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ON KALUZA-KLEIN THEORY

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ON KALUZA-KLEIN THEORY *

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ABSTRACT

Assuming the compactification of 4+K-dimensional spacetime implied in Kaluza-Klein type theories, we consider the case in which the internal manifold is a quotient space, G/H. We develop normal mode expansions on the internal manifold and show that the conventional gravitational plus Yang-Mills theory (realizing local G symmetry) is obtained in the leading approximation. The higher terms in the expansions give rise to field theories of massive particles. In particular, for the original Kaluza-Klein 4+1-dimensional theory, the higher excitations describe massive, charged, purely spin-2 particles. These belong to infinite dimensional representations of an O(1,2).

I. INTRODUCTION

The motivation for studying generally covariant field theories in spacetime of more than four dimensions is to obtain a geometrical interpretation of internal quantum numbers such as electric charge, i.e. to place them in the same context as energy and momentum [1,2]. The latter observables are associated with translational symmetry in 4-dimensional Minkowski space; the internal observables would be associated with symmetry motions in the extra dimensions.

In theories of the Kaluza-Klein type one starts with the hypothesis that spacetime has 4+K dimensions. One assumes general covariance and adopts the 4+K-dimensional curvature scalar as the Lagrangian. Next, one supposes that because of some dynamical mechanism, the ground state of this system is partially compactified [3], $M^4 \times B^K$ rather than M^{4+K}, where M^4 denotes 4-dimensional Minkowski space and B^K is a compact K-dimensional space [4]. The size of B^K must be sufficiently small to render it unresolvable at the currently available energies. For example , with the size of $B^K < 10^{-17}$ cm, it would be invisible to probes of energy $< 10^3$ GeV, this being an upper limit.

Models have been constructed which illustrate this phenomenon of spontaneous compactification [5,6]. In general, it is necessary for the system to include non-geometrical (matter) fields, coupled to the metric field, for this effect to be triggered. The presence of such ad hoc matter fields detracts from the simplicity of the purely geometrical theory and perhaps indicates that the geometrical component is not truly fundamental. An interesting exception, however, could be the extended supergravities where certain matter-like fields are justified by an underlying supersymmetry [7] and possibly the related geometry of superspace.

Assuming that the compactification is achieved, what are the implications of taking the extra dimensions seriously? By expanding the fields in a Kaluza-Klein theory in terms of normal modes on the compact space B^K, the extra dimensions make their appearance as massive multiplets of the associated symmetry group. In this paper we wish to examine these multiplets and the higher massive excitations.

To illustrate the emergence of masses, consider the original theory of Kaluza and Klein which involves one new co-ordinate, y. The 5-dimensional line element was assumed to take the form

$$ds^2 = (dx^m e_m{}^a(x))^2 - (dy - dx^m \kappa A_m(x))^2 ,$$

where $m,n,a,b,\ldots$ take the values 0,1,2,3 and $e_m{}^a(x)$ represents the vierbein field on 4-dimensional spacetime. The 4-vector $A_m(x)$ is to be interpreted as the electromagnetic potential. Gauge transformations here take the form of co-ordinate transformations

$$x^m \to x^m$$

$$y \to y + \Lambda(x)$$

with

$$e_m{}^a \to e_m{}^a$$

$$A_m \to A_m + \frac{1}{\kappa}\partial_m \Lambda$$

which leave the 1-forms, $dx^m e_m{}^a$ and $dy - dx^m \kappa A_m$, invariant. The invariant coupling to a complex scalar field, $\phi(x,y)$, would be described by the Lagrangian

$$\det e\left[|e_a{}^m(\partial_m\phi + \kappa A_m\, \partial_y\phi)|^2 - |\partial_y\phi|^2 - \mu^2|\phi|^2\right] .$$

If one supposes that the y-dependence of ϕ is such that

$$\partial_y\phi = -\frac{i}{R}\,\phi$$

then this Lagrangian reduces to that for a charged scalar,

$$\det e\left[|e_a{}^m(\partial_m\phi - i\, e\, A_m\phi)|^2 - \left(\mu^2 + \frac{1}{R^2}\right)|\phi|^2\right] ,$$

where the charge is given by

$$e = \frac{\kappa}{R} .$$

The electric charge is hereby understood in terms of the radius, R, of the extra dimension, interpreted as a circle. Conversely, given e, this equation fixes R (the size of the manifold) as κ/e where κ is Planck length. This also implies that the masses are in Planck units $\sim \kappa^{-1} \sim 10^{19}$ GeV. Because of this one is naturally interested to discover any exceptional cases where, for reasons of symmetry, the mass contribution is suppressed.

An example is the vector A_m itself which describes a massless particle, the photon. The gauge symmetry here associated with the transformation $y \to y + \Lambda(x)$ prevents it from acquiring any mass.

To consider non-Abelian symmetries one has to interpret the new co-ordinates y^κ, $\mu = 1,2,\ldots,\kappa$ as a parametrization of the manifold of a compact non-Abelian group, G. The line element is now assumed to take the form [4]

$$ds^2 = (dx^m e_m{}^a(x))^2 - \left[(dy^\mu - dx^m \kappa A_m{}^\alpha(x)\, K_\alpha{}^\mu(y))\, e_\mu{}^\beta(y)\right]^2 ,$$

where the auxiliary functions $e_\mu{}^\beta(y)$ constitute an orthonormal K-bein on the group manifold G, and the functions $K_\alpha{}^\mu(y)$ are Killing vectors on G. This formalism will be reviewed in some detail in the following sections.

A further generalization is possible where the new co-ordinates are taken to parametrize some homogeneous space, G/H, on which G can act [8,9]. A novel aspect of this kind of scheme concerns the problem of embedding. In order to give a G-invariant meaning to the ground state it is necessary to associate the motions of G/H with frame rotations (or tangent space transformations). This requires that the stability group, H, be embedded in the tangent space group, SO(K), in such a way that the K-vector of SO(K) has the H-content of G/H.

In all these theories the metric field carries an infinite number of new degrees of freedom corresponding to the propagation of excitations in the new dimensions. As a rule these excitations are ignored because of their large masses. However the massive excitations are not always ignorable: their contributions to a typical scattering amplitude are comparable to those due to graviton exchange. To see this, consider the scattering of charged scalar particles with the interactions represented schematically by

$$\mathcal{L}_{int} \sim \kappa\, h_{mn}\, \partial_m\phi^*\, \partial_n\phi + i\, e\, A_m \phi^* \overleftrightarrow{\partial}_m \phi + \cdots ,$$

where h_{mn} denotes the graviton field and A_m one of the massive excitations. The amplitude for graviton exchange is of order κ^2E^2, while that for A-exchange is of order e^2E^2/M^2, where E represents the energy scale of the scattering process. Since $M \sim e/\kappa$ these contributions are of the same order [1]. Only in the region of low momentum transfer, $t \ll E^2$, does the graviton amplitude, $\sim \kappa^2E^4/t$, become dominant.

The purpose of this study is to deal with some of the general features of Kaluza-Klein theories. These include the geometry of homogeneous spaces, the transformations of fields defined over them, the structure of normal mode expansions [2] on G/H, the embedding of Yang-Mills gauge transformations in the group of general co-ordinate transformations, and the extraction of an effective 4-dimensional Lagrangian. Such an effective action will be particularly useful for low energy effects ascribable to the existence of the microscopic and as yet unresolvable structure on space-time. Since much of the recent work in this field has been couched in the terminology of fibre bundles [10] it is not accessible to those without expertise in such matters. We shall therefore adopt a less sophisticated approach and express results in the more generally familiar component notation.

The plan of the paper is as follows. Some definitions, notational conventions and a brief description of the 4+K-dimensional Einstein theory are given in Sec.II. There follows in Sec.III a discussion of symmetry axpects of the compactification of the extra K dimensions. Harmonic expansions on G/H are introduced to describe the excitations. An important ingredient here is the embedding of the stability group, H, in the tangent space group, SO(K). Integration over G/H is used to reduce the Lagrangian to effective 4-dimensional form. Some of this is illustrated in Sec.IV where the low energy sector of the 4+K-dimensional theory is shown to contain the 4-dimensional Einstein and Yang-Mills terms. Much of the pedagogical detail is confined to the Appendices, except for Appx.5 which deals with the spectrum of 4+1 Kaluza-Klein theory.

II. EXTENDED GRAVITY

Let the 4+K-dimensional spacetime be parametrized, at least in some local region, by a set of co-ordinates, z^M. On this space let there be defined at each point a local frame of reference in the form of 4+k linearly independent covariant vectors, $E_M{}^A$.

The basic symmetries of the theory are twofold. Firstly, there must be general covariance under the co-ordinate transformations, $z^M \to z'^M$. Being covariant vectors, the $E_M{}^A$ transform according to

$$E_M{}^A(z) \to E'_M{}^A(z') = \frac{\partial z^N}{\partial z'^N} \; E_N{}^A(z) \quad . \tag{2.1}$$

Secondly, there is to be covariance under a group of z-dependent linear transformations among the vectors of the reference frame,

$$E_M{}^A(z) \to E'_M{}^A(z) = E_M{}^B(z) \; a_B{}^A(z) \quad . \tag{2.2}$$

In order to establish a pseudo-Riemannian geometry on the 4+K-space it is usual to assume that these transformations belong to the pseudo-orthogonal group, SO(1,3+K). We shall adhere to this choice.

With frame rotations belonging to SO(1,3+K) one is led to consider the frame-independent combinations,

$$g_{MN} = E_M{}^A \, E_N{}^B \, \eta_{AB} \quad , \tag{2.3}$$

which define the covariant components of the metric tensor. By an appropriate choice of basis, the SO(1,3+K) invariant tensor, η_{AB}, can be represented by the diagonal matrix,

$$\eta_{AB} = \mathrm{diag}(1, -1, -1, \ldots, -1) \quad . \tag{2.4}$$

Indices are raised and lowered in the usual way. The contravariant quantities $E_A{}^M$ and g^{MN} are defined as matrix inverses of $E_M{}^A$ and g_{MN}, respectively. (We employ the alphabetic convention that letters $M,N,\ldots$ from the middle alphabet will be world indices while those from the early alphabet, $A,B,\ldots$ shall be frame labels.)

The coupling of extended gravity to matter is conveniently illustrated by the case of the fundamental spinor of SO(1,3+K). This spinor, $\psi(z)$, has $2^{2+[K/2]}$ components where [K/2] denotes the largest integer $\leq K/2$. Under frame rotations the components of ψ transform as an SO(1,3+K) spinor,

$$\psi(z) \rightarrow \psi'(z) = S(a^{-1})\,\psi(z) \tag{2.5}$$

but, under general co-ordinate transformations they are scalars,

$$\psi(z) \rightarrow \psi'(z') = \psi(z) \quad . \tag{2.6}$$

The partial derivatives $\partial_M\psi$ make a covariant world vector but they do not transform homogeneously under frame rotations. The latter defect is cured by introducing the spin connections, $B_M(z)$, which constitute a gauge field with respect to frame rotations,

$$B_M \rightarrow S\, B_M\, S^{-1} + S\,\partial_M\, S^{-1} \tag{2.7}$$

and a covariant vector with respect to co-ordinate transformations. The covariant derivative is then defined by

$$\nabla_M\psi = (\partial_M + B_M)\,\psi \quad . \tag{2.8}$$

The spin connections belong to the infinitesimal algebra of SO(1,3+K). In the spinor representation one can write

$$B_M = \frac{1}{8}\, B_{M[AB]}\, [\gamma^A,\gamma^B] \quad , \tag{2.9}$$

where γ^A is a generalized Dirac matrix, i.e.

$$\{\gamma^A,\gamma^B\} = 2\eta^{AB} \quad . \tag{2.10}$$

With these fields it is possible to construct a suitable Lagrangian density, one which is invariant under frame rotations and a scalar density under co-ordinate transformations

$$\mathcal{L}_\psi = \frac{i}{2}\det(E)\bar{\psi}\,\gamma^A\, E_A{}^M\, \nabla_M\psi + \text{h.c.} \tag{2.11}$$

This much is nothing but a straightforward generalization to 4+K dimensions of the well known 4-dimensional structures. However, when one comes to look for the Yang-Mills potentials among the components of E and B, one will find some less familiar things. For example, the "minimal coupling" of the vectors is contained partly in the term $\bar{\psi}\gamma^A E_A{}^M\partial_M\psi$ and partly in $\bar{\psi}\gamma^A E_A{}^M B_M\psi$. The latter term may also contain a non-minimal, Pauli type of coupling.

Out of E and B and their first derivatives it is possible to construct two distinct covariant objects, or "field strengths". These are the torsion,

$$T_{MN}{}^A = \partial_M E_N{}^A - \partial_N E_M{}^A + E_N{}^C B_{MC}{}^A - E_M{}^C B_{NC}{}^A \tag{2.12}$$

and the curvature,

$$R_{MN[AB]} = \partial_M B_{N[AB]} - \partial_N B_{M[AB]} - B_{M[AC]}B_{N[CB]} + B_{N[AC]}B_{M[CB]} \tag{2.13}$$

(where early alphabet labels are raised, lowered and contracted with η_{AB} defined in (2.4)).

The curvature scalar (density) is given by

$$\det(E)R = \det(E)E_A{}^M\, E_B{}^N\, R_{MN[AB]} \tag{2.14}$$

and can serve as a Lagrangian density for the extended gravity fields. The fields E and B may be varied independently or, alternatively, one may impose the torsion constraints, $T_{MN}{}^A = 0$, and eliminate B as an independent variable. The constraints $T = 0$ are algebraic in B and can easily be solved. One finds

$$B_{M[BC]} = \frac{1}{2} E_M{}^A \left(\Omega_{[AB]C} - \Omega_{[BC]A} + \Omega_{[CA]B}\right) \quad , \tag{2.15}$$

where

$$\Omega_{[AB]C} = E_A{}^M E_B{}^N (\partial_M E_{NC} - \partial_N E_{MC}) \quad . \tag{2.16}$$

III. COMPACTIFICATION

Having set up an action principle in 4+K dimensions one might inquire into the nature of the solutions to the resulting equations of motion. In a purely geometrical theory one would expect the ground state, or vacuum, to exhibit maximal symmetry. Indeed, one should find the 4+K-dimensional Minkowski space. However, in the presence of matter fields - scalars and vectors - the situation is more complicated and it is possible to imagine that the ground state will manifest some lower symmetry. In particular, if the theory is supersymmetric the fields which accompany E and B may trigger a compactification of some of the dimensions in the ground state. We shall assume that this happens and that the ground state geometry factorizes into the product of 4-dimensional Minkowski space with a compact homogeneous space, G/H, of K dimensions. This space is invariant under the action of some group G which will be interpreted as an "internal" symmetry. We shall further assume that the compact space is small enough (size $\sim\kappa$) to justify, for most purposes, the neglect of any excitations associated with it (i.e. that such excitations would be too massive to be accessible at present energies). As mentioned in the Introduction the excitations may not always be negligible. Their contributions are generally comparable to those of gravitons and they would certainly be important if the gravitons are quantized. The aim at this stage, however, is to develop an effective 4-dimensional theory for the long range degrees of freedom.

One way to obtain the 4-dimensional theory is to expand all field variables in complete sets of harmonics on the homogeneous space G/H. The expansion coefficients are 4-dimensional fields belonging to a sequence of representations of G. Their dynamics is governed by a 4-dimensional action functional obtained by integrating out the G/H-dependence and using the orthogonality properties of harmonics on G/H. The resulting masses and couplings are necessarily F-invariant. The scale of the masses is determined by the size of the compact manifold, and we shall assume that all non-vanishing masses are therefore large. One would like to be able to extract the zero modes and discard the rest. We shall consider some aspects of this problem in the following but do not obtain a general solution. We now develop the necessary formalism.

To distinguish the compact dimensions from the remaining four of ordinary spacetime it is convenient to adopt some notational conventions. For the co-ordinates write

$$z^M = (x^m, y^\mu) \tag{3.1}$$

with middle alphabet Latin suffices, m,n,... taking the values 0,1,2,3 and their Greek counterparts, $\mu,\nu,\ldots$ taking the values 1,2,...K. The compact manifold, G/H, is parametrized by y^μ. The early alphabet frame labels are likewise divided into lower case Latin and Greek.

To implement the parametrization of G/H there are available standard group-theoretical techniques [11], one version of which we outline briefly. Suppose the co-ordinates, y^μ, label the cosets of G with respect to the subgroup, H. That is, from each coset let there be chosen a representative element, L_y. Multiplication from the left by an arbitrary element $g \in G$ will generally carry L_y into another coset, one for which the representative element is $L_{y'}$. This defines the so-called left translation

$$g\, L_y = L_{y'}\, h \quad , \tag{3.2}$$

where h is an element of H. Both y'^μ and h are determined by this equation as functions of y^μ and g. The explicit form of these functions depends, of course, on the choice of elements L_y which represent the cosets. Such details are not usually important and, here, we shall be able to manage with some general properties of the parametrization. Among these, the most important is the notion of a covariant basis on G/H.

To define a covariant basis consider the 1-form, or differential,

$$e(y) = L_y^{-1}\, d\, L_y \quad . \tag{3.3}$$

This object belongs to the infinitesimal algebra of G and therefore can be expressed as a linear combination of the generators, $Q_{\hat\alpha}$,

$$\begin{aligned} e(y) &= e^{\hat\alpha}(y)\, Q_{\hat\alpha} \\ &= dy^\mu\, e_\mu{}^{\hat\alpha}(y)\, Q_{\hat\alpha} \quad . \end{aligned} \tag{3.4}$$

The generators $Q_{\hat\alpha}$ fall in two categories: the set $Q_{\bar\alpha}$ which generates the subgroup, H, and the remainder, Q_α, $\alpha = 1,2,\ldots,K$ associated with the cosets, G/H. Correspondingly, one writes

$$e(y) = e^{\alpha}(y)\, Q_{\alpha} + e^{\bar{\alpha}}(y)\, Q_{\bar{\alpha}} \quad . \tag{3.5}$$

Of special interest are the coefficients $e_{\mu}{}^{\alpha}(y)$ which constitute a non-singular $K \times K$ matrix. The columns of this matrix provide the covariant components of the K linearly independent vectors of the covariant basis. The contravariant components of the reciprocal basis, $e_{\alpha}{}^{\mu}$, are defined by inverting the matrix $e_{\mu}{}^{\alpha}$.

The behaviour of $e(y)$ under left translations can be deduced from (3.2) and (3.3). Thus, quite generally,

$$e(y) \to e(y') = h\, L_y^{-1} g^{-1}\, d(g\, L_y\, h^{-1})$$

$$= h\, e(y)\, h^{-1} + h\, d\, h^{-1} + h\, L_y^{-1} g^{-1}\, dg\, L_y\, h^{-1} \quad . \tag{3.6}$$

On substituting the expansion in terms of generators and introducing the matrices, $D_{\hat{\alpha}}{}^{\hat{\beta}}$, of the adjoint representation of G, defined by

$$g^{-1}\, Q_{\hat{\alpha}}\, g = D_{\hat{\alpha}}{}^{\hat{\beta}}(g)\, Q_{\hat{\beta}} \tag{3.7}$$

the formula (3.6) gives

$$e^{\hat{\alpha}}(y') = e^{\hat{\beta}}(y)\, D_{\hat{\beta}}{}^{\hat{\alpha}}(h^{-1}) + (h\, d\, h^{-1})^{\hat{\alpha}} + (g^{-1}\, d\, g)^{\hat{\beta}}\, D_{\hat{\beta}}{}^{\hat{\alpha}}(L_y\, h^{-1}) \quad . \tag{3.8}$$

The components, $(h\, d\, h^{-1})^{\hat{\alpha}}$ and $(g^{-1}\, d\, g)^{\hat{\alpha}}$, of elements in the algebra are naturally defined by

$$g^{-1}\, d\, g = (g^{-1}\, d\, g)^{\hat{\alpha}}\, Q_{\hat{\alpha}}$$

$$h\, d\, h^{-1} = (h\, d\, h^{-1})^{\hat{\alpha}}\, Q_{\hat{\alpha}} = (h\, d\, h^{-1})^{\bar{\alpha}}\, Q_{\bar{\alpha}} \quad . \tag{3.9}$$

Since $h\, d\, h^{-1}$ **belongs** to the algebra of H, we must have $(h\, d\, h^{-1})^{\alpha} = 0$.

(A more detailed discussion of the properties of the "boost" elements, L_y, together with some illustrations is given in Appendix I. Curvature and torsion formulae for G/H are derived in Appendix II, and various implications of the transformation formula (3.8) are derived in Appendix III.)

Now consider the question of harmonic expansions on the internal space. To begin with, if the internal space were simply the group G itself (i.e. $H = 1$) it would be natural to employ the full set of matrices of the unitary irreducible representations of G. For the function, $\phi(g)$, one would write

$$\phi(g) = \sum_{n} \sum_{p,q} \sqrt{d_n}\; D^n_{pq}(g)\, \phi^n_{qp} \quad . \tag{3.10}$$

where D^n_{pq} is a unitary matrix of dimension d_n and the sum includes all matrix elements of all the unitary irreducible representations, $g \to D^n(g)$. The coefficients, ϕ^m_{pq}, are projected out by integrating over the group,

$$\phi^n_{pq} = \frac{\sqrt{d_n}}{V_G} \int_G d\mu\; D^n_{pq}(g^{-1})\, \phi(g) \quad , \tag{3.11}$$

where $d\mu$ is the invariant measure normalized to volume V_G.

For functions on the coset space G/H the expansion is somewhat restricted. Typically, one is concerned with functions $\phi_i(g)$ that are subject to the auxiliary symmetry

$$\phi_i(hg) = \mathbb{D}_{ij}(h)\, \phi_j(g) \quad , \tag{3.12}$$

where $h \in H$ and $\mathbb{D}(h)$ is some particular representation of H. This means that $\phi(g)$, though not strictly constant over the points in a coset, is related by a linear rule, i.e.

$$\phi_i(g_1) = \mathbb{D}_{ij}(g_1 g_2^{-1})\, \phi_j(g_2)$$

for g_1 and g_2 in the same coset. The appropriate restriction of the expansion (3.10) is clear. It must include only those terms for which

$$\mathbb{D}^n(hg) = \mathbb{D}(h)\, D^n(g) \quad .$$

For irreducible $\mathbb{D}(h)$ one should write

$$\phi_i(g) = \sum_{n} \sum_{\zeta,q} \sqrt{\frac{d_n}{d_{\mathbb{D}}}}\; D^n_{i\zeta,q}(g)\, \phi^n_{q\zeta} \quad , \tag{3.13}$$

where the sum includes all unitary irreducible representations of G for which $D^n_{i\zeta,q}$ makes sense, i.e. all those which include $\mathbb{D}(h)$ on restriction to H. If $\mathbb{D}(h)$ is contained more than once in a given $D^n(g)$ then a supplementary label, ζ, is needed to distinguish them. The dimension of $\mathbb{D}(h)$ is denoted $d_{\mathbb{D}}$. The coefficients, $\phi^n_{q\zeta}$, are projected from the sum (3.13) by integrating over the coset space,

$$\phi^n_{q\zeta} = \frac{1}{V_k}\sqrt{\frac{d_n}{d_{\mathbb{D}}}} \int_{G/H} d\mu \; D^n_{q,i\zeta}(L_y)\; \phi_i(L_y^{-1}) \quad . \tag{3.14}$$

This expression can be derived from (3.11) by resolving the general element g^{-1} into the product of appropriate elements L_y and h. Such a decomposition is unique once the form of the representative element L_y has been chosen. From (3.12) it follows that the h-dependence factorizes,

$$\phi_i(g) = \phi_i(h^{-1} L_y^{-1})$$

$$= \mathbb{D}_{ij}(h^{-1})\; \phi_j(L_y^{-1}) \quad .$$

Integration over the subgroup H can be carried out explicitly and leaves the result (3.14).

Thus, for a set of fields $\psi_i(x,y)$ the general form of the harmonic expansion on G/H is given by

$$\psi_i(x,y) = \sum_n \sum_{\zeta,q} \sqrt{\frac{d_n}{d_{\mathbb{D}}}}\; D^n_{i\zeta,q}(L_y^{-1})\,\psi^n_{q\zeta}(x) \; . \tag{3.15}$$

Here n labels the excitations.

The coefficient fields $\psi^n_{q\zeta}(x)$ are obtained by integrating over G/H as in (3.14). If the representation $\mathbb{D}(h)$ to which ψ_i belongs is reducible then it would be necessary to separate ψ_i into irreducible parts and make an expansion of the form (3.15) for each. This raises the important question of how the H content of $\psi(x,y)$ is to be determined. It will be settled by invoking the ground state symmetry.

The full set of fields in 4+K dimensions is supposed to be governed by an action functional which is invariant with respect to both the general co-ordinate transformations and the frame rotations. These symmetries will not be manifest in the ground state, however. By assumption the ground state geometry is $M^4 \times G/H$, where G/H is a compact K-dimensional manifold. In the ground state we therefore have

$$\langle E_M{}^A(x,y)\rangle = \begin{pmatrix} \delta^a_m & 0 \\ 0 & e_\mu{}^\alpha(y) \end{pmatrix} , \tag{3.16}$$

where $e_\mu{}^\alpha$ is the covariant basis on G/H defined above. The expression (3.16) is invariant under the product of the 4-dimensional Poincaré group and the local (x-dependent) transformations of G. The Poincaré invariance is exactly as in the familiar 4-dimensional theory of general relativity. The local G-invariance is compounded of the x-dependent left translations on G/H, $y \to y'(x,y)$ and the induced tangent space transformations, $D_\alpha{}^\beta(h)$, with h defined by (3.2). To see this, consider the action of the combined transformations on $E_\mu{}^\alpha(x,y)$

$$E_\mu{}^\alpha(x,y) \to E'_\mu{}^\alpha(x,y')$$

$$= \frac{\partial y^\nu}{\partial y'^\mu}\, E_\nu{}^\beta(x,y)\; D_\beta{}^\alpha \; ,$$

where $D_\beta{}^\alpha$ is a K-dimensional orthogonal matrix. In order to have

$$\langle E'_\mu{}^\alpha(x,y)\rangle = \langle E_\mu{}^\alpha(x,y)\rangle$$

it is only necessary to choose $\langle E_\mu{}^\alpha\rangle = e_\mu{}^\alpha$ and $D_\beta{}^\alpha = D_\beta{}^\alpha(h)$ This follows from (3.8) which contains the formula

$$dy'^\mu\, e_\mu{}^\alpha(y') = dy^\mu\, e_\mu{}^\beta(y)\, D_\beta{}^\alpha(h) \; .$$

Thus the embedding is fixed. With the left translation, $y \to y'$, must be associated the SO(K) transformation, $D_\alpha{}^\beta(h)$, where $h \in H$. This implies that the K-vector of SO(K) has the H-content of G/H. More precisely, the generators, $Q_{\bar\alpha}$, of H can be expressed in terms of the generators, $\Sigma^{\alpha\beta}$, of SO(K) by the formula [3)]

$$Q_{\bar\gamma} = -\frac{1}{2}\, c_{\bar\gamma\alpha\beta}\, \Sigma^{\alpha\beta} \quad , \tag{3.17}$$

where the coefficients $c_{\bar\gamma\alpha\beta}$ are taken from the set of structure constants $c_{\hat\gamma\hat\alpha\hat\beta}$ of G.

The Yang-Mills group of transformations described here, which leaves the ground state invariant also preserves the form of the expansion (3.15). Thus, with ψ transforming according to

$$\psi_i(x,y) \to \psi_i'(x,y') = \mathbb{D}_{ij}(h)\, \psi_j(x,y) \qquad (3.18)$$

it is a simple matter to derive the corresponding rule for the expansion coefficients

$$\psi^{n}_{p\zeta}(x) \to \psi'^{n}_{p\zeta}(x) = D^n_{pq}(g)\, \psi^n_{q\zeta}(x) \quad . \qquad (3.19)$$

They belong to the various irreducible representations of G which contribute to the expansion (3.15).

In order to express the full 4+K-dimensional theory in its equivalent 4-dimensional form, one should substitute expansions like (3.15) for all of the fields in the system. The 4-dimensional Lagrangian is then obtained by integrating the 4+K dimensional one over the compact manifold, G/H. Integration of products of matrices $D^n(L_y^{-1})$ reduces, in principle, to an algebraic problem. Although it is difficult to obtain results without going into the details of particular cases, the most important feature, orthogonality, can be illustrated by an example.

Consider the invariant integral

$$I^{nn'}_{p\zeta,p'\zeta'} = \int_{G/H} d^Ky \det e(y)\, D^n_{p,i\zeta}(L_y)\, D^{n'}_{i\zeta',p'}(L_y^{-1}) \quad , \qquad (3.20)$$

where the label, i, is understood to be summed. Since the measure, $d\mu = d^Ky$ det e, is invariant under left translations, one can write

$$\begin{aligned} I^{nn'}_{p\zeta,p'\zeta'} &= \int d\mu\, D^n_{p,i\zeta}(L_{y'})\, D^n_{i\zeta',p'}(L_{y'}^{-1}) \\ &= \int d\mu\, D^n_{p,i\zeta}(g\, L_y\, h^{-1})\, D^n_{i\zeta',p'}(h\, L_y^{-1}\, g^{-1}) \\ &= D^n_{pq}(g)\, I^{nn'}_{q\zeta,q'\zeta'}\, D^{n'}_{q'p'}(g^{-1}) \end{aligned}$$

because the dependence on h cancels. Hence I is an invariant tensor of G. By Schur's lemma it must take the form

$$I^{nn'}_{p\zeta,p'\zeta'} = \delta_{nn'}\, \delta_{pp'}\, k^n_{\zeta\zeta'} \quad .$$

To evaluate $k^n_{\zeta\zeta'}$, set $n = n'$, $p = p'$ and sum over p. One obtains

$$\begin{aligned} d_n\, k^n_{\zeta\zeta'} &= \int d\mu\, D^n_{i\zeta',p}(L_y^{-1})\, D^n_{p,i\zeta}(L_y) \\ &= V_K\, d_{\mathbb{D}}\, \delta_{\zeta\zeta'} \quad , \end{aligned}$$

where d_n and $d_{\mathbb{D}}$ are the dimensions of D^n_{pq} and $\mathbb{D}_{ij}$, respectively. Thus

$$I^{nn'}_{p\zeta,p'\zeta'} = V_K \frac{d_{\mathbb{D}}}{d_n}\, \delta_{\zeta\zeta'}\, \delta_{pp'}\, \delta_{nn'} \quad . \qquad (3.21)$$

This result is needed for inverting the expansion (3.15).

The argument leading to (3.21) generalizes. For example, one can write

$$\begin{aligned} \int_{G/H} d\mu\, c^{i_1 i_2 i_3}\, D^{n_1}_{i_1\zeta_1,p_1}(L_y^{-1})\, D^{n_2}_{i_2\zeta_2,p_2}(L_y^{-1})\, D^{n_3}_{i_3\zeta_3,p_3}(L_y^{-1}) &= \\ = a_{n_1n_2n_3}\, V_K\, c^{i_1i_2i_3} \begin{pmatrix} n_1 & n_2 & n_3 \\ i_1\zeta_1 & i_2\zeta_2 & i_3\zeta_3 \end{pmatrix} \begin{pmatrix} n_1 & n_2 & n_3 \\ p_1 & p_2 & p_3 \end{pmatrix} \quad , \end{aligned}$$

where $c^{i_1i_2i_3}$ is an invariant tensor of H and the 3-n symbols are essentially Clebsch-Gordan coefficients for the unitary representations of G. The factor $a_{n_1n_2n_3}$ depends on normalization conventions for the 3-n symbols.

Having given, in outline [4], the general form of the expansions to be used, we shall in the following section restrict our considerations to the leading terms. The justification for this restriction is simply that, in the regime of low energies, only zero-modes are important.

IV. EFFECTIVE LAGRANGIANS

It is guaranteed that the graviton and a set of Yang-Mills vectors are included among the zero-mass fields of the 4-dimensional effective theory. This results from the gauge symmetries of the 4+K-dimensional theory and the assumed symmetry of the ground state. It can be proved as follows. For $E_M{}^A(x,y)$ consider the ansatz [9]

$$E_M{}^A(x,y) = \begin{pmatrix} E_m{}^a(x) & -A_m{}^{\hat\beta}(x)\, D_{\hat\beta}{}^{\alpha}(L_y) \\ 0 & e_\mu{}^{\alpha}(y) \end{pmatrix} \quad , \qquad (4.1)$$

where the y-dependence is specified. This ansatz is compatible with subgroup of the 4+K-dimensional symmetries: the 4-dimensional general co-ordinate transformations,

$$x^m \to x'^m(x)$$

and the Yang-Mills transformations, or x-dependent left translations

$$y^\mu \to y'^\mu(x,y)$$

with associated frame rotations, $D_\alpha{}^\beta(h)$,

$$E'^{a}_{m}(x') = \frac{\partial x^n}{\partial x'^m} E_n{}^a(x) \tag{4.2a}$$

$$E'^{\alpha}_{m}(x',y') = \left(\frac{\partial x^n}{\partial x'^m} E_n{}^\beta(x,y) + \frac{\partial y^\mu}{\partial x'^m} e_\mu{}^\beta(y) \right) D_\beta{}^\alpha(h^{-1}) \tag{4.2b}$$

$$e_\mu{}^\alpha(y') = \frac{\partial y^\nu}{\partial y'^\mu} e_\nu{}^\beta(y) D_\beta{}^\alpha(h^{-1}) \quad . \tag{4.2c}$$

Of these rules, (4.2a) requires no comment and (4.2c) has been dealt with already in Sec.III. There remains (4.2b) which implies, in view of the ansatz,

$$- A'^{\hat\beta}_m(x') D_{\hat\beta}{}^\alpha(L_{y'}) = \left(- \frac{\partial x^n}{\partial x'^m} A_n{}^{\hat\gamma}(x) D_{\hat\gamma}{}^\beta(L_y) + \frac{\partial y^\mu}{\partial x'^m} e_\mu{}^\beta(y) \right) D_\beta{}^\alpha(h^{-1}) \ .$$

With the help of formulae derived in Appx.3 this rule reduces to the form

$$A'_m(x') = \frac{\partial x^n}{\partial x'^m} \left(g\, A_n(x)\, g^{-1} - g\, \partial_n\, g^{-1} \right) \ , \tag{4.3}$$

where $A_m = A_m{}^{\hat\alpha} Q_{\hat\alpha}$. This is precisely the rule to be expected for a Yang-Mills potential.

The zero torsion spin connections corresponding to the ansatz (4.1) are obtained by substituting in the formulae (2.15), (2.16)

$$B_{a[bc]} = \frac{1}{2} E_{[a}{}^m E_{b]}{}^n \partial_m E_{nc} - \frac{1}{2} E_{[b}{}^m E_{c]}{}^n \partial_m E_{na} + \frac{1}{2} E_{[c}{}^m E_{a]}{}^n \partial_m E_{nb}$$

$$B_{a[b\gamma]} = -B_{\gamma[ab]} = \frac{1}{2} E_a{}^m E_b{}^n F_{mn}{}^{\hat\alpha} D_{\hat\alpha\gamma}(L_y)$$

$$B_{a[\beta\gamma]} = A_a{}^{\hat\delta} \left(D_{\hat\delta}{}^{\bar\alpha}(L_y) - D_{\hat\delta}{}^\alpha(L_y)\, \pi_\alpha{}^{\bar\alpha}(y) \right) c_{\bar\alpha\beta\gamma}$$

$$B_{\alpha[\beta c]} = 0$$

$$B_{\alpha[\beta\gamma]} = \frac{1}{2} c_{\alpha\beta\gamma} + \pi_\alpha{}^{\bar\alpha}(y)\, c_{\bar\alpha\beta\gamma} \quad . \tag{4.4}$$

In these expressions the following notations are used:

$$E_{[a}{}^m E_{b]}{}^n = E_a{}^m E_b{}^n - E_b{}^m E_a{}^n$$

$$F_{mn}{}^{\hat\alpha} = \partial_m A_n{}^{\hat\alpha} - \partial_n A_m{}^{\hat\alpha} - A_m{}^{\hat\beta} A_n{}^{\hat\gamma} c_{\hat\gamma\hat\beta}{}^{\hat\alpha}$$

$$\pi_\alpha{}^{\bar\alpha}(y) = e_\alpha{}^\mu(y)\, e_\mu{}^{\bar\alpha}(y) \tag{4.5}$$

and the structure constants of G are defined by

$$[Q_{\hat\alpha} , Q_{\hat\beta}] = c_{\hat\alpha\hat\beta}{}^{\hat\gamma} Q_{\hat\gamma} \quad .$$

On substituting the expressions (4.4) into the 4+K-dimensional curvature scalar one obtains

$$R_{4+K} = R_4 - \frac{1}{4} F_{ab}{}^{\hat\alpha} F_{ab}{}^{\hat\beta} D_{\hat\alpha}{}^\gamma(L_y) D_{\hat\beta}{}^\gamma(L_y) + R_K \ , \tag{4.6}$$

where $R_4 = R_4(E)$ is the usual 4-dimensional curvature scalar, expressed in terms of the vierbein $E_m{}^a(x)$, and R_K is the constant curvature of G/H (see Appx.2). On integrating y^μ the Yang-Mills Lagrangian emerges, since

$$\frac{1}{V_K} \int d^K y \det e(y)\, D_{\hat\alpha}{}^\gamma(L_y)\, D_{\hat\beta}{}^\gamma(L_y) = k\, \delta_{\hat\alpha\hat\beta} \ ,$$

where $k = \dim(G/H)/\dim(G)$. Thus, the above assertion that the system contains the graviton and massless Yang-Mills fields is verified.

There may also be zero mass scalars in the system, like the Brans-Dicke scalar in the K = 1 theory. This point is sensitive, however, to the matter field couplings which are generally needed to force the compactification. We shall not therefore discuss the scalar modes.

Finally, to illustrate how the ansatz described above can be combined with an analogous one for a spinor field to give an effective 4-dimensional term, we consider the expression

$$\mathcal{L}_\psi = \frac{1}{2} \det E(x,y)\, \bar{\psi}\, \gamma^A E_A^{\ M} \nabla_M \psi + \text{h.c.} \tag{4.7}$$

with $\psi(x,y)$ represented by the ansatz,

$$\psi(x,y) = D(L_y^{-1})\, \psi(x) \quad , \tag{4.8}$$

This is a typical term from the harmonic expansion of $\psi(x,y)$. The spinor of SO(1,3+K) must of course be decomposed relative to the subgroup $SO(1,3) \times H$; each piece in the decomposition will yield $\psi(x)$ and the details of labelling are ignored here. One must substitute the expressions (4.1), (4.4) and (4.8) into (4.7),

$$\mathcal{L}_\psi = \frac{1}{2} \det E(x) \det e(y)\, \bar{\psi}(x)\, D(L_y) \Big[\gamma^a \Big\{E_a^{\ m}\, \partial_m + E_a^{\ \mu}\, \partial_\mu - \frac{1}{4} B_{a[BC]}\, \gamma^{BC}\Big\} + \gamma^\alpha \Big\{e_\alpha^{\ \mu}\, \partial_\mu - \frac{1}{4} B_{\alpha[BC]}\, \gamma^{BC}\Big\}\Big]\, D(L_y^{-1})\, \psi(x) + \text{h.c.} \tag{4.9}$$

To simplify this expression it is necessary to eliminate the derivative ∂_μ by means of the formula

$$\begin{aligned} \partial_\mu D(L_y^{-1}) &= - D(L_y^{-1})\, \partial_\mu D(L_y)\, D(L_y^{-1}) \\ &= - e_\mu^{\ \hat{\alpha}}(y)\, D(Q_{\hat{\alpha}})\, D(L_y^{-1}) \end{aligned}$$

which follows from the definition (3.3). Multiplication by $e_\alpha^{\ \mu}$ then gives

$$e_\alpha^{\ \mu}\, \partial_\mu D(L_y^{-1}) = -(D(Q_\alpha) + \pi_\alpha^{\ \hat{\alpha}}\, D(Q_{\hat{\alpha}}))\, D(L_y^{-1}) \quad ,$$

where π is defined in (4.5). It turns out that the π-containing terms are cancelled by similar terms in the spin connections. This cancellation is ensured by the embedding of H in the tangent space group (see Appx.4). After some tedious calculations it is found that (4.9) reduces to

$$\mathcal{L}_\psi = \frac{1}{2} \det E(x) \det e(y)\, \bar{\psi}(x)\, D(L_y) \Big[\gamma^a \Big\{E_a^{\ m}(x)\, \partial_m + \frac{1}{4} B_{a[bc]}(x)\, \gamma^{bc} - A_a^{\ \hat{\alpha}}(x)\, D(Q_{\hat{\alpha}})\Big\} + \frac{3}{8} F_{ab}^{\ \hat{\delta}}(x)\, \gamma^{ab}\, \gamma^\alpha\, D_\alpha^{\ \hat{\delta}}(L_y^{-1}) - \gamma^\alpha D(Q_\alpha) + \frac{1}{8} c_{\alpha\beta\gamma}\, \gamma^{\alpha\beta\gamma}\Big]\, D(L_y^{-1})\, \psi(x) + \text{h.c.} \tag{4.10}$$

and it remains only to integrate over y. Since γ^a must commute with $D(L_y)$, the result takes the form

$$\frac{1}{2} V_K \det E(x)\, \bar{\psi}(x) \Big[\gamma^a \Big\{E_a^{\ m}(x)\, \partial_m + \frac{1}{4} B_{a[bc]}(x)\, \gamma^{bc} - A_a^{\ \hat{\beta}}(x)\, D(Q_{\hat{\beta}})\Big\} + \frac{3}{8} F_{ab}^{\ \hat{\delta}}(x)\, \gamma^{ab}\, \Gamma^{\hat{\delta}} - M\Big]\, \psi(x) + \text{h.c.} \tag{4.11}$$

where $\Gamma^{\hat{\delta}}$ and M are matrices defined by

$$\Gamma^{\hat{\delta}} = \frac{1}{V_K} \int_{G/H} d\mu\, D(L_y)\, \gamma^\alpha\, D(L_y^{-1})\, D_\alpha^{\ \hat{\delta}}(L_y^{-1}) \quad , \tag{4.12}$$

$$M = \frac{1}{V_K} \int_{G/H} d\mu\, D(L_y)\, (\gamma^\alpha D(Q_\alpha) - \frac{1}{8} c_{\alpha\beta\gamma}\, \gamma^{\alpha\beta\gamma})\, D(L_y^{-1}) \, . \tag{4.13}$$

Note that the "Pauli-moment term" implied by non-zero $\Gamma^{\hat{\delta}}$ is a necessary consequence of our formalism, and the precise numerical expressions for $\Gamma^{\hat{\delta}}$ and M are the hall-marks of the expansion on G/H.

V. CONCLUSIONS

In this study we have attempted to analyze in some detail various formal aspects of Kaluza-Klein theories in which the ground state geometry takes the form $M^4 \times B^K$, where B^K is a K-dimensional quotient space G/H. The symmetries of this ground state are used to classify the excitations of the system, and these excitations are found to include particles of spins zero, one and two, in general. In particular, there are a set of 4 vectors belonging to the adjoint representation of G contained among the components of the metric tensor and spin connection which serve as Yang-Mills potentials.

Harmonic expansions on B^K are used to define sequences of component fields, defined on M^4 and belonging to irreducible representations of the Yang-Mills group, G. An important feature in determining the content of these expansions is the embedding of the stability group H in the tangent space group SO(K). This embedding is discussed in detail. Considerable use is made of the "boosts", L_y, in setting up the formalism. The transformation properties of L_y are used to define the action of G on the manifold G/H, and the derivatives of L_y give the covariant basis vectors on this manifold. They also are used in the construction of integral formulae for mass matrices and coupling parameters for Pauli-moment-like terms in the 4-dimensional effective theory (formulae (4.12) and (4.13)).

We remark that the ansatz for the 4-dimensional fields associated with massless particles - graviton, Yang-Mills vectors and Brans-Dicke type scalars - can be seen to arise as leading terms in the harmonic expansions. The complete spectrum for the 4+1-dimensional theory is discussed in Appendix V. It is shown there that the massive excitations are purely of spin-2 and that they can be assigned to infinite-dimensional representations of the non-compact group SO(1,2). Such non-compact symmetries are spontaneously broken and may be viewed as spectrum generating symmetries. It is interesting that, viewed in this light, Kaluza-Klein theory seems to constitute an example of a consistent theory of massive, charged spin-2 particles.

Information drawn from the structure of the harmonic expansions would be useful in testing the classical stability of Kaluza-Klein type solutions. We have not dealt with this type of investigation (except for the simple example of 4+1 theory) since it is generally necessary to introduce matter fields of some sort into the system to generate non-vanishing curvature in G/H. [5] Such matter fields, which must affect the stability considerations, represent a non-geometrical element in the theory (except, perhaps, in the case of extended supergravitities[7]) and are particular to specific models.

APPENDIX I

Non-linear realizations

To supplement the discussion of Sec.III we give here a somewhat more amplified description of the properties of non-linear realizations of continuous groups. The concepts were thoroughly exposed in the physics literature a few years ago when their appropriateness in treating spontaneously broken symmetries was noticed [12]. In that work the space on which the group acts is a set of scalar fields and the non-linearity can be expressed through algebraic constraints on these fields. Because of such constraints it is impossible for all components to vanish simultaneously, hence the absence of an invariant ground state, and hence the relevance to spontaneous symmetry breaking. From a more abstract point of view, the set of scalar fields constitute a vector space which supports a linear representation of the symmetry. The constraints serve to pick out a subspace, generally curved, which is carried into itself by the action of the group. On such a subspace the group is said to be realized non-linearly. Indeed, to characterize the fields belonging to such a curved subspace it is generally necessary to adopt a curvilinear co-ordinate system and so to express the group actions by the transformations which they induce in these co-ordinates.

The classic example of non-linear realizations is the treatment of chiral SU(2) × SU(2). This group has the infinitesimal structure of SO(4) and the scalar fields $(\sigma,\underset{\sim}{\pi})$ are assumed to transform as a 4-vector. If they are subjected to the constraint

$$\sigma^2 + \underset{\sim}{\pi}^2 = F^2 \quad , \tag{I.1}$$

where F is a fixed number, then the number of independent components is reduced to three: they belong to a 3-sphere of radius F. This 3-sphere is of course carried into itself by the transformations of SO(4). It is invariant. To parametrize the points of the 3-sphere one could simply use the 3-vector $\underset{\sim}{\pi}$ and write $\sigma = \pm\sqrt{F^2 - \underset{\sim}{\pi}^2}$. For some purposes, however, it is more useful to define a set of boosts, L_π, which are SO(4) transformations that carry a given reference point, say

$$(\sigma,\underset{\sim}{\pi}) = (F,\underset{\sim}{0}) \tag{I.2}$$

into a general point. In terms of the 4×4 orthogonal matrices $D_{\alpha\beta}(g)$ of the vector representation one writes

$$\sigma = D_{44}(L_\pi)\, F$$

$$\pi_i = D_{i4}(L_\pi)\, F \qquad . \qquad (I.3)$$

Of course there is a great deal of lattitude in the choice of L_π and it will generally be necessary to use at least two co-ordinate patches if the sphere is to be covered without generating artificial singularities.

To illustrate the parametrization of the sphere we give the example of spherical polar co-ordinates,

$$\begin{aligned}
\sigma &= F\cos\theta \\
\pi_1 &= F\sin\theta\cos\varphi \\
\pi_2 &= F\sin\theta\sin\varphi\cos\psi \\
\pi_3 &= F\sin\theta\sin\varphi\sin\psi
\end{aligned} \qquad (I.4)$$

which corresponds to the choice of L_π ,

$$L_\pi = e^{-\psi Q_{23}}\, e^{-\varphi Q_{12}}\, e^{-\theta Q_{41}} \qquad . \qquad (I.5)$$

Here the operator $Q_{\alpha\beta} = -Q_{\beta\alpha}$ is the generator of the infinitesimal rotations in the $\alpha\beta$-plane

$$[Q_{\alpha\beta}, Q_{\gamma\delta}] = \delta_{\beta\gamma}\, Q_{\alpha\delta} - \delta_{\alpha\gamma}\, Q_{\beta\delta} + \delta_{\alpha\delta}\, Q_{\beta\gamma} - \delta_{\beta\delta}\, Q_{\alpha\gamma} \qquad . \qquad (I.6)$$

On applying an SO(4) transformation, g, to the sphere, the point whose co-ordinates are (θ,φ,ψ) will be carried into one with co-ordinates (θ',φ',ψ'). A rotation through the angle ω in the 14-plane, for example, gives

$$\begin{pmatrix} \cos\theta' \\ \sin\theta'\cos\varphi' \end{pmatrix} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta\cos\varphi \end{pmatrix} \quad .$$

By solving these equations one can discover the transformed boost, $L_{\pi'}$. In general the co-ordinate transformation is given by

$$\begin{aligned}
D_{\alpha 4}(L_{\pi'}) &= D_{\alpha\beta}(g)\, D_{\beta 4}(L_\pi) \\
&= D_{\alpha 4}(g\, L_\pi) \quad .
\end{aligned}$$

From this it follows that the precise relation between L_π and $L_{\pi'}$ must take the form

$$g\, L_\pi = L_{\pi'}\, h \qquad , \qquad (I.7)$$

where h is an orthogonal transformation in the 123-subspace, i.e.

$$D_{\alpha 4}(h) = \delta_{\alpha 4} \qquad . \qquad (I.8)$$

Since $L_{\pi'}$ is known once the angles (θ',φ',ψ') have been computed, the SO(3) transformation, $h = L_{\pi'}^{-1}\, g\, L_\pi$, is well defined.

To compute the basis vectors on the 3-sphere consider the differential

$$\begin{aligned}
e &= L_\pi^{-1}\, d\, L_\pi \\
&= e^{\theta Q_{41}}\, e^{\varphi Q_{12}}\, e^{\psi Q_{23}}\, d\left(e^{-\psi Q_{23}}\, e^{-\varphi Q_{12}}\, e^{-\theta Q_{41}}\right) \\
&= -d\theta\, Q_{41} - d\varphi(Q_{12}\cos\theta + Q_{42}\sin\theta) - d\psi(Q_{23}\cos\varphi + (Q_{13}\cos\theta - Q_{43})\sin\varphi) \\
&= -d\theta\, Q_{41} - d\varphi\sin\theta\, Q_{42} + d\psi\sin\theta\sin\varphi\, Q_{43} \\
&\quad\ -d\varphi\, Q_{12} - d\psi\cos\varphi\, Q_{23} + d\psi\cos\theta\sin\varphi\, Q_{31} \qquad .
\end{aligned} \qquad (I.9)$$

From this expression one extracts the coefficients

$$e_\mu{}^\alpha = \begin{array}{c|ccc} & 41 & 42 & 43 \\ \hline \theta & -1 & 0 & 0 \\ \varphi & 0 & -\sin\theta & 0 \\ \psi & 0 & 0 & \sin\theta\sin\varphi \end{array}$$

$$e_\mu{}^{\bar\alpha} = \begin{array}{c|ccc} & 12 & 23 & 31 \\ \hline \theta & 0 & 0 & 0 \\ \varphi & -1 & 0 & 0 \\ \psi & 0 & -\cos\varphi & \cos\theta\sin\varphi \end{array} \quad , \qquad (I.10)$$

where the labelling of rows and columns is indicated. The metric tensor on the 3-sphere is given by

$$ds^2 = (d\theta\, e_\theta{}^\alpha + d\varphi e^\alpha + d\psi\, e_\psi{}^\alpha)^2$$

$$= d\theta^2 + \sin^2\theta\, d\varphi^2 + \sin^2\theta\, \sin^2\varphi\, d\psi^2 \qquad (I.11)$$

as would be expected in spherical polar co-ordinates. The spin connection $B_{\alpha[\beta\gamma]}$ is constructed out of the quantities

$$\pi_\alpha{}^{\bar\beta} = e_\alpha{}^\mu\, e_\mu{}^{\bar\beta}$$

$$= \begin{array}{c c} & \begin{array}{ccc} 12 & 23 & 31 \end{array} \\ \begin{array}{c} 41 \\ 42 \\ 43 \end{array} & \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{\sin\theta} & 0 & 0 \\ 0 & -\frac{\cot\varphi}{\sin\theta} & \cot\theta \end{pmatrix} \end{array} \qquad (I.12)$$

as discussed in the text, (4.4) (and further in Appx.2).

The case discussed above is an example of a non-linear realization in which the group $G = SO(4)$ is made to act on a 3-sphere. The points of the 3-sphere are put into correspondence with the left cosets $SO(4)/SO(3)$ through the parametrization of the bsoosts, L_π. The procedure generalizes directly to manifolds of the form G/H where H is some subgroup of G. (Right cosets can also be used. One writes

$$g : L_y \to L_{y'} = h\, L_y\, g^{-1} \ .$$

On the manifold of G itself one can define the action of $G \times G$

$$(g_1, g_2) : L_y \to L_{y'} = g_1\, L_y\, g_2{}^{-1} \ .$$

In all such cases the mapping $y \to y'$ is determined once the form of L_y has been specified.)

Global properties of a quotient space G/H are often made transparent by embedding in a flat space of higher dimension, i.e. by viewing the non-linear realization as obtained from a linear one with constraints applied. A simple illustration of this is provided by the case of $SU(3)/SU(2) \times U(1)$ which we consider briefly.

The infinitesimal transformations of $SU(3)$ are generated by the eight operators

$$Q_a{}^b = -Q_b{}^{a\dagger} \ , \qquad a,b = 1,2,3 \quad (Q_a{}^a = 0) \ .. \qquad (I.13)$$

The boosts may be given by

$$L_y = \exp(y^\alpha\, Q_\alpha{}^3 - \text{h.c.}) \ , \qquad (I.14)$$

where the co-ordinates y^α, $\alpha = 1,2$, are complex. The little group in this case, $H = SU(2) \times U(1)$, is generated by $Q_\alpha{}^\beta$. With this choice of L_y the procedures discussed above could be used to find the mapping $y^\alpha \to y'^\alpha$ and the induced $SU(2) \times U(1)$ transformation, h, corresponding to an $SU(3)$ transformation, g. However, we shall not pursue this.

The manifold $SU(3)/SU(2) \times U(1)$ clearly has two complex dimensions. In order to see that it is, in fact, $\mathbb{C}P_2$ one can proceed as follows. Let the hermitian, traceless matrices, $Y_a{}^b$, be the co-ordinates of a flat space on which $SU(3)$ acts linearly

$$Y \to Y' = g\, Y\, g^{-1} \ . \qquad (I.15)$$

An invariant subspace can be picked out by requiring the co-ordinates to satisfy the matrix equations

$$Y^2 = \frac{R}{3}\, Y + \frac{2}{9}\, R^2 \ , \qquad (I.16)$$

where R is a constant. Since Y is hermitian, it can be diagonalized by an $SU(3)$ transformation and one finds easily that its diagonalized form must be

$$\hat{Y} = \begin{pmatrix} -R/3 & & \\ & -R/3 & \\ & & 2R/3 \end{pmatrix} \ . \qquad (I.17)$$

In other words, the general solution of the constraints is given by

$$Y = L_y\, \hat{Y}\, L_y{}^{-1} \qquad (I.18)$$

with appropriate $SU(3)$ boosts, L_y. On the other hand, it is possible to express the general solution in the form

$$Y_a{}^b = \left(\frac{1}{3}\, \delta_a{}^b - \frac{\eta_a \bar\eta^b}{\bar\eta\eta}\right) R \ , \qquad (I.19)$$

where η_a is a complex triplet and $\bar{\eta}^a = \eta_a^*$. Since Y is unchanged by the complex scaling $\eta_a \to \lambda\,\eta_a$, this means that the manifold parametrized by L_y can be identified with the 2-dimensional complex projective space, $\mathbb{C}P_2$

As a supplementary remark we note that it is possible to define the action of a non-compact group on a compact manifold. A trivial example of this is provided by the group of displacements on the real line. By factoring out a discrete subgroup corresponding to displacements which are integer multiples of some unit, $2\pi R$, one obtains a compact quotient space. Formally, one writes

$$L_y = e^{yQ} \quad , \qquad 0 \leqslant y \leqslant 2\pi R \quad , \tag{I.20}$$

where Q is the generator of infinitesimal displacements. Applying an arbitrary displacement, $g = \exp\omega Q$, on the left, one obtains as usual,

$$g\,L_y = L_{y'}\,h$$

where

$$y' = y + \omega - 2\pi Rn$$

$$h = (e^{2\pi RQ})^n \tag{I.21}$$

and the integer n is determined such that y' lies in the interval $(0,2\pi R)$. If the points $y = 0$ and $y = 2\pi R$ are identified then the quotient space is a circle of radius R.

A less trivial example is the realization of the non-compact SO(1,2) by its action on the projective cone, a compact 1-dimensional manifold. That such a realization should exist is easily seen by embedding the cone in 3-dimensional Minkowski space. The null cone

$$Y_0^2 - Y_1^2 - Y_2^2 = 0 \tag{I.22}$$

is clearly invariant under the action of SO(1,2). But it is a 2-dimensional space. However, the cone is also invariant under the scaling transformation

$$Y_a \to \lambda\,Y_a \quad . \tag{I.23}$$

A 1-dimensional manifold is obtained by identifying those points which are related by scaling. (In practice one would be dealing with functions on the cone which are homogeneous of degree zero.) A possible parametrization of the manifold would involve writing

$$Y_a = \begin{pmatrix} 1 \\ \sin y \\ \cos y \end{pmatrix} \quad . \tag{I.24}$$

To find the mapping $y \to y'$ corresponding to an hyperbolic rotation in the 02-plane, for example, one would write

$$Y'_a = \begin{pmatrix} \mathrm{ch}\omega + \mathrm{sh}\omega\cos y \\ \sin y \\ \mathrm{sh}\omega + \mathrm{ch}\omega\cos y \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda\sin y' \\ \lambda\cos y' \end{pmatrix}$$

to obtain the relation

$$\sin y' = \frac{\sin y}{\mathrm{ch}\omega + \mathrm{sh}\omega\cos y} \quad . \tag{I.25}$$

This kind of realization can be formalized in the manner used previously. The generators, $Q_{ab} = -Q_{ba}$, of SO(1,2) satisfy the commutation rules

$$[Q_{ab},Q_{cd}] = \eta_{bc}\,Q_{ad} - \eta_{ac}\,Q_{bd} + \eta_{ad}\,Q_{bc} - \eta_{bd}\,Q_{ac} \quad . \tag{I.26}$$

It is possible to pick out two of these,

$$Q_{02} \quad \text{and} \quad Q_+ = Q_{01} + Q_{12} \tag{I.27}$$

which are to be identified as the generators of the (non-compact) stability group, H. They satisfy

$$[Q_{02},Q_+] = Q_+ \tag{I.28}$$

The boosts are defined by

$$L_y = e^{y\,Q_{12}} \quad . \tag{I.29}$$

Since any element of SO(1,2) can be represented unambiguously in the form

$$e^{\alpha Q_{12}} \; e^{\beta Q_+} \; e^{\gamma Q_{02}} \tag{I.30}$$

it will always be possible to solve the mapping relation

$$g\, L_y = L_{y'}\, h$$

for y' and h as functions of y and g.

Another example is the flat group of Scherk and Schwarz [13]. In its simplest version, the generator Q_1 is distinguished from the others, Q_j, $j = 2,3,\ldots,K$. The commutation rules are

$$\begin{aligned} [Q_i, Q_j] &= 0 \\ [Q_i, Q_1] &= M_{ij}\, Q_j \, , \end{aligned} \tag{I.31}$$

where M is antisymmetric, i.e. Q_1 generates orthogonal transformations while the Q_i generate translations. (For $K=3$ this is simply the 2-dimensional Poincaré algebra.) To construct a compact manifold for the group to act on, consider the elements

$$L_y = e^{y^1 Q_1}\, e^{y^j Q_j} \quad , \tag{I.32}$$

where y^1 ranges from 0 to $2\pi/m_1$ (m_1 = smallest non-zero eigenvalue of iM_{jk}) and the y^j are restricted to the ranges

$$0 \leqslant y^j \leqslant 2\pi R^j \quad , \qquad j = 2,\ldots,N \; . \tag{I.33}$$

The action of the group on L_y is easily found. Firstly, the action of $\exp\omega^1 Q_1$ is given by

$$e^{\omega^1 Q_1}\, L_y \;=\; L_{y'} \quad , \tag{I.34}$$

where $y'^j = y^j$ and $y'^1 = y^1 + \omega^1 \pmod{2\pi/m_1}$. Secondly,

$$e^{\omega^j Q_j}\, L_y \;=\; L_{y'}\, h \quad , \tag{I.35}$$

where $y'^1 = y^1$ and

$$y'^j = y^j + (e^{-y^1 M})_{jk}\,\omega^k - 2\pi R^j n^j \tag{I.36}$$

with the integers n^j chosen such that y'^j falls in the interval $(0, 2\pi R^j)$. The little group transformation associated with this is then

$$h = \prod_j \left(e^{2\pi R^j Q_j}\right)^{n_j} \; . \tag{I.37}$$

The covariant basis is defined as usual by the 1-form

$$\begin{aligned} e &= L_y^{-1}\, d\, L_y \\ &= dy^1\,(Q_1 - y^j M_{jk} Q_k) + dy^j\, Q_j \quad , \end{aligned}$$

i.e.

$$e_\mu{}^\alpha = \begin{array}{c c} & \begin{array}{cc} Q_1 & \quad Q_k \end{array} \\ \begin{array}{c} y^1 \\ y^j \end{array} & \begin{pmatrix} 1 & -y^i M_{ik} \\ 0 & \delta_{jk} \end{pmatrix} \end{array} \; . \tag{I.38}$$

To construct Killing vectors the matrices $D_\alpha{}^\beta$ of the adjoint representation are needed. These are given by

$$\begin{aligned} D_1{}^\alpha(L_y)\, Q_\alpha &= L_y^{-1}\, Q_1\, L_y \\ &= Q_1 - y^i M_{ik} Q_k \quad , \\ D_j{}^\alpha(L_y)\, Q_\alpha &= L_y^{-1}\, Q_j\, L_y \\ &= (e^{y^1 M})_{jk}\, Q_k \; , \end{aligned}$$

i.e.

$$D_\alpha{}^\beta(L_y) = \begin{pmatrix} 1 & -y^i M_{ik} \\ 0 & (\exp y^1 M)_{jk} \end{pmatrix} \; . \tag{I.39}$$

The Killing vectors for left translations are then

$$K_\alpha{}^\mu(y) = D_\alpha{}^\beta(L_y)\, e_\beta{}^\mu$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & (\exp y^i M)_{jk} \end{bmatrix} \quad . \tag{I.40}$$

(This matrix was denoted $U(y)$ in the work of Scherk and Schwarz.)

We close with the observation that the zero-torsion spin connection $B_\alpha{}^\beta$ (see Appx.2) is given for this manifold by

$$B_1{}^j = - B_j{}^1 = 0 \quad , \quad B_j{}^k = -dy^1\, M_{jk} \tag{I.41}$$

and the curvature 2-form vanishes, i.e. the manifold is indeed flat.

APPENDIX 2

Curvature and torsion on G/H

Because of their high degree of symmetry, quotient spaces have rather simple geometrical properties and it is possible to derive a number of explicit formulae to describe these. In this appendix we give derivations for such of these formulae as have been referred to in the text. All of them arise out of manipulations of the boost elements L_y which we use to characterize the manifold G/H.

As has been mentioned in Sec.III, the first and most important step is to construct the vectors of a covariant basis on the manifold. To this end one must consider the 1-form

$$\begin{aligned} e(y) &= L_y{}^{-1}\, d\, L_y \\ &= e^{\hat\alpha}(y)\, Q_{\hat\alpha} \\ &= dy^\mu\, e_\mu{}^{\hat\alpha}(y)\, Q_{\hat\alpha} \quad , \end{aligned} \tag{II.1}$$

where the infinitesimal generators of G are denoted $Q_{\hat\alpha}$. These generators satisfy the commutation rules

$$[Q_{\hat\alpha}\, , Q_{\hat\beta}] = c_{\hat\alpha\hat\beta}{}^{\hat\gamma}\, Q_{\hat\gamma} \tag{II.2}$$

with structure constants $c_{\hat\alpha\hat\beta}{}^{\hat\gamma}$.

The differential properties of $e(y)$ needed for the discussion of curvature and torsion are summed up in the 2-form

$$\begin{aligned} d\, e(y) &= -d\, L_y{}^{-1} \wedge d\, L_y \\ &= L_y{}^{-1}\, d\, L_y\, L_y{}^{-1} \wedge d\, L_y \\ &= e(y) \wedge e(y) \quad , \end{aligned} \tag{II.3}$$

the Cartan-Maurer formula. On substituting the expression $e = e^{\hat\alpha} Q_{\hat\alpha}$ and using the commutation rules satisfied by the generators, one can extract the equivalent form

$$d\, e^{\hat\alpha} = \frac{1}{2}\, e^{\hat\beta} \wedge e^{\hat\gamma}\, c_{\hat\beta\hat\gamma}{}^{\hat\alpha} \tag{II.4}$$

or, in terms of components,

$$\partial_\mu e_\nu^{\hat\alpha} - \partial_\nu e_\mu^{\hat\alpha} = - e_\mu^{\hat\beta} e_\nu^{\hat\gamma} c_{\hat\beta\hat\gamma}^{\hat\alpha} \quad . \tag{II.5}$$

As explained in the text the generators $Q_{\hat\alpha}$ separate naturally into two sets, $Q_{\bar\alpha}$ belonging to the subgroup H, and Q_α , the remainder. The coefficients e_μ^{α}, $\alpha = 1,2,\dots,N$, constitute a set of N covariant basis vectors. The reciprocal basis is denoted e_α^{μ} , i.e.

$$e_\alpha^{\mu} e_\mu^{\beta} = \delta_\alpha^{\beta} \quad , \quad e_\mu^{\alpha} e_\alpha^{\nu} = \delta_\mu^{\nu} \quad . \tag{II.6}$$

The torsion 2-form, T^α, is defined by

$$T^\alpha = d\, e^\alpha + e^\beta \wedge B_\beta^{\alpha} \quad , \tag{II.7}$$

where the 1-form B_β^{α} represents the spin connection. Once the latter quantity has been assigned it is possible to construct, in addition to T^α, the curvature 2-form R_α^{β} defined by

$$R_\alpha^{\beta} = d\, B_\alpha^{\beta} + B_\alpha^{\gamma} \wedge B_\gamma^{\beta} \quad . \tag{II.8}$$

One of the simplest possibilities is to choose B such that the torsion vanishes,

$$0 = d\, e^\alpha + e^\beta \wedge B_\beta^{\alpha} \quad . \tag{II.9}$$

On comparing this equation with the Cartan-Maurer formula and noting that $c_{\bar\beta\bar\gamma}^{\alpha} = 0$ (because H is a group) one finds immediately

$$\begin{aligned} B_\beta^{\alpha} &= e^\gamma \tfrac{1}{2} c_{\gamma\beta}^{\alpha} + e^{\bar\gamma} c_{\bar\gamma\beta}^{\alpha} \\ &= e^\gamma \left(\tfrac{1}{2} c_{\gamma\beta}^{\alpha} + \pi_\gamma^{\bar\gamma} c_{\bar\gamma\beta}^{\alpha}\right) \quad . \end{aligned} \tag{II.10}$$

In this last equation we have used

$$e^{\bar\gamma} = e^\gamma \pi_\gamma^{\bar\gamma} = e^\gamma e_\gamma^{\mu} e_\mu^{\bar\gamma} \quad . \tag{II.11}$$

From the expression (II.10) for B one obtains the curvature 2-form

$$R_\beta^{\gamma} = \frac{1}{2} e^\alpha \wedge e^\epsilon \left\{\frac{1}{2} c_{\alpha\epsilon}^{\delta} c_{\delta\beta}^{\gamma} + c_{\alpha\epsilon}^{\bar\delta} c_{\bar\delta\beta}^{\gamma} + \frac{1}{2} c_{\alpha\beta}^{\delta} c_{\epsilon\delta}^{\gamma}\right\} \quad . \tag{II.12}$$

(In deriving this we have used $c_{\alpha\bar\beta}^{\bar\gamma} = 0$ which is not generally true. See for example, the discussion of SO(2,1) in Appx.1. In the general case one would have to keep such terms.)

Another simple possibility is to choose B such that the torsion takes the form

$$T^\alpha = \frac{1}{2} e^\beta \wedge e^\gamma \; c_{\beta\gamma}^{\alpha} \quad . \tag{II.13}$$

Comparison with the Cartan-Maurer formula this time gives

$$B_\beta^{\alpha} = e^{\bar\gamma} c_{\bar\gamma\beta}^{\alpha} \tag{II.14}$$

and hence the curvature

$$R_\beta^{\gamma} = \frac{1}{2} e^\alpha \wedge e^\epsilon \; c_{\alpha\epsilon}^{\bar\delta} c_{\bar\delta\beta}^{\gamma} \quad . \tag{II.15}$$

In particular, if the manifold is G itself, i.e. H = 1, then this assignment gives vanishing curvature. (On the other hand, for some interesting manifolds such as SO(N+1)/SO(N), the structure constants $c_{\alpha\beta}^{\gamma}$ vanish and the two assignments are identical.)

APPENDIX 3

Transformation properties

Some symmetry properties of the manifold G/H are needed for an understanding of the transformation behaviour of the Yang-Mills fields. The purpose of this appendix is to discuss the derivation of the necessary formula.

Consider firstly the transformation rule (3.8) for the 1-form $e^{\hat\alpha}$. For an x-dependent left translation, $y \to y'(x,y)$, this reads

$$e^{\hat\alpha}(y') = e^{\hat\beta}(y)\, D_{\hat\beta}^{\hat\alpha}(h^{-1}) + (h\, d\, h^{-1})^{\hat\alpha} + (g^{-1}\, d\, g)^{\hat\beta}\, D_{\hat\beta}^{\hat\alpha}(L_y\, h^{-1}) \quad , \qquad \text{(III.1)}$$

where $D_{\hat\alpha}^{\hat\beta}$ is a matrix of the adjoint representation, i.e.

$$g^{-1}\, Q_{\hat\alpha}\, g = D_{\hat\alpha}^{\hat\beta}(g)\, Q_{\hat\beta} \quad .$$

The components $(g^{-1}\, d\, g)^{\hat\alpha}$ and $(h\, d\, h^{-1})^{\hat\alpha}$ are defined by

$$\begin{aligned} g^{-1}\, d\, g &= (g^{-1}\, d\, g)^{\hat\alpha}\, Q_{\hat\alpha} \, , \\ h\, d\, h^{-1} &= (h\, d\, h^{-1})^{\hat\alpha}\, Q_{\hat\alpha} \\ &= (h\, d\, h^{-1})^{\bar\alpha}\, Q_{\bar\alpha} \, , \end{aligned} \qquad \text{(III.2)}$$

which is possible since $g^{-1}\, d\, g$ belongs to the algebra of G while $h\, d\, h^{-1}$ belongs to the subalgebra associated with H . In the following we shall assume that the algebra is fully reducible, i.e.

$$D_{\alpha}^{\bar\beta}(h) = D_{\bar\alpha}^{\beta}(h) = 0 \quad . \qquad \text{(III.3)}$$

This means that (III.1) separates into components from G/H and H, respectively,

$$e^{\alpha}(y') = e^{\beta}(y)\, D_{\beta}^{\alpha}(h^{-1}) + (g^{-1}\, d\, g)^{\hat\beta}\, D_{\hat\beta}^{\alpha}(L_y\, h^{-1}) \qquad \text{(III.4)}$$

$$e^{\bar\alpha}(y') = e^{\bar\beta}(y)\, D_{\bar\beta}^{\bar\alpha}(h^{-1}) + (h\, d\, h^{-1})^{\bar\alpha} + (g^{-1}\, d\, g)^{\hat\beta}\, D_{\hat\beta}^{\bar\alpha}(L_y\, h^{-1}) \, . \qquad \text{(III.5)}$$

Under the transformation $y \to y'(x,y)$ we have

$$dy'^{\mu} = dy^{\nu}\, \frac{\partial y'^{\mu}}{\partial y^{\nu}} + dx^{m}\, \frac{\partial y'^{\mu}}{\partial x^{m}}$$

or, equivalently,

$$dy^{\mu} = dy'^{\nu}\, \frac{\partial y^{\mu}}{\partial y'^{\nu}} + dx^{m}\, \frac{\partial y^{\mu}}{\partial x^{m}} \quad .$$

Partial derivatives transform such that

$$\begin{aligned} dy^{\mu}\, \partial_{\mu} + dx^{m} \partial_{m} &= dy'^{\mu} \left(\frac{\partial y^{\nu}}{\partial y'^{\mu}}\, \partial_{\nu}\right) + dx^{m} \left(\partial_{m} + \frac{\partial y^{\nu}}{\partial x^{m}}\, \partial_{\nu}\right) \\ &= dy'^{\mu}\, \partial'_{\mu} + dx^{m}\, \partial'_{m} \end{aligned}$$

(where ∂'_{m} denotes differentiation with respect to x^{m} at fixed y'^{μ}).

Extracting the coefficients of dy'^{μ} and dx^{m} from the formulae (III.4) and (III.5) gives

$$e_{\mu}^{\alpha}(y') = \frac{\partial y^{\nu}}{\partial y'^{\mu}}\, e_{\nu}^{\beta}(y)\, D_{\beta}^{\alpha}(h^{-1}) \qquad \text{(III.6)}$$

$$0 = \frac{\partial y^{\nu}}{\partial x^{m}}\, e_{\nu}^{\beta}(y)\, D_{\beta}^{\alpha}(h^{-1}) + (g^{-1}\, \partial_{m}\, g)^{\hat\beta}\, D_{\hat\beta}^{\alpha}(L_y\, h^{-1}) \qquad \text{(III.7)}$$

$$e_{\mu}^{\bar\alpha}(y') = \frac{\partial y^{\nu}}{\partial y'^{\mu}} \left\{ e_{\nu}^{\bar\beta}(y)\, D_{\bar\beta}^{\bar\alpha}(h^{-1}) + (h\, \partial_{\nu}\, h^{-1})^{\bar\alpha} \right\} \qquad \text{(III.8)}$$

$$\begin{aligned} 0 = \frac{\partial y^{\nu}}{\partial x^{m}}\, e_{\nu}^{\bar\beta}(y)\, D_{\bar\beta}^{\bar\alpha}(h^{-1}) &+ (h\, \partial'_{m}\, h^{-1})^{\bar\alpha} + \\ &+ (g^{-1}\, \partial_{m}\, g)^{\hat\beta}\, D_{\hat\beta}^{\bar\alpha}(L_y\, h^{-1}) \quad . \end{aligned} \qquad \text{(III.9)}$$

The formula (III.7) can be arranged in the form

$$\frac{\partial y^{\mu}}{\partial x^{m}} = -\, (g^{-1}\, \partial_{m}\, g)^{\hat\beta}\, K_{\hat\beta}^{\mu}(y) \quad , \qquad \text{(III.10)}$$

where $K_{\hat\beta}^{\mu}$, the so-called Killing vector, is defined by the expression

$$K_{\hat\beta}^{\mu}(y) = D_{\hat\beta}^{\gamma}(L_y)\, e_{\gamma}^{\mu}(y) \quad . \qquad \text{(III.11)}$$

Its transformation behaviour is obtained with the help of (III.6), viz.

$$\begin{aligned} K_{\hat\beta}{}^{\mu}(y') &= D_{\hat\beta}{}^{\gamma}(L_{y'})\ e_{\gamma}{}^{\mu}(y') \\ &= D_{\hat\beta}{}^{\gamma}(g\ L_y\ h^{-1})\ D_{\gamma}{}^{\alpha}(h)\ e_{\alpha}{}^{\nu}(y)\ \frac{\partial y'^{\mu}}{\partial y^{\nu}} \\ &= D_{\hat\beta}{}^{\hat\gamma}(g)\ K_{\hat\gamma}{}^{\nu}(y)\ \frac{\partial y'^{\mu}}{\partial y^{\nu}} \quad . \end{aligned} \tag{III.12}$$

From (III.10) and (III.12) it is a simple exercise to derive an expression for $\partial y'^{\mu}/\partial x^{m}$. Thus,

$$\begin{aligned} \frac{\partial y'^{\mu}}{\partial x^{m}} &= \frac{\partial y^{\nu}}{\partial x^{m}}\ \frac{\partial y'^{\mu}}{\partial y^{\nu}} \\ &= (g^{-1}\ \partial_m\ g)^{\hat\beta}\ K_{\hat\beta}{}^{\nu}(y)\ \frac{\partial y'^{\mu}}{\partial y^{\nu}} \\ &= (g^{-1}\ \partial_m\ g)^{\hat\beta}\ D_{\hat\beta}{}^{\hat\gamma}(g^{-1})\ K_{\hat\gamma}{}^{\mu}(y') \\ &= -(g\ \partial_m\ g^{-1})^{\hat\beta}\ K_{\hat\beta}{}^{\mu}(y') \quad . \end{aligned} \tag{III.13}$$

With the help of (III.12) and (III.13) one can easily extract the transformation rule for the Yang-Mills gauge fields $A_a{}^{\hat\beta}(x)$ defined by (3.21), or

$$E_a{}^{\mu}(x,y) = A_a{}^{\hat\beta}(x)\ K_{\hat\beta}{}^{\mu}(y) \quad . \tag{III.14}$$

Since $E_a{}^{\mu}$ is part of a contravariant vector it must transform according to

$$\begin{aligned} E_a{}^{\mu}(x,y) &\rightarrow E_a'{}^{\mu}(x,y') \\ &= E_a{}^{\nu}(x,y)\ \frac{\partial y'^{\mu}}{\partial y^{\nu}} + E_a{}^{n}(x,y)\ \frac{\partial y'^{\mu}}{\partial x^{n}} \quad . \end{aligned} \tag{III.15}$$

Substitution of (III.14) into this rule gives

$$\begin{aligned} A_a'^{\hat\beta}(x)\ K_{\hat\beta}{}^{\mu}(y') &= A_a{}^{\hat\beta}(x)\ K_{\hat\beta}{}^{\nu}(y)\ \frac{\partial y'^{\mu}}{\partial y^{\nu}} + E_a{}^{n}(x)\ \frac{\partial y'^{\mu}}{\partial x^{n}} \\ &= A_a{}^{\hat\beta}(x)\ D_{\hat\beta}{}^{\hat\gamma}(g^{-1})\ K_{\hat\gamma}{}^{\mu}(y') - \\ &\quad - E_a{}^{n}(x)\ (g\ \partial_n\ g^{-1})^{\hat\beta}\ K_{\hat\beta}{}^{\mu}(y') \quad . \end{aligned}$$

Since $K_{\hat\beta}{}^{\mu}(y')$ is a common factor in this equation, it may be removed. One is left with the rule

$$A_a'^{\hat\beta}(x) = A_a{}^{\hat\gamma}(x)\ D_{\hat\gamma}{}^{\hat\beta}(g^{-1}) - E_a{}^{n}(x)\ (g\ \partial_n\ g^{-1})^{\hat\beta}$$

or, equivalently,

$$A_m'(x) = g\ A_m(x)\ g^{-1} - g\ \partial_m\ g^{-1} \quad , \tag{III.16}$$

where $A_m(x) = E_m{}^{a}(x)\ A_a{}^{\hat\beta}(x)\ Q_{\hat\beta}$.

A more direct construction of the Killing vectors makes reference to infinitesimal transformations. Thus, the infinitesimal form of the transformation rule $L_{y'} = g\ L_y\ h^{-1}$ reads

$$L_{y+\delta y} = (1 + \delta g^{\hat\alpha}\ Q_{\hat\alpha})\ L_y\ (1 - \delta h^{\bar\alpha}\ Q_{\bar\alpha}) \quad ,$$

i.e.

$$\delta y^{\mu}\ \partial_{\mu}\ L_y = \delta g^{\hat\alpha}\ Q_{\hat\alpha}\ L_y - L_y\ \delta h^{\bar\alpha}\ Q_{\bar\alpha}$$

to first order in the infinitesimals δy^{μ}, $\delta g^{\hat\alpha}$ and $\delta h^{\bar\alpha}$. Multiply on the left by $L_y{}^{-1}$ and use the definition of e,

$$\begin{aligned} \delta y^{\mu}\ e_{\mu}{}^{\hat\alpha}\ Q_{\hat\alpha} &= \delta g^{\hat\alpha}\ L_y{}^{-1}\ Q_{\hat\alpha}\ L_y - \delta h^{\bar\alpha}\ Q_{\bar\alpha} \\ &= \delta g^{\hat\alpha}\ D_{\hat\alpha}{}^{\hat\beta}(L_y)\ Q_{\hat\beta} - \delta h^{\bar\alpha}\ Q_{\bar\alpha} \quad . \end{aligned}$$

Extract the coefficient of Q_{α}

$$\delta y^{\mu}\ e_{\mu}{}^{\alpha} = \delta g^{\hat\beta}\ D_{\hat\beta}{}^{\alpha}(L_y)$$

since $\delta h^{\alpha} = 0$. Hence the infinitesimal δy^{μ} is given by

$$\begin{aligned} \delta y^{\mu} &= \delta g^{\hat\beta}\ D_{\hat\beta}{}^{\alpha}(L_y)\ e_{\alpha}{}^{\mu}(y) \\ &= \delta g^{\hat\beta}\ K_{\hat\beta}{}^{\mu}(y) \end{aligned} \tag{III.17}$$

thereby defining the Killing vectors associated with left translations on G/H.

The same procedure could be followed for constructing the Killing vectors associated with right translations. One should use the 1-form $\tilde{e} = L\, d\, L^{-1}$ instead of e to obtain $\tilde{K}_{\hat{\alpha}}{}^{\mu} = D_{\hat{\alpha}}{}^{\beta}(L_y{}^{-1})\, \tilde{e}_{\beta}{}^{\mu}$. For the special case, $H = 1$, the result is particularly simple. One finds $\tilde{K}_{\hat{\alpha}}{}^{\mu} = -e_{\hat{\alpha}}{}^{\mu}$.

APPENDIX 4

Harmonic expansions

The general form of harmonic expansions on the manifold G/H was discussed in Sec.III. In particular, for the functions $\psi_i(x,y)$ which transform under the combined left translations, $y \to y'$, and associated tangent space rotations, h, such that

$$\psi_i(x,y) \to \psi_i'(x,y') = \mathbb{D}_{ij}(h)\, \psi_j(x,y) \qquad \text{(IV.1)}$$

the appropriate expansion would be

$$\psi_i(x,y) = \sum_n \sqrt{\frac{d_n}{d_{\mathbb{D}}}} \sum_{\zeta,p} D^n_{i\zeta,p}(L_y{}^{-1})\, \psi^n_{p\zeta}(x) \; . \qquad \text{(IV.2)}$$

The sum includes all irreducible representations, D^n, of G which contain $\mathbb{D}_{ij}$ on restriction to the subgroup, H. If $\mathbb{D}_{ij}$ occurs more than once in D^n, then the supplementary label ζ is needed. The matrices $D_{qp}{}^n$ have dimension d_n . The expansion coefficients ψ^n are fields on 4-dimensional spacetime which belong to the irreducible representation D^n ,

$$\psi^n_{p\zeta}(x) \;\to\; \psi'^n_{p\zeta}(x) = D_{pq}{}^n(g)\, \psi_{q\zeta}{}^n(x) \quad . \qquad \text{(IV.3)}$$

They can be projected from $\psi_i(x,y)$ by integrating over the manifold

$$\psi_{p\zeta}{}^n(x) = \frac{1}{V_K} \sqrt{\frac{d_n}{d_{\mathbb{D}}}} \int_{G/H} d\mu\, D_{p,i\zeta}{}^n(L_y)\, \psi_i(x,y) \; . \qquad \text{(IV.4)}$$

Differentiation with respect to y^μ of the fields $\psi_i(x,y)$ is equivalent to an algebraic operation on the components $\psi^n(x)$. The simplest covariant derivative that incorporates $\partial/\partial y^\mu$ would be

$$\begin{aligned}\nabla_\alpha\, \psi_i(x,y) &= e_\alpha{}^\mu\, \partial_\mu\, \psi_i - \frac{1}{2} B_{\alpha[\beta\gamma]}\, \mathbb{D}_{ij}(\Sigma^{\beta\gamma})\, \psi_j \\ &= e_\alpha{}^\mu\, \partial_\mu\, \psi_i + e_\alpha{}^\mu\, e_\mu{}^{\bar{\beta}}\, \mathbb{D}_{ij}(Q_{\bar{\beta}})\, \psi_j \quad . \qquad \text{(IV.5)}\end{aligned}$$

In this formula we have used the expression

$$B_{\alpha[\beta\gamma]} = e_\alpha^{\ \mu} e_\mu^{\ \bar{\alpha}} c_{\bar{\alpha}\beta\gamma} \quad ,$$

which is one of the possible spin connections on G/H, derived in Appx.2.[6)] We have also used the formula (3.17) which fixes the embedding of H in the tangent space group, $SO(N)$,

$$-\frac{1}{2} c_{\bar{\alpha}\beta\gamma} \mathbb{D}(\Sigma^{\beta\gamma}) = \mathbb{D}(Q_{\bar{\alpha}}) \quad .$$

Differentiation of the boost L_y^{-1} leads back to the definition (3.3) of e, viz.

$$\begin{aligned} \partial_\mu L_y^{-1} &= - L_y^{-1} \partial_\mu L_y L_y^{-1} \\ &= - e_\mu^{\ \hat{\beta}} Q_{\hat{\beta}} L_y^{-1} \ , \end{aligned}$$

and this implies that, in any representation,

$$\partial_\mu D^n(L^{-1}) = -e_\mu^{\ \hat{\beta}} D^n(Q_{\hat{\beta}}) D^n(L_y^{-1}) \quad . \qquad \text{(IV.6)}$$

On comparing this with (IV.5) one finds for the covariant derivative of D^n,

$$\nabla_\alpha D_{ip}^{\ n}(L^{-1}) = -D_{iq}^{\ n}(Q_\alpha) D_{qp}^{\ n}(L^{-1}) \quad . \qquad \text{(IV.7)}$$

This formula was used in Sec.IV to reduce some terms from the 4+N-dimensional Lagrangian to 4-dimensional form.

Finally, the expansion formulae are sketched for two relatively simple cases.

(1) SU(2)/U(1)

This is the case of spherical harmonics on the 2-dimensional sphere. With $G = SU(2)$ generated by $Q_{ab} = -Q_{ba}$, $a,b = 1,2,3$ and $H = U(1)$ generated by Q_{12} we can take $y^\mu = (\theta, \varphi)$ with

$$L_y = e^{-\varphi Q_{12}} e^{-\theta Q_{31}} \quad . \qquad \text{(IV.8)}$$

The covariant basis vectors are contained in the 1-form

$$\begin{aligned} e &= e^{\theta Q_{31}} e^{\varphi Q_{12}} d\left(e^{-\varphi Q_{12}} e^{-\theta Q_{31}}\right) \\ &= -d\varphi \cos\theta\, Q_{12} + d\varphi \sin\theta\, Q_{23} - d\theta\, Q_{31} \quad . \qquad \text{(IV.9)} \end{aligned}$$

Expressed in matrix notation, with rows and columns indicated,

$$e_\mu^{\ \hat{\alpha}} = \begin{array}{c} \\ \theta \\ \varphi \end{array} \begin{array}{c} \begin{array}{ccc} 13 & 23 & 12 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\theta & -\cos\theta \end{pmatrix} \end{array} \quad . \qquad \text{(IV.10)}$$

The last column gives the components of $e_\mu^{\ 12}$ (= $e_\mu^{\ \bar{\alpha}}$ in the notation of Sec.III).

The tangent space group, O(2), in this case coincides with $H = U(1)$ so the embedding problem is trivial. The 2-vector ϕ_α resolves into helicity $\lambda = \pm 1$ combinations,

$$\frac{1}{\sqrt{2}} (\phi_1 \pm i\phi_2) \quad ,$$

and likewise for higher rank tensors. For a function, $\phi_\lambda(x,y)$ of helicity λ, integer of half-integer, with respect to the tangent space group we have the expansion

$$\phi_\lambda(x,y) = \sum_j \sqrt{2j+1} \sum_m D_{\lambda m}^{\ j}(L_y^{-1}) \, \phi_{m\lambda}^{\ j}(x) \ , \qquad \text{(IV.11)}$$

where j takes the values $|\lambda|$, $|\lambda|+1, \ldots$ and D^j denotes the $2j+1$-dimensional representation of SU(2),

$$\begin{aligned} D_{\lambda m}^{\ j}(L_y^{-1}) &= D_{\lambda m}^{\ j}\left(e^{\theta Q_{31}} e^{\varphi Q_{12}}\right) \\ &= d_{\lambda m}^{\ j}(\theta)\, e^{im\varphi} \ , \qquad \text{(IV.12)} \end{aligned}$$

in the notation of Wigner.

Generalization to the N-dimensional sphere SO(N+1)/SO(N) is straightforward in principle though, of course, complicated in detail. There is no embedding to be done since H = SO(N) coincides with the tangent space group. Our second example illustrates a case where this is not so.

(2) SU(3)/SU(2) × U(1)

With G = SU(3) generated by the octet of charges, $Q_a{}^b = -Q_b{}^{a*}$ $(Q_a{}^a = 0)$, a,b = 1,2,3 and H generated by the subset $Q_\alpha{}^\beta$, $\alpha,\beta = 1,2$ we can parametrize G/H by a complex 2-vector

$$L_y = e^{y^1 Q_1{}^3 + y^2 Q_2{}^3 - \text{h.c.}} \quad . \qquad \text{(IV.13)}$$

This would not be the most convenient choice for practical computations but we do not intend to present more than a sketch here. The question of interest is how to embed H = SU(2) × U(1) in the tangent space group, SO(4). The general formula for this, (3.17), could be applied but it is simpler in the present case to work *ab initio*.

The triplet of SU(3) decomposes under the subgroup SU(2) × U(1) such that

$$3 = 2_{1/3} + 1_{-2/3} \qquad \text{(IV.14)}$$

i.e. into an SU(2) doublet with U(1) quantum number 1/3 and a singlet with -2/3. For the octet the decomposition reads

$$8 = 1_0 + 3_0 + 2_1 + 2_{-1} \quad . \qquad \text{(IV.15)}$$

If the octet is real then the doublets 2_1 and 2_{-1} are related by complex conjugation. In particular, the generators $Q_\alpha{}^3$, $Q_3{}^\alpha$ associated with G/H are in the representations 2_1 and 2_{-1}, respectively. According to the general principle expressed in formula (3.17), the 4-vector of SO(4) ~ SU(2) × SU(2) must have the SU(2) × U(1) content of G/H,

$$(2,2) = 2_1 + 2_{-1} \qquad \text{(IV.16)}$$

This means that the embedding must be such that

$$(2,1) = 2_0$$
$$(1,2) = 1_1 + 1_{-1} \qquad \text{(IV.17)}$$

for the 2-spinors of SO(4). An explicit construction for the SU(2) × U(1) decomposition of a 4-vector, ϕ_α, is easily arranged. From the 2 × 2 matrix

$$\begin{pmatrix} \phi_4 + i\phi_3 & i(\phi_1 - i\phi_2) \\ i(\phi_1 + i\phi_2) & \phi_4 - i\phi_3 \end{pmatrix} \qquad \text{(IV.18)}$$

select the first column. This belongs to the representation 2_1, and so we define the spinor ϕ_λ with components

$$\phi_{1/2} = \frac{1}{\sqrt{2}} (\phi_4 + i\phi_3)$$
$$\phi_{-1/2} = \frac{1}{\sqrt{2}} (i\phi_1 - \phi_2) \quad . \qquad \text{(IV.19)}$$

Now the representation 2_1 is found in the SU(3) multiplets 8,10,27,..., with triality zero. It follows, therefore, that the harmonic expansion of the 4-vector, $\phi_\alpha(x,y)$, must take the form

$$\phi_\lambda(x,y) = \sum_{n=8,10,27,\ldots} \sqrt{\frac{d_n}{2}} \sum_p D^n_{\lambda p}(L_y^{-1})\, \phi_p{}^n(x), \qquad \text{(IV.20)}$$

where the coefficients $\phi_p{}^n(x)$ are complex. Expansions for tensors of SO(4) can be constructed by straightforward generalization of this procedure. It is interesting to note, however, that spinors of SO(4) cannot be represented in this way. This is clear from the fact that the SU(2) × U(1) content of the 2-spinors, viz. 2_0, 1_1 and 1_{-1} is not to be found in any SU(3) multiplet. In fact, it is not possible to define spinors on the manifold SU(3)/SU(2) × U(1).

APPENDIX 5

Spectrum of 4+1 Kaluza-Klein theory

Since the ground state of the K = 1 theory is flat, it can be obtained as a vacuum solution of the 4+1-dimensional Einstein equations without a cosmological term. For $K > 1$ the manifold G/H has constant curvature and it is necessary to introduce non-geometrical (matter) fields to act as sources. Because of the simplicity of the K = 1 theory one can easily obtain its excitation spectrum and relate it to an underlying non-compact symmetry, SO(1,2).

From the fact that the 4+1-dimensional Einstein theory has five degrees of freedom one expects that all massive states belong to multiplets of spin 2. This is easily verified. Write the metric tensor in the form

$$g_{MN} = \eta_{MN} + h_{MN} \quad , \tag{V.1}$$

where $M,N = 0,1,2,3,4$ and η_{MN} is the flat metric,

$$\eta_{MN} = \mathrm{diag}(+1, -1, -1, -1, -1) \quad . \tag{V.2}$$

Treating the components h_{MN} as small quantities, one obtains the second order terms in the Einstein Lagrangian

$$\mathcal{L}_2 = \frac{1}{4}\,(h_{MN,L}\,h_{MN,L} - 2\,h_{MN,L}\,h_{ML,N} + 2\,h_{MN,N}\,h_{LL,M} - h_{MM,L}\,h_{NN,L}) + \\ + \frac{1}{2}\,J_{MN}\,h_{MN} \quad , \tag{V.3}$$

where J_{MN} is an external source. The connections vanish so that simple partial derivatives are indicated, $h_{MN,L} = \partial_L h_{MN}$. The equations of motion derived from (V.3) take the form

$$h_{MN,LL} - h_{ML,NL} - h_{NL,ML} + h_{LL,MN} + \\ + \eta_{MN}\,h_{KL,KL} - \eta_{MN}\,h_{KK,LL} = J_{MN} \quad . \tag{V.4}$$

These equations are compatible only if J is conserved,

$$J_{MN,N} = 0 \quad . \tag{V.5}$$

This is a relic of the 5-dimensional general covariance. The usual way to solve (V.4), subject to the compatibility condition (V.5), is to impose a co-ordinate condition. For example, in the gauge

$$h_{MN,N} = 0 \tag{V.6}$$

they are solved by

$$h_{MN} = \frac{1}{\partial^2}\left[J_{MN} - \frac{1}{3}\left(\eta_{MN} - \frac{\partial_M\partial_N}{\partial^2}\right)J_{LL}\right] \quad , \tag{V.7}$$

where ∂^2 represents the 5-dimensional d'Alembertian.

On substituting the solution (V.7) back into the Lagrangian (V.3), discarding total derivatives, one finds that $\mathcal{L}_2$ reduces to

$$\begin{aligned} \mathcal{L}_2 &= \frac{1}{4}\,J_{MN}\,h_{MN} \\ &= \frac{1}{4}\left[J_{MN}\,\frac{1}{\partial^2}\,J_{MN} - \frac{1}{3}\,J_{MM}\,\frac{1}{\partial^2}\,J_{NN}\right] \quad . \end{aligned} \tag{V.8}$$

This expression represents the effective interaction between conserved (i.e. physical) sources due to the exchange of the Kaluza-Klein particles.

Now consider the pole, $\partial^2 = 0$, in (V.8). The residue simplifies on using the conservation of J_{MN}. Firstly, if $\partial_4 \neq 0$ we can choose a frame in which

$$\partial_0 = \partial_4 \quad \text{and} \quad \partial_i = 0, \quad i = 1,2,3 \quad . \tag{V.9}$$

In this frame we have

$$J_{0M} = J_{4M} \quad , \quad M = 0,1,2,3,4 \tag{V.10}$$

so that $J_{MM} = -J_{ii}$ and $J_{MN}^{\,2} = J_{ij}^{\,2}$. It follows that, in the neighbourhood of the pole, the effective interaction (V.8) reduces to

$$\mathcal{L}_2 \simeq \frac{1}{4}\,J_{ij}^{\,t}\,\frac{1}{\partial_0^2 - \partial_4^2}\,J_{ij}^{\,t} \quad , \tag{V.11}$$

where the physically significant part of the source is the traceless 3-dimensional tensor

$$J^{t}_{ij} = J_{ij} - \frac{1}{3}\delta_{ij} J_{kk} \quad . \tag{V.12}$$

This result indicates that the massive states ($\partial_4 \neq 0$) are indeed purely spin 2. Their masses are given by the eigenvalues of $-\partial_4^2 = (n/2\pi R)^2$, $n = 1,2,\ldots$ R is the radius of the Kaluza-Klein circle.

On the other hand, if $\partial_4 = 0$ we have the massless states and they can be analysed conveniently by choosing a frame where

$$\partial_0 = \partial_3 \quad \text{and} \quad \partial_1 = \partial_2 = \partial_4 = 0 \quad . \tag{V.13}$$

In this frame the effective interaction reduces, in the neighbourhood of the pole, to the form

$$\frac{1}{4} \sum_{\lambda=-2}^{2} J_{-\lambda} \frac{1}{\partial_0^2 - \partial_3^2} J_\lambda \quad , \tag{V.14}$$

where λ refers to the O(2) helicity and

$$\begin{aligned}
J_2 &= \frac{1}{2}(J_{11} - J_{22}) + i J_{12} \\
J_1 &= J_{41} + i J_{42} \\
J_0 &= \frac{1}{\sqrt{6}}(J_{11} + J_{22} - 2 J_{44}) \\
J_{-1} &= J_{41} - i J_{42} \\
J_{-2} &= \frac{1}{2}(J_{11} - J_{22}) - i J_{12} \quad .
\end{aligned} \tag{V.15}$$

The terms in (V.14) correspond to the exchange of graviton, photon and Brans-Dicke scalar.

Turning now to the non-compact symmetry aspect of this theory, we note that the general co-ordinate transformations in 4+1-dimensions include, in particular, the set

$$\begin{aligned}
\delta x^m &= 0 \\
\delta y &= \omega^1(x)\,\frac{\cos My}{M} + \omega^2(x)\,\frac{\sin My}{M} + \omega^3(x)\,\frac{1}{M} \quad ,
\end{aligned} \tag{V.16}$$

where $M = 1/2\pi R$. (The finite transformations are discussed in Appx.1. See, in particular (I.25).) The infinitesimal generators are given by

$$Q_1 = -\frac{\cos My}{M}\,\partial_y \quad , \quad Q_2 = -\frac{\sin My}{M}\,\partial_y \quad , \quad Q_3 = -\frac{1}{M}\,\partial_y \tag{V.17}$$

and these are easily seen to generate the algebra of SO(1,2),

$$[Q_1,Q_2] = -Q_3 \quad , \quad [Q_2,Q_3] = Q_1 \quad , \quad [Q_3,Q_1] = Q_2 \quad . \tag{V.18}$$

Moreover, the Casimir invariant vanishes,

$$\begin{aligned}
\eta^{ab} Q_a Q_b &= -\left(\frac{\cos My}{M}\,\partial_y\right)^2 - \left(\frac{\sin My}{M}\,\partial_y\right)^2 + \left(\frac{1}{M}\,\partial_y\right)^2 \\
&= 0 \quad .
\end{aligned}$$

This means that fields defined on the circle belong to an irreducible representation of SO(1,2). It is infinite dimensional but not unitary, however. Since the mass operator is given by $-\partial_y{}^2 = M^2 Q_3^2$ one sees that the spectrum of masses is simply the spectrum of Q_3^2 in the irreducible representation of SO(1,2).

Generalizations to higher dimensional manifolds are possible. For example, on the sphere S^K, which is invariant under SO(K+1), it is possible to define the action of SO(1,K+1). However, to find the excitation spectrum and perhaps relate it to the spectrum of some combination of generators of SO(1,K+1), it would be necessary to include the excitations of the matter fields which are needed for the compactification.

FOOTNOTES

1) This comparability of contributions may have relevance to the problem of renormalizability of Kaluza-Klein theories.

2) Throughout this paper B^K is assumed to be a quotient space. One may eventually contemplate internal manifolds (associated with symmetry breaking) which are less symmetrical.

3) For an infinitesimal transformation in H we have

$$D_{\alpha\beta}(h) = \delta_{\alpha\beta} + \delta h^{\bar{\gamma}} c_{\alpha\bar{\gamma}\beta}$$

$$= \varepsilon_{\alpha\beta} + \omega_{\alpha\beta}$$

where $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. Viewed as an infinitesimal SO(K) transformation this reads $\frac{1}{2}\omega_{\alpha\beta}\Sigma^{\alpha\beta} = \delta h^{\bar{\gamma}} Q_{\bar{\gamma}}$, which implies (3.17).

4) Some illustrative examples are contained in Appx.4.

5) The original Kaluza-Klein theory in 4+1-dimensions represents a case where G/H is flat. Generalizations of this have been investigated by Scherk and Schwarz [13].

6) This spin connection corresponds to non-vanishing torsion on G/H. In general the spin connection will contain terms additional to this one, including fluctuation terms for example.

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