SUPERSPACE ASPECTS OF SUPERSYMMETRY AND SUPERGRAVITY

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ABSTRACT
Superspace, as an extension of Minkowski space-time with spinning
degrees of freedom is reviewed. Superspace techniques useful for the construction
of globally and locally supersymmetric field theories are discussed.

MIRAMARE - TRIESTE
August 1979

* Invited talk at the International Conference on Mathematical Physics,
Lausanne, Switzerland, 20-25 August 1979.
The multiplets in (1) describe scalar and spinor fields. More complicated multiplets exist as well. They are used to describe supersymmetric interactions with a gauge invariance. This gauge invariance can be associated with an internal symmetry (Yang-Mills interactions) or with a space-time symmetry (supergravity) and the corresponding (gauge) field multiplets describe massless particle doublets of helicity content \((1,1)\) and \((2,2)\), respectively, for Yang-Mills and supergravity theories.

The most natural mathematical framework for the construction and investigation of supersymmetric Lagrangian field theories has been settled by Salam and Strathdee with the introduction of the concept of superspace and superfields. Superspace is an extension of ordinary space-time (Minkowski space) to spinning space-time. The base manifold of superspace has points parametrized by co-ordinates

\[
Z^M = \{ x^m, \gamma^a \}, \quad m = 0, \ldots, D - 1, \quad a = 1, 2.
\]

Latin indices denote vectors, Greek indices spinor indices. \(x\) are commuting space-time co-ordinates and \(\gamma^a \) anticommuting Grassmann variables:

\[
[x^m, \gamma^a] = [x^m, \gamma^b] = [\gamma^a, \gamma^b] = 0 \quad (4)
\]

or more symbolically \( [x^m, \gamma^a] = 0 \).

From a group-theoretical point of view, superspace may be regarded as the quotient space \(G/H\) in which \(G\) is the graded Poincaré group and \(H\) is the homogeneous Lorentz group. In extended supersymmetry with \(N\) spinorial charges \(Q^a\), \(1 = 1, \ldots, N\), the Grassmann variables are supplemented with internal symmetry indices as well \(\theta^a \) (\(\theta^a\)) and superspace is a \(N + \text{dim}\) graded manifold. However, we shall not consider extended superspace in this review.

Supersymmetry transformations are realized as motions in superspace

\[
\delta x^m = -i e^b \overline{\theta} \gamma^b, \quad \delta \gamma^a = \overline{e}^b \theta^b, \quad \delta \overline{\theta} = \varphi \theta
\]

We shall refer to (5) as a supertranslation of the point \(Z^M = (x^m, \gamma^a, \overline{\theta}^a)\). On the other hand, an ordinary translation with parameter \(a^m\) shifts the supercoordinate as follows:

\[
\delta x^m = a^m, \quad \delta \gamma^a = 0, \quad \delta \overline{\theta} = 0
\]

The composition rule of supertranslations is obtained by performing the commutator of two infinitesimal charges of parameters \(Q^a \) and \(Q^b \) as follows:

\[
\delta Q^a = \{ Q^a, \delta Q^b \} = \{ Q^a, \delta x^m \} \gamma^m + \{ Q^a, \delta \gamma^a \} = \{ Q^a, \delta x^m \} \gamma^m + \{ Q^a, \delta \gamma^a \} = (-2 \delta \gamma^b \overline{\theta}^b + 21 \overline{\theta}^b \gamma^a, 0, 0)
\]

The composition law (7) reflects the basic anticommutators given by (2). It is the manifestation of the fact that an infinitesimal space-time displacement can be obtained in superspace by performing two infinitesimal supertranslations. If we regard the supersymmetry algebra as the "square root" of the Poincaré algebra, we could equally say that superspace is the "square root" of Minkowski space-time.

II. SUPERFIELDS

We can elucidate the action of supersymmetry transformations in superspace giving more precise concepts. The superspace previously introduced furnishes a representation of the algebra given in (2) in terms of differential operators. The set of motions defined by (5) and (6) is obtained by the left-action on an element of the graded Lie group generated by \(Q^a\), \(\overline{Q}^b\), \(P^m\), \(\overline{P}^a\):

\[
L(x, \theta, \varphi) = \exp i (\theta^a Q^a + \overline{\theta}^a \overline{Q}^a - \varphi^m P^m)
\]

with another element of the group \(G(a, \theta, \varphi)\)

\[
G(a^m, \theta^a, \varphi^m) L(x^m, \theta, \varphi) = L(x^m + a^m, \theta, \varphi) + 16 \varphi^a \theta^b \delta^m_{\theta} \theta^a, \varphi^m \theta^b \theta^a + \delta \overline{\theta}^a \theta^b \varphi^m \varphi^b \theta^a + \overline{\theta}^a \theta^b \varphi^m \varphi^b \theta^a \]

where we have used the Hausdorff formula.

The infinitesimal generators of this motion are

\[
P^m = \frac{1}{2} \delta^m, \quad Q^a = \frac{1}{2} \delta^a, \quad \overline{Q}^a = -i \overline{\theta}^a \overline{P}^a, \quad \overline{P}^a = -i \theta^b \overline{P}^a
\]

A scalar superfield is a scalar function in superspace

\[
\phi(x^m, \theta^a) = \phi(x) \theta^a
\]

where in the infinitesimal

\[
\delta \phi = \frac{1}{2} [\delta x + \overline{\theta} \gamma^a \theta^a, \delta \theta^a]
\]

Because \(\theta^a \theta^b = 0\) it is equivalent to a finite collection of ordinary fields defined over Minkowski space. A scalar field unifies into a single object 16 Bose and 16 Fermi fields. Covariant derivative can also be introduced

\[
\overline{\nabla} = \frac{1}{2} \left[ \delta \theta^a - \overline{\theta}^a \overline{\theta} \delta^m + (\delta \theta^a - \overline{\theta}^a \overline{\theta}) \right] \frac{1}{2} \delta x^m
\]

Moreover, they satisfy the algebra

\[
[Q^a, \overline{Q}^b] = -2i \delta^a_{\overline{b}} \delta^b_{\overline{a}}, \quad [P^m, Q^a] = 0, \quad [P^m, \overline{Q}^a] = 0
\]
In the literature, it has been called a vector superfield (because it contains a vector field) and its general expansion is

\( \phi(x, \epsilon, q) = C + \epsilon \phi_\epsilon(x, \epsilon) + \epsilon^2 \phi_\epsilon^2(x, \epsilon) + \cdots \)  

We have so far confined our considerations to superfields with no (external) Lorentz indices. Such an index is irrelevant for the previous considerations and can be added without modifications of the previous results. We can extend our analysis to more general superfields as follows. A general (quantum) superfield can be written as

\( \Phi \Phi = \Phi(0,0) \) is the superfield at the (super) origin. The latter is an object which transforms according to a (representation) of the stability (super) algebra \( H \) of the point \( (x, \epsilon) \approx (0,0) \). In the case of the algebra given in (2)

\( H = H_m, K_n, M, N \) is sixteen-dimensional.  

If \( \Phi \) is a representation of \( H \) we can induce on \( \Phi \Phi \) a representation of the whole algebra. If \( X \) is a generator of \( H \) we can compute the action of \( X \) on \( \Phi \Phi \) as follows:

\[ X \Phi \Phi = X \Phi(0,0) + \epsilon \frac{\partial}{\partial \epsilon} \Phi(0,0) + \epsilon^2 \frac{\partial^2}{\partial \epsilon^2} \Phi(0,0) + \cdots \]

where the infinite chain of commutators which defines \( Y \) stops after a finite number of steps due to the O'Raifeartaigh theorem. The superfield formalism is particularly convenient for working out tensor products of supergravity representations. In fact multiplication of representations is merely reduced to superfield multiplications which is an almost trivial operation. Such a multiplication in conjunction with the concept of Berezin integration 6 over anticommuting variables is one of the es-

If we expand \( \Phi \) in power series of \( \epsilon \) we get

\[ \Phi(x, \epsilon) = \Phi(0,0) + \epsilon \frac{\partial}{\partial \epsilon} \Phi(0,0) + \epsilon^2 \frac{\partial^2}{\partial \epsilon^2} \Phi(0,0) + \cdots \]

and we obtain the scalar multiplet with transformation laws as given by (1).

A superfield \( \Phi \) can be taken to be real \( \Phi = \Phi^* \); in that case we can no longer impose a condition like (21) and the superfield is essentially irreducible.

The motion induced by the left-action of a group element \( C(a, \epsilon, \zeta) \) on \( \Phi \Phi \) is given by

\[ x' = x + a + 2i \epsilon a, \quad \epsilon' = \epsilon + \zeta, \quad \zeta' = \zeta + \zeta' \]

Correspondingly, we can introduce superfields \( \theta_1(2) \) transforming according to (19), (20). From (19), (20), it follows that type I (or type II) superspace has complex bosonic co-ordinates. The curved space analogue of (19), (20), has been used recently by Ogievetsky and Sokatchev 2 in order to have a geometrical description of supergravity in terms of an axial vector superfield. One of the advantages of type I or II superfields is that the expression of covariant derivatives become extremely simple in these bases. For instance

\[ \partial_a = \frac{\partial}{\partial x^a} \quad \text{on } \theta_1 \quad \text{and} \quad \partial_a = \frac{\partial}{\partial x^a} \quad \text{on } \theta_2 \]

This shows that a solution of the covariant constraint

\[ \partial_2 \Phi \Phi = 0 \quad \text{(chirality condition)} \]

is simply given by \( \theta_1(x, \epsilon, q) = \theta_1(x, \epsilon) \), so we get

\[ \theta(x, \epsilon, q) = \theta_1(x + 16 \epsilon \delta, \epsilon) \]

\[ \theta(x, \epsilon, q) = \theta_1(x + 16 \epsilon \delta, \epsilon) \]
essential ingredients for the construction of supersymmetric field theories. If we label a chiral left-handed (right-handed) superfield and vector superfield with subscripts \(L/R)\), one can check the following properties of superfield multiplications:

\[
\begin{align*}
\{\phi_L\}_{\alpha} &= \{\phi_L\}_{\alpha} \cdot \{\phi_R\}_{\beta} = \{\phi_L\}_{\alpha} \cdot \{\phi_R\}_{\beta} \\
\{\phi_L\}_{\alpha} + \{\phi_L\}_{\beta} &= \{\phi_L\}_{\gamma} \text{ and} \{\phi_L\}_{\beta} = \{\phi_L\}_{\alpha} \text{,} \\
\{\phi_R\}_{\gamma} &= \{\phi_R\}_{\gamma} \text{ and} \{\phi_R\}_{\alpha} = \{\phi_R\}_{\beta} \\
\{\phi_Y\}_{\gamma} &= \{\phi_Y\}_{\gamma} \text{ and} \{\phi_Y\}_{\gamma} = \{\phi_Y\}_{\gamma} \\
DD \phi_L = (DD \phi_L)^R \\
\end{align*}
\]

Integration of superfields is defined through the Berezin integration recipe:

\[
\int d^8 \phi \phi = \delta (\phi) 
\]

This implies that

\[
\int d^8 \phi \phi (x, e, \theta) = \phi_{\text{LAST}}(x) 
\]

where \(\phi_{\text{LAST}}\) means the coefficient of the \(e^0 e^0\) monomial in the \(e\) expansion of \(\phi\). It follows that an invariant action constructed out of a superfield \(\phi\) is given by

\[
\int d^8 \phi \phi = \delta (\phi) 
\]

Eq. (25) is the starting point for the construction of a Lagrangian in the superfield formulation of supersymmetric Lagrangian field theories.

### III. Gauge Theories

The most striking application of superspace is in the context of gauge theories. In fact, when a gauge invariance principle is imposed in supersymmetry, it turns out that the resulting geometrical structure of superspace manifests some unconventional features with respect to ordinary gauge theories over Minkowski space. The geometry is a constrained geometry. This is the case both for the gauging of an internal symmetry (Yang-Mills) and for a space-time symmetry (supergravity). Let us consider first the gauging of an internal symmetry group \(G\). Then it is known that the appropriate superfield \(\{\phi\}_{\gamma}^G\) for the description of the vector potential is a Lie algebra valued vector superfield \(V\) and the parameter is a Lie algebra valued chiral superfield \(A_{\beta} = 0\). The (finite) Yang-Mills gauge transformation on \(V\) is given as follows:

\[
V = e^{-iA^+} e^iA 
\]

and the superfield strength is given by the following spinorial chiral multiplet \(\theta\):

\[
(34) 
\]

The Yang-Mills superspace Lagrangian is given by

\[
(37) 
\]

It is simple to show that under Yang-Mills transformation \(V\) transforms as follows:

\[
(36) 
\]

The Yang-Mills superspace Lagrangian is given by

\[
(37) 
\]

and the invariant action is obtained by integration over \(d^4 x d^2 \theta\).

It is important that the vector potential \(V\), the field strength \(W\), and the chiral parameter \(A\) can be understood in terms of a genuine Yang-Mills theory in superspace with a constrained geometry. If we define (Lie algebra valued) vector potentials \(V\) and field strengths \(F\) in superspace

\[
F_{AB} = -F_{BA} = -F_{AB} 
\]

where \(a = 1, \ldots, U\) and \(\theta = 1, 2\).

The gauge parameter \(A\) is an unconstrained "vector" superfield \(A\). It turns out that in order to reduce the number of field components of \(V\) in such a way that the theory is equivalent to the previous formulation (in terms of \(V\)) the following constraints on the field strength (38) must be imposed (10, 11):

\[
(39) 
\]

These constraints imply that the remaining components of the curvature \(F_{mn}\), \(F_{mn}\) are only functions of \(V\), \(W\), \(\theta\) and that the vector potential \(V\) is only a function of a real vector superfield \(V\). The remaining gauge transformation on \(V\) restricts the superfield \(A\) to be a chiral superfield \(\bar{A}\) in such a way that the theory is equivalent to the previous formulation (in terms of \(V\)) the following constraints on the field strength (38) must be imposed (10, 11):

\[
(39) 
\]

We learn from the geometrical description of the supersymmetric Yang-Mills theory that the kinematical constraints on the curvatures given by (39) are crucial in order to lower the spin content of the vector potential from highest spin-2 to highest spin-1 as required for a consistent description of the theory.

### IV. Supergravity

The theory of supergravity in the superspace approach can be formulated in close analogy with the case of the Yang-Mills supersymmetric gauge theory. One can start with a general affine superspace whose points are parameterized by co-ordinates \(x^M\), \(\theta^\mu\) and \(\theta^\nu\) stands for \(\theta^\mu\), \(\theta^\nu\), \(\mu = 1, 2\).

Under general co-ordinate transformations we have

\[
(40) 
\]

\(r^M(\theta)\) is an arbitrary function on superspace and \(M\) refers to a (super) world index.
At each point in superspace one erects local tangent frames and defines super tetrad (super-Vierbein)

\[ \eta_{\alpha}^M = a_0, a_\alpha, \eta^{ab} \]

and their inverse \( \eta_M^A \). Flat indices are raised and lowered with the flat metric of the group of tangent space. The appropriate group for the superspace formulation of supergravity turns out to be the Lorentz group. In the tangent space there are two invariant tensors

\[ \eta_{\alpha\beta} = \begin{pmatrix} \eta_{00} & 0 \\ 0 & \eta^{00} \end{pmatrix} \text{ and } \eta^{ab} = \begin{pmatrix} \eta^{00} & 0 \\ 0 & \eta_{00} \end{pmatrix} \]

which have no inverse. Then it follows that in this approach an invertible metric cannot be defined. This is in contrast with the metric approach considered by Nath and Arnowitt in which the tangent space group is taken to be the Orthosymplectic group Osp(4|4) with invariant (invertible) metric

\[ \eta_{\alpha\beta} = \begin{pmatrix} \eta_{00} & 0 \\ 0 & \eta^{00} \end{pmatrix} \]

(\( \eta \) being a dimensional constant).

One can define on an affine space covariant derivatives

\[ D^\alpha = \partial^\alpha - \eta_{\alpha\beta} \partial^\beta \]

in terms of a superconnection \( \phi = \phi^a \sigma^a \), which is Lie algebra valued over the Lorentz Lie algebra.

Covariant derivatives with tangent-space indices \( D^\alpha = \partial^\alpha - \eta_{\alpha\beta} \partial^\beta \) obey

\[ [D_A, D_B] = -\eta_{\alpha\beta} \partial^\alpha - T^C_{AB} D_C \]

\[ \eta_{\alpha\beta} \text{ and } \eta^{\alpha\beta} \text{ are respectively, the super torsion and the super curvature. Torsion}
\]

and curvatures satisfy two sets of Bianchi identities which follow from the Jacobi identities for the \( D_A \)’s

\[ [[D_A, D_B], D_C] + \text{cyclic} = 0 . \]

(45)

Up to now the geometry is not sufficiently specified in order to correctly reproduce the supergravity theory. As first stressed by Wess and Zumino, additional kinematical contraints must be introduced in order that the super-Vierbein \( \eta^{\alpha} \) itself be suitable for the description of the component fields of supergravity. The correct constraints to be imposed are the following restrictions on the super torsion (14):

\[ T^C_{\alpha\beta} = 2 \gamma \gamma^C, \ 
\]

\[ T^C_{\alpha\beta} = T^C_{\beta\alpha} = 0 \]

(46)

For each point \( \alpha, \beta, \gamma = 1, \ldots, k \) and \( a, b, c = 1, \ldots, 4 \).

As a consequence of the Bianchi identities (44) these constraints imply (15) that all components of the supercurvature \( T^\gamma_{\alpha\beta} \) and the remaining components of the super torsion \( T^\gamma_{\alpha\beta} \) can be expressed in terms of three superfields

\[ \gamma, \gamma^\alpha, \gamma_{\alpha\beta} \text{ (symmetric) and } R \]

which satisfy the following differential identities:\( ^{6}\)

\[ \gamma^\alpha_{\beta\gamma} = 0, \ 
\]

\[ \gamma^\alpha_{\beta\gamma} = 0 \]

(48)

The analogy with the Yang-Mills case is now clear. The constraints (46) are the analogue of the constraints (39) and the three supermultiplets in (47) are the analogue of the multiplet in (35). Eqs. (48) and (49) are the analogue of \( T^\alpha_{\alpha\beta} = 0 \). We observe that the chirality condition \( \gamma^\alpha_{\alpha\beta} = 0 \) is not in general a covariant statement in curved superspace because left (right)-handed spinorial covariant derivatives no longer anticommute (see (48)). However, the kinematical restrictions on the (super-) torsion given in (46) imply \( \gamma^a_{\alpha\beta} = \gamma^a_{\beta\alpha} = 0 \) and also \( R^a_{\alpha\beta} = 0 \) (\( a, b, c = 1, 2 \)) (through Bianchi identities).

The chirality constraint is always invariant if the superfield \( \eta \) carries only undotted indices. This is in contrast with flat superspace where no restriction on the Lorentz structure of a chiral superfield exists. This result has been also derived in the component formalism in Ref.16.

The three multiplets given by (47) are the supersymmetric completion of the multiplets containing the Einstein tensors, the Weyl tensor and the curvature scalar. It is important to point out that the torsion constraints also imply constraints on the super-Vierbein \( \eta_{\alpha}^M \) itself. In a suitable supersymmetric gauge a solution of these constraints can be given in terms of a (real) axial superfield \( \eta_{\alpha}^m \) (endowed with a vector index). This superfield, proposed \( 17 \) as the natural object which is coupled to the supercurrent superfield of globally supersymmetric theories \( 18 \), also emerges in a natural way in the unconstrained geometrical approach pursued by Afinevsky and Sokatchev \( 21 \) and in another similar (although more general) approach considered by Siegel and Gates \( 19 \), \( 20 \).

The axial superfield \( \eta_{\alpha}^m \) in supergravity is the analogue of the superfield \( \eta \) (see Eq.(49)) of supersymmetric Yang-Mills theories.

Other approaches which use a set of kinematical constraints different from those of Wess and Zumino (Eq.(5)) have been considered by Brink, Dell-Mann, Ramond and Schers \( 21 \) and J.G. Taylor \( 22 \). All these approaches turn out to be equivalent as far as the description of pure supergravity is concerned. However, differences may emerge when coupling to matter multiplets or non-minimal interactions are also considered.

Finally it is important to point out that the minimal auxiliary fields which emerge in the component formulation of supergravity \( 23 \), \( 24 \) are nothing but the Lorentz irreducible parts of \( \eta_{\alpha}^M (x, \theta = 0) \), in the formulation of Wess-Zumino \( 25 \). This has recently been used, by means of a technique developed in Ref.21, to establish the connection between the gauge of the graded Poicaré group in ordinary space and (super) co-ordinate transformations in superspace \( 26 \).
ACKNOWLEDGMENTS

The author is grateful to Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

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