(A.5) was studied by Eltayeb and Kumar. It was shown that when $\gamma$ is small only one neutral wave is present. This wave corresponds to the lowest even mode of the inertial gravity waves of class $F$ studied in I but is here modified by the
presence of the magnetic field.

As \( T \) increases two more modes appear. These two modes (of Eltayeb and Kumar 1977, Equation (A.4)) are magnetic in nature in the sense that the magnetic field plays a central role in their appearance. The separation of (A.5) into its real and imaginary parts, assuming \( \varepsilon \) to be real, and the minimization of \( R \) over \( \theta_c \) and \( \xi \) gives

\[
\Delta \theta_c = \frac{1}{\sqrt{3}}, \quad \xi_c = \frac{2\pi(1+\xi)^2}{4(1-\xi)^2} Q^2, \quad (A.6)
\]

for both waves while \( R_c, \sigma_c \) depend on whether \( q \) is greater or smaller than 1.

For \( q < 1 \), we have the two waves

\[
R_{1c} = \frac{2\pi(1+\xi)^2}{4(1-\xi)^2} \xi^{-1}, \quad \sigma_{1c} = -\frac{2\pi(1+\xi)^2}{4(1-\xi)^2} \xi^{-1} Q^2,
\]

\[
R_{2c} = \frac{2\pi(1+\xi)^2}{4(1-\xi)^2} \xi^{-1} Q^2, \quad \sigma_{2c} = -\frac{2\pi(1+\xi)^2}{4(1-\xi)^2} \xi^{-1} Q^2. \quad (A.7)
\]

For \( q > 1 \), on the other hand, we have the two waves

\[
R_{1c} = \frac{2\pi(1+\xi)^2}{4(1-\xi)^2} \xi^{-1}, \quad \sigma_{1c} = +\frac{2\pi(1+\xi)^2}{4(1-\xi)^2} \xi^{-1} Q^2,
\]

\[
R_{2c} = \frac{2\pi(1+\xi)^2}{4(1-\xi)^2} \xi^{-1} Q^2, \quad \sigma_{2c} = -\frac{2\pi(1+\xi)^2}{4(1-\xi)^2} \xi^{-1} Q^2. \quad (A.8)
\]

Thus for every value of \( q \) two magnetic modes exist and all of them depend on \( q \). \( R \), \( \sigma \) decrease with the increase of the amplitude of the magnetic field (as measured by \( Q \)) for both modes. Whereas the wavenumber \( \xi_c \) for both modes is the same, the ratio of the frequencies and the Rayleigh numbers is \( q \). For \( q = 0(1) \) both frequencies and Rayleigh numbers are comparable but for \( q \) much larger or much smaller than 1, one wave has a much smaller frequency than the other. For \( q \ll 1 \), the slower wave is preferred and for \( q \gg 1 \) the faster wave is preferred.

These results were derived for the \( M \)-field. For the \( A, S \)-fields the dispersion relation (A.1) was integrated numerically using (A.7) and (A.8) as a guide to find that the dependence on \( Q \) and \( q \), hold good provided that \( Q < q^{1/2} \). These results are used to locate the critical mode of convection as well as to study the general propagation properties of these waves.
APPENDIX B

The expression for the various integrals defined throughout §§ 3 and 4 are summarized here to avoid the occurrence of long mathematical expressions in the text. The superscripts M, A, S, when they occur, will mean that the integral corresponds to the M, A, S fields respectively.

It is convenient to define the integrals

$$K_s(x, x') = \int_0^1 \frac{C_s(x, x')}{(x^2 + x'^2)^{3/2}} dx$$

which cannot be evaluated analytically ($\gamma \neq 0$) and are computed numerically.

\begin{align*}
\Lambda_n(l, 0) = \frac{a}{2} & \left\{ T + \frac{1}{2} - \frac{1}{2(\Psi_0^3 + \Psi_0)\Psi_0} \frac{T}{2} \left[ \frac{\partial}{\partial \Psi_0^3} \right] \right\} \\
= & -\frac{a}{h} \left\{ \frac{\partial \alpha}{\partial \Psi_0^3} \left[ \frac{\partial}{\partial \Psi_0^3} \left( \frac{1}{2} \frac{\partial }{\partial \Psi_0^3} \right) \right] + \frac{1}{2h} \frac{\partial}{\partial \Psi_0^3} \left( \frac{\partial}{\partial \Psi_0^3} \right) \right\} \sin 2\theta.
\end{align*}

($B.1$)

\begin{align*}
\Lambda_n(l, h) = \frac{a}{2} & \left\{ T^2 \left( \frac{1}{2} - \frac{1}{2(\Psi_0^3 + \Psi_0)\Psi_0} \frac{T}{2} \right) \right\} \cos 2\theta

+ \left\{ \frac{2}{h} \frac{\partial}{\partial \Psi_0^3} \sin 2\theta \right\} \cos 2\theta.
\end{align*}

($B.2$)

\begin{align*}
\lambda_n(l, h) = \frac{a}{2} & \left\{ T^2 \left( \frac{1}{2} - \frac{1}{2(\Psi_0^3 + \Psi_0)\Psi_0} \frac{T}{2} \right) \right\} \cos 2\theta

+ \left\{ \frac{2}{h} \frac{\partial}{\partial \Psi_0^3} \sin 2\theta \right\} \cos 2\theta.
\end{align*}

($B.3$)

where

$$\beta_n = \Theta_0 - \Psi_0^3(l, h) + \Psi_0^3.$$
\[ C_n^A(h, h) = 2 \sum_{j=2}^{n+1} \left[ C_n^2 \left( \frac{T^2}{h^2} + \frac{1}{h^2} \right) + \frac{2}{h^2} - \frac{h}{C_n^2} \left( \frac{T^2}{h^2} + \frac{1}{h^2} \right) \right] \] (B.11)

\[ C_n^B(h, h) = \frac{1}{2} \int_{-1}^{1} \left[ \frac{\partial^2 C_n}{\partial h^2} + \frac{2}{h^2} \right] \left[ C_n^2 \left( \frac{T^2}{h^2} + \frac{1}{h^2} \right) \right] + \frac{\partial^2 C_n}{\partial h^2} \left( \frac{\partial^2 C_n}{\partial h^2} + \frac{2}{h^2} \right) \] (B.17)

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Soward, A.M., 1979b, Phys. Earth Planet. Inter. 20, 134.
Table 1
The dependence of the critical mode \((s_c, M_c, c_c, R_c)\) on the Magnetic Prandtl number \(P_m\) in the limit \(q = \infty\) when \(\lambda = 5, a_1 = 1, a_2 = 0\) and the field is the S one (see § 4).

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<th>(M_c)</th>
<th>(c_c)</th>
<th>(R_c)</th>
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<td>-2.3019</td>
<td>29.9573</td>
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<td>0.9883</td>
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Table 2
A sample of the results for the dependence of the critical mode on \(\lambda\) for \(P_m = 1\) for the S-field when \(a_1 = 1, a_2 = 0\) (§ 4)

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>(s_c)</th>
<th>(M_c)</th>
<th>(c_c)</th>
<th>(R_c)</th>
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<tr>
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**FIGURE CAPTIONS**

Figure 1.: The geometry of the problem. The notation of this figure will be frequently used.

Figure 2.: Illustration of the influence of the magnetic field on \(R_m\) for the inertial modes studied in I for the case \(a_1 = 1, a_2 = 0\), here drawn for \(\gamma = 5.5\) and \(q = 10^{-6}\). Compare with figure 2 in I. For the Malkus field, (a) and (b), the illustration is for a full sphere while in the case of the symmetric field, (c) and (d), \(n\) is taken as 0.1. If \(n\) takes larger values, the figures can be obtained by 'cutting off' the part for \(s < n\). The continuous (discontinuous) curves refer to eastward (westward) propagating waves. Note that the lowest odd mode \((n = 0\) in 2(d)) has \(R_m\) too high to be included in the figure. The results for the A field are similar to those for the S-field.

Figure 3.: The dependence of the critical mode of § 3.1.1 on \(q\) for a typical value of \(\gamma(=5.5)\). The continuous, discontinuous and dotted curves refer to the S, A and M fields, respectively, and \(n\) is assumed less than 0.23. If \(n \geq 0.23\) then the critical mode for the M-field will sit on \(C_c\) for all values of \(s < n\) and consequently \(R_c\) in that range will be higher than given in (b). Similar results will hold for the S and A fields if \(n \geq 0.8\).

Figure 4.: The dependence of \(R_m\) on \(s\) for different values of \(n\) for a differentially heated shell \((a_1 = 0, a_2 = -n)\) when \(\gamma = 5.5\) and \(q = 10^{-6}\) for (a) the Malkus field and (b) the S-field (see § 3.1.2). The behaviour of \(R_m\) for the A-field is similar to that for the S-field.

Figure 5.: A comparison of the critical modes for the three cases \((a_1, a_2) = (1,0), (1, -0.2), (0, -n)\), referred to by the Roman numbers I, II, III, respectively, as a function of \(\gamma\). The continuous (discontinuous) curves refer to the S-(A-) field. For III, \(R_c\) is normalized by \(n\). It is assumed that \(n \leq 0.3\) and \(q = 10^{-6}\).
Figure 6.: Illustration of $R_m$, $\sigma_m$ for the first few even (continuous) and odd (discontinuous) modes for the magnetic modes of § 3.2 outside $C_c$ when $a = 1$, $a_2 = 0$. (a), (c) correspond to the H-field while (b), (d) refer to the S-field.

Figure 7.: The critical mode of convection, for the (a) M-field and (b) S-field; as a function of $n$ when $a = 1$, $a_2 = 0$ and $q \ll 1$ (§ 3.2). The curves I, II, III refer to $R_c$, $-100 \sigma_c$, $100 \sigma_c$ respectively. The figure in the box is a blown up version of the curve I which appears almost horizontal in the full figure; the scale for $n$ is the same in both cases.

Figure 8.: The dependence of $R$ and $-\sigma$ for the first (a) even and (b) odd modes for a differentially heated shell in the case of inertial-buoyancy waves (§ 3.2). The continuous (discontinuous) curves refer to $R_m$ ($-\sigma_m$) and the labels A, M refer to A- and M-fields. In (a) $-\sigma_m$ is magnified by 10 for both fields while in (b) it is magnified by 100 for the A-field and shown as it is for the M-field, $\eta = 0.3$.

Figure 9.: The behaviour of $\gamma_3^{(0)}$ with $\lambda$ for the (a) first four even and (b) first four odd modes at $s_0 = 0.3$ for the Kennedy-Higgins model.

Figure 10.: $R^*, -\sigma^*$ for the lowest (a) even and (b) odd modes for the Kennedy-Higgins model. In (a) both $R^*$ and $-\sigma^*$ (multiplied by 10 for the M-field and 100 for the A-field) are shown. In (b) only the results for the A-field are shown. In both (a) and (b) the continuous curves refer to $R_m$ and discontinuous ones to $-\sigma_m$. The labels A and M refer to A, M fields.

Figure 11.: The dependence of the critical mode ($s_c$, $M_c$, $\sigma_c$, $R_c$) on $\lambda$ for $p_m = 1$ and $q \gg 1$ (§ 4) for the S-field when $a_1 = 1$, $a_2 = 0$. The continuous 'horizontal' curve is for $s \propto 10^2$, the continuous steep one is for $R_c$ and its scale starts from 25 and increases by 1; the discontinuous dotted curve for $M_c$ starts from 0.9 and increases by 0.02 while the discontinuous curve for $-\sigma_c$ starts from 1.8 and increases by 0.1.
Fig. 4 (a)

Fig. 4 (b)
Fig. 6 (a)

Fig. 6 (b)
Fig. 6 (c)

Fig. 6 (d)
Fig. 10(b)

Fig. 11
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