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# TOWARDS EXACT SOLUTIONS OF THE NON-LINEAR HEISENBERG-PAULI-WEYL SPINOR EQUATION *

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ABSTRACT

In "color geometrodynamics" fundamental spinor fields are assumed to obey a $GL(2f,\mathbb{C})\otimes GL(2c,\mathbb{C})$ gauge-invariant nonlinear spinor equation of the Heisenberg-Pauli-Weyl-type. Quark confinement, assimilating a scheme of Salam and Strathdee, is (partially) mediated by the tensor "gluons" of strong gravity. This hypothesis is incorporated into the model by considering the nonlinear Dirac equation in a curved space-time of hadronic dimensions. Disregarding internal degrees of freedom, it is then feasible, for a peculiar background space-time, to obtain exact solutions of the spherical bound-state problem. Finally, these solutions are tentatively interpreted as droplet-type solitons and remarks on their interrelation with Wheeler's geon construction are made.

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## I. INTRODUCTION

Recent speculations[1] on a new geometrodynamical model of extended particles draws attention on the possibility of describing composite baryons by a $G \equiv GL(2f,\mathbb{C})\otimes GL(2c,\mathbb{C})$ gauge-invariant nonlinear spinor equation in curved space-time. In accordance with the strong gravity hypothesis[2] the curving-up of the internal space is expected to occur in dimensions characterized by the modified Planck length

$$\ell^* = (8\pi\hbar G_s/c^3)^{1/2} = \sqrt{8\pi}\,\hbar/cM^* \approx 10^{-13}\,\mathrm{cm} \tag{1.1}$$

of the order of one Fermi.

The Poincaré-invariant gauge theory of gravity with spin and torsion analyzed by Hehl et al.[3] may be generalized[4] to one incorporating the flavor and color generating group $U(f)\otimes U(c)$. Then a nonlinear spinor equation of the Heisenberg-Pauli-Weyl-typ:[5,6]

$$\left\{ i L^\mu \nabla_\mu - \frac{3}{8}\ell^{*2}\,\bar{\psi} L^5 L_\mu \psi\, L^5 L^\mu - \mu c/\hbar \right\}\psi = 0 \tag{1.2}$$

emerges which is G-gauge invariant. Unlike Heisenberg's unified field theory[6,7] in "color geometrodynamics" (CGMD)[8] the $f\times c$ fundamental spinor fields

$$\psi \equiv \left\{ \psi^{(q_f,\,q_c)} \mid q_f = 1,\dots,f\ ;\ \ q_c = 1,\dots,c \right\} \tag{1.3}$$

may be interpreted as quark fields distinguished by f flavor and c color degrees of internal freedom.

In this gauge-theory[4] the matrices $L^\mu, L^5$ are space-time dependent generalizations of the familiar Dirac matrices $\gamma^\mu$

tensored with U(f)⊗U(c) vector operators $\lambda_j$ (generalized Gell-Mann matrices).

Whereas the nonlinearities induced into (1.2) by Cartan's torsion tensor[9] are suspected to yield (classical) bound-states of quarks, their (partial) confinement is conjectured[1,10] to result from curvature barriers produced by tensor "gluon" fields $f_{\mu\nu}$. This "role of [strong] gravitation in the building-up of elementary particles" has already been envisioned by Hermann Weyl[5] who also gave the prior construction of a SL(2,ℂ) gauge-invariant nonlinear spinor equation.

The fundamental Heisenberg-Pauli-Weyl spinor equation has already been the subject of considerable work, the internal U(f) ⊗ U(c) symmetry usually being dropped for simplicity. Then, in two space-time dimensions the renormalizable theory described by the massless universal field equation[6] of Heisenberg is commonly referred to as the Thirring model[11,12]. Interpreting $\psi$ as interacting quantum fields, Glaser[13] has obtained explicit solutions (See also Ref. 14). More recently, classical instanton- and meron-type solutions of (1.2) have also been found[15] in two dimensions.

In the real world it is notoriously difficult to obtain solutions even for the classical problem. Finkelstein et al.[16,17], Soler[18], and more recently Rañada[19] as well as Takahashi[20] have obtained radially localized solutions of a nonlinear Dirac equation similar to (1.2). However, their analysis rest upon numerical calculations and in addition to that, is



restricted to flat space-time. On the other hand, it is known[21,22] that a nonlinear scalar field theory having a dynamics which is related[23,24] to the squared form of (1.2) admits exactly solvable radial solutions in an Einstein universe.

In this paper, the freedom in the choice of the background space-time again is instrumental for the construction of exact radial solutions of a nonlinear spinor equation, having a compared to (1.2) algebraically simplified self-interaction.

The paper proceeds as follows:

In Section II the set-up of a generally covariant nonlinear spinor theory in curved space-time is reviewed. Emphasis is layed on the case of a spherically symmetric background. Section III deals with the spherical spinor formalism familiar from the bound-state problem of the hydrogen atom.[25,26] This formalism allows to reduce the spinor equation to a system of two nonlinear differental equations which are of first order with respect to the radial coordinate r. Employing a stationary Ansatz, exact spherical spinor waves are explicitly constructed in Section IV. A peculiar class of spherically symmetric space-times is specifically adjointed as to allow for solutions in closed form. Although unlikely, the possible physical significance of these exact solutions and connections with Wheeler's geon concept[27] are discussed in Section V.

## II. THE MODEL

Disregarding from now on internal degrees of freedom, the self-interacting spinor model of CGMD is given[4] by the Lagrangian density

$$\mathcal{L}_{D.-W.} = \sqrt{|f|}\,\Big\{ i\bar{\Psi} L^{\mu} \nabla_{\mu}\psi - \frac{3\epsilon}{16}\,\ell^2 (\bar{\psi} L^5 L^{\mu}\psi)^2 - (\mu c/\hbar)\,\bar{\Psi}\psi \Big\}, \quad \epsilon = \pm 1 \tag{2.1}$$

defined on a curved space-time with pseudo-Riemannian metric and connection. (The basic field equation (1.2) may be derived by varying (2.1) for $\delta \mathcal{L}_{D.-W.}/\delta\bar{\psi}$ if $\epsilon = +1$ and $\ell = \ell^*$.)

In terms of the familiar Dirac matrices $\gamma_\alpha$ (conventions are those of Bjorken and Drell[25]) satisfying

$$\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2\eta_{\alpha\beta}, \qquad \mathrm{diag}\,\eta_{\alpha\beta} = (1,-1,-1,-1) \tag{2.2}$$

space-time dependent matrices

$$L_\mu \equiv L_\mu{}^{\alpha}\gamma_\alpha \tag{2.3}$$

$$L^5 \equiv \frac{i}{4!}\frac{1}{\sqrt{|f|}}\,\varepsilon^{\alpha_1\cdots\alpha_4} L_{\alpha_1}\wedge\cdots\wedge L_{\alpha_4}, \quad L^5 L^5 = 1 \tag{2.4}$$

are introduced in terms of the vierbein field $L_\mu{}^{\alpha}$. These matrices are via

$$f_{\mu\nu} = \frac{1}{4}\,\mathrm{Tr}\,(L_\mu L_\nu) \tag{2.5}$$

related to the metric tensor $f_{\mu\nu}$ of the curved manifold (of hadronic dimensions).



The Dirac operator has been generalized to a curved space-time by using the SL(2,$\mathbb{C}$) gauge-covariant derivative[28]

$$\nabla_\mu \equiv \partial_\mu + i\,\Gamma_\mu, \tag{2.6}$$

Following Brill and Wheeler[29] the spinor connection $\Gamma_\mu$ can be explicitely expressed in terms of the symmetric Christoffel symbols $\Gamma^{\tau}{}_{\mu\nu} \equiv \frac{1}{2} f^{\tau\sigma}(\partial_\nu f_{\sigma\mu} + \partial_\mu f_{\sigma\nu} - \partial_\sigma f_{\mu\nu})$ (2.7) (metrical connection coefficients) as follows

$$\Gamma_\mu = \frac{1}{4}\left(\Gamma^{\tau}{}_{\mu\nu} L^{\nu\alpha} L_\tau{}^{\beta} - L_\nu{}^{\alpha}\,\partial_\mu L^{\nu\beta}\right)\sigma_{\alpha\beta}. \tag{2.8}$$

Here

$$\sigma_{\alpha\beta} \equiv \frac{i}{2}\,[\gamma_\alpha, \gamma_\beta] \tag{2.9}$$

denote the infinitesimal generators of the covering group SL(2,$\mathbb{C}$) of the Lorentz group.

According to well-known relations[17] between scalar products of bilinear forms containing <u>identical</u> spinors the self-interaction in (2.1), which is of the axial-vector-type, may be replaced by a scalar-minus-a-pseudoscalar-type self-coupling, i.e.,

$$(\bar{\Psi}\, iL^5 L_\mu \psi)(\bar{\Psi}\, iL^5 L^\mu\psi) = (\bar{\Psi} L_\mu\psi)(\bar{\Psi} L^\mu\psi) = (\bar{\Psi}\psi)^2 - (\bar{\Psi} L^5\psi)^2. \tag{2.10}$$

Therefore it is justified to a certain extent to consider instead of (1.2) the nonlinear Dirac equation

$$\Big\{ i L^\mu \nabla_\mu + \frac{3\epsilon}{8}\ell^2\,\bar{\Psi}\psi - \mu c/\hbar \Big\}\,\psi = 0 \tag{2.11}$$

having an algebraically simplified self-interaction. (The inclusion of the pseudoscalar term would lead to a more intricate model.)

Besides zero, equation (2.11) admits for $\epsilon = +1$ the constant solution

$$\psi_c = \begin{cases} \psi_c(q_f, q_c) = 0 & q = 1, \dots, f-1 \,; \\ \psi_c(f, q_c) = \frac{2}{\ell}\sqrt{\frac{2\mu c}{3\hbar}}\, e^{i\delta} \end{cases} . \qquad (2.12)$$

If the fermionic nature of the corresponding quantum fields could be disregarded, (2.12) would signal the occurrence of a spontaneous breaking of the internal $U(f)\otimes U(c)$ symmetry, similar as in Goldstone's model field theory (See, e.g. Taylor[30]).

Since radially localized "bound-state" configurations may (classically) describe extended particles it is consistent to base our search for solutions of (2.11) on a spherically symmetric background space-time. In the "isotropic" presentation the corresponding general line element reads

$$\begin{aligned} ds^2 &= f_{\mu\nu}\, dx^\mu dx^\nu \qquad (2.13) \\ &= e^{\nu} c^2 dt^2 - e^{\lambda}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2\right), \end{aligned}$$

where the functions $\nu = \nu(\varrho)$ and $\lambda = \lambda(\varrho)$ depend solely on the dimensionless radial coordinate

$$\varrho = [illegible]\, r = [illegible]\, r \,, \qquad (2.14)$$

The spherical coordinates $r = |\vec{x}|$, $\theta$, and $\varphi$ may be expressed in terms of the usual Cartesian coordinates $\vec{x}$.



In order to facilitate the following analysis it will be assumed that the background metric $f_{\mu\nu}$ is conformally related to another metric via

$$f_{\mu\nu} = e^{\lambda}\, \bar{f}_{\mu\nu} \,. \qquad (2.15)$$

Obviously, the corresponding "Vierbeine" then can be conformally related in a similar way:

$$L_\mu{}^\alpha = e^{\lambda/2}\, \bar{L}_\mu{}^\alpha \qquad (2.16)$$

With respect to the conformal change (2.15) the (symmetric!) Christoffel symbols are related via

$$\Gamma^\tau{}_{\mu\nu} = \bar{\Gamma}^\tau{}_{\mu\nu} + \frac{1}{2}\left(\delta^\tau_\mu\, \partial_\nu \lambda + \delta^\tau_\nu\, \partial_\mu \lambda - f_{\mu\nu}\, \partial^\tau \lambda\right). \qquad (2.17)$$

(See, e.g., Ref. 31, Appendix(A.2.)). Employing also (2.16) a short calculation reveals that

$$\Gamma_\mu = \bar{\Gamma}_\mu + \frac{1}{8}\left(\delta^\beta_\mu\, \partial^\alpha \lambda - \delta^\alpha_\mu\, \partial^\beta \lambda\right)\sigma_{\alpha\beta} . \qquad (2.18)$$

Thus, as a by-product, it has been shown that the nonlinear Dirac equation (2.11) on conformally related pseudo-Riemannian manifolds takes the form

$$\begin{aligned} \Big\{ i \bar{L}^\mu \big(\partial_\mu &+ i \bar{\Gamma}_\mu + \tfrac{3}{4}\,\partial_\mu \lambda\big) \\ &+ \frac{3\epsilon}{8}\,\ell^2 e^{\lambda/2}\, \bar{\psi}\psi - e^{\lambda/2}\mu c/\hbar \Big\}\psi = 0 \,. \end{aligned} \qquad (2.19)$$

In order to proceed further, the spinor connection $\overline{\Gamma}_\mu$ in terms of a metric $\overline{f}_{\mu\nu}$ which is via (2.15) conformally related to the background (2.13) has to be ascertained. The comparison with the results (30) of Ref. 29 obtained for a similar case shows that

$$\overline{\Gamma}_0 = \frac{i}{4} e^{(\nu-\lambda)/2} \partial_r (\nu-\lambda) \gamma_0 \gamma_1 ,$$

$$\overline{\Gamma}_1 = 0 , \quad \overline{\Gamma}_2 = \frac{i}{2} \gamma_2 \gamma_1 ,$$

$$\overline{\Gamma}_3 = \frac{i}{2} ( \sin\Theta\, \gamma_3 \gamma_1 + \cos\Theta\, \gamma_3 \gamma_1 ). \tag{2.20}$$

The insertion of these expressions into (2.19) finally leads to

$$\Big\{ i \gamma^0 \partial_0 - e^{(\nu-\lambda)/2} \Big[ i \vec{\gamma}\cdot\vec{\partial} + i \gamma^r \partial_r \Big( \frac{\lambda}{2} + \frac{\nu}{4} \Big) \Big] + \frac{3\epsilon}{8} \ell^2 e^{\nu/2} \overline{\psi}\psi - e^{\nu/2} \mu c/\hbar \Big\} \psi = 0 . \tag{2.21}$$

For later convenience, the occuring spatially flat Dirac operator $i\vec{\gamma}\cdot\vec{\partial}$ has been reexpressed in terms of Cartesian coordinates.



## III. SEPARATION ANSATZ FOR SPHERICAL WAVES

It may be noticed that the curved background occurs in the "conformal" spinor equation (2.21) only in a multiplicative manner, except for the third term. However, this term may be absorbed by the factor $\exp(-\lambda/2 - \nu/4)$ in the following Ansatz:

$$\psi = \frac{4}{\ell} \sqrt{\frac{2\pi \mu c}{3\hbar}}\, e^{-\frac{\lambda}{2} - \frac{\nu}{4}}\, e^{-i\omega t \mu c^2/\hbar} \times \begin{bmatrix} i\, G(\varsigma)\, \chi_\varkappa^m \\ F(\varsigma)\, \frac{\vec{\sigma}\cdot\vec{x}}{|\vec{x}|}\, \chi_\varkappa^m \end{bmatrix} \tag{3.1}$$

Following essentially the notation of Rose[26], the spin-weighted, spherical harmonics $\chi_\varkappa^m$ of parity $P = (-1)^\ell$ are defined by

$$\chi_\varkappa^m \equiv \sum_{\overline{m}=\pm\frac{1}{2}} C(\ell \tfrac{1}{2} j ; m-\overline{m}, \overline{m})\, Y_\ell^{m-\overline{m}}(\Theta,\varphi)\, \chi^{\overline{m}} . \tag{3.2}$$

Here

$$\varkappa \equiv \mp (j + \tfrac{1}{2}) \quad \text{for } j = \ell \pm \tfrac{1}{2} , \tag{3.3}$$

and $C(\ell \frac{1}{2} j ; m-\overline{m}, \overline{m})$ are Clebsch-Gordan-coefficients. These spherical 2-spinors are known to satisfy the eigenvalue equations

$$\vec{J}^2 \chi_\varkappa^m = j(j+1) \chi_\varkappa^m , \tag{3.4}$$

$$\vec{\sigma}\cdot\vec{L}\, \chi_\varkappa^m = -(\varkappa+1)\, \chi_\varkappa^m , \tag{3.5}$$

for the operators $\vec{J}$ and $\vec{L}$ of total and orbital angular momentum and also

$$\frac{\vec{\sigma}\cdot\vec{x}}{|\vec{x}|}\ \chi^m_{\varkappa} = -\ \chi^m_{-\varkappa}\ . \tag{3.6}$$

Moreover, it can be shown[26] that

$$i\gamma^0\vec{\gamma}\cdot\vec{\partial}\begin{bmatrix} i\ G(\varrho)\ \chi^m_{\varkappa} \\ -\ F(\varrho)\ \chi^m_{-\varkappa}\end{bmatrix} = \frac{\vec{\sigma}\cdot\vec{x}}{|\vec{x}|}\left(\partial_r - \frac{1}{r}\ \vec{\sigma}\cdot\vec{L}\right)\begin{bmatrix} i F(\varrho)\ \chi^m_{-\varkappa} \\ G(\varrho)\ \chi^m_{\varkappa}\end{bmatrix}\ . \tag{3.7}$$

With respect to the Ansatz (3.1) the self-coupling term in (2.21) takes the form[17]

$$\frac{3}{8}\ell^2\bar{\psi}\psi = 4\pi\frac{\mu c}{\hbar}\ e^{-\lambda-\frac{\nu}{2}}\left(G^2 - F^2\right)\left|Y^{|\varkappa|-1}_{|\varkappa|-1}(\Theta,\varphi)\right|^2 . \tag{3.8}$$

This self-interaction potential has to be spherically symmetric in order to ensure separability. This is the case for $|\varkappa| = 1$, only. Then, the quantum numbers for the spin and angular momentum of admissible solutions are restricted to $j = 1/2$, $\ell = 0, 1$, and $m = \pm 1/2$.

With all this information at hand the insertion of (3.1) into (2.21) yields

$$\partial_{\varrho^*} G + \frac{1+\varkappa}{\varrho}\ e^{(\nu-\lambda)/2}\ G = \frac{1}{\beta}\left[\omega + e^{\nu/2} - \epsilon\ e^{-\lambda}(G^2 - F^2)\right] F\ , \tag{3.9}$$

$$\partial_{\varrho^*} F + \frac{1-\varkappa}{\varrho}\ e^{(\nu-\lambda)/2}\ F = \frac{1}{\beta}\left[-\ \omega + e^{\nu/2} - \epsilon\ e^{-\lambda}(G^2 - F^2)\right] G \tag{3.10}$$

for the remaining <u>radial equations</u>. It is convenient, but not really necessary in our case, to employ Wheeler's "tortoise coordinate"[27] $\varrho^*$ defined by the differential form

$$d\varrho^* = e^{(\lambda-\nu)/2}\ d\varrho \tag{3.11}$$

in the above equations. In (3.9) and (3.10) the ratio

$$\beta = \frac{M^*}{2\mu} \tag{3.12}$$

measures how much the "bare mass" $\mu$ of a fermion contributes to the (strong) gravitational mass $M^*$.

The nonlinear term $\sim(G^2 - F^2)$ suggest to utilize the Ansätze

$$G(\varrho) = e^{\vartheta(\varrho)}\,\mathrm{ch}\,\phi(\varrho)\ , \tag{3.13}$$

$$F(\varrho) = e^{\vartheta(\varrho)}\,\mathrm{sh}\,\phi(\varrho) \tag{3.14}$$

in order to achieve a simplification of the equations. (Similar Ansätze have already been used by Finkelstein et al.[17] and van der Merve[32] in their search for asymptotic solutions at spatial infinity.) Solving then for the first derivatives yields:

$$\partial_\varrho \phi = \frac{\varkappa}{\varrho} \operatorname{sh} 2\phi + \frac{1}{\beta}\left( e^{\nu/2} - \epsilon\, e^{2\vartheta-\lambda} - \omega \operatorname{ch} 2\phi\right) e^{(\lambda-\nu)/2}, \qquad (3.15)$$

and

$$\partial_\varrho \vartheta = \frac{\omega}{\beta}\, e^{(\lambda-\nu)/2} \operatorname{sh} 2\phi - \frac{1}{\varrho}\left(1 + \varkappa \operatorname{ch} 2\phi\right) . \qquad (3.16)$$



IV. POSSIBLE EXACT SOLUTION

As the resulting radial equations (3.15) and (3.16) form an underdetermined system, the freedom in the choice of the background space-time may be used to absorb the non-linear contribution $e^{2\vartheta}$ in equation (3.15). In particular, the assumption

$$e^{\nu/2} - \epsilon\, e^{2\vartheta-\lambda} = \omega \operatorname{ch} 2\phi \qquad (4.1)$$

will cause equation (3.15) and (3.16) to separate. Then, (3.15) reduces to

$$\frac{d\phi}{\operatorname{sh} 2\phi} = \varkappa \frac{d\varrho}{\varrho} . \qquad (4.2)$$

The left-hand side can be integrated by employing formula 2.423.1 of Ref. 33 :

$$\int \frac{dz}{\operatorname{sh} z} = \ln \operatorname{th} \frac{z}{2} . \qquad (4.3)$$

After solving (4.2) this leads to the initial information that

$$\operatorname{ch} \phi = \left(1 - \Lambda^2 \varrho^{4\varkappa}\right)^{-1/2}, \qquad (4.4)$$

$$\operatorname{sh} \phi = \Lambda \varrho^{2\varkappa} \operatorname{ch} \phi , \qquad (4.5)$$

where $\Lambda$ denotes an integration constant.

To proceed further, the differential equation

$$\frac{d\vartheta}{d\phi} = \frac{\omega}{\beta\varkappa}\, \varrho\, e^{(\lambda-\nu)/2} - \frac{1}{\varkappa \operatorname{sh} 2\phi} - \operatorname{cth} 2\phi \qquad (4.6)$$

expressing the implicit dependence of $\vartheta(\rho)$ on $\phi(\rho)$, in view of condition (4.1) can now be deduced from (3.16) and (3.15).

Under the further assumption, that

$$e^{(\nu-\lambda)/2} = \rho \; th\, 2\phi \equiv \frac{2\rho \, sh\phi \, ch\phi}{sh^2\phi + ch^2\phi} \tag{4.7}$$

holds, equation (4.6) can be easily integrated with the aid of (4.3) and

$$\int \frac{ch z}{sh z} \, dz = \ln sh z \tag{4.8}$$

(formula 2.423.33 of Ref. 33). The result can be expressed in terms of the already known functions (4.4) and (4.5) as

$$e^{\vartheta} = \frac{C}{2\Lambda}\left(\frac{1}{\Lambda} \, sh\,\phi\right)^{\frac{\omega-\beta}{2\beta\varkappa} - \frac{1}{2}} (ch\,\phi)^{\frac{\omega+\beta}{2\beta\varkappa} - \frac{1}{2}} , \tag{4.9}$$

where C is a second integration constant.

The remaining task is to determine the metric functions $e^{\nu}$ and $e^{\lambda}$. The substitution of (4.7) into the other subsidiary condition (4.1) leads merely to an algebraic equation

$$y^3 + 3py + 2q = 0 \tag{4.10}$$

for

$$y \equiv e^{\lambda/2} - \sqrt{|p|} \; . \tag{4.11}$$



With respect to the parameters

$$p = -\left[\frac{\omega\,(sh^2\phi + ch^2\phi)^2}{6\rho \; sh\phi \, ch\phi}\right]^2 \le 0 \; , \tag{4.12}$$

and

$$q = -|p|^{3/2} - \frac{\epsilon \, e^{2\vartheta}}{2\rho \; th\, 2\phi} \tag{4.13}$$

it takes the normal form (4.10) of a cubic equation. If $p < 0$, (4.10) has one or three reel roots depending whether or not the discriminant

$$\begin{aligned} D &\equiv q^2 + p^3 \\ &= \left(\frac{e^{\vartheta}}{\rho \; th 2\phi}\right)^2 \left[\frac{1}{4} e^{2\vartheta} + \frac{\epsilon}{3}|p|\,\omega \, ch\, 2\phi\right] \end{aligned} \tag{4.14}$$

is positive or not (See Ref. 34, § 59).

By introducing the complex variable

$$\alpha \equiv \frac{1}{3} \, arch\left(|q|\,|p|^{-3/2}\right) \tag{4.15}$$

$$= \frac{1}{3} \, arch\left(\left|1 - 2\epsilon \frac{3^3 \, e^{2\vartheta} \rho^2 \, sh^2\phi \, ch^2\phi}{\omega^3 \, (sh^2\phi + ch^2\phi)^5}\right|\right)$$

$$= \frac{1}{3} \, arch\left(\left| 1 - \frac{\epsilon \, 3^3 C^2}{2\,\omega^3} \rho^{2\varkappa + 2\omega/\beta} \times (1 + \Lambda^2 \rho^{4\varkappa})^{-5} (1 - \Lambda^2 \rho^{4\varkappa})^{(4\beta\varkappa - \omega)/\beta\varkappa}\right|\right)$$

with respect to the principal value of the inverse hyperbolic function, it can be inferred from the trigonometric identity

$$4\,\mathrm{ch}^3\alpha = 3\,\mathrm{ch}\,\alpha + \mathrm{ch}\,3\alpha \tag{4.16}$$

(Ref. 33, formula 1.324.2) that

$$y_1 = -2\,\mathrm{sgn}(q)\sqrt{|p|}\begin{cases}\cos\alpha & \text{for } D\leq 0, \text{ or}\\ \mathrm{ch}\,\alpha & \text{for } D>0\end{cases} \tag{4.17}$$

is a solution of (4.11) for $p<0$. Independent of the sign of D this solution is always real and therefore acceptable for the components of a (pseudo-) Riemannian metric.

Using this root in (4.11) and employing also (4.12) and (4.13) yields (below written down for $D>0$ only)

$$e^{\lambda/2} = \frac{\omega(\mathrm{sh}^2\phi+\mathrm{ch}^2\phi)^2}{6\varrho\,\mathrm{sh}\phi\,\mathrm{ch}\phi}\left[1-2\,\mathrm{sgn}(q)\,\mathrm{ch}\,\alpha\right]$$

$$= \frac{\omega}{6\Lambda}\,\frac{(1+\Lambda^2\varrho^{4\varkappa})^2}{\varrho^{2\varkappa+1}(1-\Lambda^2\varrho^{4\varkappa})}\left[1-2\,\mathrm{sgn}(q)\,\mathrm{ch}\,\alpha\right] \tag{4.18}$$

as a result for one of the metric functions, whereas $e^{\nu/2}$ may now be derived from (4.7).

Since $F=\Lambda\varrho^{2\varkappa}G$, it is sufficient to record the radial dependence of the upper spinor component of the Ansatz (3.1). After employing (4.4), (4.5), (4.9), (3.13), (4.18), and (4.7) its explicit expression reads:



$$\begin{aligned}\tilde G \equiv G\,e^{-\lambda/2-\nu/2} = {} & C\left(\frac{3}{\omega}\right)^{3/2}\varrho^{\varkappa+\omega/\beta}\\ & \times(1+\Lambda^2\varrho^{4\varkappa})^{-5/2}(1-\Lambda^2\varrho^{4\varkappa})^{(3\beta\varkappa-\omega)/2\beta\varkappa}\\ & \times\Big[1-2\,\mathrm{sgn}(q)\,\mathrm{ch}\Big\{\frac{1}{3}\,\mathrm{arch}\Big(\Big|1-\frac{\epsilon\,3^3C^2}{2\,\omega^3}\,\varrho^{2\varkappa+2\omega/\beta}\\ & \quad\times(1+\Lambda^2\varrho^{4\varkappa})^{-5}(1-\Lambda^2\varrho^{4\varkappa})^{(4\beta\varkappa-\omega)/\beta\varkappa}\Big|\Big)\Big\}\Big]^{-3/2}\end{aligned} \tag{4.19}$$

**A time-independent solution can be obtained from the Ansatz (3.1) by putting $\omega=0$. This corresponds to the case $p=0$ which, for a cubic equation, has to be treated separately.**
**However, then (4.10) admits the obvious solution**

$$e^{\lambda/2} = \left(\frac{\epsilon\,e^{2\vartheta}}{\varrho\,\mathrm{th}\,2\phi}\right)^{1/3} = \frac{C^{2/3}}{2\Lambda\varrho^{1+4\varkappa/3}}\left[\frac{1+\Lambda^2\varrho^{4\varkappa}}{\epsilon(1-\Lambda^2\varrho^{4\varkappa})}\right]^{1/3} \tag{4.20}$$

**In the deduction of the second part of this relation use has been made of the fact that the expression (4.9) is still valid for $\omega=0$. The radial dependence of the upper spinor component turns out to be comparatively simple:**

$$\tilde G = \left[\epsilon(1-\Lambda^2\varrho^{4\varkappa})\right]^{-1/2} \tag{4.21}$$

**For $\varrho<\varrho_0$ $(\equiv\Lambda^{-1/2\varkappa})$ the metric function $e^{\lambda}$ is real as required, whereas the spinor solutions are real for $\varkappa=\epsilon=\pm 1$, only. In the domain $\varrho>\varrho_0$ both spinor components $\hat G$ and $\hat F$ become imaginary. This, in effect, amounts to a change of the sign of the self-interaction of our model (2.1). Therefore, only the interior (or exterior) solution ( their domains being separated by a pole of (4.21) and (4.20) at $\varrho_0$) would commonly be acceptable for $\epsilon=+1$ (or $\epsilon=-1$).**

Some insights into the mathematical structure of these solutions may be gained by studying their asymptotic behaviour.
From the explicit expressions (4.15) it can be easily deduced that the term

$$|q||p|^{-3/2} - 1 \sim \begin{cases} \rho^{2\varkappa + 2\omega/\beta} & \text{for } \rho \to 0, \\ \rho^{-(2\varkappa + 2\omega/\beta)} & \text{for } \rho \to \infty \end{cases} \tag{5.1}$$

tends to zero with the same degree at the origin and at infinity.
Therefore, the radial solution (4.19) of the stationary case behaves as

$$\tilde{G} \sim \begin{cases} \rho^{\varkappa + \omega/\beta}\,[1 - 2\,\mathrm{sgn}(q)]^{-3/2} & \text{for } \rho \to 0, \\ \rho^{-(3\varkappa + \omega/\beta)}\,[1 - 2\,\mathrm{sgn}(q)]^{-3/2} & \text{for } \rho \to \infty . \end{cases} \tag{5.2}$$

For $q < 0$ these asymptotic expressions are reel. For the lower spinor component $\tilde{F} = \Lambda \rho^{2\varkappa} \tilde{G}$ to be asymptotically vanishing, it has to be required that $\omega/\beta > -\varkappa$.
Furthermore, only in the special case#

$$\omega = 3\beta\varkappa \tag{5.3}$$

(4.19) and $\tilde{F}$ would constitute a real, completely regular and localized solution, if the underlying space-time manifold were flat.

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\# A deviation of the effective mass $\omega\mu$ appearing in the phase of the Ansatz (3.1) from the "bare mass" $\mu$ of the fundamental spinors may be attributed to a possible "Archimedes effect"[10] of the hadronic environment. In CGMD the latter may originate from nonlinear self-interactions and the "gluonic" curvature barriers of strong gravity.



After all, it follows from (4.7) that the line element (2.13) is via

$$ds^2 = e^{\lambda}\, ds_0^{\,2} \tag{5.4}$$

conformally[31] related to the nonsingular line element

$$ds_0^{\,2} = \frac{4\Lambda^2 \rho^{4\varkappa+2}}{(1+\Lambda^2\rho^{4\varkappa})^2}\, c^2 dt^2 - \frac{\ell^{*2}}{2\pi}\left(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta\, d\varphi^2\right) \tag{5.5}$$

which is degenerate at the origin (for every $\varkappa$).
However, the conformal function $e^{\lambda}$ given by (4.18) in the stationary case and by (4.20) for a time-independent solution is singular at the origin, at $\rho = \rho_0$, as well as at infinity. This fact could spoil altogether the regularity of the solution (4.19) in the case (5.3) if everything is expressed in a different coordinate system. For these (and other) reasons the classical field energy[29] associated with these solutions may turn out not to be finite.

In the time-independent case, the solution (4.21) and a similar expression for the lower spinor component are valid for a finite radius $\rho < \rho_0$ for $\epsilon = +1$. However, the obtained background geometries given by (5.4) and (4.20) likewise are of finite extent and, therefore, may remotely resemble the de Sitter spaces[35], which have been recently considered for confining models[10].

Nevertheless, the physical meaning of the obtained exact solutions so far remains rather obscure. The direction of future work may be indicated by the following related, but speculative remarks:

For a massive nonlinear spinor theory with a peculiar self-interaction of polynomial degree $k$, $0 < k < 1$, Werle[36] found exact radial solutions in flat space-time which are also confined to the interior of sphere. By continuing with $\psi \equiv 0$ outside this sphere, these solutions are claimed to have many features of droplets or "bags"[37]. In the case of the massive Thirring model exact solutions [illegible] the properties of bound states have been constructed by Chang et al.[38] and studied as an example of chiral confinement.

Furthermore, the construction scheme for our exact solution should be also commented upon. Recall, that the method of Section IV would not work in flat space-time. The freedom in the choice of a spherically symmetric background is essential for the method presented in this paper. However, in a self-consistent approach the metric functions $\nu(\rho)$ and $\lambda(\rho)$ would have to be determined by the stress-energy content (see Section 10 of the Brill and Wheeler paper[29]) of the spinor solution via Einstein's field equations. This would lead to the construction of nonlinear spinor geons in CGMD.

Moreover, the internal symmetries inherent in the G-gauge-invariant equation (1.2) should be properly dealt with. A possible approach in this direction has been undertaken by Takahashi[39] who obtained numerically non-topological soliton solutions with vanishing (total) "color" charge. If also colored spinor solutions occur, again Wheeler's geon construction[27] may become important. The reason beeing that, according to recent speculations[8], color may become "transcendent"[40] in black hole-type geons. Expressed in a Wheeler-type phrase, there may be "color without color" in CGMD.

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