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ON PSEUDOPARTICLE SOLUTIONS IN YANG'S THEORY OF GRAVITY *

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ABSTRACT

Within the framework of differential geometry, Yang's parallel-displacement gauge theory is considered with respect to "pure" gravitational fields. In a four-dimensional Riemannian manifold it is shown that the double self-dual solutions obey Einstein's vacuum equations with cosmological term, whereas the double anti-self-dual configurations satisfy the Rainich conditions of Wheeler's geometrodynamics. Conformal methods reveal that the gravitational analogue of the "instanton" or pseudoparticle solution of Yang-Mills theory was already known to Riemann.

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I. INTRODUCTION

It is now generally believed that Yang-Mills type [55] gauge field theories eventually may provide a true unification of all fundamental interactions. Such a unification should include gravitation [34]. However, gravity is different in several aspects. First of all, it is a parallel-displacement gauge theory [54], which means in the language of fibre bundle theory that the bundle of linear frames with the non-compact structure group SL(4,R) is soldered [46] to the base manifold. By introducing a Riemannian metric the symmetry group becomes broken down to the Lorentz group SO(1,3) as a gauge group, a procedure which resembles the Higgs mechanism. Furthermore, the dynamics of Einstein's standard theory of gravity given by the familiar Hilbert action is not modeled after Maxwell's theory of electromagnetism. Therefore, Herman Weyl, who introduced the concept of gauge invariance in physics, argued at the end of his 1919 paper [47] that the most natural gravitational Lagrangian should be quadratic in Riemann's curvature tensor. Later on, this point of view has been taken up and developed further by e.g. Lanczos [29], Stephenson[45] and Yang [54].

Further justifications for this alternative choice come from quantum gravity [12]. There it is known from one-loop calculations that quadratic curvature terms must be added to the conventional Einstein-Hilbert action as counter terms. Moreover, if a Yang-Mills type action [15] is incorporated immediately from the beginning, the model is known [44] to be renormalizable, but suffers from a tensor ghost. It seems plausible that under certain conditions the Froissart unitarity boundedness may still be guaranteed [42]. By admitting also propagating torsion (compare with Hehl et al. [24]) a new class of ghost-free gravity Lagrangians have recently been found [43]. After Euclideanizing space-time and by restricting the field configurations of the SU(2) Yang-Mills theory [55] to be self-dual, exact solutions have been obtained by Belavin et al. [5], which are non-singular for imaginary time. These "instanton" - or pseudoparticle solutions have received much attention because of their significance for a non-perturbative treatment of quantum field theory [25]. In particular, they can be interpreted as evidence for a quantum-mechanical tunnelling between topologically different vacuum sectors. The Riemannian analogues of these pseudoparticle solutions should play a similarly important role for a non-perturbative method (still in need of invention) of quantizing gravity.

According to the Cambridge school [18] gravitational instantons are defined by non-singular, complete and positive-definite metrices which satisfy the vacuum Einstein equations with or without a cosmological term. Solutions having also self- or anti-self-dual curvature are distinguished among asymptotically locally Euclidean (ALE) metrics. If the "generalized positive action conjecture" [22,19] holds, these are the only solutions having zero Einstein-Hilbert action.

Since Yang's parallel displacement gauge theory [54] of gravity is formally much more related to Yang-Mills theories than Einstein's, in the former case gravitational pseudoparticles may be defined by self- or anti-self-dual field configurations. In Yang's theory however, the notion of double duality of Riemann's curvature tensor, discussed in Sec.II, turns out to be more appropriate. The reason for this deviating approach can be traced back to the special isomorphism $so(4) \approx su(2) \otimes su(2)$ of the Lie group of the $O(4)$ symmetry group in four dimensions [20]. By doing so, the search for gravitational instantons in Yang's theory without sources [15] is converted into a problem of classical differential geometry.

The main result obtained in Sec.III is that non-singular solutions of Einstein's theory are double self-dual and can therefore be rightly regarded as "instantons" in Yang's theory. In particular, the "Euclideanized" de Sitter cosmologies turn out to be the Riemannian analogues of the pseudo-particle solutions in Yang-Mills theory.

Sec.IV deals with the coupled Einstein-Yang-Mills system. For the Abelian case, it is known that the physical context of the resulting Einstein Maxwell theory can be completely coded in the geometry of a Riemannian manifold, provided that it satisfies the Rainich conditions [39] of geometrodynamics [35]. It turns out that double anti-self-dual solutions of Yang's equation fulfill just these conditions. Since these metrices are conformally flat, they constitute only a restricted class of Rainich geometries. Some concluding remarks on the main topological invariants of Riemannian manifolds i.e. on the Euler number and the Pontryagin index, are made in Sec.V.

II. FOUR-DIMENSIONAL RIEMANNIAN GEOMETRY

In this section a few important properties of the Riemannian curvature tensor $R^{\mu\nu}{}_{\rho\sigma}$ are recapitulated. All occurring geometrical objects are globally defined on an oriented, four-dimensional manifold M^4 equipped with a positive definite metric $g_{\mu\nu}$, but, for reasons of familiarity, are here regarded as local expressions of the corresponding differential forms [34] (MTW).

The double dual of $R^{\mu\nu}{}_{\rho\sigma}$ is defined as follows [34] (MTW, p.325)

$$ {}^*R^{*\alpha\beta}{}_{\gamma\delta} \equiv \frac{1}{4}\epsilon^{\alpha\beta\mu\nu} R_{\mu\nu}{}^{\rho\sigma} \epsilon_{\rho\sigma\gamma\delta} \quad . \tag{2.1}$$

The Riemannian curvature is known to satisfy identically the second Bianchi identity which may be given the more concise form [29]

$$ \nabla^{\delta}\, {}^*R^*{}_{\alpha\beta\gamma\delta} = 0 \quad , \tag{2.2}$$

In the theory of generalized curvature fields (see Ref.31 for review) tensors satisfying (2.2) are called to be proper.

It follows from (2.2) by consecutive contractions that

$$ \nabla^{\mu} R_{\mu\nu\rho\sigma} = \nabla_{\rho} R_{\nu\sigma} - \nabla_{\sigma} R_{\nu\rho} \tag{2.3}$$

and

$$ \nabla^{\mu} G_{\mu\nu} = 0 \quad , \tag{2.4}$$

where

$$ G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = - {}^*R^{*\alpha}{}_{\mu\alpha\nu} \tag{2.5}$$

is the Einstein tensor. *)

The following analysis will be facilitated upon the introduction of an orthogonal decomposition of curvature tensors in a four-dimensional Riemannian manifold **)

$$ R^{\mu\nu}{}_{\rho\sigma} = C^{\mu\nu}{}_{\rho\sigma} \oplus R^{I\mu\nu}{}_{\rho\sigma} \oplus R^{II\mu\nu}{}_{\rho\sigma} \quad , \tag{2.6}$$

where $C^{\mu\nu}{}_{\rho\sigma}$ denotes the trace-free conformal Weyl tensor. From elementary relations between permutation symbols [34] (MTW, p.87) it can be deduced that

$$ \begin{aligned} R^{I\mu\nu}{}_{\rho q} &\equiv K(\delta^{\mu}{}_{\rho}\,\delta^{\nu}{}_{\sigma} - \delta^{\mu}{}_{\sigma}\,\delta^{\nu}{}_{\rho}) \\ &= \frac{K}{2}\epsilon^{\mu\nu\alpha\beta}\epsilon_{\rho\sigma\alpha\beta} \end{aligned} \tag{2.7}$$

*) The sign conventions for the Riemann- and Ricci tensor are the same as in MTW. Due to the positive signature of the metric, the permutation symbols pick up opposite signs.

**) See, e.g. Sec.4 of Ref.31 for generalizations to arbitrary dimensions.

defines a double self-dual curvature tensor. In the case that $R^{I\mu\nu}{}_{\rho\sigma}$ is proper, i.e. satisfies (2.2), its sectional curvature

$$K = \frac{R}{12} \quad , \tag{2.8}$$

would be constant. The remaining tensor

$$R^{II\mu\nu}{}_{\rho\sigma} \equiv \frac{1}{2}(R^{\mu}{}_{\rho}\,\delta^{\nu}{}_{\sigma} - R^{\mu}{}_{\sigma}\,\delta^{\nu}{}_{\rho} + R^{\nu}{}_{\sigma}\,\delta^{\mu}{}_{\rho} - R^{\nu}{}_{\rho}\,\delta^{\mu}{}_{\sigma}) - 3R^{I\mu\nu}{}_{\rho\sigma} \tag{2.9}$$

of the decomposition (2.6) spans the orthogonal complement of the Weyl tensor $C^{\mu\nu}{}_{\rho\sigma}$ in the space of curvature tensors with vanishing scalar curvature. According to a remarkable identity of Lanczos [28] valid in four dimensions only, precisely this tensor measures the deviation of Riemann's curvature from being double self-dual *)

$$R_{\mu\nu\rho\sigma} - {}^*R^*{}_{\mu\nu\rho\sigma} = 2R^{II}{}_{\mu\nu\rho\sigma} \quad . \tag{2.10}$$

An equivalent decomposition has been employed by Atiyah et al. [1] in a profound recent analysis of the geometry of gauge theories. According to theirs, in four dimensions the conformal Weyl tensor has the further invariant splitting

$$C = C_+ \oplus C_- \quad , \tag{2.11}$$

$$C^{\mu\nu}_{\pm\,\rho\sigma} \equiv \frac{1}{2}(C^{\mu\nu}{}_{\rho\sigma} \pm {}^*C^{*\mu\nu}{}_{\rho\sigma})$$

under the operation of taking the double dual. It follows as an immediate consequence of (2.6) and (2.10) that the condition

$$C^{\mu\nu}{}_{\rho\sigma} = C^{\mu\nu}_{+\rho\sigma}\ , \text{ i.e. } C_- = 0 \tag{2.12}$$

may serve as a definition [1] for "self-dual" Riemannian manifolds.

III. GRAVITATIONAL INSTANTONS IN YANG'S PARALLEL-DISPLACEMENT GAUGE THEORY

Consider the parallel-displacement gauge theory proposed by Yang [54] as an alternative theory of gravity. The Lagrangian density is chosen to be Yang-Mills:

*) The relation (2.5) may now be derived simply by contracting (2.10).

$$\mathcal{L}_Y \equiv \frac{\hbar c}{4\ell^{*2}}\sqrt{g}\; R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$
$$= \frac{\hbar c}{4\ell^{*2}}\sqrt{g}\left\{R_{\mu\nu\rho\sigma}{}^*R^{*\mu\nu\rho\sigma} + 4R_{\mu\nu}R^{\mu\nu} - R^2\right\}. \tag{3.1}$$

The equivalent expression given above can be readily deduced by contracting (2.10) with $R^{\mu\nu\rho\sigma}$ as had already been noted by Lanczos [28]. For later convenience, the (strong) [3] gravitational coupling constant G has been absorbed into the modified Planck length $\ell^* \equiv (8\pi\hbar G/c^3)^{1/2}$. According to the Palatini method the Euler-Lagrange equations may be derived by (a priori) independent variations of (3.1) for the (affine) connection and for the metric :

Varying for $\delta\mathcal{L}_Y/\delta\Gamma^{\nu\rho\sigma}$ yields

$$\nabla^{\mu} R_{\mu\nu\rho\sigma} = 0 \tag{3.2}$$

after specializing [15] to the Riemann-Christoffel connection $\Gamma_{\nu\rho}{}^{\sigma}$. Due to (2.3), Yang's equation (3.2) is said to describe a "pure" gravitational field, i.e. one for which the Codazzi equation *)

$$\nabla_{\rho} R_{\nu\rho} = \nabla_{\sigma} R_{\nu\rho} \tag{3.3}$$

holds.

Varying for $\delta\mathcal{L}_Y/\delta g^{\mu\kappa}$, results in the condition

$$H_{\mu\kappa} \equiv R_{\mu\nu\rho\sigma}R_{\kappa}{}^{\nu\rho\sigma} - {}^*R^*{}_{\mu\nu\rho\sigma}{}^*R^*{}_{\kappa}{}^{\nu\rho\sigma} = 0 \tag{3.4}$$

for the corresponding gravitational stress-energy tensor. The relations (3.4), not noted by Yang, have in an equivalent form been derived by Stephenson [45]. The superspace **) of all Riemannian metrics $g \in \mathcal{M}^*_Y$ (modulo diffeomorphisms $\mathfrak{D}$) which satisfy the field equations (3.2) and (3.4) of Yang's theory of gravity will be denoted by $\mathcal{S}_Y \equiv \mathcal{M}^*_Y/\mathfrak{D}^*$.

*) In this context (3.3) is not to be regarded as an embedding equation.

**) The geometrodynamical notion of "superspace" (see also MTW, Chapter 43) has a completely different meaning than that used in supergravity.

In mathematical physics it is commonly desirable to reduce (3.2), which are of third order with respect to the metric, to equations of second order. In view of the Bianchi identity (2.2) it is suggested here to solve both field equations (3.2) and (3.4) by imposing the <u>necessary</u> conditions

$$R_{\mu\nu\rho\sigma} = \pm {}^*R^*{}_{\mu\nu\rho\sigma} \qquad (3.5)$$

Although the superspace ${}^*\mathcal{S}^*_\pm = {}^*\mathcal{M}^*_\pm/\mathcal{D}^4 \subseteq \mathcal{S}_Y$ of solutions of (3.5) will likely be only a <u>true</u> subspace of $\mathcal{S}_Y$, the former turns out to be of considerable geometrical significance by itself:

If the plus sign holds, a contraction reveals that the corresponding Ricci tensor is a pure trace tensor. Moreover, the direct inspection of (2.9) proves

<u>Theorem I</u>

Any solution of <u>Einstein's vacuum equations</u> *) <u>with cosmological term</u>

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \qquad (3.6)$$

is <u>double self-dual</u> and therefore also a solution of Yang's equations.

Related observations have been made by Lanczos [29], Wilczek [49] and Olesen [36]. Moreover, a modified [15] version of Yang's theory of gravity is known [11] to be entirely equivalent to Ricci flat spaces.

Consequently, e.g., the Euclideanized [21,18] Schwarzschild-de Sitter solution of (3.6) defined on $R^2 \times S^2$ can also be regarded as a <u>gravitational instanton</u> in Yang's theory. On manifolds endowed with the topology of the connected sum [30] $R^4 \;\#^n\; (R^1 \times S^1 \times S^2)$ of $n = 0,1,2,\ldots$ "<u>wormholes</u>" it should be possible to construct <u>multi</u>-Schwarzschild instanton solutions by the method of images in Euclideanized geometrostatics [33]. The reason being, that the latter is just the study of the "time"-symmetric initial-value problem [31,30]

$${}^{(3)}R = 2\Lambda \qquad (3.7)$$

of general relativity, i.e., one which corresponds to an embedding problem with <u>vanishing</u> second fundamental form. (Solutions of (3.6) on $S^1 \times S^3$, the four-dimensional analogue [19] of the wormhole topology, are ruled out if the positive action conjecture holds.)

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*) It should be noted that (3.6) as an equation of second order in the metric is unique in four dimensions [41].

Other gravitational multi-instantons of Einstein's theory have been obtained from generalizing Taub-NUT spaces [21,37]. All these solutions suffer from the deficiency of being asymptotically flat in the spatial directions only, but periodic in the imaginary time direction. In order to resemble the pseudoparticles of Belavin <u>et al.</u> [5] much more closely, the metrics should be asymptotically flat in the four-dimensional sense.

In fact it has been proven that there are no such solutions for which the curvature is self-or anti-self-dual. (The universal covering space of the K3 surface is known (see Hawking and Pope [23]) to be the only <u>compact</u> manifold admitting a self-dual metric.) The solutions found by Eguchi and Hanson [14] as well as generalization [17] thereof are asymptotically <u>locally</u> Euclidean (ALE) only, i.e., outside some compact region these solutions approach the flat metric on R^4 modulo identifications [7,19] under a discrete subgroup of $O(4)$.

In Yang's theory formally rather close analogues of the Belavin <u>et al.</u> instantons do exist, as will be discussed below. Before doing so, let us return to the full field equations (3.2). By noting (2.4) a contraction reveals that

$$\nabla_\mu R = 0 \qquad (3.8)$$

is a <u>necessary</u> condition on its solutions. This may not be a restriction at all, if the following conjecture of Yamabe [53] is true.

Any compact C^∞-Riemannian manifold of dimension $n \geq 3$ can be conformally deformed to one owing a "curvature centric" metric with <u>constant scalar curvature</u>.

However, there are topological obstructions to positive scalar curvature for certain ("exotic") spin manifolds which have to be avoided [26] (see Ref. 26 also for a partial proof of this conjecture).

Metrics with <u>constant</u> scalar curvature which are at the same time solutions to Yang's equations can be constructed by conformal techniques [31] similar to those employed in the proof of Yamabe's conjecture.

To this end, consider the task of relating a geometry of constant scalar curvature R_0 to one with another constant R by means of a conformal change

$$ds_0^2 \equiv dw^2 + \frac{6}{R_0}\,{}^{(3)}g_{ab}\,dx^a dx^b \longrightarrow ds^2 = \phi^{-2}(w)\,ds_0^2 \qquad (3.9)$$

of the Riemannian line element. Assume that ds_0^2 is written in a Gaussian co-ordinate system (as it can always be done) and that the 3-metric ${}^{(3)}g_{ab}$ as independent of w.

According to equation (A.5) of Ref.31, this amounts to solving

$$R = R_0\,\phi^2 + 6(\phi\phi'' - 2(\phi')^2) \tag{3.10}$$

for the conformal factor $\phi(w)$ regarded as a function of w only. (Similar equations will be encountered in Yang-Mills theories in its exponentiated [50] form $\phi = e^{-\psi}$ or in the ϕ^4 theory [49], even with a coupling to Einstein's theory of gravitation [3]. The latter theory results from (A.6) which is an equivalent presentation of (A.5), according to the Appendix of Ref.31.) One obvious solution *) of (3.10) is

$$\phi(w) = \left(\frac{R}{2R_0}\right)^{1/2} \mathrm{ch}\,\{w\,\sqrt{R_0/6}\,\} \quad . \tag{3.11}$$

Substituting

$$r \equiv +\sqrt{x^\mu x_\mu} = \frac{2}{\sqrt{K}}\, e^{w\sqrt{R_0/6}} \tag{3.12}$$

in (3.11) yields

$$ds^2 = \frac{R_0}{R}\left(1 + \frac{K}{4}r^2\right)^{-2}\left(dr^2 + r^2\,{}^{(3)}g_{ab}\,dx^a\,dx^b\right) \quad . \tag{3.13}$$

It generalizes the line element of a <u>space</u> of <u>constant sectional</u> (Gaussian) curvature [51] $K = R_0/12$ for which already Riemann has shown that <u>locally</u> its metric can be written in the <u>conformally flat form</u>

$$g_{\mu\nu} \equiv \phi(r)^{-2}\,\delta_{\mu\nu} = \left(1 + \frac{K}{4}r^2\right)^{-2}\delta_{\mu\nu} \quad . \tag{3.14}$$

The topology of these "Euclideanized" de Sitter cosmologies (compare with Ref.13) is S^4 or R^4, depending on whether or not the manifold has been conformally compactified. According to a classical result the curvature tensor associated with (3.14) is given by (2.7). It is therefore a straightforward matter to supplement Theorem I with the following.

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*) It has been employed in the construction of localized solutions [32] of a non-linear Klein-Gordon field over Minkowski space.

<u>Corollary</u>

Spaces of constant curvature are double self-dual solutions of Yang's equation. Furthermore, they can be regarded as the Riemannian analogue of the pseudoparticle solutions of the Yang-Mills theory as derived by Belavin <u>et al.</u>[5].

To see the latter correspondence explicitly, the SO(4) gauge connection (see e.g. Brill and Wheeler [8])

$$\Gamma_\mu = \frac{1}{4}\, g_{\varsigma\tau}\left\{(\partial_\mu e_\nu^{\ \alpha})\, \overset{-1}{e}{}^{\tau}{}_{\alpha} - \Gamma^{\tau}{}_{\mu\nu}\right\} \overset{-1}{e}{}^{\varsigma}{}_{\gamma}\, \overset{-1}{e}{}^{\nu}{}_{\delta}\, \sigma^{\gamma\delta} \tag{3.15}$$

written in terms of the generators

$$\sigma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu] \tag{3.16}$$

will be calculated. In agreement with (3.14) the <u>Vierbein</u> field (cross-section of the bundle of linear frames) is <u>locally</u> given by

$$e_\nu^{\ \alpha} = \phi(r)^{-1}\,\delta_\nu^{\ \alpha} \quad . \tag{3.17}$$

Since the Christoffel symbols for conformally flat metrics have the following structure (see, e.g. equation (A.2) of Ref.31):

$$\Gamma^{\ \tau}_{\mu\nu} = \frac{1}{\phi}\{g_{\mu\nu}\,\partial^\tau - \delta_\mu^\tau\,\partial_\nu - \delta_\nu^\tau\,\partial_\mu\}\phi \tag{3.18}$$

a short calculation yields

$$\Gamma^\mu = \frac{2}{\frac{4}{K} + r^2}\,\sigma^{\mu\nu}\,x_\nu \quad , \text{ for } K \neq 0 \quad . \tag{3.19}$$

With respect to a spherical co-ordinate system these are exactly the gauge potentials of the pseudoparticle solutions [5,25] of the SU(2) Yang-Mills theory, the structure group of which has been extended to $SU(2) \otimes SU(2) \approx SO(4)$. (If the curvature of (3.14) would not become singular, the case $K = \infty$ would correspond to the so-called "meron" solution [2].)

IV. COUPLED EINSTEIN-YANG-MILLS SYSTEM AND RAINICH GEOMETRY

Ultimately, the U(n) gauge-invariant Yang-Mills theory [55] of internal symmetries would have to be unified with gravity. In the conventional formulation of such a theory, the gauge field strength $F_{\mu\nu}$ must satisfy the Yang-Mills equations

$$\nabla^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = 0 \qquad (4.1)$$

in the Euclideanized background "space-time" self-consistently curved up by the stress energy

$$\Theta_{\mu\nu} \equiv \frac{2}{\alpha} \mathrm{Tr}(F_{\mu\alpha} F_\nu{}^\alpha - {}^*F_{\mu\alpha} {}^*F_\nu{}^\alpha) \qquad (4.2)$$

via the Einstein-type field equations

$$G_{\mu\nu} = \frac{\ell^{*2}}{\hbar c} \Theta_{\mu\nu} \quad . \qquad (4.3)$$

(Due to the presence of the Planck length ℓ^*, the Yang-Mills coupling constant e is now incorporated into a generalized dimensionless Sommerfeld constant $\alpha = e^2/\hbar c$.)

The Bianchi-type identities

$$\nabla^\mu {}^*F_{\mu\nu} + [A^\mu, {}^*F_{\mu\nu}] = 0 \qquad (4.4)$$

are identically satisfied by the field strength, where

$${}^*F_{\mu\nu} \equiv \frac{1}{2} \sqrt{g}\, \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \qquad (4.5)$$

denotes their simple dual.

In the Abelian case ($n = 1$) it is known that the Einstein equations (4.3) – on account of (4.2) – may be replaced by the algebraic Rainich conditions [39]

$$R = 0 \quad , \qquad (4.6)$$

$$R_{\mu\alpha} R_\nu{}^\alpha = \frac{1}{4} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} \quad , \qquad (4.7)$$

$$R_{00} \geq 0 \quad , \qquad (4.8)$$

of the "already unified field theory" or geometrodynamics of Misner and Wheeler [35]. The stress energy tensor (4.2) remains invariant with respect to duality rotations defined by

$$e^{*\delta} F_{\mu\nu} \equiv F_{\mu\nu} \operatorname{ch}\delta + {}^*F_{\mu\nu} \operatorname{sh}\delta \quad . \qquad (4.9)$$

However, Maxwell's equations, of which (4.1) and (4.4) are generalizations, demand that the duality rotation angle or complexion δ for Abelian gauge fields has to satisfy the differential equation

$$[\nabla_\kappa, \nabla_\lambda]\, \delta = 0 \quad . \qquad (4.10)$$

Since the gradient of δ can be expressedin terms of the Ricci curvature

$$\nabla_\lambda \delta = \sqrt{g}\, \varepsilon_{\lambda\varrho\sigma\tau} R^\tau{}_\nu \nabla^\sigma R^{\nu\varrho} / R_{\alpha\beta} R^{\alpha\beta} \qquad (4.11)$$

the physical context of Maxwell's theory may be regarded as completely coded in the geometry of the underlying Riemannian manifold and vice versa *). (In a space-time with Lorentzian signature, null fields make an exception [16] as they belong to the "kernel" of this geometrization. Their structure cannot be deduced from the geometry alone. However, this difficulty does not arise in "Euclideanized" gravity.)

With this equivalent formalism at hand, let us return to the double anti-self dual solutions in Yang's theory.

On account of (2.10) the minus sign in (3.5) leads to

$$R_{\mu\nu\rho\sigma} = R^{II}{}_{\mu\nu\rho\sigma} \quad , \qquad (4.12)$$

which, because of the defining relation (2.9) already implies (4.6). In order to see if the other Rainich conditions are valid, the following identity for "squared" curvature tensors may be borrowed from a stimulating paper by Bach [4]:

$$\begin{aligned} & R_{\mu\alpha} R_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta} \\ &= \frac{1}{2} R_{\mu\alpha\beta\gamma} R_\nu{}^{\alpha\beta\gamma} - \frac{1}{2} C_{\mu\alpha\beta\gamma} C_\nu{}^{\alpha\beta\gamma} \\ &\quad + R_{\mu\alpha\beta\nu} R^{\alpha\beta} + \frac{1}{2} R R_{\mu\nu} - \frac{1}{12} g_{\mu\nu} R^2 . \end{aligned} \qquad (4.13)$$

*) This "codification" may also be expressed in terms of the invariants associated with the Ricci Vierbein field of principal directions [40].

Insertion of (4.12) and (4.6) in the decomposition (2.6) shows that the conformal Weyl tensor $C^{\mu\nu}{}_{\rho\sigma}$ vanishes. Therefore

$$R_{\mu\alpha} R_{\nu}{}^{\alpha} - \frac{1}{4} g_{\mu\nu} R_{\alpha\beta} R^{\alpha\beta}$$

$$= \frac{1}{2} R^{I}{}_{\mu\alpha\beta\gamma} \left(R^{II}{}_{\nu}{}^{\alpha\beta\gamma} + 2\,\delta_{\nu}{}^{\gamma} R^{\alpha\beta} \right) = 0 \qquad (4.14)$$

as the terms in parenthesis are zero in this instance. Thus (4.17) is true. Furthermore, the complexion δ trivially satisfies equation (4.10). The latter result is an immediate consequence of the fact that its gradient defined by (4.11) vanishes since (3.5) implies the condition (3.3) for a "pure" gravitational field. This completes the following.

Theorem II

Double anti-self-dual solutions of Yang's equation are conformally flat and, for $R_{00} \geq 0$, fulfill the Rainich conditions of geometrodynamics. (This includes Theorem 1 of Gu et al. [20].)

An extension of the Rainich-Misner-Wheeler unification programme to the case of U(n) gauge invariant and generally covariant Yang-Mills theory ($n \geq 2$) has not yet been worked out. Nevertheless, the following remark on an interesting degenerate case of this unification with gravity can be made at this stage. Assume that the internal gauge field strength themselves are self- or anti-self dual, i.e.

$$\pm F_{\mu\nu} = {}^{*}F_{\mu\nu} \quad . \qquad (4.15)$$

Then, the field equations (4.1) are automatically satisfied because of the Bianchi-type identities (4.4). Moreover, the geometrodynamical background manifold is not deformed off the Einstein spaces (3.6) since (4.13) implies [25] the vanishing of the symmetric stress energy tensor (4.2). However, in the case that the underlying, now autonomous manifold owns a non-trivial topology, there exist exact Yang-Mills instantons which have no flat space analogue [9,38]. Furthermore, if the strong gravity hypothesis is tentatively adopted, these solutions may become important for the confinement problem [3].

V. REMARKS ON GLOBAL TOPOLOGY

Two invariants are commonly invoked in order to characterize gravitational instantons on closed manifolds topologically. The Euler number

$$\chi(M) = \frac{1}{2(4\pi)^2} \int R_{\mu\nu\rho\sigma} \, {}^{*}R^{*\mu\nu\rho\sigma} \sqrt{g} \; d^4x \qquad (5.1)$$

(see Ref.27, Vol.II, p.318 for its generalization to arbitrary even-dimensional manifolds) and the first Pontryagin index [27] (Vol.II, p.313)

$$p_1(M^4) = \frac{1}{(4\pi)^2} \int R_{\mu\nu\rho\sigma} \, {}^{*}R^{\mu\nu\rho\sigma} \sqrt{g} \; d^4x \quad . \qquad (5.2)$$

For non-compact manifolds, additional boundary terms must be considered (see Chern [10], also Gibbons and Hawking [18]. From the inequality (see Belavin and Burlankov [6], Gu et al.[20] as well as Xin [52])

$$\frac{\ell^{*2}}{\hbar c} \int \mathcal{L}_Y \, d^4x + \frac{1}{4} \int {}^{*}R^{*}_{\mu\nu\rho\sigma} \, {}^{*}R^{*\mu\nu\rho\sigma} \sqrt{g}\, d^4x - (4\pi)^2 \chi(M^4)$$

$$= \frac{1}{4} \int \left(R_{\mu\nu\rho\sigma} - {}^{*}R^{*}_{\mu\nu\rho\sigma} \right)^2 \sqrt{g}\, d^4x \geq 0 . \qquad (5.3)$$

it can be inferred that the gravitational action (3.1) is minimized by double self- or anti-self dual solutions of (3.2) and (3.4) for which the Euler number is zero. Compared to Yang-Mills theory in which simple duality is employed, the roles of $p_1(M^4)$ and $\chi(M^4)$ are interchanged with respect to this issue.

The Euler-Poincare characteristic is via

$$\chi(M^4) = \sum_{p=0}^{4} (-1)^p \, \beta_p \qquad (5.4)$$

related to the rank of the p^{th} homology groups $H_p(M^4)$, i.e. to the Betti numbers $\beta_p = \dim H_p(M^4)$. The first Pontryagin index is known, if there exists a smooth self-transversal immersion f of the oriented, compact manifold M^4 into Euclidean 6-space. Then, $p_1(M^4)$ is minus three times the number τ of triple points of f which, due to its self-transversality, are isolated and hence finite [48].

As a connected complete four-dimensional space of constant curvature $K = 1/r_0^2$ is isometric to a sphere S^4 of radius r_0 (or to a real projective space [27] if non-orientable spaces are admitted), its Pontryagin index vanishes, whereas the Euler number is two (or one, respectively). Therefore, Yang's action (3.1) is not minimized for the examples of spaces of constant curvature given above, but, e.g. for the <u>flat torus</u> $T^4 \equiv S^1 \times S^1 \times S^1 \times S^1$.

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