ON THE REPRESENTATION
OF GENERALIZED DIRAC (CLIFFORD) ALGEBRAS

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I. INTRODUCTION

We consider here the finite representations of generalized Dirac algebras, i.e., Clifford algebras over the complex field. Complete results on these were given by Brauer and Weyl in 1935 whose approach was unnecessarily sophisticated and, as we shall show, can be simplified. Our approach will be found to be a generalization of an old idea given by Jordan and Wigner. They, however, confined themselves to even \( n \) whereas this paper deals with even and odd \( n \).

The significance of the paper rests on its easy readability by physicists, resting, as it does, on a few well-known theorems on representations of finite groups.

II. THE GROUP \( G(\mathbf{n}) \)

Consider the associative algebra \( \mathfrak{g}(\mathbf{n}) \) generated by the \( n \) elements \( g^\mu (\mu = 1,2,\ldots,n) \) satisfying

\[
g^\mu g^\nu + g^\nu g^\mu = 2 e^{\mu\nu},
\]

where \( e \) is the identity.

Let \( \nu_1, \nu_2, \ldots, \nu_k \) and \( \mu_1, \mu_2, \ldots, \mu_k \) be two sets of \( k(2 < k \leq n) \) distinct integers chosen from \( 1,2,\ldots,n \). Then

\[
\sigma = \nu_1 \mu_2 \cdots \nu_k = \pm \mu_1 \nu_2 \cdots \mu_k.
\]

Thus there are \( 2^k \) distinct elements of \( \mathfrak{g}(\mathbf{n}) \) formed by taking products of \( k \) of the generators \( g^\mu \). Attaching \( \pm e \) and \( \pm g^\mu (\mu = 1,2,\ldots,n) \), we obtain a set of \( 2^{n+1} \) elements which, as can easily be demonstrated, constitute a group \( G(\mathbf{n}) \) whose group operation is the associative multiplication of the algebra.

III. CONJUGACY CLASSES OF \( G(\mathbf{n}) \)

We give below some results whose proofs are being omitted as they are either obvious or require routine checking. The main purpose of the results is to lead to Theorem (I;3) at the end of this section.

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Result (1): If two elements of \( G \), \( a_i \) and \( a_j \), are given by
\[
a_i = \beta_1 \beta_2 \cdots \beta_k
\]
and \( r \) is the number of integers common to the sets \( (\mu_1, \mu_2, \ldots, \mu_k) \) and \( (\nu_1, \nu_2, \ldots, \nu_r) \), then \( a_i \) and \( a_j \) commute if \( k-r \) is even and anticommute otherwise.

Result (2): If \( a_i \) and \( a_j \) are distinct and conjugate, then
\[
a_j = -a_i.
\]
(The converse of this is not always true as is shown below.)

Result (3): If \( a_i = \beta_1 \beta_2 \cdots \beta_k \), \( 1 \leq k \leq n \), \( a_i \) and \( -a_i \) are conjugate.

Result (4): If \( a_i = \beta_1 \beta_2 \cdots \beta_n \), \( a_i \) and \( -a_i \) are conjugate if \( n \) is even, otherwise not.

Taking into account Results (2), (3) and (4) and the fact that \( e \) and \( -e \) form separate conjugacy classes, we arrive at

Theorem (I;3): The number of conjugacy classes of \( G \) is given by
\[
(i) \ 2^n + 1 \text{ when } n \text{ is even; } \\
(ii) \ 2^n + 2 \text{ when } n \text{ is odd.}
\]

IV. COMMUTATOR SUBGROUP OF \( G \)

If \( a_i, a_j \in G \), they either commute or anticommute by Result (1) of Sec.III. Therefore, their (group) commutator
\[
a_i^{-1} a_j^{-1} a_j a_i = \text{either } e \text{ or } -e,
\]
which means that the subgroup consisting of the two elements \( e \) and \( -e \) is the commutator subgroup of \( G \). Its index in \( G \) is \( 2^n \) which gives the number of representations of \( G \) of degree 1.

We now prove

Theorem (II;4): The degree of the irreducible representation classes of \( G \) are given by the formulae
\[
(i) \ |G| = 2^{n+1} = 2^n(1)^2 + 1 \ (2^{n/2})^2 \text{ when } n \text{ is even; } \\
(ii) \ |G| = 2^{n+1} = 2^n(1)^2 + 2 \ (2^{n-1/2})^2 \text{ when } n \text{ is odd},
\]
where \( |G| \) is the order of the group and the numbers in parenthesis the degrees of irreducible representations.

Proof:
(i) Using the connection between the order of a group and the degrees of its irreducible representation classes, the fact that the number of conjugacy classes is \( 2^n + 1 \) and the number of representations of degree 1 is \( 2^n \), the formula follows.

(ii) The problem here consists of breaking up \( 2^n \) into the sum of two squares. Let, if possible,
\[
2^n = a^2 + b^2, \quad a \neq b, \quad a, b \neq 1.
\]

Since the degree of any irreducible representation should divide \( |G| \) i.e. \( 2^n \), \( a \) and \( b \) are both of the form \( 2^m \). Therefore
\[
2^n = 2^p + 2^q \quad (p, q \text{ unequal}),
\]

Let \( p < q \). Dividing by \( 2^p \),
\[
2^{n-p} = 1 + 2^{q-p},
\]
which is impossible. Hence
\[
2^n = 2 \left( \frac{n-1}{2} \right)^2
\]
is the only possible expression and this leads to formula (ii).

V. IRREDUCIBLE REPRESENTATIONS OF \( A(n) \)

Though the representations of \( A(n) \) automatically give representations of \( G \), the converse need not be true. It certainly is not for the degree 1 representations given by Theorem (II;4).

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Instead of proving a theorem on the point, we give explicitly a class of representations of $\mathcal{A}^{(2m)}$ by proceeding as below.

Case 1: $n$ even $= 2m$

Assume that there exists a $2^m \times 2^m$ irreducible representation of $\mathcal{A}^{(2m)}$ given by the matrices

$$\rho_u^m \ (u = 1, 2, \ldots, 2m)$$

the bold type denoting complex matrices as opposed to abstract elements of $\mathcal{A}^{(2m)}$. Let

$$\rho_u^m = \left[ \begin{array}{cc} \rho^m & 0 \\ 0 & -\rho^m \end{array} \right]$$

where $\rho^m$ are $2^m \times 2^m$ matrices. Set

$$\rho_u^{2m+1} = \rho_u^m \rho_u^m \ldots \rho_u^m$$

if $m$ be even

$$= i\rho_u^m \rho_u^m \ldots \rho_u^m$$

if $m$ be odd

and put

$$\rho_u^{2m+1} = \left[ \begin{array}{cc} \rho_u^{2m+1} & 0 \\ 0 & -\rho_u^{2m+1} \end{array} \right]$$

Finally, put

$$\rho_u^{2m+2} = \left[ \begin{array}{cc} 0 & \sigma_k \\ \sigma_k & 0 \end{array} \right]$$

where $\sigma_k$ is the $2^k \times 2^k$ identity matrix for any positive integer $k$.

It is easily verified that

(1) $\rho_u^{2m+1} \rho_v^{2m+1} + \rho_v^{2m+1} \rho_u^{2m+1} = 2 \delta^{uv} \rho_u^{2m+1} \ (u, v = 1, 2, \ldots, 2m, 2m+1, 2m+2)$;

(11) The above representation for $\mathcal{A}^{(2m+2)}$ is irreducible.

We have thus established that there exists an irreducible representation of degree $2^{2m+1}$ or $\mathcal{A}^{(2m+2)}$ if one of degree $2^m$ exists for $\mathcal{A}^{(2m)}$.

Thus, starting with any two of the three Pauli spin matrices, say, $\sigma_1^-$ and $\sigma_2^+$, which constitute a faithful irreducible Hermitian unitary representation of $\mathcal{A}^{(2)}$, we can build a similar representation for any $\mathcal{A}^{(2m)}$.

Case 2: $n$ odd $= 2m+1$

Assume again, as in Case 1, that $\rho_u^m \ (u = 1, 2, \ldots, 2m)$ constitute an irreducible representation of $\mathcal{A}^{(2m)}$ and that $\rho_u^{2m+1}$ is defined as there.

Then $\rho_u^{2m+1} \ (u = 1, 2, \ldots, 2m, 2m+1)$ is seen to constitute a $2^m \times 2^m$ faithful Hermitian irreducible representation of $\mathcal{A}^{(2m+1)}$, the hermiticity being there, of course, if the build up is as in Case 1.

Now put $\rho_u^{2m+1} = -\rho_u^{2m+1}$. Then $\rho_u^{2m+1} \rho_v^{2m+1} = -\delta^{uv} \rho_u^{2m+1}$. Thus, $\rho_1^{2m+1}, \rho_2^{2m+1}, \rho_3^{2m+1}$ again a representation of $\mathcal{A}^{(2m+1)}$ but, as can be seen by an elementary consideration, is equivalent to the earlier one. We thus have the second class of representations foreshadowed by Theorem (II:1).

Finally, if for some reason of physics, we have to preclude a multiplicative relation between the basic elements $\sigma_i$ of $\mathcal{A}^{(2m+1)}$, the representations may be obtained by regarding it as a subalgebra of $\mathcal{A}^{(2m+2)}$ when they will, of course, consist of $2^{m+1} \times 2^{m+1}$ matrices. For reasons outlined above, the correct description of the Pauli algebra is

$$\sigma_1 \sigma_1^+ + \sigma_2 \sigma_2^+ = 2 \delta^{ij} \sigma_i$$

rather than

$$\sigma_1 \sigma_1^+ + \sigma_2 \sigma_2^+ = 2 \delta^{ij} \sigma_i$$

The same is true for any $\mathcal{A}^{(2m+1)}$.

We summarize the considerations of this section in the form of a theorem. 2)

*) It is a simple matter to show that hermiticity is preserved in passing from the $\mathcal{A}^{(2m)}$ representation to the $\mathcal{A}^{(2m+2)}$ one.

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Theorem (III.5): For an arbitrary positive integer \( n \), \( A_2^{(2m)} \) has only one class of irreducible representations of degree \( 2^m \), whereas \( A_2^{(2m+1)} \) has two classes of the same degree.

We end this section by stating that establishing the linear independence of the \( 2^m \) elements of \( G^{(2m)} \), which are selected by taking arbitrarily one of each positive, negative pair, presents neither any problems nor, indeed, any special interest. Since the irreducible representation is of degree \( 2^m \), it follows that \( A_2^{(2m)} \) is isomorphic to the complete matrix algebra of degree \( 2^m \).

VI. SOME PROPERTIES OF DIRAC MATRICES

The subject is too fully discussed in the literature \(^3\) to \(^1\) to call for any further treatment except to shorten it. Setting

\[
X^0 = 1, X^k = i \delta^{k+1} (k = 1,2,3),
\]

where the \( \delta^k \)'s are as in Sec. V with \( m = 2 \), thus obtaining \( 4 \times 4 \) matrices satisfying

\[
X^u X^v + X^v X^u = 2 \delta^{uv} S_0 (u,v = 0,1,2,3)
\]

with \( X^0 \) Hermitian and the rest anti-Hermitian, and then using Theorem (III.5), it is a simple matter to establish the so-called fundamental theorem of Pauli \(^6\) on the existence of a unique pair of \( 4 \times 4 \) matrices, \( iS \), for every element of the extended Lorentz group such that

\[
X^u = S^{-1} \lambda^u, \quad X^v = S^{-1} \lambda^v
\]

\( (\delta^{uv} \lambda^u \lambda^v = \delta^{uv}, \quad g^{uv} \) is the Minkowski metric tensor).
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