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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON WITTEN'S CHARGE FORMULA *

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ABSTRACT

Witten's formula for the dyon charge in the presence of a CP violating surface term is derived by a canonical argument.

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Witten [1] has observed an interesting effect caused by the presence of a surface term in the action functional – a 4-divergence in the Lagrangian density. He finds that the electric charge of a magnetic monopole or dyon is perturbed from its naively quantized value. This is a surprising result and one which deserves to be further explored. In this note we should like to point out that Witten's formula can be understood in the standard terms of canonically quantized field theory through the effect that a surface term can have on the canonical commutators.

To illustrate the point we consider the SO(3) gauge system with a triplet of scalar fields. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}\underset{\sim}{W}^2_{\mu\nu} + \frac{1}{2}(\nabla_\mu\underset{\sim}{\phi})^2 - V(\underset{\sim}{\phi}) + \frac{\beta}{2}\,\underset{\sim}{W}_{\mu\nu}\cdot\underset{\sim}{\widetilde{W}}_{\mu\nu} \quad , \tag{1}$$

where the field strengths $\underset{\sim}{W}_{\mu\nu}$ and covariant derivatives $\nabla_\mu\underset{\sim}{\phi}$ are defined as usual,

$$\begin{aligned}\underset{\sim}{W}_{\mu\nu} &= \partial_\mu\underset{\sim}{W}_\nu - \partial_\nu\underset{\sim}{W}_\mu + g\,\underset{\sim}{W}_\mu \times \underset{\sim}{W}_\nu \\ \nabla_\mu\underset{\sim}{\phi} &= \partial_\mu\underset{\sim}{\phi} + g\,\underset{\sim}{W}_\mu \times \underset{\sim}{\phi}\end{aligned} \tag{2}$$

and the dual field is denoted by $\underset{\sim}{\widetilde{W}}_{\mu\nu}$. For canonical quantization it is convenient to introduce the electric- and magnetic-type field components by the definitions

$$\begin{aligned}\underset{\sim}{E}_i &= \underset{\sim}{W}_{0i} = -\frac{1}{2}\epsilon_{ijk}\underset{\sim}{\widetilde{W}}_{jk} \\ \underset{\sim}{B}_i &= \underset{\sim}{\widetilde{W}}_{0i} = \frac{1}{2}\epsilon_{ijk}\underset{\sim}{W}_{jk} \quad .\end{aligned} \tag{3}$$

The last term in (1), with $\beta = \theta g^2\hbar/16\pi^2$, is the 4-divergence whose influence we wish to trace. Notice firstly that it turns up in the definition of canonical momentum,

$$\begin{aligned}\underset{\sim}{\pi}_i &= \frac{\partial\mathcal{L}}{\partial\underset{\sim}{W}_{i,0}} = \underset{\sim}{E}_i - \beta\,\underset{\sim}{B}_i \\ \underset{\sim}{p} &= \frac{\partial\mathcal{L}}{\partial\underset{\sim}{\phi}_{,0}} = \nabla_0\underset{\sim}{\phi} \quad .\end{aligned} \tag{4}$$

The Hamiltonian is given by

$$H = \frac{1}{2}(\underset{\sim}{\pi}_i + \beta \underset{\sim}{B}_i)^2 + \frac{1}{2}\underset{\sim}{B}_i^2 + \frac{1}{2}\underset{\sim}{p}^2 + \frac{1}{2}(\nabla_i \underset{\sim}{\phi})^2 + V -$$

$$- \underset{\sim}{W}_0 \cdot (\nabla_i \underset{\sim}{\pi}_i + g\, \underset{\sim}{\phi} \times \underset{\sim}{p}) \quad . \tag{5}$$

Observe that this expression becomes identical with the usual Hamiltonian density if one eliminates $\underset{\sim}{\pi}_i$ in favour of $\underset{\sim}{E}_i$. This is of course as it should be if the classical field equations are to be unaffected by the surface term. However, there is a significant change in the commutation rules so that quantum properties will be sensitive to β.

In order to fix the commutators one must choose a gauge. One possibility would be to follow the traditional prescriptions for canonical quantization in the Coulomb gauge. However, it appears to be much simpler to employ the temporal gauge,

$$\underset{\sim}{W}_0 = 0 \quad . \tag{6}$$

The classical equation of constraint $\partial H/\partial \underset{\sim}{W}_0 = 0$ in this gauge becomes a constraint on the physical states, $|P\rangle$, (rather than on observables) and takes the form

$$0 = \left(\nabla_i \underset{\sim}{\pi}_i + g\, \underset{\sim}{\phi} \times \underset{\sim}{p}\right) |P\rangle \quad . \tag{7}$$

The canonical commutation rules are

$$[p^a(x), \phi^b(y)] = \frac{\hbar}{i}\delta^{ab}\,\delta_3(x-y)$$

$$[\pi_i^a(x), W_j^b(y)] = \frac{\hbar}{i}\delta^{ab}\,\delta_{ij}\,\delta_3(x-y) \tag{8}$$

with other commutators vanishing. It is easy to show that $[E,E] = [\pi + \beta B, \pi + \beta B] = \beta[\pi,B] + \beta[B,\pi]$ does not vanish at equal times if β is not zero.

Now consider the operator (for arbitrary time-independent $\underset{\sim}{\omega}$)

$$Q(\underset{\sim}{\omega}) = \frac{1}{\hbar}\int d^3x \left(\frac{1}{g}\underset{\sim}{\pi}_i \cdot \nabla_i \underset{\sim}{\omega} + \underset{\sim}{p}\cdot\underset{\sim}{\phi} \times \underset{\sim}{\omega}\right) \quad . \tag{9}$$

According to the canonical rules this operator, if it exists, should generate a local SO(3) transformation, i.e.

$$\frac{1}{i}[\underset{\sim}{\phi}, Q(\underset{\sim}{\omega})] = \underset{\sim}{\phi} \times \underset{\sim}{\omega}$$

$$\frac{1}{i}[\underset{\sim}{W}_i, Q(\underset{\sim}{\omega})] = \frac{1}{g}\nabla_i \underset{\sim}{\omega} \quad , \tag{10}$$

etc. The action of $Q(\underset{\sim}{\omega})$ on a physical state can be expressed in terms of the asymptotic parts of the field strengths. Thus, taking note of (7) one writes

$$Q(\underset{\sim}{\omega})|P\rangle = \frac{1}{\hbar}\int d^3x \left[\frac{1}{g}\partial_i(\underset{\sim}{\pi}_i \cdot \underset{\sim}{\omega}) - \underset{\sim}{\omega}\cdot(\frac{1}{g}\nabla_i \underset{\sim}{\pi}_i + \underset{\sim}{\phi} \times \underset{\sim}{p})\right]|P\rangle$$

$$= \frac{1}{g\hbar}\oint dS_i \underset{\sim}{\pi}_i \cdot \underset{\sim}{\omega}|P\rangle$$

$$= \frac{1}{g\hbar}\oint dS_i\, (\underset{\sim}{E}_i - \beta \underset{\sim}{B}_i)\cdot\underset{\sim}{\omega}|P\rangle \quad , \tag{11}$$

where the integral is over the surface of an infinitely large sphere. The matrix elements of $Q(\underset{\sim}{\omega})$ in the subspace of physical states are thus expressible in terms of the long-range behaviour of the matrix elements of the electric- and magnetic-type fields. If the gauge symmetries are all spontaneously broken then there will be no such long-range fields and the operators $Q(\underset{\sim}{\omega})$ will vanish in the physical subspace.

The most interesting case concerns the breakdown $SO(3) \to U(1)$ in which the surviving $U(1)$ symmetry is associated with electromagnetism. In the vacuum sector one can choose the Higgs field such that

$$\langle \phi^a \rangle = C\,\delta^{a3} \quad , \tag{12}$$

where C is a constant. This sector is invariant under transformations $\omega^a = \omega\,\delta^{a3}$. The generator, Q^3, of these transformations, e.g.

$$[\phi^1 \pm i\phi^2, Q^3] = \pm(\phi^1 \pm i\phi^2)$$

clearly takes integer eigenvalues. Since there is no long-range magnetic field in this sector it follows that the electric charge, defined as the integral of electric flux, takes the values

$$q_e = \frac{1}{4\pi}\oint dS_i\, E_i^3$$

$$= \frac{ng\hbar}{4\pi} \tag{13}$$

where n is an integer.

On the other hand, in a sector with non-vanishing long-range magnetic flux one may choose the Higgs field such that

$$\langle \phi^a \rangle \simeq c \, \frac{r_a}{r} = c \, \hat{r}_a \tag{14}$$

for $r \to \infty$. This sector is invariant under the U(1) group $\omega^a = \omega \, \hat{r}_a$ generated by $Q(\hat{r})$. The operator $Q(\hat{r})$ also takes integer values since it is included in the algebra of SO(3) and generates rotations around the $\hat{r}$ direction. With electric and magnetic charge defined now by the respective flux integrals,

$$q_e = \frac{1}{4\pi} \oint dS_i \, E_i^a \, \hat{r}_a$$

$$q_m = \frac{1}{4\pi} \oint dS_i \, B_i^a \, \hat{r}_a \tag{15}$$

one gets the Witten formula,

$$q_e - \beta q_m = \frac{ng\hbar}{4\pi} \quad . \tag{16}$$

In addition to the charge quantization realized in the formulae (13) and (16) for vacuum and monopole sectors, respectively, there is a further, strictly classical "quantization" of the magnetic flux. Indeed, for topological reasons the monopole sectors are characterized by

$$q_m = \frac{4\pi m}{g} \quad , \tag{17}$$

where m is an integer or half-integer. (The well-known Dirac quantization condition results on multiplying (16) – with $\beta = 0$ – by (17),

$$q_e q_m = \frac{ng\hbar}{4\pi} \, \frac{4\pi m}{g}$$

$$= \ell\hbar$$

where $\ell = nm$ is an integer or half-integer.)

The parameters q_e and q_m are "apparent charges" such as would be observed in static interactions at large distances. The departure of q_e from the familiar $ng\hbar$ does not imply any violation of charge conservation.

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