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REMARKS ON HIGH ENERGY STABILITY AND RENORMALIZABILITY OF GRAVITY THEORY

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ADDENDUM

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Abdus Salam and J. Strathdee

A D D E N D U M

After this work was issued as a preprint, we received an article by Julve and Tonin [20] in which a similar approach is developed. In this work the one-loop contributions to the Callan-Symanzik functions are evaluated. There are altogether four independent running parameters, $g(\kappa)$, $g'(\kappa)$, $K(\kappa)$ and $\lambda(\kappa)$ corresponding to the (unrenormalized) Lagrangian

$$\mathcal{L} = \sqrt{-g}\left[\lambda_0 + \frac{1}{K_0^2}R + \frac{1}{g_0^2}\left(\frac{1}{3}R^2 - R_{\mu\nu}R^{\mu\nu}\right) + \frac{1}{6g_0'^2}R^2\right] . \quad (24)$$

According to Julve and Tonin the Callan-Symanzik equations read (using $16\pi^2\tau = \ln\kappa^2$)

$$\frac{dg^2}{d\tau} = -\frac{98}{15}g^4 , \quad (25)$$

$$\frac{dg'^2}{d\tau} = -\frac{1}{6}g'^4 + \frac{5}{2}g'^2g^2 + \frac{5}{3}g^4 , \quad (26)$$

$$\frac{dK^2}{d\tau} = \left(\frac{1}{4}g'^2 + \frac{13}{12}g^2 - \frac{5}{6}\frac{g^4}{g'^2}\right)K^2 , \quad (27)$$

$$\frac{d\lambda}{d\tau} = \left(\frac{1}{3}g'^2 + \frac{14}{3}g^2\right)\lambda + \frac{1}{4}\left(g'^4 + 5g^4\right)K^{-4} . \quad (28)$$

The dimensionless couplings g and g' can be related to the masses of the tensor ghost and scalar particles, respectively,

$$g^2 = K^2M^2 \quad \text{and} \quad g'^2 = K^2M'^2 , \quad (29)$$

although we do not wish to prejudge the signs of M^2 and M'^2. These quantities could be negative, in which case the bare propagator would exhibit tachyon poles.

The general form of Eqs.(25)-(28) is not difficult to understand. The parameters g^2, g'^2, K^2 and λ carry the respective dimensions 0, 0, -2 and 4 in units of mass. Hence, if subtractions are made according to the prescriptions of 't Hooft (see Ref.9), Eqs.(25) and (26) must be independent of K^2 and λ, while (27) must be linear in K^2 and independent of λ and, lastly, (28) must be linear in λ and K^{-4}. Although we have treated g and g' as independent couplings and they appear in Eqs.(25)-(28) as if they are on the same footing, this is in fact not the case. One should write $g'^2 = g^2/\omega$ and treat g as the unique dimensionless expansion parameter. In this expansion (around $g = 0$) the coefficients have an exactly computable ω dependence. The appearance in (27) of $g^4/g'^2 = \omega g^2$ reflects this secondary role of g', and one should expect powers of g/g' in the higher loop contributions.

Now, in agreement with the discussion of Sec.IV of this paper, Julve and Tonin surmise that the tensor ghost singularity of the free field approximation may be driven to infinity (when the interactions are properly taken into account) if the effective dimension of $M(\kappa) = g(\kappa)/K(\kappa)$ is negative. By this one means

$$\lim_{\kappa\to\infty} \kappa\frac{d}{d\kappa}\ln M(\kappa) > 0 . \quad (30)$$

This condition could be realized, should there be an ultraviolet stable fixed point

$$g(\kappa) \to g_\infty , \quad g'(\kappa) \to g'_\infty ,$$

with g_∞ and g'_∞ non-vanishing. It is clear from (23) (or (27)) that unless at least one of the parameters g_∞, g'_∞ is non-zero, there is no hope of satisfying the condition (30).

However, we differ from Julve and Tonin in our suggestion (Footnote 17) that the dimension of $M(\kappa)$ should be not just negative but in fact < -1 in order to have $\kappa/M(\kappa) \to 0$ when $\kappa\to\infty$. If $g(\kappa)\to g_\infty \neq 0$ this is equivalent to $\kappa K(\kappa)\to 0$. (Such an outcome is made plausible already by the calculations of Fradkin and Vilkovisky [10] - cf. Eqs.(17) and (18), above - who give $K^2(k^2) \sim (k^2 \ln k^2)^{-1}$.) A limiting behaviour like

$$K^2 \sim (k^2)^{-\alpha} \quad , \quad \alpha > 1 \tag{31}$$

would guarantee that the n-graviton amplitudes decrease asymptotically like $k^{4-n\alpha}$ (see Eq.(11) and the theory would respect Froissart's unitarity bounds [17].

It may prove sensible to have $g_\infty = 0$ but $g'_\infty \neq 0$. In the one-loop Callan-Symanzik functions appearing in (25)-(28) there are no negative powers of g and we believe that this may be true also of higher orders. Choosing $g^2 > 0$ then implies $g_\infty = 0$ and one must choose $g'^2 < 0$ in order to have $g'_\infty \neq 0$. From (27) one finds

$$k^2 \frac{dK^2}{dk^2} = \frac{1}{16\pi^2} \left(\frac{g'^2}{4} + \dots \right) K^2 \quad ,$$

i.e. K^2 scales according to (31) with

$$\alpha = - \frac{g'^2_\infty}{64\pi^2} + \dots \tag{32}$$

Of course the one-loop contribution cannot be taken seriously here since, in order to have $\alpha > 1$, we need $|g'_\infty| > 8\pi$ and power series methods cannot be used in finding such a fixed point.

To conclude, we have made it plausible that if $K^2(\kappa) \sim (\kappa^2)^{-\alpha}$, $\alpha > 1$, the Einstein part R/K^2 of the Lagrangian dominates over the ghost-producing quadratic part. This is what has essentially guaranteed Froissart unitarity boundedness [21]. The quadratic part of the Lagrangian plays little role, for ultraviolet behaviour of the theory, except to motivate and justify an orderly renormalization group approach, towards the relation $K^2(\kappa) \sim (\kappa^2)^{-\alpha}$, $\alpha > 1$, through g_∞ and g'_∞ non-vanishing.

REFERENCES AND FOOTNOTES

20) J. Julve and M. Tonin, "Quantum gravity with higher derivative terms", Padua preprint IFPD 2/78.

21) In a separate note, Abdus Salam and J. Strathdee, Appendix III ("Higgs couplings and asymptotic freedom", ICTP, Trieste, preprint IC/78/44), a dispersive technique has been suggested to establish Froissart boundedness which uses only the Einstein part of the Lagrangian (or a supersymmetric extension of it, if matter is to be included). This technique relies on the finiteness of the on-shell one-loop amplitudes and the presumed relation $K^2(\kappa) \sim (K^2)^\alpha$, $\alpha \geqslant 1$, plus dispersion theory.

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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

REMARKS ON HIGH ENERGY STABILITY AND RENORMALIZABILITY OF GRAVITY THEORY *

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ABSTRACT

Arguing that high-energy (Froissart) boundedness of gravitational cross-sections may make it necessary to supplement Einstein's Lagrangian with terms containing R^2 and $R^{\mu\nu}R_{\mu\nu}$, we suggest criteria which, if satisfied, could make the tensor ghost in such a theory innocuous.

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I. PROPOSALS FOR RENORMALIZING GRAVITY

At present there are two views about renormalization prospects of quantum gravity.

(I) S-matrix elements, as contrasted from Green's functions, may be finite. This result, substantiated at the two-loop level for the S matrix in extended supergravities, may hopefully hold also for Green's functions, once supergravities are formulated within a superfield formalism [1].

(II) Gravity may be renormalizable, but non-perturbatively. Two non-perturbative techniques have been suggested: (IIA) the non-polynomial technique,[2] which relies on a summation of "cocoon" graphs, using the formula $\langle \phi^n(x)\, \phi^n(0)\rangle \approx n!\left(\frac{1}{x^2}\right)^n$; (IIB) the gauge technique [3], which relies on a solution of Dyson-Schwinger [4] equations, by making use of a non-perturbative solution of gauge identities connecting the inverse Green's function Δ^{-1} with the vertex operators Γ .

Both proposals (II) and (IIA) but not (IIB) suffer from one serious defect. The high-energy behaviour of matrix elements in each order of approximation increases like $(K^2k^2)^n$. Thus any (Froissart) boundedness of cross-sections [5] can become manifest only after a further summation of the perturbation series – a task surely not to be undertaken lightly.

In order to improve high-energy behaviour, we wish to revive the suggestion [6] that the Einstein Lagrangian (R) should be supplemented by higher derivative Lagrangians containing terms of the type [7] $R^{\mu\nu}R_{\mu\nu}$ and R^2 . Such Lagrangians have been shown to be renormalizable. [6] However, they contain ghosts. Based on a renormalization group investigation, we suggest criteria which, if satisfied, could make the ghosts innocuous.

II. STABLE HIGHER DERIVATIVE THEORIES

Since the Lagrangians we wish to consider contain higher than second-order derivatives, we first examine these for high-energy stability. A theory is stable if, in each order of a perturbation expansion, the high-energy behaviour in momenta k does not increase, except to the extent of powers of logarithms ($\log k^2$). Conventional renormalizable theories are stable [8]. So are higher derivative theories, provided the number of derivatives in the interaction Lagrangian does not exceed the number in the free Lagrangian.

A. Conventional renormalizable theories: Prototype $L = \frac{1}{2}(\partial\varphi)^2 - \lambda\phi^4$.

Since $(\varphi\varphi) \approx \frac{1}{x^2}$, $\varphi \sim \frac{1}{x}$ for $x \to 0$ in the Wilson product expansion sense, and L is no more singular than $1/x^4$. For such theories, matrix elements $\Gamma^E(k)$ with E external lines are stable and behave like k^{4-E} (barring logarithmic factors). A φ^3 theory ($\varphi^3 \sim \frac{1}{x^3}$) is superstable with $\Gamma^E(k) \sim k^{4-E-n}$, where n is the order of perturbation.

B. Higher derivative theories: $L = L_I + L_{II} + L_{III}$, (1)

$$L_I = \frac{1}{2}(\partial\varphi)^2 + \frac{1}{2}g_1(\partial\varphi)^2 \left(\frac{\varphi}{M}\right)^{\ell_1}, \tag{2}$$

$$L_{II} = \frac{1}{2}\left(\frac{\partial^2\varphi}{M}\right)^2 + g_2\left(\frac{\partial^2\varphi}{M}\right)^2 \left(\frac{\varphi}{M}\right)^{\ell_2} + g_3\left(\frac{\partial\varphi}{M}\right)^4 \left(\frac{\varphi}{M}\right)^{\ell_3}, \tag{3}$$

$$L_{III} = (g_4 M^4)\left(\frac{\varphi}{M}\right)^{\ell_4}. \tag{4}$$

All g's are dimensionless. The theory contains a positive-norm massless and a negative-norm massive particle of mass M. Since

$$(\varphi\varphi) \underset{x\to 0}{\approx} M^2 \log x^2, \text{ i.e. } \left(\frac{\varphi}{M}\right) \sim (\log x^2)^{1/2},$$

barring logarithms, L_I behaves like $\frac{1}{x^2}$ $(x \to 0)$ (superstable), L_{II} like $\frac{1}{x^4}$ (stable), and L_{III} like ≈ 1 (superstable). The high-energy behaviour of matrix elements with E external lines is given by

$$\phi^E\Gamma^E(k) \approx k^4 \left(\frac{\phi}{M}\right)^E \sum_{n_1,n_2,n_3,n_4} C_{n_1,n_2,n_3,n_4}\,[\log(k^2/M^2)] \left(g_1\frac{M^2}{k^2}\right)^{n_1} g_2^{n_2} g_3^{n_3} \left(g_4\frac{M^4}{k^4}\right)^{n_4}. \tag{5}$$

1) Note the difference from the conventional renormalizable case; here $\Gamma^E \sim k^4$ rather than k^{4-E} with no variation with E. On the face of it, this behaviour (k^4) is non-Froissart for the scattering process ($E = 4$). For gravity theory, however, there may be an amelioration on account of gauge invariance, since at least two of the k factors must refer to the external-line momenta.

2) The theories are "renormalizable"; infinities for all matrix elements are quartic (unless gauge invariance diminishes them), but absorbable in the type of terms shown, provided $\ell_1, \ell_2, \ell_3, \ldots$ range over all $\ell \geq 1$ – i.e. provided such theories are intrinsically non-polynomial and there are an infinity of coupling constants $g_1^{(\ell)}, g_2^{(\ell)}, \ldots$

3) One may set up renormalization group equations in the conventional manner. Write $\varphi/M = \varphi'$, so that L reads:

$$L = \left[\frac{1}{2}(\partial^2\varphi')^2 + g_2(\partial^2\varphi')(\varphi')^{\ell_2} + g_3(\partial\varphi')^4(\varphi')^{\ell_2}\right] + \left[\frac{1}{2}M^2(\partial\varphi')^2\left(1 + g_1(\varphi')^{\ell_3}\right)\right] + \left[(g_4M^4)(\varphi')^{\ell_4}\right]. \tag{6}$$

The "super-renormalizable" terms in the second and third brackets may be treated formally as perturbations in M^2, though this procedure, emphasising as it does the dipole-ghost produced by the first bracket, militates against the physical acceptability of the theory, for which renormalized M^2 must $\to \infty$ (freedom from negative norms). We come back to this problem.

The renormalization group equations we shall need are similar to those written down by Weinberg, Collins and Macfarlane, and Lee [9]. We follow, in Sec.IV, Lee's treatment, which relies on the dimensional regularization [9] method.

III. THE GRAVITY THEORY

The stable gravitational Lagrangian we wish to work with is given by

$$L = \sqrt{-g}\left[R/K_0^2 + \frac{1}{g_0^2}\left(\frac{1}{3}R^2 - R^{\mu\nu}R_{\mu\nu}\right) + \frac{6}{g_0'^2}R^2\right]. \tag{7}$$

As shown by Stelle [6], the theory possesses a tensor ghost of mass

$$M_0^2 = g_0^2/K_0^2 \qquad (8)$$

and a good scalar positive norm particle of mass

$$m_0^2 = g_0'^2/K_0^2 \quad . \qquad (9)$$

The first term in the Lagrangian has the form of L_I of Eq.(1), the remaining terms have the form of L_{II} . (If we had started with a cosmological term, this would have resembled L_{III} of Eq.(1).) [10] To simplify discussion and to bring out the main points we shall, for the time being, neglect the good positive-norm scalar particle $\left(m_0 = \infty\ ,\ \frac{1}{g_0'^2} = 0\right)$, ignore gauge breaking terms needed to define the theory, as well as the Faddeev-Popov terms [6]. Nothing essential is lost so far as problems discussed in this note are concerned, though the exact counterparts of Eqs.(13)-(16) below are much more complicated if this is not done.

Writing [10] $g_{\mu\nu} = \eta_{\mu\nu} + K_0\ \phi_{0\mu\nu}$, we can estimate the high-energy behaviour of matrix elements using Eq.(5). Set

$$g_1^{(\ell)} \approx (K_0M_0)^{\ell}\ ,\quad g_2^{(\ell)} \approx (K_0M_0)^{\ell}\ ,\quad g_3^{(\ell)} \approx (K_0M_0)^{\ell}\ , \qquad (10)$$

we obtain

$$\phi_0^E\Gamma^E \approx (K_0\phi_0)^E k^4 \sum_{\ell} (g_0^2)^{\ell-1} \sum_n \left(\frac{M_0^2}{k^2}\right)^n C_{n,\ell}\left[\log K_0^2k^2\right] \quad . \qquad (11)$$

Here n is the order of perturbation for $L_{Einstein} = \sqrt{-g}\ R/K_0^2$ and ℓ is the number of loops. Note that on account of (8), the matrix elements depend on two independent parameters only; either g_0 and M_0 or equivalently g_0 and K_0 . (For the one-loop case, the explicit dependence of (11) on g_0^2 drops out.) Since the theory is renormalizable, these constants, after renormalizations are replaced by their renormalized counterparts, g_R and M_R (or g_R and K_R). In the simplified version of ignoring all but the spin-2 parts of the propagator (and suppressing the indices), the relation between renormalized and unrenormalized parameters is given by the spectral function for the spin-2 inverse propagator. Thus write

$$\Delta_0^{-1} = k^2 - \frac{k^4}{M_0^2} + k^2 \int \frac{\sigma(\mu^2)}{\mu^2-k^2}\, d\mu^2 \quad , \qquad (12)$$

$$K_0 = Z^{-1/2} K_R\ ,\quad \varphi_0 = Z^{1/2}\varphi_R \quad . \qquad (13a)$$

Eq.(13a), a consequence of gauge invariances of the theory, ensures that

$$g_{\mu\nu} = \eta_{\mu\nu} + (K_0\phi_0)_{\mu\nu} = \eta_{\mu\nu} + (K_R\phi_R)_{\mu\nu} \quad . \qquad (13b)$$

The integration in (12) may be expected to range from 0 to ∞, assuming that there are no tachyons. The negative norm of the massive spin-2 ghost (as well as the gauge character of the theory, and the presence of Faddeev-Popov ghosts) implies that no statement can be made about the sign of $\sigma(\mu^2)$.

Writing

$$\Delta_0^{-1} = Z^{-1}\Delta_R^{-1} = Z^{-1}\left\{k^2 - \frac{k^4}{M_R^2} + k^6 \int \frac{\sigma_R}{(\mu^2-k^2)\mu^4}\, d\mu^2\right\} , \qquad (14)$$

one may infer

$$Z = \frac{K_R^2}{K_0^2} = 1 - \int \frac{\sigma_R}{\mu^2}\, d\mu^2 \quad , \qquad (15)$$

$$\frac{1}{g_0^2} = \frac{1}{g_R^2} + \frac{1}{K_R^2}\int \frac{\sigma_R}{\mu^4}\, d\mu^2 \quad . \qquad (16)$$

If $\mu^2\sigma_R(\mu^2) \sim \mu^4$ for large μ^2 [11] (as expected from Eq.(11)), Z (or, equivalently, $\frac{1}{K_0^2}$ or M_0^2) would be quadratically divergent and g_0^{-2} logarithmically divergent (cf. Eqs.(15) and (16)). Fradkin and Vilkovisky [10] have computed σ_R for the pure Einstein Lagrangian and give

$$Z = K_R^2/K_0^2 = 1 + \frac{23}{96\pi^2}\, K_R^2L^2 \qquad (17)$$

and the leading terms of the spin-2 part of the inverse propagator as

$$k^2\left\{1 - \frac{7}{320\pi^2} K_R^2k^2\ \ln K_R^2|k^2| + i\pi\theta(k^2)\, K_R^2k^2\right\} \quad . \qquad (18)$$

Here L^2 is the quadratic infinity. Note $Z > 1$, and (relatedly) there is no (CDD) zero [12] for spacelike k^2 ($k^2 < 0$).

(There is of course no real zero for $k^2 > 0$, since the physical threshold lies at $k^2 = 0$.)

IV. THE GHOST PROBLEM AND THE RENORMALIZATION GROUP

We are now in a position to consider the ghost problem [13]. The measure of the problem is this. We would like to compute physical quantities as functions of K_R^2 and g_R^2, with possibly large (but finite) values of external momenta, and then take the limit $g_R^2 \to \infty$, corresponding to the ghost mass $M_R^2 = g_R^2/K_R^2 \to \infty$. Clearly, from (11), this is not possible, except in a non-perturbative sense. That such a limit may be feasible, if one does sum the perturbation series, can be seen by examining the expression (14) for Δ^{-1}, where the limit $M_R^2 \to \infty$ can indeed be taken, with σ_R self-consistently computed from the Einstein part of the Lagrangian alone. Starting, for example, in the one-loop approximation, we would get the CDD-zero-free expression (18) for the leading part of the inverse spin-2 propagator. The important point is that this still exhibits a k^4 dependence [14] for large k, so that the use of the corresponding (ghost-free) propagator in a Dyson-Schwinger scheme would continue to produce a renormalizable set of Green's functions. (The gauge technique of Ref.2, would be needed to compute self-consistently the corresponding vertex functions Γ.)

Unfortunately, a scheme of the type described above has not been developed sufficiently far to constitute a basis for a claim that A) the ghost in (7), B) the presumed unboundedness from below of the corresponding Hamiltonian, as well as C) the infinities of the Einstein Lagrangian have all been laid to rest, by taking M_R^2 to infinity self-consistently, at the end of the calculations with the Lagrangian (7).

But what we can examine, with more confidence, are the criteria which may ensure that when all momenta in the theory grow by scaling ($k \to \kappa k$) in a renormalization group sense, the corresponding effective mass $M^2(\kappa)$ should also grow to infinity when $\kappa \to \infty$. That is to say, while we are, for technical reasons, unable to make a dent on the problem of whether the limit of the theory exists when $M_R^2 \to \infty$ for momenta large but yet smaller than M_R, we may be able to answer the question one way or another of whether $M^2(\kappa) \to \infty$, when the momenta and the effective mass are examined for growth

together: this to be accomplished through using the superior techniques of the renormalization group method, and the perturbation summation implied by their use. Naturally this will entail calculations of the appropriate renormalization group functions with the Lagrangian (7), or possibly its supersymmetric variants [15].

To formulate these criteria, we follow the procedure and notation of Lee [9]. If $g(\kappa)$ and $M^2(\kappa)$ are the effective parameters of the theory, and μ a reference mass, we expect

$$\Gamma'^E(\kappa k, g_R, M_R^2, \mu) = \kappa^4\, \Gamma'(k, g(\kappa), M^2(\kappa)) \quad . \tag{19}$$

Here Γ' is Γ of Eq.(11) with K_0^E left out. (On account of (13b) it is advantageous to leave out this factor and also the Z factors for field renormalization.) From the remarks made at the end of Sec.II and the degree of divergence revealed by relations (15) and (16) for the relevant functions, we may write:

$$g_0 = \mu^{\frac{1}{2}(4-n)} \left[g_R + \sum_{\ell=1} \frac{R^\ell(g_R)}{(n-4)^\ell} \right] \quad , \tag{20}$$

$$M_0^2/M_R^2 = \mu^2 \left[1 + \sum \frac{R_m^\ell(g_R)}{(n-4)^\ell} \right] \quad . \tag{21}$$

Defining the usual quantities

$$\alpha = \mu \frac{\partial M_R^2}{\partial \mu} \quad , \quad \beta = \mu \frac{\partial g_R}{\partial \mu} \quad , \tag{22}$$

one can show with Lee that [16]

$$\frac{M^2(\kappa)}{M_R^2} = \kappa^{-2} \exp\left[- \int_1^\kappa H_m(g(\kappa')) \frac{d\kappa'}{\kappa'} \right] \quad , \tag{23}$$

where

$$H_m = \frac{1}{2} g_R \frac{\partial R_m^1}{\partial g_R} \quad .$$

Our criterion for the innocuousness of the tensor ghost then is that the exp factor in (22) should grow at least as [17] fast as $\kappa^2 (\log \kappa^2)^\epsilon$, $\epsilon > 0$ (for example, $H_m(g_\infty) < -2$ where g_∞ is a zero of $\beta(g_\infty)$), so that $M^2(\kappa^2) \to \infty$ as $\kappa \to \infty$. The task of future calculations then is to see if this criterion is met [18] and, if it is, to set up a non-perturbative calculational scheme [3] with $M_R^2 \to \infty$ and finite k $\left[\text{i.e. } \frac{k^2}{M_R^2} \to 0\right]$ (that is, not just a scheme for the high-energy behaviour of the matrix elements, as is accomplished, e.g., by the relation (19), where the weaker condition $M^2(\kappa) \to \infty$, $\kappa \to \infty$, replaces $M_R^2 \to \infty$).

Before concluding, it is perhaps worth remarking that Lagrangians like (7), when suitably supplemented by appropriate scalar and vector fields, can admit of an exact or a spontaneously broken Weyl invariance [19]. This - together with supersymmetries - may provide welcome - perhaps even necessary - restrictions for the realization of the criterion stated above.

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4) For a manageable set of Dyson-Schwinger equations, it is preferable to use a first-order formalism for quantum gravity; e.g. $L_{\text{Einstein}} = \text{Tr}\, \epsilon^{\mu\nu\rho\kappa} (L_\mu L_\nu B_{\rho\kappa} \gamma_5)$, where L is the vierbein and B the spin connection.

5) Remark, however, that Froissart boundedness has not been demonstrated for field theories containing massless particles. Also, it may be argued that the theory is likely to be physically safe till we reach Planck energies of the order of $1/K_R \sim 10^{19}$ GeV. This, however, is not a good argument; if it were, one should not take the infinities of the theory seriously either, for they begin to affect the predictions of the theory around the same energy.

6) R. Utiyama and B.S. DeWitt, J. Math. Phys. 3, 608 (1962);
K.S. Stelle, Phys. Rev. D16, 953 (1977), where additional references are given.

7) This implies that besides the Newtonian constant $K_R^2 (= 32\pi G)$ there may be further constants, to be determined from experiments, in gravity theory, in particular the completely innocuous $g_R'^2$. Since in the theory considered (ignoring g'^2) we have just two constants g^2 and K^2, there are intrinsically two infinities (Eqs.(15) and (16) or Eqs.(20) and (21)). This should be realized in a gauge analogous to the axial gauge in Yang-Mills theories and would justify the appearance of the same Z renormalizing K as well as ϕ (cf.(13a), (13b)). In other gauges further infinite constants will appear as in Yang-Mills theories.

8) S. Weinberg, Phys. Rev. 118, 838 (1960).

9) S. Weinberg, Phys. Rev. D8, 3497 (1973); J.C. Collins and A.J. Macfarlane, Phys. Rev. D10, 1201 (1974); S.Y. Lee, Phys. Rev. D10, 1103 (1974). In the dimensional regularization method care is needed with counter-terms involving $R^{\mu\nu\rho\kappa} R_{\mu\nu\rho\kappa}$, which in four dimensions are absent on account of the Gauss-Bonnet theorem. These terms may give rise to new anomalies. We are indebted to Dr. D.M. Capper for this remark.

10) E.S. Fradkin and G.A. Vilkovisky ("On renormalization of quantum gravity in curved spacetime", Berne preprint, October 1976) have argued that in a massless theory "radiational cosmological term cannot arise at all". Even if a cosmological counter-term is needed we shall take the renormalized cosmological constant to equal zero, in order that a Minkowskian background can be used. Otherwise one would have to face the problem of quantum corrections to the de Sitter world.

11) In a non-perturbative technique, based for example on using self-consistently the modified propagator (14), $\mu^2\sigma(\mu^2)$ may behave like $\mu^4/(\log \mu^2)^{1+\epsilon}$, $\epsilon > 0$. In this case $\int \sigma_R/\mu^4 \, d\mu^2$ could be finite with no need to renormalize g_0^2. This is similar to the situation shown to hold, in the context of non-perturbative gauge technique for electrodynamics of spin-zero charged particles, discussed in the second paper of Ref.2, where it is shown that there is no infinity corresponding to $\lambda\phi^{*2}\phi^2$ - the conventional infinity being an artifact of the perturbation treatment.

12) As noted by these authors [10], this characteristic of no CDD zeroes in the inverse propagator for spacelike momenta $k^2 < 0$, whenever $Z > 1$, is also true of the asymptotically free Yang-Mills theory. One of the shortcomings of the gauge technique, as noted in Ref.2, was the presence of CDD zeroes in Δ^{-1} (or, equivalently, CDD poles in the propagator Δ), the removal of which produced ambiguities in the theory and necessitated introduction of new parameters. From the above indications, such problems may not be present in a gauge technique applied to gravity theory. This point (i.e. whether Δ^{-1} has CDD zeroes) can, however, only be settled after a further investigation with the full Lagrangian (7) rather than with just the Einstein part of the Lagrangian (as in Ref.10).

13) Since the object at $k^2 = M^2$ must possess an intrinsic width, we are dealing with a ghost resonance - a pole on a non-physical sheet, representing a state which does not belong to the set of incoming or outgoing states.

14) From this point of view, it is of advantage to retain the positive norm good spin-zero particle in the theory (i.e. keep $\frac{1}{g_0'^2} \neq 0$). The relevant part of Δ^{-1} is then always of order k^4, and this would help with high-energy behaviour of some of the graphs (see F. Englert, E. Gunzig, C. Truffin and P. Windey, Phys. Letters 57B, 73 (1975).

15) In supersymmetric gravities, the cancellation of infinities between fermion and graviton loops may even make the integral $\int \sigma_R/\mu^4 \, d\mu^2$ in (16) finite. If this happens, with the assumption of an infinite bare ghost mass ($M_0^2 \to \infty$, $g_0^{-2} \to 0$) the relation (16) may then be used to express the new constant g_R^2 of the theory in terms of K_R^2 (and possibly the good scalar mass m_R^2).

16) If matter is present, Eq.(23) will contain further contributions (cf. Lee's equation (52)). These may in fact be extremely important for the prospects of the theory.

17) Barring logarithmic factors, ideally one would like the growth of $M^2(\kappa)$ to be as fast as $\kappa^2 (\log \kappa^2)^\epsilon$ $\epsilon > 0$, so that $M(\kappa)$ rises faster than the momenta. (This could happen if $H_m(g_\infty) < -4$.) In this case the effective propagator of the theory $\Delta(k) \approx (k^2 - k^4/M^2)^{-1} + \frac{1}{\kappa^2}\Delta(k)$ as $k \to \kappa k$ and the theory would indeed most likely be Froissart bounded.

18) A fair statement about this criterion is that, if it is not satisfied, it is unlikely that the theory makes sense.

19) P.G.O. Freund, Ann. Phys. (NY) 84, 440 (1974).