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RENORMALIZATION OF THE NEW TRAJECTORY
IN THE UNITARIZED CONVENTIONAL DUAL MODEL

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I. INTRODUCTION

The analytic properties of reggeon-reggeon amplitudes, considered as asymptotic limits of many-particle amplitudes, have recently received a great amount of attention in several models, in order to evaluate unitarity integrals. Since the conventional dual resonance model (CDRM) is the simplest known model having multi-Regge behaviour, it seems very appropriate to study the properties and asymptotic behaviour of reggeon-reggeon amplitudes. In particular, Royer, Torner and Weber realized that the multi-Regge limit of the six-particle amplitude, for tree diagrams in the CDRM, corresponding to ordinary trajectories, \( T_6^{(a)} \), has poles at nonsense wrong-signature points \( a(t) = -2n + \frac{1}{2}(1 + r) \) \((n = 1, 2, \ldots)\). Since the full amplitude is regular at these points there must be an additional contribution \( T_6^{(b)} \), coming from a new Regge trajectory, that cancels the poles in \( T_6^{(a)} \) and restores the proper asymptotic behaviour. The contribution of the new trajectory \( \delta(t) = \frac{1}{2} (a(t) - 1) \) was obtained by Royer et al. analysing the double helicity pole limit of \( T_6 \). They showed that the asymptotic behaviour of \( T_6^{(b)} \) was dominant at \( a(t) < -1 \) and that the full amplitude is regular at \( a(t) = -1 \). More recently, Zakrzewski studied the six-particle tree diagrams in a linear triple-Regge limit exhibiting the contribution of the \( S \) trajectory. In that paper they show the cancelation of all nonsense wrong-signature poles and argue that the existence of unconventional Regge behaviour, for some values of the transferred momenta, is not confined to the CDRM but must hold in more general models.

However, if dual resonance models are to be taken seriously as a theory of strong interactions, the tree diagrams must be considered as a kind of Born approximation to the amplitude, and loop corrections must be computed. They produce normal threshold singularities, demanded by unitarity and the renormalization of Regge trajectories, determining in this way the spectrum of particles. It seems thus important, in order to analyse reggeon amplitudes, to consider, besides the tree diagrams, also unitarity corrections, coming from loop diagrams, in the multi-Regge limit. In fact, when the contribution of loop diagrams to reggeon amplitudes is considered, several interesting points must be clarified: a) The nature of the asymptotic contribution in the \( a \) sector. It is expected to give, as in the four-particle amplitude, a renormalization contribution to the \( a \) trajectory. b) The nature of the asymptotic contribution (if it exists) in the \( S \) sector. c) Integrals with \( \{ \) planar \( \} \) loops are exponentially divergent and dual theories must be renormalized. The renormalization of \( N \)-particle amplitudes has been performed by Neveu and Scherk, who showed that infinite quantities could...
be subtracted from the a sector in a unique way by means of a renormalization of the coupling constant and wave functions. However, Neveu and Scherk's prescription does not ensure that divergent quantities disappear from the 8 sector also. Thus the validity of the Neveu-Scherk renormalization must be re-analysed when reggeon amplitudes are considered.

In this paper we shall analyse the contribution of planar loops in the CDMK to the (unsignedurized) two-reggeon + two-particle amplitude corresponding to Fig.1 and, in particular, its asymptotic behaviour corresponding to the linear triple Regge limit of the 'tree-particle' diagram. The full amplitude receives contributions also from non-planar orientable 8 diagrams (non-orientable diagrams being excluded in a theory with U(n) invariance). We shall consider in this paper only the contribution of planar diagrams, where divergences appear and where the renormalization of Regge trajectories is the most delicate point. Non-planar contributions, exhibiting other kinds of difficulties, mainly those associated with the appearance of the Pomeron singularity, will be discussed in a separate paper.

The momenta should be considered as all incoming, as in Fig.1, and independent kinematical invariants defined as

\[ s_1 = (p_2 + p_3)^2, \]
\[ s_2 = (p_2 + p_3 + p_4)^2 = x_2 s_3 s_4, \]
\[ s_3 = (p_3 + p_4)^2, \]
\[ s_4 = (p_4 + p_5)^2, \]
\[ s_5 = (p_2 + p_5)^2 = x_1 x_2 s_1 s_5, \]
\[ t_1 = (p_1 + p_2)^2, t_2 = (p_1 + p_2 + p_3)^2, t_3 = (p_2 + p_6)^2. \]

There is a complicated relation among \( s_1, s_2, s_3, t_1, t_2, t_3, x_1, x_2 \) and \( \phi \), leaving eight independent invariants as it must be.

This paper is organized as follows. In Sec.II we briefly review the results concerning the linear triple-Regge limit of tree diagrams. The mechanism for extracting \( \alpha \) and \( \beta \) contributions to the amplitude has been usually traced from the behaviour of the generalised hypergeometric function \( {\cal G}(a,b;\alpha,c;z) \), obtained by integration of a confluent hypergeometric function \( F(a,b;\alpha;z) \). An alternative method based upon a standard decomposition of confluent hypergeometric functions, \( \phi = \phi(a) + \phi(b) \), Eq.(7), is described in this section. This method will be useful in Sec.III, where more complicated expressions are involved in the integral, and the integration over confluent hypergeometric functions cannot be explicitly performed.

In Sec.III the linear triple-Regge limit of one-loop planar diagrams is computed. In the \( \alpha \) sector we find a typical renormalization effect, to order \( g^2 \), of the leading a trajectory in the t channel. This is exactly the same result obtained by Neveu and Scherk studying the simple Regge limit of the four-particle one-loop planar diagram. The only discrepancy between our result and theirs (see Eq.(6) of Ref.6) comes from the fact that they take a different value for the slope \( \alpha' \). In the \( \beta \) sector, the behaviour of the function governing the asymptotic behaviour, \( \exp(-s_{\beta}^2) \), shows the same mechanism which led to the appearance of the \( \beta \) trajectory in tree diagrams. Thus linear terms in the relevant variables are cancelled in \( s_{\beta} \) so that quadratic terms lead, asymptotically, to a renormalization effect, to order \( g^2 \), of the leading \( \beta \) trajectory.

In Sec.IV the renormalization of the dual theory is discussed. We find that the Neveu and Scherk renormalization prescription renders finite the renormalised Regge trajectories both in the \( \alpha \) and in the \( \beta \) sector. The intercept of the \( \beta \) trajectory is shifted from its bare value \( \beta(0) = 0 \) by the renormalization procedure, while that of the \( \alpha \) trajectory is not shifted, as it was required by the gauge invariance of the dual theory in the Virasoro case \( a(0) = 1 \). Some possible phenomenological implications of this fact are briefly drawn in Sec.V.

II. MULTI-REGGE LIMIT OF TREE DIAGRAMS

Let us consider the tree diagram shown in Fig.2. It gives the following contribution to the amplitude:

\[ T_6 = \frac{G}{4} \int_0^1 dx_1 dx_2 dx_3 \frac{1}{x_4-x_5-x_6-x_7} \]
\[ \frac{1}{x_4-x_5-x_6-x_7} \frac{1}{x_4-x_5-x_6-x_7} \]
\[ \frac{1}{x_4-x_5-x_6-x_7} \frac{1}{x_4-x_5-x_6-x_7} \]
\[ \frac{1}{x_4-x_5-x_6-x_7} \frac{1}{x_4-x_5-x_6-x_7} \]
\[ \frac{1}{x_4-x_5-x_6-x_7} \frac{1}{x_4-x_5-x_6-x_7} \]

(1)

There is a complicated relation among \( s_1, s_2, s_3, t_1, t_2, t_3, x_1, x_2 \) and \( \phi \), leaving eight independent invariants as it must be.

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where \( a_i = a(t_i) \) and linear Sege trajectories with unit intercept and unit slope are assumed. Using standard methods, the behaviour of \( T_6 \) when \( s_2 \rightarrow \pm \) can be written as

\[
T_6 \sim \Phi^{	ext{alt}} \left( -s_2 \right) \Gamma(-s_2) \Gamma(-s_3) \Gamma(-s_4)
\]

\[
\int_{-s_2}^{s_2} x e^{-s_2} \left( 1 - x + x \gamma_x \gamma_s \right)^{s_2} \left( 1 - x + x \gamma_s \gamma_x \right)^{s_2} F(z)
\]

where \( F(z) \) is the following function:

\[
F(z) = \frac{\Gamma(s_2) \Gamma(-s_2) \Gamma(-s_3) \Gamma(-s_4)}{\Gamma(-s_2)} \Phi^{	ext{alt}}(z)
\]

where \( \Phi(z) \) is the following function:

\[
\Phi(z) = z^{s_2} \psi(-s_2, s_2 - s_2 + 1, z)
\]

and \( z \) is given by

\[
z = (2 - x + x \gamma_x \gamma_s)/(2 \gamma_x \gamma_s - 2).
\]

The Tricomi function \( \psi \) is not analytic in the \( z \) plane but has a cut along the negative real axis. Using the relation of \( \psi \) to confluent hypergeometric functions \( \phi \),

\[
\psi(a, c, x) = \frac{\Gamma(-c)}{\Gamma(-a - c + 1)} \phi(a, c, x) + \frac{\Gamma(-c)}{\Gamma(-a)} \phi(a - c + 1, 2 - c, x),
\]

we can write

\[
\Phi(z) = z^{s_2} \frac{\Gamma(s_2) \Gamma(s_3)}{\Gamma(-s_3)} \phi^{	ext{alt}}(-s_2, s_3 - s_4 + 1, z) + \{ z \rightarrow z_x \}
\]

The integral (3) is defined for \( Re(a(s_2)) < 0 \) and this prevents us from doing \( \psi \rightarrow \pm \), corresponding to the diagram shown in Fig.3, along the real positive axis. On the other hand, \( F(z) \) is also \( s_2 \) dependent and we must make an asymptotic estimation on it. To this end we shall use the following decomposition of confluent hypergeometric function

\[
\phi(a, c, x) = \frac{\Gamma(a)}{\Gamma(a - c)} e^{	ext{ic} \exp\{ i \text{en} \}} \Psi^{	ext{alt}}(a, c, x),
\]

\[
\phi(a, c, x) = \frac{\Gamma(a)}{\Gamma(a - c)} e^{	ext{ic} \exp\{ i \text{en} \}} \Psi^{	ext{alt}}(a, c, x),
\]

and \( \varepsilon = \text{sign}(\text{Im} x) \), together with the asymptotic behaviour of \( \phi \).

Thus, if \( Re x \rightarrow + \), only the contribution of \( \phi^{	ext{alt}} \) survives, while if \( Re x \rightarrow - \), only \( \psi^{	ext{alt}} \) should be kept. If we wish to keep both contributions, the limit must be made along a direction parallel to the imaginary axis, i.e. \( Re x \) fixed and \( |\text{Im} x| \rightarrow \). This gives us the clue to find a suitable direction along which the limit \( s_2 \rightarrow \) must be performed. Parallel to decomposition (7) for confluent hypergeometric functions there exists a similar one for \( F(z) \) and, thus, for the amplitude, as

\[
F(z) = F(a(z)) + F(b(z)),
\]

\[
\gamma_6 = \gamma_6^{	ext{alt}} + \gamma_6^{	ext{alt}}.
\]

We shall choose to compute the asymptotic behaviour of \( T_6 \) along the direction \( s_2 = Re s_2 - iv \) with \( Re s_2 \) inside the region of convergence of (3) and \( v \rightarrow \), and then to continue analytically the result to the \( s_2 \) physical region. We are therefore led to study the asymptotic behaviour of Fourier integrals, which is determined by the behaviour of the integrand near some critical points. This powerful method has been explained in detail in Ref.13 and will be widely used in the following section to compute the asymptotic behaviour of planar loops.
Using (7a) in (3) we can see that the asymptotic behaviour of \( \eta_0^{(s)} \) is determined by the behaviour of the function \( \exp(-s_2 \sinh(1-x_0)) = \exp(s_2(x_0 + \frac{1}{2} x_0^2 + \cdots )) \), which gives a vanishing contribution unless \( x_0 \approx 0 \). Thus, changing variables in (3) as \( x_0 = y \{ -s_2 \} \) we obtain

\[
T_6^{(s)} = \int_0^1 dy_2 y^{-s_2-1} \exp \left( -y(1-x_2 y)(1-x_2 y)^{s_2} \right) F^{(s)}(y),
\]

where the function \( F^{(s)}(y) \) is defined by

\[
F^{(s)}(y) = \frac{(-1)^s (1-y^2)^{s_2}}{\phi(s_2,2s_2)}.
\]

On the other hand, the behaviour of \( \eta_0^{(s)} \), as \( s_2 \to \infty \), is governed by the function \( f(x_0) = -x_0^2 - \sinh(1-x_0) = \frac{1}{2} x_0^2 + \cdots \). This function has a stationary point of first order at \( x_0 = 0 \) and the integral is dominated by the values of the integrand at \( x_0 \approx s_2^{-1} \), so that we have

\[
F^{(s)}(x_0) \sim \frac{1}{2} \phi(-s_2,2s_2) \exp(-s_2 x_0^2),
\]

where use has been made of the analytically of \( \phi \) functions and

\[
\phi^{(s)}(x_0) = \phi^{(s)}(-s_2) \phi^{(s)}(-s_2,1) + \phi^{(s)} \left( x_0 + s_2 \right).
\]

As has been discussed in detail \( 3^{,4,11} \), in the limit corresponding to the configuration of Fig.3, the Toller variable \( s = 1 \) so that \( \eta_0^{(s)} \) vanishes, as can easily be shown using the properties of \( \Gamma \) functions, and \( \eta_0^{(s)} \) does not give any contribution to the linear triple-Regge limit. However, when both reggeons \( a_1 \) and \( a_2 \) are twisted, as in the configuration shown in Fig.4, \( \phi \) must be continued analytically to \( \phi = e^{2\pi i} \) and the two terms in (13) do not cancel, giving the value \( \phi \exp(-s_2(1-a_1 + a_2)) \). Henceforth we shall write \( \eta_0^{(s)} \) in general, and the above discussion should be understood.

When we consider the expression (13) for \( \eta_0^{(s)}(x) \) in (3) we are led to compute the asymptotic behaviour of the following integral:

\[
I(s_2) = \int_0^1 dy_2 y^{-s_2-2} \exp \left( -y(1-x_2 y)(1-x_2 y)^{s_2} \right),
\]

which can easily be estimated as \( 3^{,5} \)

\[
I(s_2) \sim \frac{1}{2} \phi(-s_2,2s_2) \exp(-s_2 x_2),
\]

and the asymptotic behaviour of (3) in the \( \Theta \) sector is given by the following expression:

\[
T_6^{(s)} \sim \frac{1}{2} \phi^{(s)}(-s_2,2s_2) \exp(-s_2 x_2),
\]

which shows the contribution to the double twisted configuration of Fig.4.

III. MULTIPLE-REGGE LIMIT OF ONE-LOOP PLANE DIAGRAMS

The amplitude corresponding to the six-particle one-loop planar diagram shown in Fig.5 has been computed by Gross, Neveu, Scherk and Schwars \( 16 \) as

\[
T_c = (2\pi)^{3/2} \phi^{(3)} \int_0^1 dx_2 \exp \left( -s_2 x_2 \right) \exp \left( -s_2 x_2 \right),
\]

where \( \phi^{(3)}(w) \) is the usual partition function, \( w = \sum x_i \) and \( x_{i,j} = x_i \cdots x_{i-1} \). The function \( g(x) \) can be expressed in terms of the Jacobi elliptic theta function \( \theta_1 \) as follows:

\[
g(x) = \theta_1 \left( x \right).
\]
\[ \psi(x) = -\text{Im} \frac{\{\text{Im} \frac{1}{\xi_i(0)}\}}{1} \quad (19) \]

\[ D \text{ is the dimension of spacetime and the additional factor } (1-\omega)^F \zeta(\omega)^E \text{ comes from the elimination of spurious states. If } D = 4 \text{ one finds } E = F = 1 \]

whereas in the case \( D = 26 \) (critical dimensionality) the value \( E = 2, F = 0 \) is obtained \( \& \) the whole set of longitudinal modes decouples. In the following we shall not adhere to any particular value of spacetime dimensionality but, instead, we shall keep \( D \) dependence through the function

\[ a(\omega) = (2\pi)^{D/2} \omega^{2D} (1-\omega)^F \zeta(\omega)^E / (\text{Im} \xi_i(0))^{D/2} \quad (20) \]

The various possibilities will be discussed in the following section, where the renormalization of the theory is considered.

Expressing the invariants \( p_i, p_j \) in terms of the independent invariants of Eq.(1) we can cast the amplitude (18) into the following form:

\[ p = g^6 \int_{\mathcal{L}^D} \prod \omega(x) \frac{\psi(\xi_i, \xi_j) \psi(\xi_{i'}, \xi_{j'}) \psi(\xi_{i''}, \xi_{j''})}{\psi(\xi_i, \xi_{i'}) \psi(\xi_j, \xi_{j'}) \psi(\xi_{i''}, \xi_{i'''})} \]

\[ \exp \left\{ - \sum_{i} \xi_i \cdot U \right\} \exp \left\{ - \sum_{i} \xi_i \cdot V \right\} \]

\[ \exp \left\{ - \xi_1 \cdot \xi_2 \ U_1 - \xi_3 \cdot \xi_5 \ U_2 - \xi_1 \cdot \xi_6 \ \phi \ \xi_3 \cdot \xi_5 \ U_3 \right\} \quad (21) \]

where functions are defined as follows:

\[ U_1 = \frac{\psi(\xi_i, \xi_j) \psi(\xi_{i''}, \xi_{j''})}{\psi(\xi_i, \xi_{i'}) \psi(\xi_j, \xi_{j'})} \quad (22) \]

Let us consider now the double-regge limit, \( s_1, s_3 \rightarrow \), corresponding to the two-reggeon - two-particle amplitude shown in Fig.1. We change the variables of integration as

\[ x_1 = 1 - y_1/(s_1), \quad x_2 = 1 - y_2/(s_2), \]

and take the limit in the integrand, with the usual prescription.

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As can be seen from (22), $V_1$ and $W_1$ vanish at $x_1 = 1$, while $V_3$ and $W_3$ vanish at $x_3 = 1$ and we can write the following asymptotic estimations for these functions:

\[ s_1 V_1 = y_1 V_1 + O(s_1^{-1}) \]
\[ s_1 W_1 = y_1 W_1 + O(s_1^{-1}) \]
\[ s_3 V_3 = y_3 V_3 + O(s_3^{-1}) \]
\[ s_3 W_3 = y_3 W_3 + O(s_3^{-1}) \]

where the new functions appearing in (23) are defined as

\[ v_1(x_2, x_3, x_4, x_6) = -\frac{1}{\ln u} (D(x_2) + D(x_4 x_6)) \]
\[ v_2(x_2, x_3, x_4, x_6) = -\frac{1}{\ln u} (D(x_6) + D(x_2 x_3)) \]
\[ v_3(x_2, x_3, x_4, x_6) = v_1(x_4, x_3, x_2, x_6) \]
\[ v_3(x_2, x_3, x_4, x_6) = v_1(x_4, x_3, x_2, x_6) \]

and $D(x)$ is the logarithmic derivative of the Jacoby function, whose properties are listed in the appendix.

In the same way, using (22), it can be shown that $v_2 = 3W_2/3x_1 = -3W_2/3x_5 = 0$ at $x_1 = x_5 = 1$, and thus

\[ s_1 s_3 W_2 = y_1 y_3 W_2 + O(s_1^{-1} s_3^{-1}) \]

where the function $D^{(2)}(x)$ being defined by (A.3).

Using (23)-(26) and the property that $y(x)$ behaves like $-\ln x$ near $x = 1$, we can write the following asymptotic behaviour for the amplitude (21):

\[ P_n \sim q^n (-s_1)^{n_1} (-s_2)^{n_2} \int_0^1 dx_2 dx_3 dx_4 dx_6 \propto (\omega) \]

\[ \frac{\psi'(x_2) + \psi'(x_4)}{\psi(x_2) + \psi(x_4)} \left\{ \frac{\psi(x_2) + \psi(x_4)}{\psi(x_2) + \psi(x_4)} \right\}^{-\frac{3}{2}} \propto (\omega) \]

where $\omega = x_2 x_3 x_4 x_6$, $v_2 = v_2(x_1 = x_5 = 1)$, $u_2 = u_2(x_1 = x_5 = 1)$ and

\[ u_2(x_2, x_3, x_4, x_6) = \ln \left[ \frac{\psi(x_2) + \psi(x_4)}{\psi(x_2) + \psi(x_4)} \right] \]

\[ u_3(x_2, x_3, x_4, x_6) = u_1(x_4, x_3, x_2, x_6) \]

Next we introduce complex helicity-like integrations applying the identity

\[ e^x = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} d\lambda \Gamma(-\lambda) (-x)^\lambda \]
to the last exponential in (27). This allows to perform $y_1$ and $y_2$ integrations in (27) ending up with

$$P_e \sim q^6 (-s_3)^{y_1} (-s_3)^{y_2} \int_0^m dx_2 dx_3 dx_4 dx_6 \alpha(w)$$

$$\Psi(x_3) \Psi(x_4) \{\Psi(x_2) \Psi(x_6)\}^{-1} e^{\int - \sum \gamma_i \alpha_i} \epsilon^{z_2 y_2}$$

$$\left(\frac{1}{2 \pi i}\right) \int_{\gamma \rightarrow \infty} d \alpha \Gamma(\alpha) \Gamma(\alpha - y_1) \Gamma(\alpha - y_2) \Gamma(y_1 + \alpha \gamma_2 \omega_2) (\gamma_2 + \alpha \gamma_2 \omega_2)^{y_2 - 1} \gamma_1 \gamma_2 \omega_2 \omega_3$$

Using the identity (28)

$$\int_{t = -1}^{t = \infty} ds s^{-a} \Gamma(a) \Gamma(a + s) = \Gamma(a) \Gamma(a + 1) = x^{a} \psi(x, s)$$

A integration can be performed in (29), getting thus the following asymptotic behaviour:

$$P_e \sim q^6 (-s_3)^{y_1} (-s_3)^{y_2} \int_0^m dx_2 dx_3 dx_4 dx_6 \alpha(w)$$

$$\Psi(x_3) \Psi(x_4) \{\Psi(x_2) \Psi(x_6)\}^{-1} e^{\int - \sum \gamma_i \alpha_i} \epsilon^{z_2 y_2}$$

$$\left(\frac{1}{2 \pi i}\right) \int_{\gamma \rightarrow \infty} d \alpha \Gamma(\alpha) \Gamma(\alpha - y_1) \Gamma(\alpha - y_2) \Gamma(y_1 + \alpha \gamma_2 \omega_2) (\gamma_2 + \alpha \gamma_2 \omega_2)^{y_2 - 1} \gamma_1 \gamma_2 \omega_2 \omega_3$$

(30)

where $F(z)$ is the function defined in Eq.(6) and $z$ is a function of the integration variables, given by

$$z(x_2 x_3 x_4 x_5 x_6) = (y_3 + x_1 y_2 x_1) (y_3 + x_2 y_3 x_2)$$

From (30) we can extract the two-reggeon $\rightarrow$ two-particle amplitude, corresponding to the diagram shown in Fig.1, as given by

$$P_e (-s_1)^{-y_1} (-s_2)^{-y_2} \Gamma(\alpha - y_1) \Gamma(\alpha - y_2)$$

and whose Regge behaviour we are going to consider now.

At this point the discussion follows parallel to that of tree diagrams in Sec.II. Integral (30) is defined for $-t_2 < Re z < \mu^2$, between normal thresholds in the direct and crossed channels, and we shall choose to take the limit $\alpha_0 = \alpha$, like in tree diagrams, for Re $z_2$ fixed, and inside the strip of convergence, and Im $\alpha_2$ $\to \infty$ Thus, decomposition (9) for $F(z)$ holds and we can write the amplitude, as in Eq.(10),

$$P_6 = p_6 + p_6'$$

(32)

Let us first investigate the $\alpha$ sector. The asymptotic behaviour of $P_6$ is dominated by the contribution of the integrand near some critical points. In order to locate critical points let us study the function $Y_2(z_2 x_1 x_4)$ (disregarding $x_3$ and $x_6$ dependence). It is easily proved, with the aid of properties summarised in the appendix, that

$$\frac{3y_2}{\alpha x_2} (x_2 + x_4) = 0$$

so that the whole boundaries $x_2 = 1$ and $x_4$ arbitrary are made up of critical points. The integrand must be expanded around $x_2 = x_4 = 1$ and the contribution of both critical points added. The vanishing of $V_2$ at $x_2 = x_4 = 1$ ensures the regularisation of the amplitude. The following expansions around the critical points can be obtained.
where $h(x_3,x_5) = \omega_2(1,x_3,x_5)$ as given by (26). On the other hand, using (A.2) we have

\begin{align*}
v_2 &\sim (x_2^{-1} \cdot x_4^{-1} \cdot h(x_3,x_5) ,
\end{align*}

\begin{align*}v_3 &\sim (x_2^{-1} \cdot h(x_3,x_5) ,
\end{align*}

\begin{align*}(A.2)
\end{align*}

so that the amplitude $P_6^{(a)}$ can be written as

\begin{align*}P_6^{(a)} \sim \Delta^{(a)} \frac{\Delta \cdot \Gamma(-\lambda)}{(-\lambda)^{k_0} \cdot (-\lambda)^{k_0} \cdot (-\lambda)^{k_0}} \cdot \int_0^1 dx_3 dx_5 \alpha(x)
\end{align*}

the function $I^{(a)}$ defined by

\begin{align*}I^{(a)}(x_3,x_5) = \int_0^1 dx_3 dx_5 \left( \frac{\Delta \cdot \Gamma(-\lambda)}{(-\lambda)^{k_0} \cdot (-\lambda)^{k_0} \cdot (-\lambda)^{k_0}} \right) \cdot \int_0^1 dx_3 dx_5 \alpha(x)
\end{align*}

\begin{align*}(A.23)
\end{align*}

with $\Delta^{(a)}$ given by Eq.(12). In order to compute the asymptotic behaviour of (36) we shall proceed as in Ref.14 and change variables as $x_{k_1} = \cdots $

\begin{align*}I^{(a)}(x_3,x_5) = \int_0^1 dx_3 dx_5 \left( \frac{\Delta \cdot \Gamma(-\lambda)}{(-\lambda)^{k_0} \cdot (-\lambda)^{k_0} \cdot (-\lambda)^{k_0}} \right) \cdot \int_0^1 dx_3 dx_5 \alpha(x)
\end{align*}

\begin{align*}(A.26)
\end{align*}

\begin{align*}
\Delta^{(a)} \left( \frac{\Delta \cdot \Gamma(-\lambda)}{(-\lambda)^{k_0} \cdot (-\lambda)^{k_0} \cdot (-\lambda)^{k_0}} \right) \cdot \int_0^1 dx_3 dx_5 \alpha(x)
\end{align*}

\begin{align*}(A.27)
\end{align*}

and the last integral in (37) can be evaluated asymptotically making the change of variable $y = x_2h$ and getting thus

\begin{align*}I^{(a)}(x_3,x_5) \sim \Delta^{(a)} \left( \frac{\Delta \cdot \Gamma(-\lambda)}{(-\lambda)^{k_0} \cdot (-\lambda)^{k_0} \cdot (-\lambda)^{k_0}} \right) \cdot \int_0^1 dx_3 dx_5 \alpha(x)
\end{align*}

\begin{align*}(A.28)
\end{align*}

Substitution of (38) into (35) allows us to write the asymptotic behaviour of $P_6^{(a)}$ as

\begin{align*}P_6^{(a)} \sim \Delta^{(a)} \left( \frac{\Delta \cdot \Gamma(-\lambda)}{(-\lambda)^{k_0} \cdot (-\lambda)^{k_0} \cdot (-\lambda)^{k_0}} \right) \cdot \int_0^1 dx_3 dx_5 \alpha(x)
\end{align*}

\begin{align*}(A.29)
\end{align*}

Let us remark that the Regge trajectory $\alpha_2$ has a residue singular at $\omega_2 = 0,1,2,\ldots$ even if the usual function $\Gamma(-\lambda)$ cannot be extracted from the last integral in (39). Expression (39) is a typical renormalization of the Regge trajectory $\alpha_2$ and the Regge residue, as given by tree diagrams in Eq.(11). In Sec.V we shall comment on some features of this renormalization.

In the $\Delta$ sector, using (7b) we can write

\begin{align*}F^{(1)}(a) = \sum_{n=0}^{\infty} \left[ \Gamma(a-n,\lambda) \cdot \Gamma(a-n,\lambda) \cdot \Gamma(a-n,\lambda) \cdot \exp \left\{ i\pi n \cdot (-\lambda) \right\} \right] \cdot \sum_{m=0}^{\infty} \left( \frac{\Delta \cdot \Gamma(-\lambda)}{(-\lambda)^{k_0} \cdot (-\lambda)^{k_0} \cdot (-\lambda)^{k_0}} \right) \cdot \int_0^1 dx_3 dx_5 \alpha(x)
\end{align*}

\begin{align*}(A.30)
\end{align*}
where $z' = z(\phi \equiv 1)$ and $z$ is given by Eq. (31). In Eq. (40) use has been made of the analyticity of $\phi$ functions.

In this way we can see that the asymptotic behaviour of $p(\phi)$ is governed by the exponential $\exp(-\phi f(x_2 x_3 x_6))$ with the function $f$ given by

$$f(x_2 x_3 x_6) = v_2 - v_1 w_3 (w_3)^{-1},$$  \hspace{1cm} (41)$$

whose critical points must be found in order to expand the integrand around them. As we have proved above, $x_2 = 1$ and $x_4 = 1$ are critical points of the function $v_2$. Furthermore, using definition (24) it is easily proved that

$$\frac{\partial}{\partial x_2} \{v_2 v_2^{-1} (x_2 x_6 = 1) = \frac{\partial}{\partial x_4} \{v_2 v_2^{-1} (x_2 = 1, x_4) = 0,$$

so that $x_2 = x_4 = 1$ are the critical points of the exponent $f$ and we must expand function $f$ in a double series expansion around $x_2 = 1$ and $x_4 = 1$ as

$$f(x_2 x_3 x_6) = A(x_3 x_6) (x_2 - 1) (x_4 - 1) + B(x_3 x_6) (x_2 - 1)^2 (x_4 - 1) +$$

$$+ C(x_3 x_6) (x_2 - 1)^2 + D(x_3 x_6) (x_2 - 1)^3 (x_4 - 1)^2 + \cdots$$  \hspace{1cm} (42)$$

As can easily be seen, terms in $(x_2 - 1) (x_4 - 1)$ coming from $v_2$ and from $v_1 w_3 (w_3)^{-1}$ cancel, so that $A = 0$ in (42). This cancellation reminds us of a similar one in tree diagrams where the linear term in $-\phi_A A(x_1 x_2)$ is cancelled by $-\phi_A x_2$ giving rise to the $\phi$ contribution. The mechanism of cancellation is identical, although more complicated, in the planar loop we are considering. Moreover, function $f$ can be expanded in a simple power series around $x_4 = 1$ as

$$f(x_2 x_3 x_4 x_6) = (x_2 - 1) \frac{\partial f}{\partial x_2} (x_2 - 1) + \mathcal{O}((x_2 - 1)^2)$$

and using the property (A.6) we find

$$\frac{\partial f}{\partial x_2} (x_2 = 1) = - \frac{1}{\ln^2} \{D(x_2) + D(x_4),$$

$$\frac{\partial f}{\partial x_4} (x_4 = 1) = \frac{1}{\ln^2} D(1) (x_4),$$

so that

$$\frac{\partial f}{\partial x_2} (x_2 = 1) = \frac{\partial f}{\partial x_4} (x_4 = 1) = \frac{\partial f}{\partial x_2} (x_2 = 1) = \frac{\partial f}{\partial x_4} (x_4 = 1) = 0,$$

proving that there are no linear terms in $(x_2 - 1)$. In particular, it implies that coefficients of $(x_1 - 1) (x_2 - 1)^n$ ($n = 1, 2, 3, \ldots$) in the expansion (42), vanish, so that $B(x_3 x_6) = 0$.

In the same way, expanding $f$ around $x_2 = 1$, it can be proved that coefficients of $(x_2 - 1) (x_4 - 1)^n$ ($n = 1, 2, 3, \ldots$) are cancelled and, in particular, $C(x_3 x_6) = 0$. We can thus write

$$f(x_2 x_3 x_4 x_6) = (x_2 - 1)^2 (x_4 - 1)^2 E(x_3 x_6),$$  \hspace{1cm} (43)$$

and expand the rest of the integrand around $x_2 = 1$ and $x_4 = 1$, bearing in mind that the dominant part of $p(\phi)$ will be given by points $x_2$ and $x_4$ such that $(x_2 - 1) (x_4 - 1) \sim \phi^{S^2/2}$. Thus, using (8), the function $F(\phi) (\phi)$ can be estimated, near the critical point, as
where $\mathcal{P}(s)$ is given by Eq. (14), and the whole discussion following this formula could now be repeated.

Using (43), (44), (33) and (34) the amplitude $\mathcal{P}_6(s)$ can be shown to have the following asymptotic behaviour:

$$
\mathcal{P}_6(s) \sim 3^6 \mathcal{P}_1(s) \left(\begin{array}{c}
\gamma_1 \gamma_2
\end{array}\right) e^{-\frac{s}{2}} \left(\begin{array}{c}
\gamma_1 \gamma_2
\end{array}\right)^3 
\int_0^1 \frac{\delta x_5 \delta x_6 \mu(\omega)}{[\sigma(s,x_5,x_6)]^2} \mu(\omega) \Gamma_1^{\frac{1}{2}} \Gamma_2^{\frac{1}{2}} \Gamma_1^{\frac{1}{2}} \Gamma_2^{\frac{1}{2}}
$$

where the $\Delta$-dependent function $\mathcal{P}_6(s)$ is given by

$$
\mathcal{I}_6(s_x,s_y) = \frac{1}{d_x d_y} \mu(s_x,s_y) \left(\begin{array}{c}
\gamma_x \gamma_y
\end{array}\right) E(s_x,s_y)
$$

The asymptotic behaviour of $\mathcal{I}_6(s)$ is easily estimated using the same methods as those used to compute $\mathcal{I}_6(s)$ in Eqs. (37) and (39), leading thus to

$$
\mathcal{P}_6(s) \sim \frac{1}{s} \frac{d_x d_y}{\Delta_x \Delta_y} \left(\begin{array}{c}
\gamma_x \gamma_y
\end{array}\right) E(s_x,s_y)
$$

or

$$
\left\{ \delta x_5 \delta x_6 \mu(\omega) \right\} \left(\begin{array}{c}
\gamma_1 \gamma_2
\end{array}\right) e^{-\frac{s}{2}} \left(\begin{array}{c}
\gamma_1 \gamma_2
\end{array}\right)^3 
$$

where

$$
\mathcal{P}_6(s) \sim 3^6 \mathcal{P}_1(s) \left(\begin{array}{c}
\gamma_1 \gamma_2
\end{array}\right) e^{-\frac{s}{2}} \left(\begin{array}{c}
\gamma_1 \gamma_2
\end{array}\right)^3 
$$

$$
\int_0^1 \frac{\delta x_5 \delta x_6 \mu(\omega)}{[\sigma(s,x_5,x_6)]^2} \mu(\omega) \Gamma_1^{\frac{1}{2}} \Gamma_2^{\frac{1}{2}} \Gamma_1^{\frac{1}{2}} \Gamma_2^{\frac{1}{2}}
$$

Substitution of (47) into (45) gives for the amplitude

$$
\mathcal{P}_6(s) \sim 3^6 \mathcal{P}_1(s) \left(\begin{array}{c}
\gamma_1 \gamma_2
\end{array}\right) e^{-\frac{s}{2}} \left(\begin{array}{c}
\gamma_1 \gamma_2
\end{array}\right)^3 
$$

$$
\int_0^1 \frac{\delta x_5 \delta x_6 \mu(\omega)}{[\sigma(s,x_5,x_6)]^2} \mu(\omega) \Gamma_1^{\frac{1}{2}} \Gamma_2^{\frac{1}{2}} \Gamma_1^{\frac{1}{2}} \Gamma_2^{\frac{1}{2}}
$$

Eq. (48) is the contribution to the asymptotic behaviour coming from a double pole in the complex $\delta$ plane at $\delta = \delta(s_x)$. It can be interpreted as a renormalisation effect, to order $\delta^2$, of the leading $\delta$ trajectory in the $\delta_x$ channel. The explicit calculation of the function $H(s_5,\delta_x)$ is straightforward but lengthy. We shall need it to discuss the renormalisation properties of the amplitude in the next section. Making use of the properties quoted in the appendix the result can be written as

$$
E(s_x,s_y) = \frac{1}{s} \frac{d_x d_y}{\Delta_x \Delta_y} \left(\begin{array}{c}
\gamma_x \gamma_y
\end{array}\right) E(s_x,s_y)
$$

where functions $D^{(2)}$ and $D^{(3)}$ are explicitly given by (A.1) and (A.2).
IV. AMPLITUDE RENORMALIZATION

The results concerning linear triple-Regge limits computed in Secs. II and III can be summarized as follows.

In the a sector the amplitude \( A_6 = T_6 + P_6 \) can be written as

\[
A_6^{(a)} \sim q^2 \left( \beta_2 \right)^2 \Gamma(\beta_2) \Gamma(\beta_3) \Gamma(\beta_4) Y^{(a)}(\beta_2) (-s_2) \\
+ \left( \beta_2 \right)^2 \frac{d^4 \alpha}{d^4 \Omega} \frac{d^4 \Omega}{d^4 \alpha} (-s_2) Y^{(a)}(\beta_2) (-s_2)
\]

(50)

where \( \alpha_2(t_2) = \alpha_2 + g^2 \beta_2(t_2) + O(g^4) \) is the new trajectory and \( \beta_2^{(a)}(t_2) \)

\[
= 1 + g^2 \beta_2(t_2) + O(g^4)
\]
is the new residue, and

\[
\sum \psi^{(a)}(t_2) = \int_0^1 d\alpha_3 d\alpha_4 \alpha(\alpha) Y^{(a)}(\alpha) \left\{ - \frac{1}{\Delta_2 \omega} J^{(1)}(\alpha_3) \right\} \xi^{-1} \\
\left( \frac{1}{\Delta_2 \omega} \right) J^{(1)}(\alpha_4) \\
(51a)
\]

\[
\psi^{(a)}(t_2) = \int_0^1 d\alpha_3 d\alpha_4 \alpha(\alpha) \left\{ Y^{(a)}(\alpha) \left\{ - \frac{1}{\Delta_2 \omega} J^{(1)}(\alpha_3) \right\} \xi^{-1} \\
\right\} \left\{ \frac{1}{\Delta_2 \omega} \right\} J^{(1)}(\alpha_4)
\]  

(51b)

Let us remark that \( A_6^{(a)} \), as given by Eq. (50), is equal to the sum of \( A_6^{(a)} \),
Eq. (11), and \( P_6^{(a)} \), Eq. (39), only to order \( g^2 \), so that the contribution of planar loops to the two-reggeon \(-\) two-particle amplitude must be interpreted, as it has been usually recognized in N-particle amplitudes (6), (7), in the a sector as a renormalization to order \( g^2 \) of the leading Regge trajectory \( a(t_2) \).

In particular, the function \( Y^{(a)}(t_2) \) can be written as

\[
\sum \psi^{(a)}(t_2) = \int_0^1 d\alpha_3 d\alpha_4 \alpha(\alpha) \left\{ \beta_2^{(a)}(t_2) \right\} \left\{ - \frac{1}{\Delta_2 \omega} J^{(1)}(\alpha_3) \right\} \xi^{-1} \\
\left( \frac{1}{\Delta_2 \omega} \right) J^{(1)}(\alpha_4)
\]

(52)

where we changed variables from \((x_3, x_4)\) to \( \omega = x_3 x_4 \), \( \theta = \pi \) in \( x_6/\pi \omega \) and the functions \( \psi(\beta, \omega) \) and \( D^{(1)}(\beta, \omega) \) are easily obtained from \( \psi \), Eq. (19), and \( D^{(1)} \), Eq. (A.3). However, the integral (52) is badly divergent at the corner \( \omega = 1 \) because the measure of integration \( \alpha(\omega) \) and, in particular, the partition function \( Z^{(1)}(\omega) \), diverges as \( \omega^{-1/2} \exp\left[ \frac{1}{6}(1-\omega) \right] \) when \( \omega = 1 \).

In this way the amplitude must be renormalized and the infinite part of the integrand of \( \bar{z}^{(a)} \) subtracted. It is easily proved that

\[
\left( \frac{e^{2}\rho^{(a)}(\beta, \omega)}{2\omega} \right)^{\frac{1}{\rho^{(a)}(\beta, \omega)}} = 1 + 0(g^4),
\]

(53)

and thus the divergent part of \( \bar{z}^{(a)} \) is independent of \( t_2 \) and a single subtraction is enough to make \( \bar{z}^{(a)}(t_2) \) finite.

The renormalization program for N particle amplitudes has been performed by Neveu and Scherk (5, 7). They introduce a cut-off \( \theta(N-\epsilon) - \sum x_i \), which does not change the singularity structure of the loop, and subtract from the amplitude \( P_N \), a counter-term \( P_{\epsilon} \), so that the difference \( P_N - P_{\epsilon} \) is finite in the limit \( \epsilon \to 0 \), and such that the properties of crossing symmetry, duality and Regge behavior should be preserved. They explicitly construct a counter-term with these properties, given by

\[
P_{\epsilon} = P_N \left( \frac{\epsilon}{\epsilon - \rho} \right),
\]

that means that \( \rho \) functions entering into the definition of \( P_N \) must be replaced by

\[
\rho(x_4) = - \frac{\Delta u}{\lambda} \sin \left( \frac{\sin x_4 \Delta u}{\lambda} \right).
\]

Using Neveu and Scherk's prescription to renormalize \( P_N \), Eq. (18), we find Eq. (50) for the amplitude \( A_6 \) in the a sector, but here \( \alpha(\beta, t_2) = a(t_2) + g^2 \beta_2(t_2) - \psi(a) \), \( \beta_2^{(a)}(t_2) = 1 + g^2 \beta_2(t_2) \psi(a) \) and

-22-
with the notation \( S_n^{(a)}(t) = \Xi_n^{(a)}(t) \). In particular, at \( t_2 = 0 \)

\[
\sum \langle \omega \rangle - \bar{\Xi}^{(a)}(\omega) = \frac{1}{n^2} \int d\omega \frac{\omega(\omega)}{\cos^2 \omega} \int d\phi \left\{ \bar{I}^{(a)}(\phi, \omega) - \bar{\Xi}^{(a)}(\phi, \omega) \right\},
\]

and using (A.3) we have that

\[
\bar{I}^{(a)}(\phi, \omega) - \bar{\Xi}^{(a)}(\phi, \omega) = \frac{1}{n^2} \sum_{m=1}^{\infty} \frac{m q^{\Delta_m}}{1 - q^{2m}} \cos 2m \theta,
\]

which vanishes under \( \theta \) integration. The intercept \( a(0) \) of the leading trajectory is not shifted from its bare value in the renormalization procedure. This has been recognized by Neveu and Scherk to be a consequence of the gauge invariance of dual theories in the Virasoro case \( a(0) = 1 \).

Next, let us see what happens in the \( \theta \) sector. The amplitude \( A_\theta \) can be cast into the following form:

\[
A_\theta^{(a)} \sim \frac{1}{\eta^4} \bar{I}_1^{(a)}(\omega) - \bar{I}_1^{(a)}(\omega) \sim \frac{1}{\eta^4} \bar{I}_1^{(a)}(\omega) \sum_{m=1}^{\infty} \frac{q^{\Delta_m}}{1 - q^{2m}} \cos 2m \theta,
\]

The divergent part of \( \bar{I}_1^{(a)}(\omega) \) is independent of \( t_2 \), and again a single subtraction is enough to render \( \bar{I}_1^{(a)}(t_2) \) finite. Thus, using the Neveu-Scherk renormalization prescription, we get

\[
\bar{I}_1^{(a)}(t_2) = \frac{1}{\eta^4} \int d\omega \frac{\omega(\omega)}{\cos^2 \omega} \int d\phi \left\{ \bar{I}^{(a)}(\phi, \omega) - \bar{\Xi}^{(a)}(\phi, \omega) \right\},
\]

and a similar expression for \( \rho^{(a)}(t_2) \).

The integral (57) is also divergent due to the behaviour of the measure \( a(0) \) at the corner \( \omega = 1 \). It is thus important to know whether the Neveu-Scherk renormalization is able to render the theory finite in the \( \theta \) sector.

To this end we must analyse the \( t_2 \)-dependent quantity in the integrand of (57). Using the explicit forms (A.3)-(A.5) we find

\[
\bar{I}_1^{(a)}(t_2) = \frac{1}{\eta^4} \int d\omega \frac{\omega(\omega)}{\cos^2 \omega} \int d\phi \left\{ \bar{I}^{(a)}(\phi, \omega) - \bar{\Xi}^{(a)}(\phi, \omega) \right\},
\]

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\[
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\]

and a similar expression for \( \rho^{(a)}(t_2) \).

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\]

and a similar expression for \( \rho^{(a)}(t_2) \).

The integral (57) is also divergent due to the behaviour of the measure \( a(0) \) at the corner \( \omega = 1 \). It is thus important to know whether the Neveu-Scherk renormalization is able to render the theory finite in the \( \theta \) sector.
by the renormalization procedure, we must evaluate \( I^a(0) - I^a(B) \). As can easily be seen from (57) and (60) the mechanism leading to the vanishing of \( \chi(a(t) - \Gamma^a) \) does not hold in this case, so that \( B(0) \neq B(0) = 0 \) and the intercept of the leading \( S \) trajectory is renormalized by the unitarity corrections to the Born approximation. We shall briefly comment, in the following section, on some possible phenomenological implications of this fact.

We think there is no conflict between the renormalization of the intercept of leading \( S \) trajectories and the gauge invariance of dual theories for \( a(0) = 1 \). In this section we made no distinction between different dimensionalities of spacetime \( D \), and subsequent values of \( E \) and \( F \), determining the precise functional form of the measure of integration \( a(\omega) \), as given by Eq.(20). In particular, one can see that near \( \omega = 1 \), \( a(\omega) \sim q^{-1/3} \), so that the renormalization prescription renders the theory finite in the \( \alpha \) and the \( S \) sector. However, for \( D = 4, E = 1 \), we have \( a(\omega) \sim q^{-1/3} \), so that the renormalization prescription renders the theory finite in the \( \alpha \) and the \( S \) sector. Nevertheless, in \( D = 26, E = 2, a(\omega) \sim q^{-2} \), and, using (55) and (58), there still remains a logarithmic divergence in \( \chi(a(t) - \Gamma^a) \) and in \( \chi(\beta(t) - \Gamma^\beta) \).

It is usually thought that this divergence is entirely due to the existence of massless particles. Thus in a more realistic model with \( a(0) < 1 \), no such particles would appear, and the theory should be finite both in the \( \alpha \) and in the \( S \) sector.

V. CONCLUSION

In this paper we derive the contribution of one-loop planar diagrams in the CDRM to two-reggeon two-particle amplitudes and analyse its Regge limit. This is done, as usual, by taking the linear triple-Regge limit of the six-point one-loop planar diagram. The result splits into two separate contributions, which we call \( \alpha \) and \( S \) sectors, in an analogous way to what happens for tree diagrams. The contribution in the \( \alpha \) sector can be interpreted as a renormalization, to order \( q^2 \), of the \( \alpha \) trajectory. On the other hand, in the \( S \) sector, we find strong cancellations in the function governing the asymptotic behaviour, like in tree diagrams, so that it can be interpreted also as a renormalization, to order \( q^2 \), of the \( S \) trajectory. Both renormalizations are divergent due to the divergence of the measure of integration at the corner \( \omega = 1 \). We have proved that the renormalization prescription introduced by Neveu and Scheck is able to render finite both renormalizations so that the usual regularization of dual amplitudes must not be enlarged in order to contain also the \( S \) sector.

Recently Hoyer 20 analysed the behaviour of the eight-particle amplitude, for tree diagrams, in a helicity pole limit and found, besides the \( \alpha \) and \( S \) trajectories, the contribution of a further trajectory, \( \gamma = \frac{1}{3}. \) This led him to speculate about the existence of a whole family of trajectories, \( \alpha_0 = \frac{1}{3} (K-1), K = 1,2,3,..., \) such that the coupling of \( \alpha_0 \) to states of no more than \( K \) particles vanishes. Some of these trajectories were detected and studied by Sarbushaei, Zakrzevski and Barrett 11 in the multi-Regge limit of seven- and eight-particle amplitudes. In view of the simplicity of the mechanism which led to the results of our paper, we are convinced that unitarity corrections to seven-, eight-..., particle amplitudes have the effect of renormalizing \( \alpha_0 \) trajectories. However, an explicit proof of that assertion is lacking for the moment.

Another open question is related to one-loop non-planar orientable diagrams which are usually associated with the exchange of the Pomeron trajectory in the crossed channel. The structure of the six-point function in the Shapiro-Virasoro model has recently been investigated by Barrett 21 in the multi-Regge limit, providing evidence for the existence of a Pomeron sister trajectory, related to the usual trajectory by \( \beta_0 = \frac{1}{3} \alpha_0 = 1 \). Thus the appearance of a Pomeron sister trajectory in one-loop non-planar orientable diagrams is expected and investigation of these diagrams in a multi-Regge limit is necessary in order to get a deeper insight into the intriguing connection between non-planar models and the corresponding planar ones, which suggests that non-planar models may describe Pomeron states.

To conclude this section we shall remark that \( \alpha \) and \( S \) trajectories, \( \alpha_0 = \alpha_0(q^2 a(\alpha), \beta_R = \beta_R(q^2 c(\beta)) \), are not renormalized in the same way: the functions \( a(\alpha) \) and \( c(\beta) \) are indeed very different. Thus the \( t \) region over which the \( \beta \) contribution dominates is changed by the renormalization. In particular, we have found that \( a(\alpha)(0) = 0 \) while \( c(\beta)(0) \neq 0 \), so that at values \( |t| \approx 0 \) the renormalization of the \( S \) trajectory should be taken into consideration. To decide whether \( c(\beta) \) is greater or less than \( a(\beta) \), a detailed analysis of \( c(\beta) \) must be made and the "reggeonic" part of non-planar graphs, \( c(\beta) \), added. However, the dominance of \( \beta(\beta) \) over \( \alpha(t) \) in the CDRM is effective in a region where the contribution of Regge cuts is expected to become relevant and renormalisation effects are not expected to be so strong as to shift this region sufficiently towards \( t = 0 \). Nevertheless, in more realistic dual models, as the Neveu-Schwarz model, the presence of new
trajetories in multi-Regge limits has been equally found. The phenomenological implications of the new trajectories has recently been discussed by Sarbishai, Zakszewski and Barrett in the framework of the Neveu-Schwarz model, and a way to detect the $B$ trajectory experimentally is proposed.

Thus, if the renormalization of the $B$ trajectory does not vanish at $t=0$, as we have proved to happen in the CDMN, then renormalization effects could enhance or depress the $B$ trajectory, and they should be considered.

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APPENDIX

In this appendix we shall enlist the definition and main properties of the functions used throughout this paper.

Functions $D^{(n)}(x)$ are defined as follows:

$$D^{(n)}(x) = \frac{\theta}{\theta_{1}(v)} \ln \theta_{1}(v),$$

where $\theta_{1}$ is the usual Jacobi function, $v = \sin x / \sin \omega$ and $D^{(0)}(x) = D(x)$.

An explicit series representation for $n = 0, 1, 2$ and 3 is given by

$$D^{(0)}(x) = \pi \cosh \pi x + 4 \pi \sum_{m=1}^{\infty} \frac{q^{2m}}{\sin^{2} \pi x} \sin 2m \pi x,$$

$$D^{(1)}(x) = -\frac{\pi^{2}}{\sin^{2} \pi x} + 8 \pi^{2} \sum_{m=1}^{\infty} \frac{m^{2} q^{2m}}{1 - q^{2m}} \cos 2m \pi x,$$

$$D^{(2)}(x) = 3 \pi^{2} \frac{\cos \pi x}{\sin^{2} \pi x} - 16 \pi^{2} \sum_{m=1}^{\infty} \frac{m^{2} q^{2m}}{1 - q^{2m}} \sin 2m \pi x,$$

$$D^{(3)}(x) = -3 \pi^{2} \frac{1 + 2 \cos \pi x}{\sin^{2} \pi x} - 32 \pi^{2} \sum_{m=1}^{\infty} \frac{m^{2} q^{2m}}{1 - q^{2m}} \cos 2m \pi x,$$

where $q = \exp(\pi^{2} / \ln \omega)$.

A very useful property, which can be inferred from (A.1)–(A.5), is

$$D^{(n)}(\omega / x) = (-1)^{n+1} D^{(n)}(x).$$
REFERENCES

9) M. Quiro, in preparation.
Fig. 1

Two-Reggeon two-particle amplitude as a double-Regge limit of the six-particle amplitude.

Fig. 2

Tree graph used in the calculation of Sec.II.

Fig. 3

Multi-Regge limit of the six-particle amplitude in the untwisted configuration.

Fig. 4

Multi-Regge limit of the six-particle amplitude in the double twisted configuration.

Fig. 5

Dual loop used in the calculation of Sec.III.
IC/78/28 * B.S. LEE: Helical ordering in the Jahn-Teller cooperative phase transition.

IC/78/29 S.S. AHMAD: The iterative R-matrix method for the accurate calculation of nuclear reaction cross-sections.

IC/78/30 J.E. HSSBIGI: G-B structures and space-time geometry - I: Geometric objects of higher order.

IC/78/31 S.C. LIM: Euclidean massive spin-3 field which is Markovian.

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