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ABSTRACT

It is argued that in a non-Abelian gauge theory with no bare Higgs self-couplings, asymptotic freedom is not disturbed. This is because the effective Higgs couplings (induced through gauge particle exchanges) are shown to be convergent and gauge invariant (on mass shell) provided we use renormalization group estimates for the asymptotic behaviour of vertex and self-energy insertions. We show that these insertions are well approximated, so far as potential infinities of scalar-scalar amplitudes are concerned, by the (gauge-invariant) running coupling constants replacing renormalized constants in the unmodified sets of graphs. In the case of weak coupling, we find that Higgs and vector meson masses are in general of the same order of magnitude.

I. INTRODUCTION

Renormalizable gauge theories need scalar Higgs fields in order to realize spontaneous symmetry breaking in the context of a perturbative scheme. Strict perturbative renormalizability implies independent (non-computable) quartic couplings among the scalars. It is well known that these couplings tend to spoil asymptotic freedom.[1)]

Assume that these bare quartic couplings are missing, e.g. (temporarily ignoring fermions) let the Lagrangian be $-\frac{1}{4}F_{\mu\nu}F_{\mu\nu}+\frac{1}{2}(\nabla_\mu\phi)^2 - \frac{1}{2}\phi^T m_0^2\phi$, where $\nabla_\mu\phi = \partial_\mu\phi - g_0 A_\mu^\alpha\theta_\alpha\phi$. There are no bare terms of the type $\lambda_0\phi^4$. Scalar-scalar amplitudes are induced through exchanges of gauge mesons. If such amplitudes were truly infinite, renormalized $\lambda\phi^4$ terms would be needed to absorb the logarithmic infinities of the 4-scalar amplitudes, induced by gauge particle exchange. In this paper we show that the conventional infinities associated with these amplitudes are an artifact of a strict perturbation expansion. The result is proved by using in the unmodified graphs vertex and self-energy insertions as improved by the renormalization group. We show that so far as the potential infinities of scalar-scalar amplitudes are concerned, the result of these insertions is well approximated to, in asymptotically free theories, by the ansatz of replacing the renormalized coupling constants by the running coupling constants in the unmodified diagrams. The induced effective Higgs couplings thus computed are (on shell) gauge-invariant and finite (but not analytic in the gauge couplings).

The proof that self-energy and vertex insertions in the unmodified graphs[2)] are equivalent to the replacement $g_{\text{renormalized}} \to \bar{g}(k)$ (so far as infinity-counting estimates for the scalar-scalar amplitude are concerned) is presented in the Appendix. The proof relies on showing that in a gauge theory, Ward's identities ensure that vertex insertions essentially cancel with the scalar meson self-energy insertions, leaving behind only the insertions associated with the gauge particle propagator. The proof is carried through for a U(1) gauge theory and in an axial gauge for non-Abelian theories. We remark now that - following the original definition of Gell-Mann and Low[3)] - the gauge propagator modifications are precisely what defines the ratio of the running gauge coupling constant $\bar{g}(k)$ to the renormalized constant g. Specializing to the non-Abelian case, the fact that the running constant $\bar{g}(k)$ is driven to zero, for asymptotically free theories, implies that in the deep Euclidean region, the resulting (Wick-rotated) integrals converge. (Since the running coupling constants are thought to be gauge independent, these replacements do not disturb on-shell gauge invariance. [4])

Consider as an illustration the 2-gluon contribution to the lowest-order scalar-scalar amplitude (Fig.1). With the modification $g \to \bar{g}(k)$ each of these graphs makes a contribution of the form

$$\int d^4k \quad \frac{\bar{g}(k)^4}{k^4}$$

(where we have neglected both masses and external momenta in comparison to the loop momentum k). In the unimproved version with renormalized constant g instead of $\bar{g}(k)$ these integrals diverge. Our main point is to remark that such a pure perturbation approximation is a poor approximation: according to renormalization group estimates - which are reliable if the theory is asymptotically free - the effective coupling $\bar{g}(k)$ behaves like $(\ln k^2)^{-1/2}$ and the above integrals therefore converge (in the ultraviolet)

$$\int^{\infty} d^4k \, \frac{(\ln k^2)^{-2}}{k^4} < \infty \quad .$$

In the Appendix we shall show that the same type of convergence is obtained after the replacements $g \to \bar{g}(k)$ in higher-order unmodified graphs.

II. SPECTRAL REPRESENTATION FOR THE RUNNING COUPLING PARAMETER

So much for the estimate of the ultraviolet behaviour. What should one use for the running parameter $\bar{g}(k^2)$ in the finite k^2 plane? Consider for the moment the U(1) theory. From the pioneering work of Gell-Mann and Low the effective coupling is tied to the transverse photon propagator [3], $D(k^2,g^2)$, viz.

$$\frac{\bar{g}^2(k^2)}{k^2} = g^2 D(k^2,g) \tag{1}$$

and the structure of D is well understood. It is analytic in the cut k^2 plane with branch points at the various 1^- states to which the photon can couple. (For U(1) there is no asymptotic freedom, but we ignore this for the moment and return to asymptotically free theories in Sec.III.)

To obtain an approximation to the right-hand side of (1), which coincides in the ultraviolet with the estimates obtained by using the Callan-Symanzik [3] formalism for $\bar{g}^2$, write

$$\frac{g^2}{\bar{g}^2} = (k^2 D)^{-1} = 1 + k^2 \int \frac{\rho(\kappa^2)\, d\kappa^2}{\kappa^4 |D(\kappa^2)|^2 (\kappa^2 - k^2)} \quad , \tag{2}$$

where

$$\rho = -\frac{1}{\pi k^2} \operatorname{Im} k^2 D \quad .$$

The spectral representation is a simple consequence of the assumed analyticity, asymptotic behaviour and absence of CDD zeroes for D . (In writing (2), the reference mass of the renormalization group has been taken equal to zero, i.e. $\bar{g}^2(k^2 = 0) = g^2$.)

Now, as is well known, in the deep Euclidean region, the one-loop approximation $\rho \approx \rho_{\text{one-loop}}$ (with the estimate $\kappa^2 D(\kappa^2) = 1$ used in the denominator of (2)) yields for $1/\bar{g}^2$ an expression which, for large k^2 , coincides with the leading term obtained from renormalization group considerations, i.e. $(g^2/\bar{g}^2) = 1 - bg^2 \ln k^2$, $(b > 0)$, for $k^2 \to \infty$. We shall assume henceforth that, to the *one-loop approximation*, $\bar{g}^2(k^2)$ is given completely by this ansatz of substituting $\rho = \rho_{\text{one-loop}}$ and $\kappa^2 D(\kappa^2) = 1$ in the integrand of (2). (One might in fact have thought of this as a good "small g^2 " approximation for $\bar{g}^2$, if the theory had been asymptotically free. As it is, the lack of asymptotic freedom hits us, in that the spectral function $\rho(\kappa^2)$ for a U(1) theory is positive definite - in contrast to the case of non-Abelian gauge theories, where it has the opposite sign. This means that expression (2) contains a spurious CDD pole in $\bar{g}^2(k^2)$ for large spacelike $k^2 \approx \exp[1/bg^2]$. As we shall see, it is precisely this defect which asymptotic freedom cures.)

III. NON-ABELIAN THEORIES

Consider now a non-Abelian gauge theory. To obtain the analogue of (1), the simplest gauge to work with is the axial guage, where - just like the U(1) case - the renormalization of the coupling constant is associated solely with the gauge-particle propagator. Using the analysis and notation (apart from the metric convention) of Frenkel and Taylor [5], we surmise, in analogy with (1) and (2), that

$$\frac{g^2}{\bar{g}(k^2)} = \lim_{k_L^2 \to 0} \left[1 + (k^2 - M^2) \int \frac{\rho_1(\kappa^2, k_L^2) d\kappa^2}{|D_1(\kappa^2, k_L^2)|^2 (\kappa^2 - k^2)}\right] . \qquad (3)$$

Here $k_{L\mu} = \frac{(n \cdot k)}{n^2} n_\mu$, n_μ is a unit (spacelike) 4-vector, used to specify the axial gauge $(n_\mu A_\mu = 0)$, and M^2 is the reference mass $(\bar{g}^2(M^2) = g^2)$, which may coincide with the gauge particle mass $(M^2 = g^2\langle\phi^2\rangle$, see Sec.IV). The functions ρ_1 (and ρ_2) are the decompositions of the spectral function $\rho_{\mu\nu}$ for the propagator $D^{ij}_{\mu\nu} = \delta^{ij}[D_1 P_{\mu\nu} + D_2 \delta^T_{\mu\nu}]$ defined such that

$$\rho^{ij}_{\mu\nu} = \delta^{ij} [\rho_1(\kappa^2, k_L^2) P_{\mu\nu} + \rho_2(\kappa^2, k_L^2) \delta^T_{\mu\nu}] , \qquad (4)$$

where

$$P_{\mu\nu} = \delta_{\mu\nu} - \frac{k_\mu n_\nu + k_\nu n_\mu}{(n \cdot k)} + \frac{k_\mu k_\nu n^2}{(n \cdot k)^2} , \qquad (5)$$

$$\delta^T_{\mu\nu} = \delta_{\mu\nu} - \frac{n_\mu n_\nu}{n^2} \qquad (6)$$

and

$$D_1 = 1 - (k^2 - M^2) \int \frac{\rho_1 d\kappa^2}{\kappa^2 - k^2 - i\epsilon} .$$

Frenkel and Taylor show that in the one-loop approximation (with zero-mass gauge particles, $M^2 = 0$)

$$\rho_1(\kappa^2, k_L^2) = \frac{1}{2\pi\kappa^2} [a^2 C_1 + (a^2 - 1)(3a^2 - 2) C_2] , \qquad (7)$$

where

$$C_1 = \frac{C(G) g^2}{16\pi^2} \left\{1 + \frac{1-a^2}{a^2}\left[\frac{1+a^2}{2a} \ln\frac{1+a}{1-a} - 1\right] + 8\left[a \ln\frac{1+a}{1-a} - 2\right]\right\} ,$$

$$C_2 = \frac{C(G) g^2}{16\pi^2} \left\{\frac{a^2}{1-a^2}\left[1 + \frac{1-a^2}{a^2}\left(\frac{1+a^2}{2a} \ln\frac{1+a}{1-a} - 1\right)\right] + \frac{a^2-1}{a} \ln\frac{1+a}{1-a} + \frac{5-3a^2}{3(a^2-1)}\right\} . \qquad (8)$$

Here C(G) is the Casimir operator for the group and

$$a^2 = \left[1 - \frac{\kappa^2}{k_L^2}\right]^{-1} . \qquad (9)$$

The spectral function ρ_1 is formally positive; however, implied in its definition is a principal value prescription - part of the axial gauge recipe - for defining poles like $\frac{1}{k \cdot n} = \frac{1}{2}\left(\frac{1}{k \cdot n + i\epsilon} + \frac{1}{k \cdot n - i\epsilon}\right)$. Keeping this prescription in mind, Frenkel and Taylor in their important paper show that the Callan-Symanzik (one-loop approximation to) $\bar{g}^2/g^2$ is expressible in terms of the gauge-invariant quantity

$$\tilde{\rho}_1 = \lim_{k_L^2 \to 0} \rho_1 = -\frac{11}{48\pi^2} \frac{C(G) g^2}{\kappa^2} . \qquad (9a)$$

It is the essential (and still somewhat mysterious) negativity [5] of $\tilde{\rho}_1(\kappa^2)$ which is the hall-mark of asymptotic freedom and which guarantees that for expression (3) the right-hand side is positive for spacelike k^2 , so that $\bar{g}(k^2)$ has no spacelike spurious CDD poles.

To summarize, for non-Abelian theories, the general expression for $g^2/\bar{g}^2$ is given by (3). The one-loop renormalization group estimate is obtained by using for $\rho_1(\kappa^2)$ the expression $-\frac{11}{48\pi^2} \frac{C(G) g^2}{\kappa^2}$ (with $D_1(\kappa^2) = 1$) in the integrand. This is valid for the gauge mass $M^2 = 0$. For $M^2 \neq 0$ there are appropriate and computable one-loop threshold modifications to the spectral functions (7) and (8), which will automatically restrict the threshold mass and the range of integration in (3) in such a manner that (with $\tilde{\rho}_1(\kappa^2) < 0$) there are no unphysical CDD singularities. This should be done, but for the illustrative estimates of this paper, where masses are important only to circumvent problems of infra-red and other singularities which are not really there, we shall be content to approximate to the spectral functions (7) and (8), for $M^2 \neq 0$, by a simple ansatz, obtained, e.g., by replacing $1/\kappa^2$ in (9a) by $\frac{1}{\kappa^2 - M^2}$ and introducing a threshold mass $4m^2 > M^2$:

$$\frac{g^2}{\bar{g}^2(k^2)} = 1 - \frac{11}{48\pi^2} C(G) g^2 (k^2 - M^2) \int_{4m^2}^{\infty} \frac{1}{\kappa^2 - M^2} \frac{1}{\kappa^2 - k^2} d\kappa^2$$

$$= 1 + \frac{11}{48\pi^2} C(G) g^2 \ln \frac{-k^2 + 4m^2}{-M^2 + 4m^2} \quad . \tag{10}$$

Now, apart from the appearance of masses in the arguments of the logarithmic function, (10) coincides with its conventional renormalization group one-loop estimate. Using (10), we are in a position to estimate the induced scalar-scalar amplitudes and the effective potential resulting therefrom - for use, in their turn, to estimate vacuum expectation values of Higgs scalars.

IV. THE COMPUTATION OF THE EFFECTIVE HIGGS POTENTIAL

The end result of Sec.III was that the induced self-coupling of scalars with scalars is essentially given by an integral of the form

$$\lambda = \frac{1}{i} \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - M^2)^2} \cdot \bar{g}^4 \quad , \tag{11}$$

where M^2 is the mass of the gauge particle retained in order to avoid infrared effects. For $\bar{g}^2$ one may use the approximation of Sec.III:

$$\frac{\bar{g}^2}{g^2} = \left[1 + bg^2 \ln \frac{-k^2 + 4m^2}{-M^2 + 4m^2}\right]^{-1} , \quad b = \frac{1}{16\pi^2} \left(\frac{11}{3} C(G) - \frac{1}{6} S_3\right) , \tag{12}$$

with [1] S_3 defined through the relation $\mathrm{Tr}\,\theta_a\theta_b = S_3\delta_{ab}$. One can easily generate an asymptotic series for λ ,

$$\lambda = \sum_{k=1}^{\infty} C_k \left(\frac{M^2}{4m^2} - 1\right) \left[\frac{1}{bg^2} - \ln\left(1 + \frac{M^2}{4m^2}\right)\right]^{-k} , \tag{13}$$

where $C_1 = 1$ and, for $k \geq 2$,

$$C_k(u) = (-)^k (k-1)! \sum_{n=1}^{\infty} \frac{(-u)^n}{n^{k-1}} . \tag{14}$$

For g^2 sufficiently small we obtain the surprising result:

$$\lambda = \frac{g^2}{b} + O(g^4)$$

(provided $bg^2 \ln(1 + \frac{M^2}{4m^2}) << 1$). This result is a consequence of lack of analyticity of (11) in g^2 plane. (As will be seen later, this implies that the Higgs masses are comparable to the vector masses.) [6]

The utility of the computation of λ , resides in the estimate it provides for an expression for the now purely radiative effective potential. The basic formula for the one-loop contribution is

$$V(\phi) = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \ln |\det \Delta(k,\phi)| \quad ,$$

where Δ denotes the propagation matrix for the entire boson system moving in a background of constant scalar fields, ϕ . (Fermions make a similar contribution with the overall sign reversed.) The matrix Δ depends on the constant g and the suggestion is that this coupling should be replaced by $\bar{g}(k^2)$.

An approximate evaluation proceeds as follows. [7] it is convenient to work in Landau gauge, where

$$V = \frac{1}{2i} \int \frac{d_4k}{(2\pi)^4} \ln \det(-k^2 + \bar{M}^2)^3 \quad ,$$

with the gluon mass matrix given by

$$\bar{M}^2_{ab} = \frac{1}{2} \bar{g}^2 \phi^T \{\theta_a, \theta_b\} \phi \quad , \quad a,b = 1,2,\ldots,N,$$

in which the <u>running</u> coupling is used. The gluon propagator refers to three independent polarization states as well as N internal degrees of freedom. After a Wick rotation the integral takes the form

$$V = \frac{3}{64\pi^2} \int_0^{\Lambda^2} dK^2\, K^2\, \ln \det(K^2 + \bar{M}^2) \qquad (15)$$

but cannot be evaluated explicitly owing to the K^2 dependence of $\bar{M}^2$. The cut-off can be eliminated by subtracting terms quadratic in ϕ yielding, after an integration by parts,

$$V = -\frac{3}{64\pi^2} \int_0^\infty dK^2\, \mathrm{Tr}\Big[\bar{M}^4 (K^2 + \bar{M}^2)^{-1} + b\bar{g}^2 (K^2 + 4m^2)^{-1} \bar{M}^4 - b\bar{g}^2 (K^2 + 4m^2)^{-1} (K^2 + \bar{M}^2)^{-1} \bar{M}^6\Big] \quad . \qquad (16)$$

The second and third terms are of order g^4 and g^8, respectively, while the first term yields a contribution of order g^2. This is given by

$$V \simeq \frac{3}{64\pi^2} \frac{g^2}{b} \phi^T \theta_a \theta_b \phi\, \phi^T \theta_a \theta_b \phi \quad . \qquad (17)$$

Remark that the structure of the induced couplings (17) may not always be the most general, allowed by the symmetry considerations. This will have consequences for the pattern of spontaneous symmetry breaking likely to emerge using this potential.

To complete the expression for the effective potential we must introduce fermions and their couplings with Higgs', of the form $h\bar{\psi}\Gamma_i \psi \phi^i$.

As shown by Cheng, Eichten and Li [1], in a gauge theory, the fermion coupling parameter h is driven to zero even faster than the gauge coupling g. In fact for the case of a single Yukawa coupling the Callan-Symanzik equations read

$$\frac{dg^2}{dt} = -bg^4 \;,\; \frac{dh^2}{dt} = Ah^4 - Bh^2 g^2 \;;\; (A,B,b > 0) \quad .$$

Solving these one finds

$$\bar{h}^2/h^2 \approx \left(1 - \frac{A}{B-b} \cdot \frac{h^2}{g^2}\right)^{-A} \left(\frac{\bar{g}^2}{g^2}\right)^{B/b} \;,$$

provided $B > b$. (With the inclusion of the fermions, b now equals

$$\frac{1}{16\pi^2} \left\{\frac{11}{3} C(G) - \frac{1}{6} S_3 - \frac{4}{3} F\right\} \;, \qquad (17a)$$

where F is defined by $\mathrm{Tr}\, t^a t^b = F\delta^{ab}$ and t^a is the matrix appearing in the covariant derivative for fermions.)

Using the Cheng, Eichten and Li result stated above, clearly the fermion loop contribution to the scalar-scalar amplitude is convergent provided we replace the renormalized parameter h by its running analogue $\bar{h}(k^2)$. For the effective potential this means replacing h by $\bar{h}$ in the fermion mass matrix $\bar{\mu} = \bar{h}\, \Gamma_i\, \phi_i$. The fermion contribution to the effective potential is then given by

$$V_f = -\frac{2}{i} \int \frac{d^4k}{(2\pi)^4} \ln \det(k^2 \bar{\mu}^2) \approx \frac{1}{16\pi^2} \int_0^\infty d\kappa^2\, \mathrm{Tr}[\bar{\mu}^4 (k^2 + \bar{\mu}^2)^{-1} + \;\ldots .]$$

$$\approx \frac{1}{16\pi^2} \int_0^\infty \frac{d\kappa^2}{k^2 + \mu^2} \bar{h}^4\, \mathrm{Tr}\, (\Gamma\phi)^4 \;, \qquad (18)$$

which finally works out as:

$$V_f \approx \frac{1}{16\pi^2} \frac{h^4}{bg^2} \left(1 - \frac{A}{B-b} \frac{h^2}{g^2}\right)^{-2A} \mathrm{Tr}(\Gamma\phi)^4 \qquad (19)$$

to the leading order in g^2.

Collecting terms (17) and (19) and taking these together with the Higgs mass term $-\frac{1}{2} \phi^T m_s^2 \phi$, we obtain for the ratio of the scalar to vector $(\text{mass})^2$, the order of magnitude estimate $\approx 6\left[\frac{11}{3} C(G) - \frac{4}{3} F - \frac{1}{6} S_3\right]^{-1}$ times a group-theoretic factor which depends on the details of the representation content of the Higgs and the fermions.

V. RENORMALIZATION OF QUANTUM GRAVITY

We conclude with a few remarks regarding the use of the running coupling constant for renormalization prospects of quantum gravity. Let us assume that one could formulate Feynman rules for gravity in such a way that the Dyson insertions into unmodified graphs could be unambiguously made, and without any double counting. (This appears possible in a first-order formalism.) Assume also that a consistent renormalization group [8] can be set up to estimate the high-energy behaviour of the running gravitational coupling constant $\bar{\kappa}^2(\kappa^2)$. With these assumptions it seems reasonable that the running constant has the form $(16\pi K^2 = G_N)$,

$$\frac{\overline{K}^2(k^2)}{K^2} \approx \left[1 + \frac{23}{96\pi^2} K^2 k^2 \ln|k^2|\right]^{-1} \quad . \tag{20}$$

This result is implicit in the work of Fradkin and Vilkovisky [9]. From the estimates given in Ref. 8, one can show that the insertion of $\overline{K}^2(k^2)$ in place of K^2 inside loop integrals for unmodified graphs (thereby simulating self-energy and vertex insertions) will render quantum gravity finite and renormalizable. Since the coefficient of K^2k^2 in (2) is positive, the theory is likely to be asymptotically free, with the effective gravitational coupling diminishing in magnitude as $k^2 \to \infty$ (or equivalently, in configuration space, $\overline{K}^2(r)$ decreasing like r as $r \to 0$).

This consequence of asymptotic freedom of gravity (*diminishing gravitational constant as distance decreases*) is likely to have a profound influence on the very early history of the universe, i.e. when the radius of the universe approaches Planck length. To state the result differently, asymptotic freedom implies that an effective short-range repulsive interaction compensating the Newtonian attraction makes its appearance at short distances, such that the Newtonian potential - if such language is still appropriate - is changed to

$$V(r) \sim \frac{K^2}{r}\,(1 - \exp[-r/r_0]) \approx \frac{K^2}{r_0} \quad . \tag{21}$$

Here $r_0 = \sqrt{\frac{23}{96}}\,\frac{K}{\pi}$. Clearly - depending on the matter and radiation density of the universe - this implies that the final $r = 0$ singularity, as the ultimate fate of the universe, is by no means inevitable.

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We much appreciate discussions with Steven Weinberg.

APPENDIX I

As shown in Sec.II, for the U(1) theory, the replacement $g \to \overline{g}(k^2)$ in the unmodified scalar-scalar amplitudes is by definition tied to the insertions in the photon (inverse) propagator. What about the other insertions - the vertex and the meson self-energy? We shall show that, on account of Ward identities, these other insertions essentially cancel one another out, so far as the potential infinities of the scalar-scalar meson amplitudes are concerned. Among the vertex insertions are included both the two-meson-one-photon renormalized insertions $\Gamma_\mu(p,p-k;k)$ and the sea-gull two-meson-two-photon renormalized insertions $C_{\mu\nu}(p,p+k-k';k,k')$. The Ward-Takahashi identities read

$$k_\mu\Gamma_\mu = \Delta^{-1}(p) - \Delta^{-1}(p-k) \quad ,$$

$$k_\mu C_{\mu\nu} = \Gamma_\nu(p+k,p+k-k';k') - \Gamma_\nu(p,p-k';k') \quad ,$$

$$k'_\nu C_{\mu\nu} = \Gamma_\mu(p,p+k;-k) - \Gamma_\mu(p-k',p+k-k';-k) \quad .$$

Their differential form is

$$\Gamma_\mu(p,p,0) = \partial_\mu\Delta^{-1}(p) \quad ; \quad C_{\mu\nu}(p,p+k;k,0) = \partial_\nu\Gamma_\mu(p,p+k;-k) \quad .$$

Here Δ is the complete renormalized meson propagator.

Now all unmodified graphs for meson-meson scattering in a U(1) gauge theory are made up of two meson lines running throughout the graphs with photon lines exchanged between them, plus a number of closed meson loops with four or more photon lines attaching these closed loops to the running meson lines. These closed meson loops can be ignored from consideration so far as potential overall infinities of the meson-meson scattering graphs are concerned. This is because subgraphs with four or more photon lines are comfortably convergent themselves and also do not enhance (by contributing any logarithmic factors) potential infinities of the meson-meson scattering graphs we are considering. (This can be substantiated by using renormalization group techniques, which show that the only logarithmic factors which such subgraphs can contribute are those associated directly with their external photon lines. These are, in any case, taken care of by the replacements $g \to \overline{g}(k)$.) We are thus left with consideration of only those unmodified graphs which are made up of the two original running meson lines with n photons exchanged between them. Fig.1 gives the lowest order example of this class.

Now, without loss of generality - so far as ultraviolet infinity counting is concerned - we may, as in Sec.I, take the external meson momenta and the masses of the mesons as zero. The types of unmodified graphs we are considering possess n (internal) photon lines with momenta, which we shall label as $k_1, k_2, \cdots, k_n$, and altogether $(n-1)$ loops $(k_1 + k_2 + \cdots + k_n = 0)$. These graphs, after vertex and self-energy insertions, correspond to an integral of the type:

$$\int \delta(\Sigma k)(d^4k)^n [D(k_1) \cdots D(k_n)] \times g^n \times$$
$$\Big(\{\Gamma(0,-k_1;k_1)\Delta(-k_1)\Gamma(-k_1,-k_1-k_2;k_2)\ \Delta(-k_2)\ \cdots\ \Delta(k_n)\Gamma(k_n,0;k_n)\}$$
$$\times \{\text{a similar string of } (\Gamma\Delta\Gamma \cdots \Delta\Gamma) \text{ with } k\text{'s permuted}\}$$
$$+ \text{terms like the above with } C\text{'s replacing the combinations } (\Gamma\Delta\Gamma) \text{ in all possible ways}\Big) .$$

To study the potential infinities of this integral, we wish to let all or a subset of the k's grow large and use renormalization group estimates for the ultraviolet behaviour of Δ, Γ, C and D. Using these estimates (or simply the Ward-Takahashi identities), it is easily seen that in the deep Euclidean region, the symmetrical combination $\sqrt{\Delta(k_1)}\ \Gamma(k_1,k_2;k_1-k_2)\ \sqrt{\Delta(k_2)}$ tends to its unmodified value $(k_1 + k_2)$, for large k_1 and k_2, with similar results for the sea-gull insertion. Provided that we encounter no problems of extra logarithmic factors from the non-symmetrical combinations like $\Gamma(0,-k;k)\sqrt{\Delta(k)}$ and $\sqrt{\Delta}\Gamma(k,0;k)$ (where $\Gamma(0,-k;k)$ and $\Gamma(k,0,-k)$ are those end vertices where the external meson debouches), we will have shown that so far as infinity counting of scalar-scalar amplitudes is concerned, it is just the photon-modified propagators which determine what the final situation is. (If, for example, $k^2D(k) \approx (\log k^2)^{-1}$, there will be a comfortable margin of logarithmic convergence factors to ensure finiteness. This margin is the bigger, the larger the number of loops.) Thus, all in all, the worst graphs from the point of view of potential infinities are the graphs of Fig.1, with just one loop, only two photon lines and with vertices of the type $\Gamma(k,0;k)$, where one of the two meson lines has zero external momentum [10]. The modified integral has the form:

$$\int d^4k\ [\Gamma(0,-k,k)\Delta(k)\Gamma(k,0,-k) + C(0,0;k,-k)]^2[D(k)]^2 .$$

Now the expression within the brackets in the integral above is the matrix element of two current operators $(\text{meson}|j(x)j(y)|\text{meson})_+$. Remarking

that the current operators possess no anomalous dimensions and using the Wilson-product expansion methods and the renormalization group machinery, one can show that for large k, the bracket simply contributes a factor of order unity. (In fact, a similar argument can be made repeatedly for all contiguous pairs of space-time points, where photons impinge on the running meson line; the crucial element of the argument being absence of anomalous dimensions of current operators.) Thus the entire ultraviolet behaviour of the final integral is governed by the modified photon propagators. Since the modified photon propagator defines the running coupling constant, we have proved the result that altogether the photon and the meson self-energy as well as the vertex insertions are equivalent for scalar-scalar amplitudes to the replacements $g \to \bar{g}(k)$ (where k's are photon momenta) for infinity counting purposes.

The considerations above have so far been for the Abelian gauge group U(1). As explained in Sec.II, this unfortunately is precisely the gauge group for which we cannot assert that $(\bar{g}(k)/g)^2 \sim k^2D(k)$ falls as $(\log k^2)^{-1}$ for large k^2, without encountering inconsistencies with the use of perturbation estimates for $\bar{g}$ as well as problems with CDD singularities. Thus gauge propagator modifications are not expected to produce convergence in this theory. The question therefore arises: can our considerations be extended to non-Abelian gauge theories with photons replaced by "gluons"? The answer appears to be in the affirmative, with even the formal steps being similar for the Abelian and the non-Abelian cases, in the axial gauge. The only change is that now the closed meson loops with three and four gluons attaching such loops to the running meson lines need also to be kept in the analysis presented above, since these contribute to the renormalization of g itself. [11] Technically, however, this presents no difficulty. An important remark in this context is that even though the axial gauge is a non-covariant gauge, the expressions for infinite constants are Lorentz covariants.

Throughout this paper we have considered the case of bare $\lambda_0 = 0$. If this is not assumed - and if the results of this paper's analysis regarding the finiteness of induced scalar-scalar coupling do not change (and they do not if the theory remains asymptotically free) - then the Callan-Symanzik function $\beta_\lambda(\bar{\lambda},\bar{g}^2)$ $(= d\bar{\lambda}/dt)$ has the property that $\beta_\lambda(0,\bar{g}^2) = 0$. To take an example, for the one-loop case in an O(N) gauge theory with one (N-fold) Higgs multiplet, $\frac{d\bar{\lambda}}{dt} = \frac{1}{16\pi^2}\,[(N+8)\bar{\lambda}^2 - 3(N-1)\bar{\lambda}\bar{g}^2]$. For the theory to be asymptotically free, it can now be shown [1] that $N \geq \frac{b}{3} + 1$,

$b > 0$, with b defined in Eqs.(12) and (17a). If b is small (e.g. if the theory contains a fair number of fermions), the internal symmetry group need not be excessively large in order to secure asymptotic freedom. Following the analysis of Cheng, Eichten and Li in Ref.1, one can examine more elaborate situations, but the general conclusion seems to be that the prospects for asymptotic freedom are improved.

APPENDIX II

A manifestly gauge-independent proof of the finiteness of induced Higgs couplings can be based on the use of dispersion integrals. In this approach one represents the on-shell Higgs scattering amplitudes $A(E)$ (forward scattering for simplicity) by a dispersion integral. If there are no fundamental Higgs coupling parameters, then such integrals must require no subtractions. That this is indeed the case in theories which are otherwise asymptotically free can be seen without difficulty.

Consider the n-gluon contribution to the absorptive part $\mathrm{Im}A(E)$ in the meson-meson annihilation channel. This is given by phase space integrals of functions of the form

$$\mu^{4-2n}\, F(\frac{E}{\mu}\, , X\, , \bar{g}(\mu)) \quad ,$$

where E denotes the total centre-of-mass energy and X a set of dimensionless variables needed to specify the multiparticle configuration. The reference mass μ can be chosen at will since this expression must not depend on it. In particular one may choose $\mu = E$ so as to obtain [12]

$$E^{4-2n}\, F(1,\, X\, ,\, \bar{g}(E)).$$

This quantity, being the square of an on-shell S-matrix element is certainly gauge independent. Moreover, it is entirely unambiguous: the dependence of the running coupling on the total energy, E, in the annihilation channel has been chosen for convenience - it is constant with respect to the phase space integration. If the theory is asymptotically free, then $\bar{g}(E) \sim (\ell n E)^{-1/2}$. The large E behaviour of F can be estimated perturbatively. In this way one finds that the required absorptive part is dominated at high energy by the 2-gluon Born terms,

$$\mathrm{Im}A(E) \sim (\ell n E)^{-2} \quad .$$

(The n-gluon contribution to the absorptive part goes like $(\ell n E)^{-n}$.)

The contributions of fermion + antifermion states fall off even faster since the effective Yukawa coupling parameter is generally driven to zero faster than $\bar{g}(E)$. The only terms which could fail to vanish this rapidly would be those contributed by scalar + scalar intermediate states <u>if</u> a non-vanishing bare coupling were included. In the absence of such a coupling the absorptive part decreases with sufficient rapidity to obviate the need for subtractions. The induced Higgs self-couplings are thus intrinsically finite.

APPENDIX III

The dispersive method of Appx.II can perhaps be used to elucidate the asymptotic behaviour of Einstein's gravity theory and to prove its convergence. Consider firstly the one-loop graphs with n-external gravitons in pure gravity theory (or extended supergravity if "matter" is to be included). The on-shell amplitudes are finite and behave like $\kappa^n E^4$ as the energy E becomes large.

Anticipating the invention of a renormalization [8] group type methodology for systematically estimating the high-energy behaviour, we shall assume that the running coupling $\bar{\kappa}(E)$ can be defined and, moreover, that

$$\bar{\kappa}(E) \sim E^{-1}(\ell nE)^{-1/2} \quad .$$

This estimate is supported by the work of Fradkin and Vilkovisky,[9] who show that the spin-2 part of the graviton coupling is suppressed by such a factor (cf. Eq.(20)), just as in asymptotically free Yang-Mills theories, where the effective gluon coupling is proportional to $(\ell nE)^{-1/2}$. In effect we are supposing that the zero-loop (tree) contribution is misleading. Rather, the n-graviton amplitude should behave like $\bar{\kappa}(E)^n E^4 \sim E^{4-n} (\ell nE)^{-n/2}$. Apart perhaps from the logarithmic factors, this is what one would expect in a renormalizable Froissart bounded theory. Using the method of Appx.II, and starting with the on-shell one-loop estimate above, the power behaviour E^{4-n} will now certainly be reproduced in dispersive computations of many-loop higher order graphs, with any possible subtraction constants, absorbable in the renormalization of κ. (However, it is not clear whether the logarithmic factors may add up to produce an anomalous power behaviour of the on-shell amplitudes.)

REFERENCES AND FOOTNOTES

1) It was pointed out by D.J. Gross and F. Wilczek (Phys. Rev. D8, 3633 (1973)) and H.D. Politzer (Phys. Rep. 14C, 4 (1974)) that scalars tend to destabilize the asymptotic freedom of gauge theories because of their quartic couplings. The exact nature of the instability depends on the gauge group and on the representation content of the scalars. A number of cases were considered by T.P. Cheng, E. Eichten and Ling-Fong Li (Phys. Rev. D9, 2259 (1974)), who observe a general pattern. As the number of scalar fields increases, the dimension of the group and therefore the number of gauge bosons must also be increased to maintain asymptotic freedom. They found no examples of ultraviolet stable systems where the symmetry is broken down to an Abelian (electromagnetic) symmetry.

2) It is important that for scalar-scalar amplitudes there are no problems connected with overlapping insertions. Hence the ansatz of using $\bar{g}$ in place of g, as an approximation to the self-energy and vertex insertions into the unmodified perturbation graphs is unambiguous. In principle there is no reason why a similar self-consistent methodology could not be used for improving on the renormalization group estimates for $\bar{g}$, as well as the self-energy and vertex insertions themselves. This should be carried out. However, here the problem of overlaps will be severe. Relevant in this context is a calculation by E.C. Poggio and G. Pollack (Harvard preprint HUTP-77/A067), who have computed within the framework of QCD, to the fourth order in perturbation theory, the leading and next-to-leading contributions to the asymptotic behaviour of the colour-singlet-exclusive quark vertex function. They find that the contributions appear to add up to an exponential form, with the exponent given by the lowest order contribution, evaluated replacing g by $\bar{g}$. We are indebted to Dr. V. Elias for showing us this paper.

3) M. Gell-Mann and F.E. Low, Phys. Rev. 95, 1300 (1954);
N.N. Bogolubov and D.V. Shirkov, Theory of Quantized Fields (Interscience, New York 1959).

4) The suggestion that self-energy and vertex insertions may render scalar-scalar amplitudes finite in gauge theories was first made in Abdus Salam and R. Delbourgo, Phys. Rev. 135, 1398 (1964), and R. Delbourgo, Abdus Salam and J. Strathdee, Il Nuovo Cimento 23A, 237 (1974). As contrasted with the present note, where these insertions are approximated to by what one hopes is a gauge-invariant quantity at least to the two-loop level (i.e. the running coupling constant), the suggested expressions for insertions in these earlier papers were non-gauge-invariant.
It is worth remarking that for scalar-scalar amplitudes of order higher than g^4 the replacement $g \to \bar{g}(k)$ (with k representing gauge particle momenta) implies a very comfortable overall convergence, as well as a convergence of the 4-scalar subgraphs, contained inside the overall graph (see Appx.I).

5) J. Frenkel and J.C. Taylor, Nucl. Phys. B109, 439 (1976). These authors define $\tilde{\rho}_1$ as the limit of ρ_1 for $\kappa^2 \to \infty$. This definition is equivalent to ours for the one-loop case when $M^2 = 0$. The important points are the gauge invariance and the negativity of $\tilde{\rho}_1(\kappa^2)$ (or at least of its dispersion integral over κ^2). It would be good to carry through the analysis of Frenkel and Taylor for an arbitrary gauge and to trace the origin of these characteristics of $\tilde{\rho}_1$. We believe that the negativity of $\tilde{\rho}_1$ (defined as the spectral function for $g^2/\bar{g}^2(k)$ for an arbitrary gauge) or at least of its dispersion integral over κ^2 like (3) may be connected with the fact that all components of the gauge fields in a non-Abelian gauge theory possess derivative couplings. This is not true of the photon in U(1), where the analogue of a non-Abelian theory would have been a Pauli-like magnetic-moment derivative-containing coupling of the photon with the charged spin-one source particles (e.g. $ieF_{\mu\nu}(A^+_\mu A^-_\nu - A^-_\mu A^+_\nu)$). (See Abdus Salam and J. Strathdee, Nucl. Phys. B90, 203 (1975).)

6) This is to be contrasted with the result of S. Coleman and E. Weinberg (Phys. Rev. D7, 1888 (1973)), who obtain $M^2_{scalar}/M^2_{vector} \sim g^2$ in cases where this ratio can be computed perturbatively. It is amusing to remark that in supersymmetric theories where Higgs particles and fermions are placed in the same (matter) multiplet and where a $\lambda\phi^4$ potential automatically appears when this multiplet is gauged, the supersymmetric value of λ is precisely g^2. (See, for example,

Abdus Salam and J. Strathdee, ICTP, Trieste, preprint IC/76/12, to appear in Fortschritte der Physik.) It is well known that super-symmetric gauge theories are asymptotically free, a circumstance which may be related - as in this paper - to λ being of order g^2 rather than of order g^4. It is also interesting that for theories - and some examples are now known - where the Callan-Symanzik function β vanishes to one-or two-loop level, the corresponding λ, computed using the running coupling constant as in this paper, will start with g^4 or g^6 rather than with a g^2 term. Perhaps the hierarchy of Higgs masses suspected to obtain in physics may be traced to this circumstance.

7) The effective potential applies to zero four-momentum external scalars. Thus the result is gauge-invariant only for massless scalars; such objects being automatically on-mass-shell. This is the well-known weakness of the effective potential method.

8) This can be done if one supplements Einstein's Lagrangian with terms containing R^2 and $R^{\mu\nu}R_{\mu\nu}$. See Abdus Salam and J. Strathdee, ICTP, Trieste, preprint IC/78/12, (February 1978), to be published, for a discussion and earlier references. The criterion for the absence of negative norm objects stated in this paper (i.e. $\bar{K}^2 k^2 \to 0$, Footnote 17) appears to be satisfied for $\bar{K}^2$ given by (20). See also J. Julve and M. Tonin, " Quantum gravity with higher derivatives", University of Padua preprint, February 1978.

9) E.S. Fradkin and G.A. Vilkovisky, "On renormalization of quantum gravity in curved space-time", Berne University preprint, October 1976.

10) Notice that from the Ward-Takahashi identity $k_\mu \Gamma_\mu(k,0;k) = \Delta^{-1}(k)$ for massless mesons, we may infer that $\Gamma_\mu(k,0;k) = k_\mu k^{-2}\Delta^{-1}$.

11) In fact, possibly the most appropriate definition of $\bar{g}^2(k)$ for non-Abelian theories may be obtained in terms of gluon-gluon forward scattering amplitude on the mass shell.

12) The scaling behaviour sketched here is explained in the elegant presentation of S. Weinberg, "Ultraviolet divergences in quantum theories of gravitation", to be published in the Einstein centenary volume, Gravitational Theories since Einstein, ed. by S.W. Hawking and W. Israel (C.U.P.). We are ignoring the Higgs mass which should be represented by a running parameter in these formulae. This is reasonable insofar as we are discussing the ultraviolet regime.

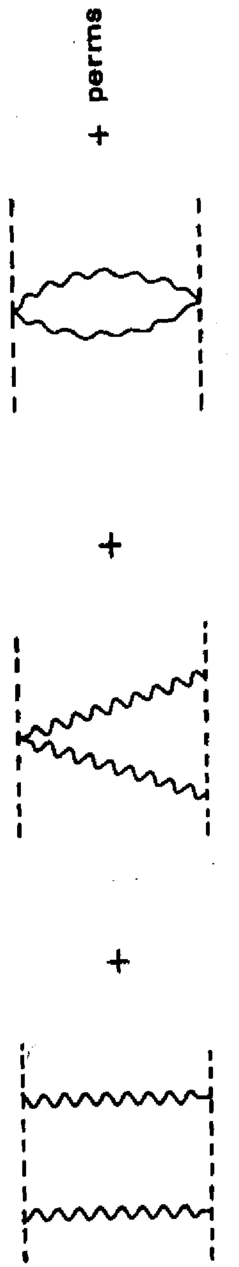

Fig.1 The dotted lines are scalars; the wavy lines gauge particles.

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