IC/83/75

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INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1983 MIRAMARE-TRIESTE

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International Atomic Energy Agency

and

United Nations Educational Scientific and Cultural Organization

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MIRAMARE – TRIESTE
4 July 1983

* To be submitted for publication.

ABSTRACT

The question of fermion chirality in Kaluza-Klein theories with coupling to Yang-Mills fields is discussed. The argument is illustrated in eight dimensions where an SU(2) Yang-Mills field assumes the 1-instanton form on the internal space. This serves not only to trigger spontaneous compactification of the internal space but will ensure the emergence of $n_L - n_R = \frac{2}{3} t(t+1)(2t+1)$ zero modes in an irreducible 8-spinor belonging to the (2t+1)-dimensional representation of SU(2).

1. One of the outstanding problems of spontaneously compactified higher dimensional field theories is the determination of the spectrum of massless fermions in the effective 4-dimensional theory [1),2)]. It is possible to ensure the existence of such fermionic zero-modes by coupling to gauge fields with non-trivial topology. This has been pointed out before [3),3a)] and, in a previous paper, we have applied the idea to the case of 6-dimensional Einstein-Maxwell theory coupled to fermions [4)]. We computed the spectra and verified the presence of fermions in complex representations of the effective local symmetry, SU(2) × U(1). The source of this chirality was the non-vanishing ground state value of the Maxwell field on the "internal" space, a 2-sphere, where it assumed a magnetic monopole configuration [*)]. The non-vanishing gauge field triggers spontaneous compactification by providing the stress components which are needed to generate the curvature of the 2-sphere [5)]. The non-vanishing Chern character ensures its stability [3)], as well as the fermion chirality.

The aim of this note is to provide another example of the mechanism: 8-dimensional gravity coupled to SU(2) Yang-Mills fields [**)]. Spontaneous compactification results in an internal 4-sphere on which the SU(2) gauge field takes the 1-instanton configuration. This configuration is [illegible]-invariant and the effective 4-dimensional theory which emerges has [illegible].

On the issue of introducing elementary gauge fie[illegible] dimensional action (which can lead, not only to stable compactification but also play a rôle in ensuring the chirality of the fermionic spectrum in the effective 4-dimensional theory) we adopt the point of view that these could perhaps naturally arise in a supergravity theory, starting from a still higher dimension.

2. Consider compactification induced by matter fields. Quite generally, start from the (4+d)-dimensional Einstein field equations coupled to matter: (metric: -1,1,1,1,...,1)[***)]

$$R_{AB} = -\frac{\kappa^2}{2}\left(T_{AB} - \frac{1}{2+d}\eta_{AB}T + \frac{2}{2+d}\lambda\,\eta_{AB}\right) . \qquad (1)$$

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*) The effect is possibly only if the compactified internal space has an even number of dimensions.

**) This theory is free of anomalies after fermions are introduced. We are grateful to E. Witten for a discussion on anomalies in Kaluza-Klein theories.

***) Our notation is the same as in Ref.4. Unless otherwise stated we use an orthonormal basis in the manifold.

Here T_{AB} is the energy momentum tensor of the non-gravitational fields $T = \eta^{AB} T_{AB}$ and λ is a constant. We are interested in those solutions of Eq.(1) which split the (4+d)-dimensional space into a product space $M_4 \times B_d$, where M_4 is Minkowski space-time and B_d is a compact, Riemannian manifold. If we substitute $R_{ab} = 0$ in Eq.(1) we get

$$T_{ab} - \frac{1}{2+d} \eta_{ab} T + \frac{2}{2+d} \lambda \eta_{ab} = 0 \quad . \tag{2}$$

From 4-dimensional Lorentz invariance we conclude that

$$T_{ab} = \frac{c}{\kappa^4} \eta_{ab} \quad , \tag{3}$$

where c is a constant. Now the assumption that the internal d-dimensional space is an Einstein space implies that

$$T_{\alpha\beta} = \frac{c'}{\kappa^4} \delta_{\alpha\beta} \qquad c' = \text{constant} \; . \tag{4}$$

Upon substitution of (3) and (4) into (2) and into (α,β) components of (1), we get

$$\lambda\kappa^4 = c + (c' - c) \frac{d}{2} \quad , \tag{5}$$

$$R_{\alpha\beta} = \frac{-1}{2\kappa^2} (c' - c) \delta_{\alpha\beta} \quad ; \tag{6}$$

For a symmetric Riemannian space, compactness is ensured if $c'-c > 0$. These constants will depend on the details of the individual models *). For the case of gauge fields alone, due to Lorentz invariance, Eqs.(3)-(6) reduce to

$$\lambda = \frac{1}{4} \bar{F}^2, \; R_{\alpha\beta} = \frac{-\kappa^2 \bar{F}^2}{2d} \delta_{\alpha\beta}, \quad c = -\frac{\kappa^4}{4} \vec{F}^2, \quad c' = \kappa^4 \bar{F}^2 \frac{4-d}{4d} \quad , \tag{7}$$

where $\vec{F}^2 = \vec{F}_{\alpha\beta} \cdot \vec{F}^{\alpha\beta}$. The negativity of the Ricci tensor, $R_{\alpha\beta}$, ensures compactification. Thus the problem reduces to finding appropriate solutions of the Yang-Mills equations in the internal space. These solutions must have a constant $\vec{F}^2$. In the case of a homogeneous internal space, G/H, there exist standard solutions of the gauge field equations, where all components of the Yang-Mills fields are zero, except those which belong to H or any of its subgroups. (Recall that H must be contained in the Yang-Mills gauge group.) Such solutions have been discussed in Ref.7; they have the characteristic property that $T_{\alpha\beta}$ is proportional to (d-4) and vanishes when the internal space is 4-dimensional (as for example in the case of the internal BPST instanton).

*) The source of compactification may be a quantum loop in which case one would have to make a self-consistent computation [6] of these.

3. Now consider the case of 8-dimensional gravity coupled to SU(2) Yang-Mills. One can demonstrate the existence of a vacuum solution with the geometry $M^4 \times S^4$, where the 4-dimensional internal space, S^4, supports an SU(2) instanton. We would like to establish the local symmetry of this background by identifying the massless vector states.

It is well known that the single instanton solution on S^4 is invariant under the combined action of O(5) and SU(2). [8] More precisely, under the isometric action of O(5) on S^4 the instanton configuration changes by an SU(2) gauge transformation. Therefore it is natural to classify the spectrum of small oscillations into multiplets of O(5). To start with the spectral analysis we substitute

$$\begin{aligned} g_{MN} &= \bar{g}_{MN} + h_{MN} \\ \vec{A}_M &= \vec{\bar{A}}_M + V_M \end{aligned} \tag{8}$$

into the action integral and expand it in a power series of h_{MN} and $\vec{V}_M$. Here $\bar{g}_{MN}$ and $\vec{\bar{A}}_M$ indicate the background classical solution and h_{MN} and $\vec{V}_M$ are small perturbations which depend on all the eight co-ordinates in a completely arbitrary way. As a result of the general co-ordinate and the SU(2)-gauge invariance of the 8-dimensional theory, the linearized theory will be invariant under the following set of local gauge transformations:

$$\begin{aligned} \delta h_{MN} &= \xi_{M \cdot N} + \xi_{N \cdot M} \\ \delta \vec{V}_M &= \vec{\Omega}_{\cdot M} + \vec{F}_{MN} \xi^N \end{aligned} \quad , \tag{9}$$

where ξ_M and $\vec{\Omega}$ are arbitrary parameters and "$\cdot N$" refers to the covariant differentation relative to the background connection. To make the theory well defined we impose the following gauge conditions:

$$(h_{MN} - \frac{1}{2} \bar{g}_{MN} h_{PP})_{\cdot M} = 0, \quad \vec{V}_{N \cdot M} = 0 \quad . \tag{10}$$

Subject to these constraints, the spin-1 modes in the bilinear action are given by

$$S_2(\text{spin-1}) = \frac{V(S^4)}{\kappa^4}\int d^4x \left[\sum_{n\geq 1}\sum_q \left\{\frac{1}{2} h^{n1}_{a,q}(x)\left[\partial^2 - \frac{n}{a^2}(n+3) - \frac{2}{a^2}\right] h^{n1}_{a,q}(x)\right.\right.$$

$$+ \frac{1}{2} h^{n0}_{a,q}(x)\left[\partial^2 - \frac{n}{a^2}(n+3)\right] h^{n0}_{a,q}(x)$$

$$+ \frac{1}{2} V^{n1}_{a,q}(x)\left[\partial^2 - \frac{n}{a^2}(n+3) + \frac{2}{a^2}\right] V^{n1}_{a,q}(x)$$

$$\left.\left. - \frac{\sqrt{2}}{a^2}\sqrt{(n+1)(n+2)}\; h^{n1}_a V^{n1}_a\right\}\right] \quad . \qquad (11)$$

Here (n_1,n_2) specify the O(5)-representation of dimension $d(n_1,n_2)$ given by

$$d(n_1,n_2) = \frac{1}{6}(n_1+n_2+2)(n_1-n_2+1)(2n_1+3)(2n_2+1) \qquad (12)$$

and q can be assumed to range from 1 to $d(n_1,n_2)$.

It is easily seen that S_2 (spin-1) contains two classes of representations characterized by $(n,0)$ and $(n,1)$, with $n \geq 1$. In the first class all modes are massive with their $(\text{mass})^2$ given by

$$\frac{n}{a^2}(n+3)\,, \quad (n \geq 1) \qquad (13)$$

while in the second class the $(\text{mass})^2$ matrix is as follows:

$$M_n^2 = \begin{pmatrix} \frac{n}{a^2}(n+3) + \frac{2}{a^2} & \frac{\sqrt{2}}{a^2}\sqrt{(n+1)(n+2)} \\ \frac{\sqrt{2}}{a^2}\sqrt{(n+1)(n+2)} & \frac{n}{a^2}(n+3) - \frac{2}{a^2} \end{pmatrix} . \qquad (14)$$

It follows that

$$\text{Det}\, M_n^2 = \frac{1}{a^4}(n-1)(n^3 + 7n^2 + 14n + 6) \quad . \qquad (15)$$

Since $n \geq 1$, the <u>only</u> zero of $\text{Det}\, M_n^2$ is located at $n = 1$. The formula for $d(n_1,n_2)$ indicates that the multiplicity of this mode is ten. The massless spin-1 fields are thereby seen to belong to the adjoint representation of O(5). We conclude that the effective 4-dimensional theory is an O(5) Yang-Mills theory. Notice that the SU(2) Yang-Mills symmetry of the 8-dimensional Lagrangian is spontaneously broken *). From (11) it is clear that the spin-1 sector is stable against small oscillations.

*) This point was overlooked in Ref.7 where the statement with regard to the number of massless spin-1 fields is incorrect.

4. We now consider the question of chirality. Classical backgrounds can be topologically non-trivial and may therefore induce chiral fermion spectra. More precisely, it can be asserted that, if the internal space is even-dimensional then there will emerge a set of fermionic zero-modes whose number is governed by the Atiyah-Singer index theorm [9]. To see this consider the Dirac Lagrangian in (4+d)-dimensional space time,

$$\mathcal{L}_D = i\bar{\psi}\not{\nabla}\psi \qquad (16)$$

in which $\not{\nabla} = \Gamma^A \nabla_A = \Gamma^A(e_A + \omega_A + A_A)$ where ω_A and A_A denote the Riemann and Yang-Mills connections, respectively.

We shall assume that d is <u>even</u> and consider the following realization of the Dirac matrices, Γ_A,

$$\Gamma_a = \gamma_a \times 1 \qquad a = 0,1,2,3$$

$$\Gamma_\alpha = \gamma_5 \times \gamma'_\alpha \qquad \alpha = 5,6,\ldots,4+d \;, \qquad (17)$$

where γ_a and γ'_α are the 4- and d-dimensional Dirac matrices, respectively. This realization is appropriate for treating the decomposition of the SO(1,3+d) spinor under the subgroup SO(1,3) × SO(d), the product of the tangent space groups on the Minkowski and internal spaces. The two eigenvalues of the SO(1,3+d) invariant matrix,

$$\bar{\Gamma} = \Gamma_0\Gamma_1 \cdots \Gamma_{4+d} \qquad (18)$$

distinguish two inequivalent irreducible representations of the tangent space group, SO(1,3+d). In the representation (17) we have

$$\bar{\Gamma} = \gamma_5 \times \bar{\gamma}' \quad , \qquad (19)$$

where $\bar{\gamma}' = \gamma'_5\gamma'_6 \cdots \gamma'_{4+d}$ is SO(d) invariant, and γ_5 is the familiar SO(1,3) invariant which distinguishes chirality. It is clear that an irreducible spinor of SO(1,3+d), say $\bar{\Gamma} = 1$, must branch into two irreducible pieces under the subgroup SO(1,3) × SO(d),

$$\psi_L \text{ with } \gamma_5 = 1 \text{ and } \bar{\gamma}' = 1$$

$$\psi_R \text{ with } \gamma_5 = -1 \text{ and } \bar{\gamma}' = -1 \quad . \qquad (20)$$

The main point to emphasize is that the left and right components belong to <u>inequivalent</u> representations of SO(d). (It can be shown [10] that the representations corresponding to $\bar{\gamma}' = 1$ and $\bar{\gamma}' = -1$ are conjugate complex if $d = 4k + 2$, and real or pseudoreal but inequivalent if $d = 4k$.) The theory possesses a global U(1) invariance.

The mass operator takes the form

$$\bar{\psi}_L \, i \, \Gamma^\alpha \, V_\alpha \, \psi_R + \text{h.c.} \tag{21}$$

where V_α operates in the internal space. This operator will have zero eigenvalues as the index theorem dictates. For example, in the model discussed above where the internal space is the 4-dimensional sphere, S^4, and the background gauge field takes an instanton configuration, let ψ belong to the $(2t+1)$-dimensional representation of the Yang-Mills SU(2). Then

$$N_L - N_R = \frac{2}{3}\, t(t+1)\,(2t+1)\, k \quad , \tag{22}$$

where N_L and N_R are the numbers of left and right-handed zero modes and k is the instanton number.

The modes can be found explicitly if the internal space is a quotient space G/H and the gauge field configuration is G-invariant. The modes belong to irreducible representations of G and the zero modes correspond to those representations of G which appear in the harmonic expansion of either ψ_L or ψ_R (but not both).

In the above example, if $k = 1$ then the background has SO(5) invariance and the zero modes must belong to a representation of this group. To find the zero-modes it is necessary only to examine the harmonic expansions of spinors on S^4. A general technique for constructing these expansions has been described elsewhere [11]. It involves decomposing the spinor which belongs to an irreducible representation of the group $SO(1,7)_{\text{tangent space}} \times SU(2)_{\text{gauge}}$, into pieces according to the chain

$$\begin{aligned} SO(1,7)_{t.s.} \times SU(2)_{g.} &\to (SL(2,C) \times SU(2)_A \times SU(2)_B)_{t.s.} \times SU(2)_{g.} \\ &\to SL(2,C) \times SU(2)_A \times SU(2)_{B+g} \end{aligned}$$

That is, from the tangent space group (or rather its covering group) is picked out the subgroup associated with the product manifold $M^4 \times S^4$. The factor associated with S^4 has been denoted as $SU(2)_A \times SU(2)_B$. The instanton is supposed to lie in the algebra of $SU(2)_B$. It is invariant with respect to the diagonal subgroup of $SU(2)_B \times SU(2)_g$ in which tangent space rotations

are accompanied by gauge transformations. Finally, the pieces which are irreducible with respect to $SU(2)_A \times SU(2)_{B+g}$ can be expanded in SO(5) harmonics.

Now consider the 8-spinor of SO(1,7) which also belongs to $2t+1$ of $SU(2)_g$. With respect to $SL(2,C) \times SU(2)_A \times SU(2)_B$ it branches into two pieces, ψ_L and ψ_R. The $SU(2)_A \times SU(2)_B \times SU(2)_g$ content is

$$\psi_L \sim (2,1,\ 2t+1) \quad \text{and} \quad \psi_R \sim (1,2,\ 2t+1) \quad . \tag{23}$$

With respect to $SU(2)_A \quad SU(2)_{B+g}$, we have

$$\begin{aligned} \psi_L &\sim (2,\ 2t+1) \\ \psi_R &\sim (1,\ 2t) + (1,\ 2t+2) \quad . \end{aligned} \tag{24}$$

To facilitate the expansion we re-express these SO(4) representations in the notation $[m_1, m_2]$ of Gel'fand and Zetlin [12],

$$\begin{aligned} \psi_L &\sim [t + 1/2,\ t - 1/2] \\ \psi_R &\sim [t - 1/2,\ t - 1/2] + [t + 1/2,\ t + 1/2] \quad . \end{aligned} \tag{25}$$

The SO(5)-harmonics are labelled by a pair of integers (or half integers) (n_1, n_2) and the expansion of an SO(4) piece $[m_1, m_2]$ must include all (n_1, n_2) subject to the inequalities

$$n_1 \geqslant m_1 \geqslant n_2 \geqslant |m_2| \quad . \tag{26}$$

It is easy to verify that the only harmonic which fails to appear in both ψ_L and ψ_R is

$$n_1 = n_2 = t - 1/2 \tag{27}$$

which is present only in ψ_R. This must constitute the zero-mode multiplet. Its dimensionality is given by Eq.(20)

$$d\,(t - 1/2,\ t - 1/2) = \frac{2}{3}\, t(t+1)\,(2t+1) \tag{28}$$

which agrees with the prediction (22) of the index theorem.

ACKNOWLEDGMENT

We are appreciative of discussions with E. Witten.

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