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THE METHOD OF BOSON EXPANSIONS IN QUANTUM THEORY

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ABSTRACT

We give a review of boson expansion methods applied in quantum theory, e.g., expansions of spin, bifermion and fermion operators in cases of finite and infinite number of degrees of freedom. The basic purpose of the paper is to formulate the most general criteria allowing us to obtain the so-called finite spin approximation of any given Bose field theory and the class of fermion theories associated with it. On the other hand, we also need to be able to reconstruct the primary Bose field theory, when any finite spin or Fermi systems are given.
INTRODUCTION

Within the last twenty years, the method of boson expansions has proved useful in the quantum theory of many-fermion systems. For example, it allowed a contemporary theory of spin waves to be built in the low-temperature description of the Heisenberg ferromagnet, 1-9,15, where for a long time it was known that the ideal magnon gas perfectly simulates the behaviour of the Heisenberg crystal itself. A similar situation appears in the study of the weak excitation limit of atomic nuclei in the microscopic model, where the spectra of low-lying excited states are similar to those of the excited system of weakly coupled quadrupole bosons, 7,18-26.

On the other hand, recent developments on the connection between the Thirring and Sine-Gordon systems in the two space-time dimensions, resulted in a couple of papers connected with the boson expansion methods in quantum field theory, see, e.g. [38,41-46,105-128], and also [16,17,27-35,76].

All the approaches mentioned above were developed independently and as applications to quite different physical phenomena. Therefore, there appears the fascinating problem of considering boson expansion methods in the quantum theory of fermion systems from a unified point of view, i.e. to establish whether there exist any global physical conditions under which this method admits concrete applications, governing all fermion systems, independent of the number of degrees of freedom in the theory. Here there appears also the question of the mutual relations between the different approaches proposed so far.

We claim that the physical essence of the boson expansion methods reads: Any quantum boson in the weak excitation limit can exhibit fermion properties, which then prevail the original boson ones. We call it a fermion-like behaviour. This is the reason for which boson systems, in sufficiently low temperatures, can be used to approximate properties of fermion systems. Quite conversely, if the higher excitations, such as, e.g. the weak coupling limit of the theory, are omitted, then the initial fermion system can exhibit the boson properties. We call it a boson-like behaviour.

The above statement is obviously not true for isolated systems, but if the contact with a suitable environment is taken into account, then, in quantum mechanics, many-body theory and quantum field theory, quite serious treatment of the question of metamorphosis of fermions into bosons is justified, and conversely, see e.g. [44]. Obviously such a metamorphosis cannot always appear for the case when both the initial boson and the final fermion are physical objects. It may happen (especially in quantum field theory, where the spin-statistics theorem should be taken into account if the number of space-time dimensions is equal to four) that the starting point is an ideal non-physical, ghost boson, whose weak excitation limit acquires the properties of a physical fermion.

From the mathematical point of view, the majority of the essential results are based on the boson expansion theory developed by the present author, 16,17,27-35,76, but in the course of the paper we give a review of the related topics which seem essential for a better understanding of the method. Our statement on the fermion-boson reciprocity is formulated on the basis of the equivalence theorems proved in [17] for the example of the Heisenberg ferromagnet, and, in what follows, for the microscopic model of the atomic nuclei. The latter result from the projection theorems proved in [76] and collected in the Appendix.

In the case of the Heisenberg lattice the projection theorems imply that its Hamiltonian can be received as a reduction, , to the Hilbert space of spin states of a suitable pure boson (magnon gas) Hamiltonian, where projects onto the spin space in the boson Fock space , A similar situation appears in the case of the atomic nuclei (see also [133]).

In application to quantum field theory, we were able to prove that with each normal ordered operator series: of free asymptotic Dirac fields one can associate the corresponding functional power series of the classical (commuting ring of functions) spinor fields. The transition from fermions to classical spinors is realized through the subsidiary (but unphysical, as involving the spinor fields which obey Bose statistics) boson level: . The boson spinors satisfy the free Dirac equation, and, the equivalence relation between the bosons and fermions on the Fermion Fock space. On the boson level one has the concept of the coherent state expectation value of the operator which realizes a correspondence principle. To the mediating bosons we have assigned their classical images: .
1. **BOSON EXPANSIONS OF SPIN OPERATORS**

Let $H$ denote a finite-dimensional Hilbert space: $\dim H = 2s + 1$ with $s = 0, 1/2, 3/2, \ldots$. Assume we have in $H$ an irreducible unitary representation of the SU(2) group whose infinitesimal generators obey the relation:

$$s = 1, 2, 3, \mathbf{s} = \{s_a\}_a = 1, 2, 3$$

$$[s_a, s_b] = i\epsilon_{abc} s_c, \quad s_a = (1/\sqrt{2}) (s_1 + i s_2)$$

$$s_3 = i s_+ s_- + s_+ s_+ + s_- s_- = s(s+1) 1.$$  (1.2)

Given, further an infinite-dimensional Hilbert space $\mathcal{F}$ and a Fock representation of the $\text{CCR}$ (canonical commutation relations algebra) in it, realized by the triple $(a^*_n, a_n, \Omega)$ with

$$[a_n, a_n^*] = 1, \quad a_n = 0.$$  (1.3)

We pose the following problem: find the representation of (1.1) in a certain finite-dimensional subspace of $\mathcal{F}$. By virtue of the irreducibility of the pair $(a^*_n, a_n)$ we can expect the operator $\mathbf{s}$ to be fully expressed in terms of boson generators.

Let us begin from the Holstein-Primakoff [1] solution

$$\mathbf{s}^H: \quad \mathbf{s}_+ = s_+ + i s_0 = i\sqrt{s} \quad \alpha \left( 1 - \alpha^2 A / 4s \right) ,$$

$$\mathbf{s}_- = s_- = i\sqrt{s} \quad \alpha^2 \left( 1 - \alpha^2 A / 4s \right) \alpha ,$$

$$\mathbf{s}_0 = s_0 = s - \alpha^2 A.$$  (1.4)

The square roots are understood formally as infinite series with respect to $a^*_n/\sqrt{s}$. The Hilbert space $\mathcal{F}$ here consists of functions of the occupation number parameter $n$ (the eigenvalue of the operator $\mathbf{s} = a^*_n$). One can easily check that the operators $s_\pm, s_0$ obey the commutation relations (1.1). However, the condition of mutual adjointness for $s_+ s_-$ and the self-adjointness for $s_0$, as well as (1.2), do not hold on the whole of $\mathcal{F}$.

The operators (1.4) leave invariant subspaces of $\mathcal{F}$ consisting of functions which depend either on the occupation number parameter $n = 0, 1, \ldots, 2s$ or $n \geq 2s + 1$. The first, $2s+1$-dimensional, subspace we denote by $H$. The orthogonal complement of $H$ in $\mathcal{F}$ is called a non-physical space, as it involves spin values greater than $s$.

In practical applications considerations must be restricted to $H$ only, where the functional argument $n$ does not exceed $2s$. However, this subsidiary condition is not easy to realize directly. Here one can use an indirect method [2]. Namely, in place of boson operators let us consider quasi-boson ones, obeying the relations:

$$[a_n, a_m^*] = \left( 1 - \frac{1}{2s} \right) (a_n^2 \alpha_m^* - a_n^* \alpha_m) ,$$

$$a_n^2 = \alpha_n^2 (n^2 - 1) = 0.$$  (1.5)

Then, by virtue of the relation

$$a_n^2 = \prod_{p=0}^{2s-1} (n^2 - p) = \prod_{p=0}^{2s-1} (\alpha_n^2 - 1) = \prod_{p=0}^{2s-1} (\alpha_n^2 - p)$$

and

$$\mathbf{r}_n f(n) = n f(n) ,$$  (1.7)

we get

$$\prod_{p=0}^{2s-1} (\alpha_n^2 - p) f(n) = \prod_{p=0}^{2s-1} (\alpha_n^2 - p) f(n) ,$$  (1.8)

which implies that for $n \leq 2s - 1$, one of factors appearing on the right-hand-side of (1.6), necessarily vanishes.

In consequence, the action of (1.5) onto any function from $\mathcal{F}$ gives a non-zero result only if $f \notin H$. In that case, the quasi-boson operators differ from boson operators outside $H$ only. This is the case when the higher excitations of the quantum system $\{a^*_n, a_n, \Omega \}$ can appear with considerable probability.

The approximation of spin operators by quasi-bosons becomes better with increasing spin value $s$.

One can also make use of the power series expansions of (1.4), (compare [3]):

$$\mathbf{s}_+ = \mathbf{i} \sqrt{s} \left( a_n - \frac{1}{4s} \right) \alpha = \frac{1}{\sqrt{2s}} (\alpha \alpha - 1) ,$$

$$\mathbf{s}_- = \mathbf{i} \sqrt{s} \left( a_n^* - \frac{1}{4s} \right) \alpha = \frac{1}{\sqrt{2s}} (\alpha \alpha^* - 1) ,$$

and neglect all terms except a few.

The approximate formulas for the spin operators obtained in this way become good for $s \gg 1$, and are surely not exact for $s = 1/2$. 
In connection with Dyson's theory of spin waves in a Heisenberg ferromagnet \([18,49]\) the so-called Dyson-Maleev solution \([4]\) was introduced:

\[
\mathcal{P}: \quad \begin{align*}
\Delta S_x &= (1 - \frac{1}{2}\hat{a}^\dagger\hat{a}) \cdot \alpha^x , \\
\Delta S_y &= (1 - \frac{1}{2}\hat{a}^\dagger\hat{a}) \cdot \alpha^y , \\
S_z &= s - \hat{a}^\dagger\hat{a} .
\end{align*}
\]

(1.10)

The operators \(S_x, S_y\) are never mutually adjoint inside \(\mathcal{F}\) (hence the operator \(\mathcal{P}\) cannot be Hermitian) though the commutation relations (1.1) obviously hold in \(\mathcal{F}\). Moreover, the eigenvalues of \(S_z\) are equal to \(-s, -s+1, \ldots\) and are not bounded from above, while for a real spin operator we would have the upper bound equal to \(s\).

The difficulties with adjointness and the spectrum of \(S_z\) can be removed \([5-7]\), by a suitable choice of the metric in \(\mathcal{F}\). Namely, \(\mathcal{F}\) is a Hilbert space with respect to a scalar product:

\[
\mathcal{F}\ni \xi, \eta \Rightarrow (\xi, \eta) = \mathcal{F}_n = \frac{1}{\sqrt{\Omega}} \mathcal{D}^\dagger \mathcal{D} \Omega ,
\]

(1.11)

\[
(\xi, \eta) = \mathcal{F}_n = \mathcal{F}_m .
\]

Let us consider in \(\mathcal{F}\) a sesquilinear form:

\[
(\xi, \eta) = (\xi, \mathcal{F}_n \eta) ,
\]

(1.12)

where \(\mathcal{F}\) is a Hermitian operator. We impose on \(\mathcal{F}\) the additional restriction:

\[
(\xi, \mathcal{D}^\dagger \mathcal{D} \xi) = (\mathcal{D}^\dagger \mathcal{D} \eta) ,
\]

(1.13)

where \(\mathcal{D}^\dagger\) is the Dyson-Maleev spin operator.

By virtue of (1.13) we have satisfied in \(\mathcal{F}\) the two identities:

\[
\mathcal{F} S_x = S_x \mathcal{F} , \quad \mathcal{F} S_y = S_y \mathcal{F} ,
\]

(1.14)

which, by taking into account the boson expansions (1.10), lead to:

\[
\alpha^x (\mathcal{D} - \alpha^x \mathcal{D}^\dagger \mathcal{D}) = \mathcal{F} \alpha^x ,
\]

(1.15)

\[
\alpha^\dagger \mathcal{F} = \mathcal{F} \alpha^\dagger ,
\]

proving that \(\mathcal{F}\) can be diagonalized in \(\mathcal{F}\) together with \(\hat{n} = \alpha^\dagger \alpha\).

Hence:

\[
(\mathcal{D} - \hat{n}(\mathcal{D})_n \mathcal{D}^\dagger \mathcal{D}) = \mathcal{F} \alpha^x ,
\]

(1.16)

where:

\[
\mathcal{F}_n = (\xi, \eta) = (\xi, \mathcal{F}_n \eta) ,
\]

(1.17)

\[
\mathcal{F}_n = \langle \mathcal{D} - \hat{n}(\mathcal{D})_n \mathcal{D}^\dagger \mathcal{D} \rangle , \quad \mathcal{F}_n = \langle \mathcal{D} - \hat{n}(\mathcal{D})_n \mathcal{D}^\dagger \mathcal{D} \rangle .
\]

It is obvious that for non-physical spin values, namely for \(n \geq 2s+1\), the matrix element \(\mathcal{F}_n\) of \(\mathcal{F}\) vanishes, because, by virtue of (1.11) and (1.12), we have:

\[
(\xi, \mathcal{F}_n \eta) = \mathcal{F}_n \xi = 0 ,
\]

(1.18)

The metric defined by the scalar product \((\xi, \eta)_{\mathcal{F}}\) is indefinite:

\[
(\xi, \mathcal{F}_n \eta)_{\mathcal{F}} = 0 , \quad \text{for } n \geq 2s .
\]

However, in \(\mathcal{F}\) it distinguishes a 2s-dimensional proper subspace \(\mathcal{H}\) of physical states, on which \((\xi, \eta)_{\mathcal{F}}\) defines a Hilbert space topology.

The two solutions discussed, \(S^N\) and \(S^D\), are not fully independent, and a connection between them can be found. Namely \([5-9]\), the square root \(\mathcal{F}^{1/2}\) of \(\mathcal{F}\) induces the following identities on \(\mathcal{H}\):

\[
S^N = \mathcal{F}^{1/2} \mathcal{S}^D \mathcal{F}^{-1/2} ,
\]

(1.19)

\[
S^N = \mathcal{S}^D \mathcal{F}^{1/2} \mathcal{F}^{-1/2} \mathcal{F}^{1/2} \mathcal{S}^D \mathcal{F}^{-1/2} .
\]

(1.20)

More detailed considerations on this subject can be found in \([7-9]\). Together with \(S^N\) and \(S^D\), one can imagine the more general Cooke-Loly solution, given in \([10, 11]\):

\[
\mathcal{S}^C: \quad \begin{align*}
\Delta S_x &= \mathcal{F} \Delta S_x (\mathcal{D} - \alpha^x \mathcal{D}^\dagger \mathcal{D})^{1/2} , \\
\Delta S_y &= \mathcal{F} \Delta S_y (\mathcal{D} - \alpha^y \mathcal{D}^\dagger \mathcal{D})^{1/2} , \\
S_z &= s - \Delta S_x .
\end{align*}
\]

(1.20)
One can develop a few approaches, such as, e.g., Schwinger's method of paired bosons [12,13], the use of two sets of bosons, being responsible for the dynamics and the kinematics, respectively, of the spinning system [14], as well as the introduction of two sets of fermions [15]

However, all these approaches meet difficulties analogous to these exposed connection with $\mathcal{S}^\mathbb{R}$ and $\mathcal{S}^\mathbb{P}$.

A concrete algebraic realization of the infinitesimal generators of the SU(2) group, still does not guarantee physically correct results, and subsidiary conditions (including a proper definition of the state space and the scalar product in it) are unavoidable.

Recently, in Refs. 16, 17, another boson expansion for spin operators was suggested, giving exact results for spin-1/2, and which is free of all disadvantages discussed above. We have

\begin{equation}
\begin{aligned}
\mathcal{T} S_+ &= \dot{a} : \exp(-a^2 \dot{a}) : a, \\
\mathcal{T} S_- &= : \exp(-a^2 \dot{a}) : a, \\
S_3 &= (-4/3) \Delta F + \dot{a} : \exp(-a^2 \dot{a}) : a,
\end{aligned}
\end{equation}

where $:\ldots:\$ symbolizes the normal ordering of generators $a^2 \dot{a}$, and one can equivalently introduce $S_2 = (4/3) \Delta F - a^2 : \exp(-a^2 \dot{a}) : a$

Here.

\begin{equation}
\Delta F = : \exp(-a^2 \dot{a}) : + \dot{a} : \exp(-a^2 \dot{a}) : a
\end{equation}

so that

\begin{equation}
S_2 = (-4/3) \{ : \exp(-a^2 \dot{a}) : - \dot{a} : \exp(-a^2 \dot{a}) : a \}
\end{equation}

Because the operator $\exp(-a^2 \dot{a})$ projects onto the ground state $\Omega = f_0$.
Obviously one can equally well use $-S_z$ in place of $S_z$, (1.25b) is a projected set of operators (1.25a).

The question of higher spins will be considered below in connection with boson expansions of fermion operators. The All above considerations were pure quantum mechanical in spirit. The transition to the description of the infinite assembly of spins is immediate here. If we consider a Fock representation of the CCR, generated by the triple \( \{ \Omega_k, \Omega^*_l \} \) for all \( k = 1, 2, \ldots \), and repeating arguments given previously for each single \( l = 1, 2, \ldots \), we get a corresponding sequence of bosonized spin operators \( \{ S_k \} \) for \( k = 1, 2, \ldots \).

\[
[S_{ka}, S_{kb}] = S_{kc} [S_{ka}, S_{kb}] = S_{kc} S_{kb} = \delta_{d}^{abc} S_{d}. \tag{1.27}
\]

This sequence is, in fact, used in practical applications.

2. BOSON EXPANSIONS OF DIMENSION OPERATORS

Let the triple \( \{ b^*_1, b_1, \Omega_F \} \) generate a Fock representation of the CAR (canonical anticommutation relation algebra):

\[
\{ b^*_k, b_k, \Omega_F \} = S_{k} A_F, \quad b_k \Omega_F = 0 \quad \text{for all} \ k = 1, 2, \ldots \; \tag{2.1}
\]

\[
[S_{ka}, S_{kb}] = S_{kc} [S_{ka}, S_{kb}] = S_{kc} S_{kb} = \delta_{d}^{abc} S_{d}. \tag{1.27}
\]

\( A_F \) denotes here an operator unit in the algebra. Each element of the representation, being quadratic in \( b^*_k, b_k \), we call a bifermion operator. Following papers [18-25], we restrict our considerations to operators \( b^*_k b_k \) and \( b^*_k b_k \), which together with their adjoints satisfy the commutation relations:

\[
[b^*_k b_k, b^*_l b_l] = \delta_{d}^{abc} b^*_k b_k - \delta_{d}^{abc} b^*_k b_k ,
\]

\[
[b^*_k b_k, b^*_l b_l] = \delta_{d}^{abc} b^*_k b_k - \delta_{d}^{abc} b^*_k b_k ,
\]

\[
[b^*_k b_k, b^*_l b_l] = \delta_{d}^{abc} b^*_k b_k + \delta_{d}^{abc} b^*_k b_k . \tag{2.2}
\]

\( W \) denotes here an operator unit in the algebra. Each element of the representation, being quadratic in \( b^*_k, b_k \), we call a bifermion operator. Following papers [18-25], we restrict our considerations to operators \( b^*_k b_k \) and \( b^*_k b_k \), which together with their adjoints satisfy the commutation relations:

\[
[b^*_k b_k, b^*_l b_l] = \delta_{d}^{abc} b^*_k b_k - \delta_{d}^{abc} b^*_k b_k ,
\]

\[
[b^*_k b_k, b^*_l b_l] = \delta_{d}^{abc} b^*_k b_k - \delta_{d}^{abc} b^*_k b_k ,
\]

\[
[b^*_k b_k, b^*_l b_l] = \delta_{d}^{abc} b^*_k b_k + \delta_{d}^{abc} b^*_k b_k . \tag{2.2}
\]

We are interested not only in concrete \( b^*_k b_k, b^*_k b_k \) but in the whole class of operators \( U b^*_k b_k U^*, U b^*_k b_k U \) determined by (2.2) up to a unitary transformation. Now, let us assume to have given a Fock representation of the CAR algebra. We wish to prove that there really exists a unitary transformation \( U \) such that the boson expansions

\[
U b^*_k b_k U^* = \sum_{n=0}^{\infty} \sum_{\alpha} \sum_{\beta} \sum_{\gamma} P_n (k, \alpha, \beta, \gamma) \alpha^* \beta^* \gamma^* \delta_{\alpha^* \beta^* \gamma^*}, \tag{2.3}
\]

hold on a suitable domain. Here, \( \mathbb{F} = (r_1, \ldots, r_n) \).
In the above conjecture, we have combined the original idea of Belov, Znilevinsky [16, 23] with the Marumori [19-21, 25] approach, giving compact formulas for boson expansions of bifermion operators. The latter approach allows us to avoid a wearisome use of the iteration procedure extensively applied in (18).

Let \( f_i \) \( i = 1, 2, \ldots \) constitute a complete orthonormal system in \( L^2(\mathbb{R}^n) \),

\[
\sum_i \delta_i (\overrightarrow{p}) \delta_i (\overrightarrow{q}) = \delta (\overrightarrow{p} - \overrightarrow{q}) ,
\]

\[
\int d\overrightarrow{p} \delta_i (\overrightarrow{p}) \delta_i (\overrightarrow{q}) = \delta_{\overrightarrow{p} \overrightarrow{q}}.
\]

It allows us to consider in place of discretely indexed operators \( \tilde{a}_k, \tilde{a}_k \), see (1.26), continuously indexed ones:

\[
\tilde{a}_k (\overrightarrow{p}) = \delta_{\overrightarrow{p} \overrightarrow{q}} \delta_{i} (\overrightarrow{q}) ,
\]

\[
\tilde{a}_k (\overrightarrow{p}) = \delta_{\overrightarrow{p} \overrightarrow{q}} \delta_{i} (\overrightarrow{q}).
\]

Let us write \( \mathbb{R}^n = \mathbb{R}^i \otimes \mathbb{R}^j \), \( i + j = n \), \( \overrightarrow{p} = (p_1, \ldots, p_i, \overrightarrow{p} \in \mathbb{R}^i, \overrightarrow{q} \in \mathbb{R}^i \).

We denote the basis in \( L^2(\mathbb{R}^i) \) and \( L^2(\mathbb{R}^j) \) by \( (k, l)^i, \ldots, \) and \( (l, k)^j, \ldots, \) respectively. Then the basis in \( L^2(\mathbb{R}^i) \otimes L^2(\mathbb{R}^j) \) is given by \( |k, l> \otimes |l, k>. \)

Let us consider the set of antisymmetric basis elements in \( L^2(\mathbb{R}^i) \otimes L^2(\mathbb{R}^j) \) given by \( \{ f_{kl} \} \), \( k, l = 1, 2, \ldots \), where

\[
f_{kl} = \frac{1}{k!} (q_k \otimes \overrightarrow{h_l} - q_l \otimes \overrightarrow{h_k}) ,
\]

\[
f_{kl} (\overrightarrow{q}, \overrightarrow{r}) = \frac{1}{k!} [ q_k (\overrightarrow{q}) \overrightarrow{h_l} (\overrightarrow{r}) - q_l (\overrightarrow{q}) \overrightarrow{h_k} (\overrightarrow{r}) ] .
\]

We have

\[
f_{kl} (\overrightarrow{q}, \overrightarrow{r}) = f_{kl} (\overrightarrow{r}, \overrightarrow{q}) .
\]

Hence, the operators (2.5) can be transformed into the form:

\[
\int d\overrightarrow{p} \tilde{a}_k (\overrightarrow{p}) f_{kl} (\overrightarrow{p}) = \delta_{kl} ,
\]

\[
\int d\overrightarrow{p} \tilde{a}_k (\overrightarrow{p}) f_{kl} (\overrightarrow{p}) = \delta_{kl} .
\]

and, by virtue of:

\[
(f_{kl}, f_{kk}) = \int d^2 q \int d^2 p \ f_{kl} (\overrightarrow{q}, \overrightarrow{p}) \ f_{kk} (\overrightarrow{q}, \overrightarrow{p}) = 0 ,
\]

the operators (2.6) satisfy

\[
[f_{kl}, f_{kl}] = (\delta_{kl} - \delta_{kk}) \ f_{kk} ,
\]

\[
[f_{kl}, f_{kl}] = (\delta_{kl} - \delta_{kk}) \ f_{kk} .
\]

We denote the boson Fock space generated by (2.10) by \( \mathcal{F}_B \). Its most general element can be written in the form:

\[
|\Omega_B> = \sum (\tilde{a}_{kl})^m \tilde{a}^*_{kl} |\Omega_B> ,
\]

where

\[
(m)_B = (\tilde{a}_{kl})^m \tilde{a}^*_{kl} |\Omega_B> .
\]

and we have

\[
(F_m, l m>) = \sum_{m'} F^m_{m'} (l m') \Omega_B .
\]

Here \( F^m_{m'} \) is a tensor, totally symmetric in the variables \( n_1, n_2, \ldots, n_m \).

Let us consider the antisymmetric basis system in \( H \):

\[
|l m> = \sum_{m'} (-1)^m \tilde{a}^*_{kl} |l m'> ,
\]

and

\[
|\Omega_B> = \sum (4!)^m (m, m') \tilde{a}^*_{kl} |l m'.
\]

We denote the fermion Fock space for the representation \( \mathcal{F}_F \).

and

\[
(F_m, l m>) = \sum_{m'} F^m_{m'} (l m') \Omega_B .
\]

Let us further denote by \( \mathcal{F}_F \) a fermion Fock space for the representation.
The basis vectors in $\mathcal{F}_F$ are totally antisymmetric and by analogy to (2.13) can be defined in the form:

$$|\omega\rangle_F = b^*_{u_1} b^*_{u_2} ... b^*_{u_n} |\Omega_F\rangle$$

(2.14)

Now, we take into consideration the tensor product space $\mathcal{F}_B \otimes \mathcal{F}_F$, the vacuum in which is denoted: $\Omega_B \otimes \Omega_F = |\Omega_B\rangle |\Omega_F\rangle$.

Furthermore, we restrict $\mathcal{F}_B \otimes \mathcal{F}_F$ to $H \otimes \mathcal{F}_F$.

Here $H \otimes \mathcal{F}_F$ is spanned by vectors $|\omega\rangle_H |\Omega_F\rangle$, and $\Omega_B \otimes \mathcal{F}_F$ by $|\omega\rangle_B |\Omega_F\rangle$.

Let us define the operator $U_M$:

$$U_M = |\Omega_B\rangle |\Omega_F\rangle \langle 0| \sum_{m=0}^{\infty} \frac{1}{(2m-1)!} \left( \sum_k a^*_{kk} b_k b_k \right)^m |\omega_F\rangle$$

(2.15)

which, if acting in $H \otimes \mathcal{F}_F$, has the remarkable property of realizing the map:

$$U_M : \Omega_B \otimes \mathcal{F}_F \rightarrow H \otimes \Omega_F.$$

(2.16)

In this connection compare [7, 19, 20]:

$$U_M |\omega_B\rangle |\omega_F\rangle = |\omega_H\rangle |\Omega_F\rangle$$

(2.17)

$$f(\alpha_H |\omega|) U_M^* = f(\alpha_B |\omega|)$$

$$f(\alpha_B |\omega| U_M |\omega_B\rangle |\omega_F\rangle = f(\alpha_B |\omega_F\rangle)$$

which proves that $U_M^* U_M$ plays the role of the unit operator on $\Omega_B \otimes \mathcal{F}_F$.

An arbitrary operator $T$ acting in $\Omega_B \otimes \mathcal{F}_F$ has its image $\mathcal{U}$ in $H \otimes \Omega_F$:

$$U_M (U_M^* T U_M) U_M^* = f(\alpha_B |\omega_F\rangle)$$

(2.18)

Let us note that a projection onto $\Omega_B \otimes \mathcal{F}_F$,

$$U_M U_M^* = U_M^* U_M = \frac{1}{|\Omega_B\rangle |\Omega_F\rangle}$$

(2.19)

which is a projection onto $H \otimes \Omega_F$.

Because of

$$U_M^* \hat{P} = U_M^*, \quad \hat{P} U_M = U_M, \quad \hat{P}^2 = \hat{P}^* = \hat{P}$$

(2.20)

we get

$$\mathcal{U} \hat{P} = \mathcal{U}^* \hat{P} = \mathcal{P} \mathcal{U} \hat{P},$$

(2.22)

which proves that $\mathcal{U}$ possesses non-zero matrix elements between physical states in $\mathcal{F}_B \otimes \mathcal{F}_F$ only, i.e. between elements of $H \otimes \Omega_F$.

Following [7], we shall give an explicit formula for the boson image $\mathcal{U}$ of the operator $T$:

$$\mathcal{U} = \sum_{m=0}^{\infty} \frac{1}{(2m)!} \sum_{(k,l)} \frac{1}{(2m-1)!} \left( \sum_{k,l} a^*_{kk} b_k b_k \right)^m |\omega_F\rangle$$

(2.23)

$$\sum_{m=0}^{\infty} \frac{1}{(2m)!} \left( \sum_{k,l} a^*_{kk} b_k b_k \right)^m |\omega_F\rangle$$

$$\sum_{m=0}^{\infty} \frac{1}{(2m)!} \left( \sum_{k,l} a^*_{kk} b_k b_k \right)^m |\omega_F\rangle$$

$$\sum_{m=0}^{\infty} \frac{1}{(2m)!} \left( \sum_{k,l} a^*_{kk} b_k b_k \right)^m |\omega_F\rangle$$

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$$\sum_{m=0}^{\infty} \frac{1}{(2m)!} \left( \sum_{k,l} a^*_{kk} b_k b_k \right)^m |\omega_F\rangle$$
The infinite boson expansions (2.27)–(2.26) are not convenient for practical applications, especially because the convergence of these series is not sufficiently quick. Therefore, one usually either neglects all expansion terms except a few, or one tries to develop a separate theory of finite boson expansions, see e.g. [26].

Let us consider in place of $U_M$ the operator $U$,

$$U = 10_F \exp \left( \sum_{k \lambda} \hat{a}_{\lambda k} b_k b_{\lambda} \right) 10_F$$

This operator, though mapping $Q_B \otimes \Sigma_F$ onto $H \otimes Q_F$, spoils the normalizability of the basis vectors.

$$\langle m \rangle_B 10_F = \frac{1}{\sqrt{2n-1}} U \langle m \rangle_F$$

To improve this defect, let us define an operator $\tilde{U}$:

$$\tilde{U} = 10_{BB} \langle m \rangle_F \sum_{m=0}^{\alpha+1} \left( \sum_{k \lambda} \hat{a}_{\lambda k} b_k b_{\lambda} \right)^m 10_{BF}$$

(note that $\tilde{U}$ is a slight modification of $U$), satisfying:

$$\langle m \rangle_B 10_F = \frac{1}{\sqrt{2m-1}} \tilde{U} \langle m \rangle_B 10_F$$

One can easily check that the operator $\tilde{U}$ is a unit operator in $Q_B \otimes \Sigma_F$, while $\tilde{U} = \tilde{U}^2$ in $H \otimes Q_F$, respectively. By analogy to previous considerations, for an arbitrary operator $T$ acting in $Q_B \otimes \Sigma_F$, we can easily get its boson image $\tilde{T}$ acting in $H \otimes Q_F$:

$$\tilde{T} = U T \tilde{U}$$

By virtue of the relation $\tilde{U}^2 = U$, $T^* = T$ here does not imply $\tilde{T}^* = \tilde{T}$. 

---

This lengthy formula for a special case of bifermion operators reads:

$$U_M b_k^* b_{\lambda}^* U_M^* = \hat{a}_{\lambda k} = \sum_{m=0}^{\infty} \hat{a}_{\lambda m} a_{m m} + \ldots$$

In [22] there were derived formal, but compact, formulas for bifermion operators of interest:

$$U_M b_k^* b_{\lambda}^* U_M^* = \hat{a}_{\lambda k} = \sum_{m=0}^{\infty} \hat{a}_{\lambda m} a_{m m} a_{m m} + \ldots$$

where

$$\hat{3}_{kl} = \sum_{m} \hat{a}_{m m} a_{k m}$$

We have thus found the Hilbert space in which the operators $U_M b_k^* b_{\lambda}^* U_M^*$ satisfy the commutation relations (2.2) and, moreover, admit the boson expansions (2.23)–(2.26), which by virtue of (2.5)–(2.8) prove our starting conjecture. In this way we have further disclosed the conditions under which Belav-Zelezinsky and Karunori
By making a few, not too difficult calculations, one can derive finite boson expansions for bifermion operators:

\[ U b_k^* b_l^* \bar{U} = (\delta_{kl} - \sum_m \alpha_{km} \alpha_{lm}) \bar{\hat{p}}, \]

\[ U b_k b_l \bar{U} = \alpha_{kl} \bar{\hat{p}}, \]

\[ U b_k^* b_l \bar{U} = \sum_r \alpha_{kr} \alpha_{lr} \bar{\hat{p}}, \]

obviously implying the commutation relations (2.2).

In connection with both infinite and finite Marumori expansions, one can express a few objections, being close analogies with those appearing in the discussion of boson expansions of spin operators.

A practical use of the infinite case meets essential difficulties, though it gives a Hermitian Hamiltonian. Usually one takes into account the first two terms of the expansions, as e.g. in [18, 20, 23, 24], not worrying whether this assumption is mathematically and physically correct. On the other hand, even the application of full infinite expansions can appear to be doubtful, because there is no rigorous proof of their convergence.

In case of finite expansions, it is not possible to get a Hermitian Hamiltonian, by virtue of the fact that for example \( U b_k^* b_l^* \bar{U} \) cannot be transformed into \( U b_k^* b_l \bar{U} \) by the use of the \( \star \) operation; compare also a discussion of the Dyson-Maleev expansion, where an analogous question appeared.

3. BOSON EXPANSIONS OF FERMIOM OPERATORS

To explain the leading idea motivating approaches to the question of boson expansions for fermions, let us quote from Ref. 33:

"It has been taken for granted that a fundamental theory of elementary particles must involve Fermion fields in the basic formalism, but need not involve bosons. The spinor theory of Heisenberg [36] is the most developed example of this philosophy (compare also considerations on the two-neutrino theory of photons [93]). More recently the opposite point of view has been proposed: that is, a theory in which only observable fields, necessarily uncharged bosons, occurring in the basic formalism, might be capable of describing fermions, or uncharged bosons. An early paper of Skyrme [37] goes so far as to give explicit formulas for the fermion field in terms of the boson one in a two-dimensional field theory."

3.1 Representations of the CAR generated by representations of the CCR

Now, we shall collect a few results of the boson expansion theory developed in the series of papers [27-34, 16, 17, 76]. As its extremely exciting feature there appears the fact that we deal with a kind of universal boson expansions for fermion operators, involving applications in quantum theory as a whole, beginning from quantum mechanics through many-body problems and ending in quantum field theory. From a physical point of view, this theory governs the behaviour of systems whose excitation level is so lowered that a probability of occupying other than the 0-th and 1-st energy levels of each single degree of freedom (normal mode) is very small.

Let us begin from the case of the infinitely many degrees of freedom. We denote \( \mathcal{K} \) a complex Hilbert space. By \( \mathcal{U}_C (\mathcal{K}) \) we denote a Fock representation of the CAR (canonical anticommutation relations algebra) over
K, acting on the representation space $F^F$, and defined by the triple

$[a(f),a(q)^*] = (f;q)D_f$, \hspace{1cm} (3.1)

$[a(f),a(q)] = 0 = [a(f)^*,a(q)^*]$. \hspace{1cm} (3.2)

By $U_B(K)$ we denote a Fock representation of the CCR (canonical commutation relations algebra) over $K$, generated by the triple $\{a(f),a(q)^*,\Omega^F\}$.

The Fock representation $U_B(K)$ of the CCR algebra acts in the domain $\Phi \subset F^F$. We denote $\langle \bar{a}^n, a^m \rangle = \delta^m_n \delta^{\frac{m}{2}}_n \delta^{\frac{n}{2}}_n \phi$. Then the operators $b(f)$, $b(q)^*$ with:

$b(f) = \exp\{-\bar{a}(f)a(f)\} = \sum_{m=0}^{\infty} \frac{1}{m!} \bar{a}(f)^m a(f)^m$, \hspace{1cm} (3.7)

where

\[
\phi_{m,n}(E_m, E_n) = \int dE_m dE_n \int dE_p dE_q \phi_{m,n}(E_m, E_n, E_p, E_q), \hspace{1cm} (3.8)
\]

generate a Fock representation $U_B(K)$ of the CAR algebra acting on the following subspace of $F^G$.

The Boson expansions of Fermion operators (3.7)-(3.9) can be applied in...
quantum field theory. Now let us study a transition to a finite number of degrees of freedom.

We have proved in [32, 16, 17] that the boson expansions (3.7)-(3.9) can be reduced to a finite number of degrees also. Namely, let $K = \sum_{k=0}^{\infty} K_k$, where $\dim K_k < \infty$ for each $k$. Then we can construct what we call the truncated representation of the CAR, consisting of mutually commuting segments, inside which the usual CAR hold. The $l$-th segment is generated by the triple $\{ b^l, b^l, \Omega^l \}$, given by:

$$b_{\alpha}^l := \exp \left( -\sum_{\alpha=0}^{N_l} a_{\alpha}^l a_{\alpha}^l \right), \quad \sum_{\alpha=-N_l}^{N_l} \sum_{\beta=-N_l}^{N_l} E_{\alpha \beta} a_{\alpha}^l a_{\beta}^l = 0$$

with

$$f_{\lambda \mu} (\alpha, \beta, \gamma) := \sum_{x=0}^{N_l} \sum_{y=0}^{N_l} E_{xy} a_{\alpha}^l a_{\beta}^l$$

and $K = \sum_{l=1}^{N_l} f_{\lambda \mu} (\alpha, \beta, \gamma)$ being the basis system in $K_{l+1}$.

Furthermore,

$$E_{\alpha_0 \ldots \alpha_{N_l}} = \varepsilon_{\alpha_0 \ldots \alpha_{N_l}} \varepsilon_{\alpha_0 \gamma_0} \ldots \varepsilon_{\alpha_{N_l} \gamma_{N_l}}$$

where $\varepsilon_{\alpha_0 \ldots \alpha_{N_l}}$ is the generalized Levi-Civita tensor vanishing if any two indices coincide and taking the value $\pm 1$ depending on the odd or even permutation of indices.

We then get:

$$[b_{\alpha}^l, b_{\beta}^l] = \left( \frac{\gamma}{\alpha \beta} \right) [b_{\alpha}^l, b_{\beta}^l]$$

$$[b_{\alpha}^l, b_{\beta}^l] = 0$$

where $\Delta_{p}^l$ is a unit operator in the $l$-th segment of the truncated representation of the CAR, projecting onto the subspace $\mathcal{F}_{p} = \mathcal{F}_{p}^l \mathcal{F}_{p}^l$ of $\mathcal{F}_{p}$ with

$$\Delta_{p}^l := \exp \left( -\sum_{\alpha=0}^{N_l} a_{\alpha}^l a_{\alpha}^l \right), \quad \sum_{\alpha=-N_l}^{N_l} \sum_{\beta=-N_l}^{N_l} E_{\alpha \beta} a_{\alpha}^l a_{\beta}^l = 0$$

and in $\mathcal{F}_{p}$, there obviously holds:

$$[b_{\alpha}^l, b_{\beta}^l] = \left( \frac{\gamma}{\alpha \beta} \right) [b_{\alpha}^l, b_{\beta}^l]$$

The simplest example here is $\dim K = 1$. If $\Omega_{p}^l = \Omega_{p}$ for all $l = 1, 2, \ldots$ we get:

$$b_{\alpha}^l = a_{\alpha}^l$$

and in $\mathcal{F}_{p}$, there obviously holds:

$$[b_{\alpha}^l, b_{\beta}^l] = \left( \frac{\gamma}{\alpha \beta} \right) [b_{\alpha}^l, b_{\beta}^l]$$

The simplest example here is $\dim K = 1$. If $\Omega_{p}^l = \Omega_{p}$ for all $l = 1, 2, \ldots$ we get:

$$b_{\alpha}^l = a_{\alpha}^l$$

and in $\mathcal{F}_{p}$, there obviously holds:

$$[b_{\alpha}^l, b_{\beta}^l] = \left( \frac{\gamma}{\alpha \beta} \right) [b_{\alpha}^l, b_{\beta}^l]$$
If with each l-th segment we associate a particular l-th vacuum $\Omega^l_k$ and then restrict our considerations to a concrete single segment, we deal with a one-dimensional quantum mechanical system. Omitting the now superfluous index l, we get the following identities for $b, b^\dagger$:

\[ [b, b^\dagger]_+ = \delta_F, \quad b^\dagger b = 0 = b b^\dagger, \]

\[ b\Omega = 0, \quad \delta_F = \exp(-a^\dagger a) + a^\dagger : \exp(-a^\dagger a) : a, \]

which thus define an irreducible representation of the CAR in the two-dimensional subspace $F_F$ of the original Hilbert space $F$ generated by $\{a^\dagger_0, a_0, \Omega_a\}$. $F_F$ is spanned by the ground state $\Omega_a$ and the one-particle state. For more details see also [16,17]: $a^\dagger_0 \Omega_a = b^\dagger \Omega_a$ (3.17) is the case when the correct boson expansions for spin-1/2 can be proposed.

In the case of $\dim K = 2$ it is possible to find the correct boson expansions for spin 0 and 1 operators. We deal here with a two-particle quantum system, where by (3.10)-(3.14) a boson with two internal degrees of freedom $\{a^\dagger_0, a_0, \Omega_{a^2}\}$ generates a corresponding fermion

\[ \{b^\dagger_2, b_2, \Omega_{b^2}\} \]

with

\[ b^\dagger_2 = \exp(-\frac{3}{4} a^\dagger a) \cdot \{a^\dagger_2, a_2, \Omega_{b^2}\} \]

(3.18)

which by virtue of (3.12) gives:

\[ b_2 = \exp(-\frac{3}{4} a^\dagger a) \cdot \{a_2, a^\dagger_2, a^\dagger a\} \]

(3.19)

and

\[ \delta_F = \exp(-\frac{3}{4} a^\dagger a) \cdot [1 + a^\dagger_0 a^\dagger a + a^\dagger_0 a^\dagger a a^\dagger 2 a_2], \]

(3.20)

Here, the anticommutation relations can be easily checked through an immediate calculation, by taking into account that $\exp(-\frac{3}{4} a^\dagger a)$ projects onto $\Omega_{b^2}$.

The representation space $F_F$ is now four-dimensional.

The basis vectors in $F_F = F_a F_a$ can be chosen here in the form:

\[ a^\dagger_0 = a^\dagger_0 a^\dagger a \Omega_{b^2} = \Omega_{b^2} \]

\[ a^\dagger_2 = (a^\dagger_0) \cdot (a^\dagger_0 a^\dagger a) \Omega_{b^2} = (a^\dagger_0) \cdot (a^\dagger_0 a^\dagger a) \Omega_{b^2} \]

(3.21)

where $\Omega_{b^2} = \Omega_{b^2} \cdot \Omega_{b^2}$, while in the Hilbert space of a single boson, we would have:

\[ \Omega_{b^2} = \Omega_{b^2} \cdot \Omega_{b^2} \]

In (3.21) we have indicated spin properties of the quantum system (3.18)-(3.20). Namely, by [17], we have defined in $F_F$ the representation of the $SU(2)$ group, whose infinitesimal generator $\mathfrak{g}$ is given by:

\[ \mathfrak{g}_\alpha \cdot S_+ = \frac{\alpha}{2} a^\dagger a \cdot \exp(-a^\dagger a) \cdot \exp(a^\dagger a) \]

\[ \mathfrak{g}_\alpha \cdot S_- = \frac{\alpha}{2} a^\dagger a \cdot \exp(-a^\dagger a) \cdot \exp(a^\dagger a) \]

(3.22)

where $\{a^\dagger a\} = \sum_\alpha \alpha^\dagger a^\dagger a$ and

\[ \delta_F = \exp(-a^\dagger a) \cdot [1 + a^\dagger a a^\dagger a] \]

(3.23)

One can check that the first number on the left-hand side of (3.22) indicates the eigenvalue of $S^2$, while the second number indicates that of $S_z$, respectively.
Hence the representation \( \mathcal{H} \) naturally splits into the two irreducible components corresponding to spin 1 and spin 0 respectively. Let us note that the formula (3.22) was presented in such a way as to allow an apparent extension onto an arbitrary many-boson, \( \text{dim} K = n \), case.

Now let us consider the case of boson expansions of multi-fermion operators. We assume that a finite, \( \text{dim} K = 1 + N \), number of bosons is involved. Then, we can apply expansions (3.10)-(3.14) to find the corresponding expansions for bifermions \( \alpha_{k} b_{\alpha} \), \( \beta_{k} b_{\beta} \) with \( \alpha_{k} = 0,1, \ldots, N \).

To make an explicit calculation is not difficult, but rather tedious, and therefore let us mention a useful method of functional representations of the CAR and CCR, developed in [27,28,76], see also the Appendix. In [28], in the proof of Theorem 1, we have listed formulas needing a few minor modifications to solve the problem under consideration.

Namely, we have:

\[
\begin{align*}
\zeta_{k} = (\alpha_{k}, \ldots, \alpha_{k}), \\
\zeta_{k}^{*} = (\beta_{k}, \ldots, \beta_{k}), \\
b_{k} \cdots b_{k} &= \exp(-\zeta_{k}^{*} \zeta_{k}), \\
E_{k+n}(\zeta_{k+n}^{*} \zeta_{k+n}), E_{m}(\zeta_{m}^{*} \zeta_{m}) &\quad \alpha_{k} \cdots \alpha_{m} := 0,1, \ldots, N, \\
b_{k} \cdots b_{k} &= \exp(-\zeta_{k}^{*} \zeta_{k})), \\
E_{k+n}(\zeta_{k+n}^{*} \zeta_{k+n}), E_{m}(\zeta_{m}^{*} \zeta_{m}) &\quad \alpha_{k} \cdots \alpha_{m} := 0,1, \ldots, N, \\
E_{k+n}(\zeta_{k+n}^{*} \zeta_{k+n}), E_{m}(\zeta_{m}^{*} \zeta_{m}) &\quad \alpha_{k} \cdots \alpha_{m} := 0,1, \ldots, N, \\
\zeta_{k} = (\alpha_{k}, \ldots, \alpha_{k}).
\end{align*}
\]

Identities make sense only under the sign of the bilinear form, where, after multiplying by the antisymmetric tensors, summations over indices are performed.

The restriction of (3.24)-(3.26) to \( b_{k}^{*} b_{k}^{*} b_{k} b_{k} \) of obvious. By taking into account (3.12) we can easily get expansions in terms of boson operators for concrete values of \( \alpha_{k} = 1,2, \ldots, N \), where tensors \( E_{k+n}, E_{n+k} \) involve suitable sign factors.

Let us note that the expansions, though infinite, are well defined as operators in \( \mathcal{H} \). No convergence questions appear here in contrast to the Marumori or Belav-Zelevinsky approach.

The original boson expansions of fermion operators constitute, of course, infinite series. However, for practical purposes, it is useful to know that one can always restrict one's considerations to a few lowest terms only. If we remember that full operators make invariant the representation space \( \mathcal{H} \), and we take care not to obtain vectors from beyond \( \mathcal{H} \), no explicit use of the infinite expansions is needed. Let us, for example, notice that under the sign of the bilinear form, the following identity holds:

\[
b_{k} \cdots b_{k}^{*} \Omega_{B} = \sum_{m=0}^{N} a_{m}^{*} \cdots a_{m}^{*} E_{m} \Omega_{B} = \sum_{m=0}^{N} a_{m}^{*} \cdots a_{m}^{*} E_{m} \Omega_{B} \quad (3.27)
\]

It suggests that pure boson operators can sometimes be used in place of pure Fermion operators. In fact (3.27) is a special example of the quite general correspondence relation between fermion and boson algebras.

Let us consider an arbitrary operator,

\[
F(b_{k}^{*} b_{k}): = \sum_{m=0}^{N} a_{m}^{*} \Omega_{B} = \sum_{m=0}^{N} a_{m}^{*} \Omega_{B} \quad (3.28)
\]

whose generating triple \( \{b_{k}^{*} b_{k}, \Omega_{B}\} \) is associated with the...
initial boson triple \{a^+,a,\Omega_0\} used to perform the construction of the Fock representation of the CAR in this of the CCR. Here \(f_{nm}\) is a totally antisymmetric \(n+m\) point function (distribution in general).

We have satisfied the following identity:

\[
F(b^*,b) = \exp(-\{\xi^a,a\}) \cdot \frac{1}{\sum_{n+m} \Omega_{nm}} \cdot \sum_{k} \frac{1}{\Omega_k} \cdot (\xi_{nk} f_{nm} \xi_{mk} a^{*km}) ,
\]

(3.29)

where \(E_n(\xi^*)\) is the \(n\)-point Friedrichs-Klauder sign (alternating) function, being a continuous generalization of the Levi-Civita tensor:

\[
E_n(\xi^,*^*) = \sum_{k} \xi_{nk} f_{nm} \xi_{mk} a^{*km} ,
\]

(3.30)

and \(m\) denotes that the order of variables is reversed:

\[
f_{nm}(\xi^*,\xi^*) = f_{nm}(k_1,\ldots,k_n;\xi^*) .
\]

One further identity can be derived:

\[
F(b^*,b) = \exp(-\{\xi^a,a\}) \cdot \sum_{n} \frac{1}{\Omega_n} \cdot \sum_{k} \xi_{nk} f_{nm} \xi_{mk} a^{*km} \cdot \Omega_k ,
\]

(3.31)

where \(E_n(\xi^*,\xi^*) = \sum_{k} \xi_{nk} f_{nm} \xi_{mk} a^{*km} \cdot \Omega_k\).

Holds for all operators \(F(b^*,b)\) and \(F(\xi^a,a)\) related by (3.31).

As a specialization of this result, one finds at once that the canonical anticommutation relations hold on \(F^*\) for operators \(\Lambda_F a(\xi^*) \Lambda_F^*\) and \(\Lambda_F a(\xi^*) \Lambda_F\). The corresponding representation of the CAR is called a projected representation. Note here that the formal operator expressions for \(\Lambda_F a(\xi^*) \Lambda_F^*\) and \(\Lambda_F a(\xi^*) \Lambda_F\), respectively, are quite different from those for \(b(\xi^*)\) and \(b(\xi^*)^*\) and their action on vectors from the domain is essential.

In general, if the number of \(N\) bosons \{a_{\alpha}^*,a_\alpha\}_{\alpha=1}^{N-1}\) is involved, boson expansions of multifermion operators (3.24)-(3.26) can be rewritten in the form:

\[
b_\alpha^* \ldots b_{\alpha}^* \xi = \exp(-\{\xi^a,a\}) \cdot \sum_{n} \frac{1}{\Omega_n} \cdot \sum_{k} \xi_{nk} f_{nm} \xi_{mk} a^{*km} \cdot \Omega_k .
\]

(3.32)

where \(E_n(\xi^*,\xi^*) = \sum_{k} \xi_{nk} f_{nm} \xi_{mk} a^{*km} \cdot \Omega_k\).

One can, by using (3.12), be rewritten in the form:

\[
b_\alpha^* \ldots b_{\alpha}^* \xi = \exp(-\{\xi^a,a\}) \cdot \sum_{n} \frac{1}{\Omega_n} \cdot \sum_{k} \xi_{nk} f_{nm} \xi_{mk} a^{*km} \cdot \Omega_k .
\]

(3.33)

where \(E_n(\xi^*,\xi^*) = \sum_{k} \xi_{nk} f_{nm} \xi_{mk} a^{*km} \cdot \Omega_k\).

As a specialization of this result, one finds at once that the canonical anticommutation relations hold on \(F^*\) for operators \(\Lambda_F a(\xi^*) \Lambda_F^*\) and \(\Lambda_F a(\xi^*) \Lambda_F\). The corresponding representation of the CAR is called a projected representation. Note here that the formal operator expressions for \(\Lambda_F a(\xi^*) \Lambda_F^*\) and \(\Lambda_F a(\xi^*) \Lambda_F\), respectively, are quite different from those for \(b(\xi^*)\) and \(b(\xi^*)^*\) and their action on vectors from the domain is essential.

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\[
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\]

(3.32)

where \(E_n(\xi^*,\xi^*) = \sum_{k} \xi_{nk} f_{nm} \xi_{mk} a^{*km} \cdot \Omega_k\).

One can, by using (3.12), be rewritten in the form:

\[
b_\alpha^* \ldots b_{\alpha}^* \xi = \exp(-\{\xi^a,a\}) \cdot \sum_{n} \frac{1}{\Omega_n} \cdot \sum_{k} \xi_{nk} f_{nm} \xi_{mk} a^{*km} \cdot \Omega_k .
\]

(3.33)
By using (3.31), we get:

\[ b_{\alpha}^* \cdot b_{\alpha} \in F \]

In the above, the complete formulas \( F(\alpha, \beta) \) are called the infinite boson expansions equivalent on \( F \) to \( F(\beta^*, \beta) \).

while operators \( F(\alpha, \beta) \) are called the finite boson expansions corresponding to \( F(\beta^*, \beta) \).

Note that in general, \( F(\alpha, \beta) \) can appear as the infinite operator series, and the word finite indicates that only finite parts of boson expansions for \( \beta \) and \( \beta^* \) were used to construct \( F(\alpha, \beta) \).

### 3.2 Jordan-Wigner representation

In the many-body problems, people frequently employ the Jordan-Wigner construction of the representation of the CAR algebra, which is based on the use of an infinite family:

\[ [\sigma_k^+, \sigma_l^-] = 0 \quad \text{for} \quad k \neq l \]

\[ [\sigma_k^+, \sigma_l^+] = \delta_{F} \quad \left( \sigma_k^{+\dagger} \right)^2 = 0 = \left( \sigma_k^- \right)^2 \]

of spin-1/2 raising and lowering operators, see e.g. [100-105].

In practical applications, the above part-boson, part-fermion nature of operators stands for a difficulty in the theory, because no simple linear transformation between \( \sigma_k^+ \)'s and \( \sigma_k^- \)'s such as would be required to diagonalize a quadratic form (the Hamiltonian), leaves these rules invariant.

However, it is not difficult to transform rules (3.35) into a complete set of the CAR. The famous Jordan-Wigner trick is in order here, where the fermion creation and annihilation operators appear according to

\[ c_\alpha^+ = \exp \left( i \pi \sum_{j=1}^{N} \delta_{j} \sigma_j^+ \right) \sigma_\alpha^- \]

\[ c_\alpha^- = \exp \left( i \pi \sum_{j=1}^{N} \delta_{j} \sigma_j^- \right) \sigma_\alpha^+ \]

where

\[ c_\alpha^+ c_\alpha^- = \delta_{F} \]

and the inverse transformation reads:

\[ \delta_{\alpha}^- = \exp \left( i \pi \sum_{j=1}^{N} \delta_{j} \sigma_j^- \right) c_\alpha^- \]

\[ \delta_{\alpha}^+ = \exp \left( i \pi \sum_{j=1}^{N} \delta_{j} \sigma_j^+ \right) c_\alpha^+ \]

The above method was invented many times since the basic paper [100] appeared. It has been used to change spin operators into fermions in [103], to change electrons into bosons with a "hard core" [103] and to change the hard-core bosons into fermions [104]. See also [19, 120, 123-136].

The above considerations are closely connected with the application of the boson expansion methods of the previous section to the so-called Jordan-Wigner representation of the CAR algebra [100, 43, 74, 83].

Namely, by the use of operators (3.35), we can introduce the following fermion operators:

\[ -32- \]
This is a Fock representation if $\sigma_k^-\Omega = 0$ for all $k=1,2,...$

If we consider spin operators $\sigma^\pm$ as constructed in the Fock representation of the CCR algebra, we have:

$$\sigma_k^- = \exp(-a_k^*a_k) : a_k :$$

$$\sigma_k^+ = a_k^* : \exp(-a_k^*a_k) :$$

where $a_k = \{ dp\rho(p) \}_{k=1,2,...}$ is the basis system in $L^2(\mathbb{R}^3)$.

If we note that in the case of dim $K = 2$ for all $k=1,2,...$:

$$A_k^\pm = \exp(-a_k^*a_k) : (1 \pm a_k^*a_k) :$$

we at once get:

$$b_i = \prod_{k=1}^i \left[ A_k^\pm - 2a_k^* : \exp(-a_k^*a_k) : a_k : \right] : \exp(-a_i^*a_i) : a_i =$$

$$= \exp\left(-\sum_{k=1}^i a_k^*a_k\right) \prod_{j=k}^i (1-a_j^*a_j) : a_i =$$

$$b_i^* = a_i^* : \exp\left(-\sum_{k=1}^i a_k^*a_k\right) \prod_{j=k}^i (1-a_j^*a_j) :$$

which gives one more example of the CAR induced by the CCR, and is a complete "bosonization" formula for the Jordan-Wigner representation [23].

Let us now present another approach to the question of boson expansions of Fermi operators, in which, in addition to the bosons of interest, one introduces a finite number of subsidiary fermion operators.

Original considerations can be found in [40-43, 34] as well as in references listed in these papers.

Let there be given a Fock representation of the CAR algebra realized by an infinite sequence $\{ F_i \}_{i=1,2,...}$ of matrices:

$$( (F_i, F_j)_\tau )_{\tau \in \mathbb{Z}} = \sum_{n=0}^\infty ( F_i^\tau, F_j^\tau )_{\tau} = \sum_{n=0}^\infty ( F_i^n, F_j^n )_{\tau} =$$

$$= ( F_i^n, F_j^n )_{\tau} + ( F_i^n, F_j^n )_{\tau} = \delta_{ij} \delta_{\tau} = \delta_{ij} \Delta_{\tau}$$

where $\Delta_{\tau}$ is the vacuum vector for the representation. Let us write

$$K = \bigoplus_{\tau \in \mathbb{Z}} K^\tau (\mathbb{R}^3)$$

Then $K$ is spanned by a complete orthogonal system $\{ \phi^\tau_n(p) \}_{n\in\mathbb{Z},\tau}$:

$$\sum_{n=0}^\infty \delta_{\tau} \phi^\tau_n(p) \phi_n^\tau(q) = \delta_{\tau} \delta(p-q),$$

$$\sum_{n=0}^\infty \phi^\tau_n(p) \phi_n^\tau(p) dp = \delta_{\tau} \delta(p), \quad p, q \in \mathbb{R}^3.$$
Let there be given a Fock representation of the CCR algebra (1.26).

By the use of the basis system \((3.44)\) we can consider in the representation space \(F^*_B\) a continuously indexed set of generators, according to

\[
\alpha^*_m(p) = \sum \gamma \alpha^*_m(\gamma)(p),
\]

\[
\alpha_m(p) = \sum \gamma \alpha_m(\gamma)(p),
\]

where

\[
[\alpha_m(p), \alpha^*_n(q)] = \delta(p-q) \delta_{mn} A_B,
\]

\[
[\alpha_m(p), \alpha_n(q)] = 0 = [\alpha^*_m(p), \alpha^*_n(q)],
\]

\[
\alpha_m(p) \Omega_B = 0 \quad \text{for all } m,p.
\]

By using the above formulas one can easily check that the operators

\[
b_i = \sum \sum F_i^m(p,q) \alpha^*_m(p) \alpha_n(q) \quad \text{dp dq},
\]

\[
b^*_i = \sum \sum F_i^m(p,q) \alpha^*_m(p) \alpha_n(q) \quad \text{dp dq},
\]

satisfy

\[
[\gamma_i, \gamma^*_j] = \delta_{ij} A_B,
\]

\[
[\gamma_i, \gamma^*_i] = 0 = [\gamma^*_i, \gamma^*_i],
\]

the identities \((3.50)\) are valid in the one-particle sector \(B^1\) of the boson Fock space \(F_B^0\).

Moreover, by using the vector \(\Theta = \{ \Theta_j \}_{j=1,2,\ldots}\) we can construct

\[
\Theta_m(p) = \sum \Theta_j \Theta^*_j \gamma^*_m(p),
\]

and further

\[
\Omega_F = \sum \Theta_j \Theta^*_j \gamma^*_m(p) \Omega_B,
\]

the vacuum vector for the representation \((3.50)\), which obviously belongs to \(B^1\): \(\gamma_i \Omega_F = 0\) for all \(i=1,2,\ldots\).

Hence, we have defined the triple \(\{b_i, b^*_i, \Omega_F\}_{i=1,2,\ldots}\) which in \(B^1\) defines a Fock representation of the CAR algebra.

In the above we have tacitly assumed that the objects \((3.45)\) are coefficients. In this connection let us add that we have shown elsewhere \((3.4)\), that there exists a broad class of coefficients \((3.45)\) which are operator-valued and can be derived through a simple construction from an arbitrary Fock representation of the CAR algebra: the matrix representation is then explicitly constructed from the initial (not necessarily matrix) representation.

Kálmán's theory thus admits Fock representations of the CAR algebra, which are directly boson constructed, but indirectly through trilinear functions, which can appear as operator-valued, depending on certain starting Fock representation of the algebra, quite different from the derived one.
4. FERMION-BOSON RECIPROCITY IN QUANTUM FIELD THEORY

4.1 Relations between fermions and bosons

Let us start from the two arbitrary families $Q_F$ and $Q_B$ of fermion
and boson quantum fields respectively. We can consider the following
question: do there exist relations assigning to one or more elements
of $Q_B$ one or more elements of $Q_F$.

Quite popular recently supersymmetry approach [67,68] seems to offer an
example of the a priori requested relation between fermions and bosons.
The fields are formed into irreducible supermultiplets, allowing us to
deduce the conservation laws, which reflect the mentioned relation.

This is obviously a result of the supersymmetry requirement and not of
the intimate structure of $Q_B$ and $Q_F$.

Another possibility is concerned with field theories of several quantum
fields obeying the abnormal commutation relations. In this case one
introduces so-called Klein transformations [69], changing commutation
properties of fields under consideration.

The approach developed in [36,70,71,91] can be summarized in the
notion of the boson fermionization programme, where all boson and fermion fields
appearing in Nature are believed to be derivable from a single non-linear
spin-1/2 "urfield".

In the framework of the conventional quantum field theory, the fermionization programme was realized in connection with the Thirring model,
see e.g. [77,116,117]. Namely, to define a current one needs a Fock
representation of the CCR algebra over $\mathcal{L}(\mathbb{C}^n)$ constructed in terms of
the Fock representation of the CAR algebra over $\mathcal{L}(\mathbb{R}^n)$, hence
with a doubled number of the internal degrees of freedom in the theory.

Quite the converse trend is realized within the programme of fermionization of bosons, where one wishes to obtain fermion fields in the field
algebras of the given boson fields. This is the place where the boson
expansion methods are applied.

It was conjectured in [37] that it is more difficult to construct
half-integral representations of rotation groups than conversely, and it seems to be practically impossible within the limitations of the polynomial expansion. The author even tried to derive the explicit formulas for the fermion field in terms of the boson field, when the number of space-time dimensions was reduced to two.

This approach was further analyzed and generalized in [38] with the
use of the $C^*$-algebraic techniques that allowed to construct a field algebra
of the boson field, which exhibited the anticommutation rules
for certain values of the charge.

Another realization of the programme is based on the idea of boson
expansions of fermion operators. The approaches of previous sections
give here the solutions in an indirect way. Namely, the relations
between quantum fields appear on the level of the Fock representations
of the CCR and CAR algebras, hence free fields. Through the Fourier
analysis one goes from the SL(2C) covariant to the SU(2) covariant
objects. Because the Haag expansions of interacting fields and scattering
operators are power series of normal ordered operator expressions,
which include the "bare" (free) images of the "dressed" (interacting)
field only, one can always express any Fermion field in terms of the
appropriate free boson fields. See e.g. [50,31,106-120,45,46].

Recently, papers [105-107], threw some light on the possibility of the
fermion-boson metamorphosis in gauge theories with the magnetic monopole.

Namely, in the SU(2) quantum gauge field theory, with the isospin
symmetry broken spontaneously by a triplet of scalar mesons, isospin-scalar
degrees of freedom are converted into the spin degrees of freedom under the influence of the magnetic monopole field [105]. Then in
the transition formfactor, spin and isospin form an antisymmetric singlet, which implies the non-vanishing of matrix elements of the spinor field between the spinless states.

On the other hand, it was argued that as a consequence of the spin-statistics theorem, in the SU(2) theory of isospin bosons, in the field of the magnetic monopole, one can get fermions. In the analogous direction, it was shown [107] that an object composed of the spinless electrically charged particle and the spinless magnetically charged particle may bear net half-integer spin, while the two-cluster wave function is symmetric. The study of a relative motion of these clusters proves that this symmetry condition does not violate the spin-statistics theorem. An intuitive explanation of Goldhaber says: perhaps an object whose half-integer spin comes from the charge-pole contribution obeys Fermi-Dirac statistics so that a fermion can be constructed from bosons.

As a solution to this question, it is found that the anomalous relation between the cluster spin and the permutation symmetry of a two-cluster wave function produced by the static fields of charge and pole in a given cluster, is compensated by the anomalous relation between the wave-function symmetry and the quantum numbers which correspond to physical observables. All that follows from the long-range interactions of charges with poles in the different clusters. So indeed, the two anomalies combine in such a way that fermions can be constructed from bosons.

4.2 Field theories in the two space-time dimensions

The method of boson expansions is not a strange concept for people working in the domain of low-temperature description of the (anti)ferromagnetic crystals or the atomic nucleus. There is, however, not widely known, its close connection with the so-called lattice approximations of boson (quantum) field theories, see e.g. [109–120] as well as its relation to the investigations of the fermion-boson correspondence, especially for the Thirring and Sine-Gordon systems [44–46, 108–120]. The famous Coleman conjecture [44] on the possible metamorphosis of fermions into bosons, results in the statement that for mass zero space-time dimension two, the Fock space of a massless Dirac field contains massless Bose particles. In the study of equivalences between the appearing fermions and bosons, the explicit constructions of field operators were performed, see e.g. [45, 46, 109, 114, 115, 116, 118, 126, 130]. An example of the Mandelstam solution reads:

\[
\psi(x) = \{ \bar{\psi}(x) \}_{x \in \Lambda},
\]

\[
\bar{\psi}_1(x) = (c^\mu x^{\mu} \exp(\frac{i}{\hbar} \phi(x))) \exp[-\frac{i}{\hbar} \int_{x \in \Lambda} dx \bar{\psi}(x)(\frac{i}{\hbar} \phi(x) - \frac{1}{2} \beta \bar{\psi}(x))],
\]

\[
\bar{\psi}_2(x) = -i (c^\mu x^{\mu} \exp(\frac{i}{\hbar} \phi(x))) \exp[-\frac{i}{\hbar} \int_{x \in \Lambda} dx \bar{\psi}(x)(\frac{i}{\hbar} \phi(x) + \frac{1}{2} \beta \bar{\psi}(x))],
\]

(4.1)

where \( \phi(x) \) satisfies the so-called quantum Sine-Gordon equation:

\[
\Box \phi(x) = (\frac{\hbar^2}{24} - \frac{\hbar^2}{24}) \phi(x,t) = (\mu^2 \phi(x,t), \phi(x,t) + \int_{x \in \Lambda} dx \bar{\psi}(x)(\frac{i}{\hbar} \phi(x) - \frac{1}{2} \beta \bar{\psi}(x))],
\]

(4.2)

and the canonical commutation relations, while \( \bar{\psi}(x) \) is proved to be a fermion, which under suitable restrictions becomes a fermion of the massive Thirring model.

It is instructive to know that the fermion-boson correspondence as introduced results also in the equivalence of the Thirring and
Sine-Gordon model. Hamiltonians, see e.g. \([16,115]\). In \([11\text{j}]\) it is shown that any interacting spinor system in the two space-time dimensions can be equivalently described by the scalar system: both theories have a common Hamiltonian. The correspondence of this kind is further extended on the case of the vector-spinor systems and the Yukawa interacting systems, which can both be related to the Sine-Gordon model.

The transition from fermions to bosons in the two-dimensional quantum field theory follows from the fact that the Fock space of the free massless Fermion field contains in every charge sector mass zero bound states, so that the two fermion bound states are connected with the massless boson field. This result was at the root of the neutrino theory of light, \([93]\). On how to get bosons from the two-component fermions see also Ref. \([6]\).

Let us add that the structure of the fermion Fock space, with special respect to the fermion-boson reciprocity, was studied in detail, in Uhlenbrock's papers \([116,117]\). The transition from bosons to fermions is realized by the use of the exponentiated field. Here, we shall only mention an interesting feature of this approach resulting in the so-called Kronig identity between the Hamiltonians of the free massless fermion and boson fields:

\[
H_F = H_B + \frac{1}{2} \left( Q^2_+ + Q^2_- \right),
\]

with \(Q_+, Q_-\) being the suitably defined charge operators.

In the case of the Thirring model, the authors get the generalization of the Kronig identity in which the renormalized Hamiltonians \(H_F^\omega\) and \(H_B^\omega\) appear in place of \(H_F\) and \(H_B\).

In the massive case, an analogous identity can be proved if we use the concept of the dressing transformation. It allows to reformulate the physical Hilbert space problem in the Fock space, by the use of pure boson operators. Denoting the scaling dimension by \(d\), \(m\) being an index of the charge sector, and denoting the mass by \(m\), we obtain:

\[
H_F^m(d,m,n) = H_B^m(d,m,n) + \frac{1}{2} \left( Q^2_+ + Q^2_- \right).
\]

Some efforts are connected with the "choice of quasi-local Fermi fields instead of quasi-local Bose fields so that they are interpolating fields for the same one-particle states, whose corresponding multi-particle states will obey either Fermi or Bose statistics." \([14]\). The special notion of "schizon" was introduced in this connection.

Then, if starting from bosons,

\[
[a(p), a^*(q)] = p \delta(p-q),
\]

the authors define Fermions by

\[
b^*(p) = a^*(p) \exp(-i \pi \int_{p}^{\infty} \overline{n}(q) dq),
\]

\[
n(q) = (a(q) a^*(q) + a(q) a^*(q)).
\]

Note here a close analogy with the Jordan—Wigner trick.

In addition to the pure boson approach to fermions, it is useful to mention the study of the fermion-boson correspondence which is performed on the level of the Lagrangians for the simplest models, at the price, however, of having to introduce the additional, quite formal (elements of the Grassmann algebra), degrees of freedom, which are reminiscent of the supersymmetry approach. The use of Grassmann tools makes the boson-constructed fields anticommute, see e.g. \([112,113]\).
The Hamiltonian of the massive Thirring model on a one-dimensional lattice with spacing \( a \) and \( \mathcal{N} \) sites is given by\(^{19,12}\):
\[
H = \frac{i}{2a} \sum_{n=1}^{\mathcal{N}} V(G) \left( \Phi_n^+ \Phi_n^--\Phi_{n+1}^+ \Phi_{n+1}^- - (-1)^n \mathcal{M}_2 \right) + G \sum_{n=1}^{\mathcal{N}} \left( \Phi_n^+ \Phi_{n+1}^- - \Phi_{n+1}^+ \Phi_n^- \right) - \frac{G}{2a} \left( \Phi_n^+ \Phi_n^- - \frac{1}{2} \right)^2
\]
\[
\times \left( \Phi_n^+ \Phi_{n+1}^- + \Phi_{n+1}^+ \Phi_n^- \right) - \frac{G}{2a} \left( \Phi_n^+ \Phi_n^- - \frac{1}{2} \right)^2
\]
\[
g \left( \Phi_n^+ \Phi_{n+1}^- - \frac{1}{2} \right)^2 - V_0.
\]
(5.1)

The \( \Phi_n \)'s are Fermion operators \( \Phi_n = \Phi_n^+ \Phi_n^- \), \( G \) is a renormalized coupling constant, \( V(G) \) is the finite renormalization constant needed to make the speed of light equal to unity. Up to first order in \( G \), we have \( V(G) = 1 + \frac{G}{2} \) and \( \mathcal{M}_0 \) denotes the bare mass and \( V_0 \) is included to make the ground state energy vanish; \( \mathcal{M}_0 \) depends on the lattice constant \( a \).

With the help of the Jordan-Wigner trick we can relate the above model to the Heisenberg spin-chain problem, the XYZ-model with periodic boundary conditions are implied.

The appropriate version of the Jordan-Wigner trick reads:
\[
\Phi_n^+ = e^{i\pi/4(N+1)} \cdot \Phi_n^+ (i \mathcal{E}^z_n),
\]
\[
\mathcal{G}_n^+ = \frac{1}{2} (\mathcal{E}_n^+ + i \mathcal{E}_n^y).
\]
(5.3)

So that, under this transformation,
\[
H = H_{\text{XYZ}} + \frac{1}{2} (1 + (-1)^{n+F}) \cdot \left( \sum_{n=1}^{\mathcal{N}} \mathcal{E}_n^x \mathcal{E}_{n+1}^x + \mathcal{E}_n^y \mathcal{E}_{n+1}^y + \mathcal{E}_n^z \mathcal{E}_{n+1}^z \right) + \text{const},
\]
(5.4)

with \( F = \frac{1}{2} \cdot \sum_{n=1}^{\mathcal{N}} \mathcal{E}^z_n \) and \( \mathcal{G}^z = e^{i\pi/4(N+1)} \cdot \Phi_n^+ \Phi_n^- \).

Here \((\pi/4)^F\) commutes with \( H_{\text{XYZ}} \) and \( H \). It proves that in the sector \((\pi/4)^F = (\pi/F)^{\mathcal{N}+1}\) we have \( H = H_{\text{XYZ}} \), while if \((\pi/4)^F = (\pi/F^+)\), \( H \) equals to another Hamiltonian \( \mathcal{H}_{\text{XYZ}} \) given by the Heisenberg ferromagnet formula with the anticyclic boundary conditions:
\[
\mathcal{E}_n^x = -\mathcal{E}_{n+1}^x, \mathcal{E}_n^y = -\mathcal{E}_{n+1}^y, \mathcal{E}_n^z = -\mathcal{E}_{n+1}^z.
\]

Unfortunately for \( F = 0 \), which corresponds to the free model, the three Hamiltonians \( H_{\text{XYZ}}, \mathcal{H}_{\text{XYZ}}, \) and \( H \) have different spectra, which slightly spoils the obtained equivalence. Here the strong dependence of the theory while formulated in the Heisenberg language, on the choice of the boundary conditions, should be emphasized.

If to introduce the projection operators: \( P_\pm = \frac{1}{2} (1 + (-1)^{n+F}) \cdot \mathcal{P}_{\pm} \cdot \mathcal{P}_{\pm} \) we have \( \mathcal{P}_{\pm} \cdot \mathcal{H}_{\text{XYZ}} \cdot \mathcal{P}_{\pm} \cdot \mathcal{P}_{\pm} = \mathcal{H}_{\text{XYZ}} \), \( \mathcal{P}_{\pm} \cdot \mathcal{H}_{\text{XYZ}} \cdot \mathcal{P}_{\pm} \cdot \mathcal{P}_{\pm} \cdot \mathcal{P}_{\pm} = \mathcal{H}_{\text{XYZ}} \), and hence:
\[
\mathcal{H} = \mathcal{P}_+ \mathcal{H}_{\text{XYZ}} + \mathcal{P}_- \mathcal{H}_{\text{XYZ}} + \text{const},
\]
\[
\mathcal{P}_+ \cdot \mathcal{H}_{\text{XYZ}} \cdot \mathcal{P}_+ \cdot \mathcal{P}_+ \cdot \mathcal{P}_+ = \mathcal{H}_{\text{XYZ}},
\]
\[
\mathcal{P}_- \cdot \mathcal{H}_{\text{XYZ}} \cdot \mathcal{P}_- \cdot \mathcal{P}_- \cdot \mathcal{P}_- = \mathcal{H}_{\text{XYZ}}.
\]

a) For large \( N \) the spectra of \( \mathcal{P}_+ \mathcal{H}_{\text{XYZ}} \) and \( \mathcal{P}_- \mathcal{H}_{\text{XYZ}} \) are the same, which partly solves the above-mentioned spectrum problem.
Let us consider again the spatial lattice with spacing $a$ and the integer lattice label $n$. If, with each lattice site we have associated a complex scalar field $\phi_n$ satisfying the canonical commutation relations:

$$[\phi_n^{\dagger}, \phi_m] = \delta_{nm}, \quad [\phi_n, \phi_m^{\dagger}] = 0,$$

then in the corresponding Fock space we can construct the fermion unit operator where in the place of creation and annihilation operators $a^+_n, a_n$ we put respectively:

$$a^+_n, a_n^-, \text{ for all } n = 1, 2, \ldots$$

By the transformation $a^+_n \phi_n = \phi_n$, we have associated with each lattice site a single component fermion field $\phi_n$:

$$[\phi_n, \phi_m^{\dagger}] = \delta_{nm} a^+_n, \quad [\phi_n, \phi_m] = 0.$$

Let us consider the Hamiltonian, compare in this connection (5.1). We have,

$$i[H, \phi_n] = \phi_n = \frac{\phi_n^{\dagger} - \phi_n}{2a} = \frac{\Delta \phi_n}{\Delta x},$$

so that the time dependence of $\phi_n$ at even (odd) sites is determined by the spatial difference of $\phi_n^{\dagger}$ at odd (even) sites. Following the prescription of [127], let us define a two-component field $\psi_n$:

$$\psi_n = \begin{pmatrix} \phi_e^\dagger & \phi_o \end{pmatrix}, \quad \psi_o = \phi_o, \quad \psi_e = \frac{\phi_e^{\dagger} - \phi_e}{2a}.$$

The components of $\psi_n$ satisfy:

$$\dot{\psi}_e = \frac{\Delta \phi_e}{\Delta x}, \quad \dot{\psi}_o = \frac{\Delta \phi_o}{\Delta x},$$

which, at the continuum limit, becomes the massless Dirac equation:

$$\frac{\partial}{\partial t} \psi = \left( \frac{\partial}{\partial x} \right)_y \psi$$

in the standard representation $\sigma_0 = (1_4, 0)$. By using the Jordan-Wigner trick,

$$\psi_n = \prod_{k < n} \left( i e_b^{\frac{1}{2}} \sigma_n^{\dagger} - e_b^{\frac{1}{2}} \sigma_n \right),$$

$$\psi_n^{\dagger} = \prod_{k < n} \left( i e_b^{\frac{1}{2}} \sigma_n^{\dagger} - e_b^{\frac{1}{2}} \sigma_n \right),$$

the one-dimensional fermion problem can be rewritten as the one-dimensional spin problem: $\sigma_n^{\dagger}$ is the spin matrix at the $n$-th site, $l = 1, 2, 3, \ldots$ so that:

$$H_F = \frac{1}{2a} \sum_n \left( \sigma_n^{\dagger} \cdot \sigma_n + \sigma_{n+1}^{\dagger} \cdot \sigma_n + \sigma_{n-1}^{\dagger} \cdot \sigma_n \right)$$

which describes the XY antiferromagnetic chain.

As we know from previous considerations, each of Pauli operators can be bosonized so that one can once more pose the question of the boson translation of the same (see (5.8)) physical problem.
of the continuous one. However, further approximations can be made within the lattice formulation.

Let us note that in (5.19) the different single-site terms are coupled by the gradient terms only, which in fact carry interactions in the model. If we neglect the gradients, we obtain the so-called single-site approximation, where at each site we have the identical Schrödinger problem of a particle in an anharmonic potential. In this approximation, the eigenstates of the Hamiltonian are formed by the product of single-site eigenstates. The lowest energy eigenstate is $|\psi_0\rangle = \prod_j |\phi_0\rangle_j$, where $|\phi_0\rangle_j$ is the $j$-th site ground state.

The next energy level is achieved if, at most, one of the oscillators is in its first excited state; huge degeneracy appears here.

Introducing the annihilation and creation operators at each site $j$: $a_j^\dagger$, $a_j$, we can get the most general, in the single-site approximation, state function of the system:

$$|\psi\rangle = \prod_j |\psi_j\rangle = \prod_j \langle \psi_j | \psi_j \rangle = \delta_{jk},$$

where $|\psi_j\rangle$ is the $j$-th site eigenstate.

In the so-obtained single-site basis, one can approximately calculate the energy of the original, interacting, system:

$$H = \Lambda \sum_j \left\{ \frac{1}{2} \frac{\partial^2}{\partial x_j^2} + \lambda (x_j^2 - \frac{x_j^4}{4}) \right\},$$

so that its expectation value in the trial state $|\psi\rangle$:

$$C \langle \psi | H | \psi \rangle = H_{\psi} = \Lambda \sum_j \left\{ \langle \psi_j | H_{\psi} | \psi_j \rangle + \frac{1}{2} \langle \psi_j | \frac{\partial^2}{\partial x_j^2} | \psi_j \rangle \right\}.$$

In case of the translation invariant ground state, requiring $\Sigma_j D(-j) = 0$, so that $\Sigma_j D(-j) = -\Sigma_j D(0) = -\Lambda D(0)$, implies:

$$E_0 = H_0 = \Lambda \sum_j \left\{ \frac{1}{2} \frac{\partial^2}{\partial x_j^2} | \psi_j \rangle \langle \psi_j | + \lambda (x_j^2 - \frac{x_j^4}{4}) \right\}.$$
where, see, e.g. [125,129] \( \mathcal{E} = (E_a - E_b) \) \( \Delta \equiv |(0| \times |1|)^2 \),

\( \mathbf{c}_N \equiv \mathbf{c}_N \) and \( \mathbf{c}_q \) are the ordinary Pauli matrices.

Since only the two states are considered for each oscillator, they can be represented by the presence or absence of a fermion, which motivates the translation of \( H \) to the pure fermion language: the Jordan-Wigner trick:

\[
\begin{align*}
\mathbf{c}_+^i &= \prod_{j<i} (-\mathbf{1})^{n_j} \mathbf{b}_j^i, \\
\mathbf{c}_-^i &= \prod_{j>i} (-\mathbf{1})^{n_j} \mathbf{b}_j^i, \\
\mathbf{c}_0^i &= 2n_i - 1,
\end{align*}
\]

allows us to derive:

\[
H = \sum_{i} \epsilon_i \mathbf{c}_+^i \mathbf{c}_-^i - \Delta \sum_{i<j} \left( (\mathbf{b}_-^i \mathbf{b}_-^j) (\mathbf{b}_+^j \mathbf{b}_+^i) + (\mathbf{b}_+^i \mathbf{b}_+^j) (\mathbf{b}_-^j \mathbf{b}_-^i) \right) + \Delta (\mathbf{c}_-^i \mathbf{c}_-^j) (\mathbf{c}_+^j \mathbf{c}_+^i) (\exp(\text{i} \mathbf{m}) + 1),
\]

which finally enables us to associate with the starting boson system, the corresponding lattice fermion, here once one can obviously pose the question whether (5.30) can be further connected, in the sense of the lattice approximation, with some continuous (quantum field theory) Fermion system.

6. **Heisenberg Ferromagnet in the Low-Temperature Limit**

6.1 Spin-1/2 Approximation in Quantum Mechanics

Let us consider an elementary quantum system in one dimension. Such a system is completely determined by an irreducible pair \( \{ F, Q \} \equiv \mathbf{A}_B \), \( \mathbf{F} \in \mathbb{R}^2 \) of the momentum and position operators, which are self-adjoint in the suitable Hilbert space \( \mathcal{T}_F \).

We assume the quantum motion of the system to be governed by the Hamiltonian \( H \), whose complete eigenfunction system \( \{ \phi_n \}_{n=0,1,\ldots} \)

\[
\mathcal{H}_F = \mathcal{F}_F \quad \text{spans} \quad \mathcal{T}_F.
\]

We have then given the operators \( \mathcal{A} = (Q-iF), \quad \mathcal{A} = (Q+iF) \) so that \( \mathbf{q}_F^0 = 0 \), and \( \mathbf{q}_F = (Q-iF) / \Delta F \).

Hence, the triple \( \{ \mathbf{A}, \mathcal{O}_F, \mathbf{F}_F \} \) generates a Fock representation of the CAR in \( \mathcal{T}_B \). In accordance with considerations of Secs. 1, 3.2, in a two-dimensional subspace \( \mathcal{T}_F \) of \( \mathcal{T}_B \), which is spanned by vectors \( \phi_0, \phi_1 \), we have given a Fock representation of the CAR algebra \( \{ \mathbf{A}, \mathbf{c}_F, \mathbf{f}_0 \} \) with:

\[
\begin{align*}
b &= \mathbf{A} \mathbf{b} \mathbf{A}, \\
b^* &= \mathbf{A} \mathbf{c}_F \mathbf{A}, \\
a &= \mathbf{A} \mathbf{a} \mathbf{A}, \\
a^* &= \mathbf{A} \mathbf{c}_F \mathbf{A}, \\
\mathbf{c}_F &= \mathbf{A} \mathbf{c}_F \mathbf{A}, \\
\mathbf{c}_F^* &= \mathbf{A} \mathbf{c}_F \mathbf{A}, \\
\mathbf{f}_0 &= \mathbf{A} \mathbf{f}_0 \mathbf{A}, \\
f_0^* &= \mathbf{A} \mathbf{f}_0 \mathbf{A},
\end{align*}
\]

and furthermore, an infinitesimal generator \( \mathbf{g} \) of the representation of the SU(2) group irreducible in \( \mathcal{T}_F : \quad \mathbf{g}^2 = 0 \).

Finally, (5.31) can be further connected, in the sense of the lattice approximation, with some continuous (quantum field theory) Fermion system. Therefore, if the description of our elementary quantum system can be restricted to \( \mathcal{T}_F \) only, we can in principle characterize it fully by a complete family \( \{ \mathbf{f}_F, \mathbf{s}_F, \mathbf{s}_0 \} \) of the observables commuting in \( \mathcal{T}_F : \quad \mathbf{h}_F = \mathbf{c}_F \mathbf{c}_F \).

The only thing is to disclose the physical conditions, under which such a restriction is possible. It is obviously non-realizable for the isolated system. Let us therefore assume it to be in contact with a suitable, low-temperature environment (reservoir). In such a case, any pumping of the system to energies exceeding \( E_1 \) can be made negligibly probable if compared with this to produce either \( E_0 \) or \( E_2 \). Then, the description of any elementary quantum system, with good accuracy can be reduced to \( \mathcal{T}_F \) only, where a complete family of observables \( \{ \mathbf{h}_F, \mathbf{s}_F, \mathbf{s}_0 \} \) is given.

Another mechanism of this kind can be the strong coupling potential forbidding the system to occupy higher energy levels. This phenomenon seems to be of special importance in the interacting many-particle systems.

The above considerations can be summarized in the following conjecture: the quantum motion of a one-dimensional spinless system, which is governed by the Hamiltonian \( H \), provided the system is in contact with the low-temperature environment, in the weak excitation limit perfectly simulates the internal spin-1/2 quantum motion of a spinning object, in its own reference rest frame.

Furthermore, we can treat each spinless elementary quantum system as a superposition of the bosonic and fermionic "phases." If higher excitations are allowed, the bosonic one prevails; however, in the weak excitation limit, the fermionic one becomes prevailing.

One can even try to establish a certain critical temperature of the reservoir, beginning from which our boson can be considered with a good accuracy as the spin-1/2 fermion. Things would become still more exciting if there existed a large energy gap between \( E_2 \) and higher energy levels of the Hamiltonian. In that case it would even be possible to get the highly stable fermionic "phase."
of our elementary quantum system, in a large range of energies perhaps.

6.2 Bosonization of fermions on the isotropic lattice

We are interested in the special class of quantum spin systems called the isotropic Heisenberg models, whose specialized case is the famous Ising model. Let the isotropic spin lattice consist of the number $N$ of equivalent sites, each one occupied by identical atoms: each one with spin $s$ and magnetic moment $\mu$. With the $1$-th site of the lattice we associate the $1$-th copy $T^a_{\mu}$ of the finite dimensional Hilbert space $F$, with a unitary irreducible representation of the SU(2) group, with the satisfying:

$$\{S^a_{\mu}, S^b_{\nu}\} = i\epsilon_{abc} S^c_{\nu} S^a_{\mu} S^b_{\nu}, \quad \text{for} \quad a,b,c = 1,2,3 \quad (\text{or} \quad x,y,z), \quad S^a_{\nu} = s(a+1) S^a_{\nu} \quad \text{and} \quad S^a_{\nu} \text{ is a unit operator in } F_{\nu}.$$

We denote $\Xi_{kl} = I(k-1), \quad \Xi_{kk} = 0, \quad \Xi_{kk} = 1,2,\ldots, N$, the exchange integral of the lattice, and $\Xi = \Xi(0,0,\Xi)$ is the magnetic field oriented along the $z$-axis of the reference rest frame. Then the general Hamiltonian of the lattice [9], reads:

$$H = \mathcal{G}_0 - \mu \sum_{k} \mathcal{R} \cdot \mathcal{S}_k - (4i) \sum_{k,a} \mathcal{I}_{kl} \mathcal{S}_k^a \mathcal{S}_l^a,$$

and can further be written in the form:

$$H = \mathcal{G}_0 + H_2 + H_4,$$

$$\mathcal{G}_0 = -\mu \mathcal{R} \mathcal{S}_k - (4i) \mathcal{I}_{kl} \mathcal{S}_k \mathcal{S}_l,$$

$$H_2 = \left(\mu \mathcal{R} + 4 \mathcal{I}_{kl} \mathcal{S}_k \mathcal{S}_l - (4i) \mathcal{I}_{kl} \mathcal{S}_k \mathcal{S}_l \right),$$

$$H_4 = -\mu \sum_{k} \mathcal{I}_{kl} \left(\mathcal{S}_k - \mathcal{S}_l\right) \left(\mathcal{S}_k - \mathcal{S}_l\right),$$

where $\mathcal{I}(0) = \sum \mathcal{I}(0)$. Moreover, to be in agreement with the notation commonly accepted in solid state physics, $s^2$ must be identified with the notation $F S^i$, of previous sections. The description of the Heisenberg ferromagnet in the low temperature limit (Curie point) involves bosons in the basic formalism [48,49]. Namely, in that case, a collection of the ideal spin waves comprising a non-momam gas is believed to simulate perfectly the behavior of the crystal transition probabilities for the ideal spin wave processes with a good accuracy approximate these for a real system. A conventional computational tool at this point is the use of formal substitutions, as e.g. those of Holstein and Primakoff or of Dyson and Maleev. They allow us to consider in place of (6.2) a pure boson Hamiltonian of the ideal spin waves defined in the appropriate space of spin states. The use of the Holstein-Primakoff prescription for spin values $s \gg 1$ allows us to consider $H$ in the form:

$$H = H_2 + H_4,$$

while by using the Dyson-Maleev prescription we get:

$$S^a_k \to \frac{1}{2} S^a_k,$$

$$S^a_k \to \frac{1}{2} S^a_k \left(1 - a^a_k a^a_k \right) a^a_k,$$

$$S^a_k \to -\frac{1}{2} \left(1 - a^a_k a^a_k \right) a^a_k.$$

In both cases $H_2$ is interpreted to describe the non-interacting spin waves $\mathcal{H}^d$ their dynamical interaction, $\mathcal{H}^d$ being responsible for the kinematical corrections. $\mathcal{H}$ and $\mathcal{H}^d$ appear here as small perturbations of $H_2 + E_0$. In the above the condition $s \gg 1$ automatically excludes lowest spin lattices from considerations, therefore it seems reasonable to apply here the rigorous approach of Sec. 3.1. Let the triple $\{C^a_k, \mathcal{Q}, C^a_k \mathcal{Q} \mathcal{Q}_k\}$ generate a Fock representation of the CCR algebra over the complex separable Hilbert space $K = \bigotimes_{l=1} K_l$ with $\dim K_l = n$ for all $1, n = 1,2, \ldots$.
The basis system in $K$ we denote by $|\phi_k\rangle$, so that the indexing $a(\phi_k) = a_{\alpha k}$ of modes induces
\[ [a_{\alpha k}, a^*_{\alpha' k'}] = \delta_{\alpha k} \delta_{\alpha' k'} \Delta_{\alpha k} , \]
\[ [a_{\alpha k}, a_{\alpha' k'}] = 0, \quad a_{\alpha k} \Omega_{\alpha k} = 0 \quad \text{for all } \alpha, k. \] (6.7)

The underlying Fock space we denote $\mathcal{F}_B$.

Let us now assume we have defined the subsidiary boson lattice consisting of a finite (large) number of identical cells: $l = 1, 2, \ldots, N$, each one occupied by the $n$-mode cluster: $\alpha = 1, 2, \ldots, n$, enumerating the components of the cluster. We define the following operators:
\[ \hat{\delta}_c^+ = \delta_c^* + i \delta_c^\dagger, \]
\[ \hat{\delta}_c^- = \delta_c^* - i \delta_c^\dagger, \]
\[ \hat{\delta}_c^z = (\alpha/2) (a_c^* a_c^\dagger), \] (6.8)

with $(a_c^* a_c^\dagger) = \frac{1}{2} a_c^* a_c^\dagger$, and introduce the Hamiltonian of our lattice boson; notations are taken from the above, and $I_{\alpha k} = 0$ for all $k$,
\[ H_B = G_0 - \frac{\mu}{2} \sum_{k, a_c^*} \delta_c^z k^2 - (4\mu) \sum_{k, \alpha} I_{\alpha k} \delta_c^z k^2. \] (6.9)

This operator is well defined in the suitably chosen domains in $\mathcal{F}_B$.

If we admit a contact of the subsidiary boson lattice with an appropriate thermostat (reservoir), one can imagine the retaliation mechanism for the excitation level of the lattice, through the raising or lowering of the temperature.

Temperature changes influence the structure of the set of transition probabilities between lattice states: certain transitions become more probable than others.

Let us choose a discrete set $\{ T_i \}_{i = 0, 1, \ldots}$ of points along the temperature scale, each one with a corresponding neighbourhood $\Delta T_i$.

The separation intervals between the neighbouring points are assumed to be sufficiently large if compared with the corresponding $\Delta T$.

With each discrete $T_i$, let us associate a projector $\hat{P}_i$ with the property
\[ \hat{P}_i^2 = \hat{P}_i, \]
\[ \text{prob}[\hat{P}_i] = 1, \quad \text{prob}[\hat{P}_i] = 0, \]
\[ \Lambda \gg \text{prob}[\hat{P}_i] \gg \text{prob}[\mathcal{F}_B \setminus \mathcal{F}_i], \] within the interval $\Delta T$.

where the notation $\text{prob}[\mathcal{F}_i]$ denotes the probability with which the transitions between lattice states from $\mathcal{F}_i$ are realized inside $\mathcal{F}_B$ in the chosen temperature range $T_i \pm \Delta T/2$. To remove the arbitrariness connected with the relation $\gg$, one can try to associate temperature values with concrete probability values, as e.g. $\text{prob}[\mathcal{F}_i] \leq 0.001$, say.

In that case, the boson lattice, can be described with a good accuracy in terms of states from $\mathcal{F}_i$ and no necessity to consider the whole of $\mathcal{F}_B$ appears.

By virtue of considerations of Sec. 3.1 for a certain critical temperature $T_0$ ($T_0 < T_C$) and a corresponding temperature range $\Delta T$, there exists a projector $\hat{P}_0$ such that within the interval $T_0 \pm \Delta T/2$ the following operator identity:
\[ \hat{P}_0 H_B \hat{P}_0 = H = G_0 - \frac{\mu}{2} \sum_{k, \alpha} \delta_c^z k^2 - (4\mu) \sum_{k, \alpha} I_{\alpha k} \delta_c^z k^2, \] (6.11)

holds on the Hilbert space of spin states $\mathcal{F}_0 = \mathcal{P}_0 \mathcal{F}_B$ which is a finite dimensional subspace of $\mathcal{F}_B$.

Furthermore, each of operators $\hat{P}_0 = \mathcal{P}_0 \mathcal{P}_B$ is an infinitesimal generator of the representation of the $SU(2)$ group reducible on $\mathcal{F}_0$, whose irreducible components induce a corresponding splitting of $\mathcal{F}_0$ into a direct sum of suitable spin spaces. They are parametrized by the spin values associated with each of the $N$ sites of the Heisenberg lattice.

A few comments are now in order.

The identity (6.11) establishes a connection of the subsidiary boson lattice with the isotropic lattice in the weak excitation limit. It clearly exhibits the limitations (projectors $\hat{P}_i$) under which the use of boson expansions in the theory is justified. Formally one can consider a straightforward equivalence relation between $H_B$ and $H$.

However, in that case the essential domain questions arise. $H_B$ appears here as the finite boson expansions corresponding to the infinite expansion for $H$.

The cluster structure of the subsidiary boson lattice, introduced by us as the limitation of the theory, plays an essential role. Namely, in the case of $\text{dim} k = 1$, for all $l$, we get the spin-1/2 lattice. If $\text{dim} k = 2$ for all $l$, then $\mathcal{F}_0$ can be reduced and with each lattice site one can associate either spin-1 or spin-0 quantum excitation.
Our main task now is to construct explicitly the operator \( P_0 \). Let us note that \( \Omega_0 \) is a common ground state for all clusters of the subsidiary boson lattice. We denote by \( \Omega^\text{F} \) a Fock space associated with the \( k \)-th cluster, and hence determined by the triple:

\[
\{ a^*_k, a_k, \Omega^\text{F} \}_{1 \leq k \leq n}.
\]

From now on we assume the spin-1/2 approximation of our quantum system is allowed, so that the two lowest energy states corresponding to each single normal mode of the system (including the single components of the clusters) are essential. Then with each single degree of freedom we have associated the fermion:

\[
b^\alpha = a^\alpha : \exp(-a^\alpha a^\alpha) ;
\]

and furthermore the induced spin-1/2 operator \( \mathbf{S} \).

If, further, we consider the \( n \)-mode cluster, and repeat above consideration for each single component, then the operator

\[
\mathbf{S} = \sum_{\alpha} \mathbf{S}_\alpha
\]

is an infinitesimal generator of the SU(2) group reducible on \( \mathbb{F}_F \). It follows from the fact that for a free \( \mathbf{S} \), the operators \( b^\alpha \), \( \mathbf{S}_\alpha \) do commute,

\[
[ b^\alpha, b^\beta ] = 0 \Rightarrow [ b^\alpha, \mathbf{S}_\alpha ] = \sum \mathbf{S}_\beta,
\]

while

\[
[ b^\alpha, b^\beta ] = \mathbf{A}_F \quad \Rightarrow \quad b^\alpha + b^\beta = 0 \Rightarrow b^\alpha + b^\beta = 0
\]

so that the operators \( \mathbf{S}_\alpha \) obey:

\[
[ \mathbf{S}_\alpha, \mathbf{S}_\beta ] = \mathbf{S}_\gamma \mathbf{S}_{\alpha \beta},
\]

\[
a, b, c = 1, 2, 3, \quad \varepsilon_{abc} = 4, 2, 3.
\]

Bearing in mind the well-known addition theorems for the angular momenta we can add those generators, getting the new infinitesimal generator \( \mathbf{S}_h = \sum \mathbf{S}_\alpha \) of the SU(2) group being assigned to the \( k \)-th n-mode cluster under consideration. Obviously, this generator is defined on \( \mathbb{F}_F^k \), and decomposes this Hilbert space into a direct sum of spin subspaces corresponding to the irreducible components of the representation.

In the spin-1/2 approximation of the boson system we can apply the boson expansion formulas to the above operators according to:

\[
S_+^k = \sum \mathbf{S}^\alpha_{\alpha + 1} \exp(-a^\alpha a^\alpha) ;
\]

\[
S_-^k = \sum \mathbf{S}^\alpha_{\alpha - 1} \exp(-a^\alpha a^\alpha) ;
\]

\[
S_0^k = (a^\alpha a^\alpha) \mathbf{S}_\alpha^\alpha_{\alpha - 1}(a^\alpha a^\alpha) - \mathbf{S}^\alpha_{\alpha + 1}(a^\alpha a^\alpha) - \mathbf{S}^\alpha_{\alpha - 1}(a^\alpha a^\alpha) - \mathbf{S}^\alpha_{\alpha - 1}(a^\alpha a^\alpha)
\]

\[
[S_\alpha^k, S_\beta^k] = i \varepsilon_{\alpha \beta \gamma} S_\gamma^k
\]

The Fermion unit operator \( \mathbf{F}_F \) selects in \( \mathbb{F}_F^k \) a subspace of spin states corresponding to the \( k \)-th cluster with \( \dim_k = n \).

In the above, the operator \( \exp(-a^\alpha a^\alpha) \Rightarrow (a^\alpha a^\alpha) = \sum \mathbf{S}^\alpha_{\alpha - 1}(a^\alpha a^\alpha) \) is responsible for projecting onto the ground state of the cluster. A global projection onto the ground state for the whole of the lattice is then obtained by the multiplication of all particular cluster projections (by virtue of the direct product structure of \( \mathbb{F}_F \)), and reads:

\[
: \exp(-a^\alpha a^\alpha) : \Rightarrow (a^\alpha a^\alpha) = \prod (a^\alpha a^\alpha)
\]

Let us now define the following operators:

\[
\mathbf{P}_\alpha = \mathbf{F}_F \mathbf{F}_F - \exp(-a^\alpha a^\alpha) ;
\]

\[
\mathbf{P}_\alpha^k = \mathbf{F}_F \mathbf{F}_F - \exp(-a^\alpha a^\alpha) ;
\]

\[
\mathbf{P}_\alpha^k = \mathbf{F}_F \mathbf{F}_F - \exp(-a^\alpha a^\alpha)
\]

Each \( k \)-th operator projects onto the non-zero-mode subspace of \( \mathbb{F}_F^k \). Then, the operator

\[
\mathbf{P}_0 = \mathbf{F}_F \mathbf{F}_F - \exp(-a^\alpha a^\alpha) ;
\]

\[
\mathbf{P}_0 = \mathbf{F}_F \mathbf{F}_F - \exp(-a^\alpha a^\alpha)
\]

is a projector and selects in \( \mathbb{F}_F \) a subspace \( \mathbb{F}_0 = \mathbf{P}_0 \mathbb{F}_F \) which is a closed factor-theoretic union of all particular spin spaces, being the Hilbert space of spin states for the subsidiary boson lattice.

Now it is quite a trivial exercise to check that on \( \mathbb{F}_F^k \) the following operator identities hold:
where \( P_{\text{orr}} \) or \( v \) is from \( \text{Lenvin}^e \) and \( Q \) or \( \text{bosons} \) only.

Moreover, for \( k \neq 1 \) there obviously holds on \( T \):

\[
\hat{P}_0 \hat{X}_k \hat{P}_0^{-1} = \hat{P}_0^{-1} \hat{X}_k \hat{P}_0,
\]

which follows from the commutativity of the components of the operators \( \hat{X}_k \) and \( \hat{P}_0^{-1} \) respectively.

Because \( \hat{P}_0 \) is the unit operator in \( T \), we get the desired property, \( \hat{P}_0 \rightarrow \hat{P}_0 \hat{H} \hat{P}_0 = H \), to hold in \( T \).

Let us summarize the obtained results: our starting point was the subsidiary boson lattice, whose description is in the spin-\( 1/2 \) approximation reduced to a particular subspace \( T_0 \) of \( T \), so that the restricted lattice Hamiltonian \( \hat{H}_B = \hat{P}_0 \hat{H}_B \hat{P}_0 \) is in fact the Hamiltonian of the isotropic Heisenberg lattice.

The physical conditions under which a restriction to \( T_0 \) is possible, are clarified by the concept of the spin-\( 1/2 \) approximation. We need a temperature \( T \) of the thermostat so low that each single mode of the subsidiary boson lattice is either singly excited or not excited at all, so that

\[
\frac{1}{\exp(T)} \gg \frac{1}{\exp(T_0)}.
\]

Probabilities with which transitions to the higher energy levels of the system participate in (6.22) are then negligible.

In the pure boson language we deal here with a kind of condensation of the magnon gas (its single degrees of freedom, in fact) around the lattice sites, so that the magnon condensate perfectly imitates the structure of spin interactions inside the crystal.

Moreover, by the use of spin-\( 1/2 \) approximation of the boson system, we obtained what in Sec. 5 corresponded to the finite spin approximation.

The last step in these considerations should now be a comparison of results (Hamiltonians \( I \) mean) obtained by the use of either of the methods presented. Calculations here are straightforward, and to have a comparison with the Holstein-Primakoff and Dyson-Maleev expansions, we put \( \hat{X}_k^2 \rightarrow \hat{X}_k \), which implies \( \hat{S}_k^2 \rightarrow \hat{S}_k \); \( \hat{S}_k^z = \hat{a}_k^\dagger \hat{a}_k \), and results in:

\[
\hat{H}_B^0 = \left( \mu \Sigma \chi + \left( \chi \right)_0 \right) \hat{I}_{\hat{S}k} \left( N - \sum \Sigma n_k \right) - \left( \chi \right)_0 \hat{I}_{\hat{S}k} \hat{a}_k^\dagger \hat{a}_k,
\]

\[
\hat{H}_B^B = -\left( \chi \right)_0 \hat{I}_{\hat{S}k} \sum \Sigma \hat{I}_{\hat{S}k} \left( n_k + n_k \right) - \left( \chi \right)_0 \hat{I}_{\hat{S}k} n_k n_k.
\]

After the reordering of terms with respect to powers in which the operators \( \hat{a}_k^\dagger \) and \( \hat{a}_k \) appear, we get:

\[
\hat{E}_0^B = \left( \chi \right)_0 \left( N \mu \Sigma \chi + \left( \chi \right)_0 N \right) - \left( \chi \right)_0 \hat{I}_{\hat{S}k},
\]

\[
\hat{H}_B^B = -\left( \chi \right)_0 \hat{I}_{\hat{S}k} \sum \Sigma \hat{I}_{\hat{S}k} \left( n_k + n_k \right) - \left( \chi \right)_0 \hat{I}_{\hat{S}k} n_k n_k.
\]

If compared with the \( H_P \) and \( D-M \) expansions, we have modified \( \hat{E}_0^B \) and \( H_B^B \), and no kinematical term at all.

Obviously, the Hamiltonian \( \hat{H}_B = \hat{E}_0^B + \hat{H}_B^B \) of the magnon condensate perfectly imitates the structure of spin interactions inside the crystal.

6.3 The anisotropic crystal

Let us consider the generalized Heisenberg model with the diagonal interaction exchange tensor and the anisotropy with respect to all three crystallographic co-ordinate axes:

\[
\hat{I}_{\hat{S}k}^x = \delta_{\hat{x} \hat{k}}, \quad \hat{I}_{\hat{S}k}^y = \delta_{\hat{y} \hat{k}}, \quad \hat{I}_{\hat{S}k}^z = \delta_{\hat{z} \hat{k}},
\]

where \( \delta_{\hat{x} \hat{k}}, \delta_{\hat{y} \hat{k}}, \delta_{\hat{z} \hat{k}} \) are the anisotropy parameters: \( |\delta_{\hat{x} \hat{k}}|, |\delta_{\hat{y} \hat{k}}|, |\delta_{\hat{z} \hat{k}}| \leq 1 \).

For the case of a ferromagnetic crystal, we additionally need \( \delta_{\hat{k} \hat{k}} > 0 \), note that \( \delta_{\hat{k} \hat{k}} < 0 \) implies the antiferromagnetism.

The external magnetic field is again oriented along the z-axis: \( \hat{H}_z = (0, 0, \mu \Sigma \chi) \).

Then the Hamiltonian of our spin \( \cdots \) system reads:

\[
\hat{H}_B = \hat{E}_0^B + \hat{H}_B^B + \hat{H}_B^H.
\]
where we admit $1 \leq \xi \leq 1$.

If we introduce the operators $S_k^x = S_k^x + iS_k^y$, we get further:

$$
H(\xi, \eta) = -\mu \mathcal{H} \sum_{k} S_k^x = -\left(\mathcal{H}\right) \sum_{k} \left(\xi S_k^x + \eta S_k^y + S_k^z S_k' \right), \quad (6.26)
$$

Because of the translational invariance of the Hamiltonian (6.27), we can make a transition to the Fourier images of the operators $S_k^x$ in the momentum space:

$$
S_k^x = \frac{1}{2\pi} \mathcal{H} \sum_{k} S_k^x \exp(i\mathbf{p} \cdot \mathbf{p}), \quad S_k^x = \frac{1}{2\pi} \mathcal{H} \sum_{k} S_k^x \exp(-i\mathbf{p} \cdot \mathbf{p})
$$

$$
I_{k,l} = I(l) \exp(i\mathbf{p} \cdot \mathbf{p}), \quad I(l) = \frac{1}{2} I(l) \exp(i\mathbf{p} \cdot \mathbf{p}), \quad (6.28)
$$

getting instead of $H(\xi, \eta)$,

$$
H = \mu \mathcal{H} \sum_{k} S_k^x = -\left(\mathcal{H}\right) \sum_{k} \left(\xi S_k^x + \eta S_k^y + S_k^z S_k' \right)
$$

where

$$
-\gamma \leq \xi \leq \gamma, \quad 0 \leq \eta \leq 1,
$$

$$
I(l) = \prod_{l=0}^{l} I(l) = 0, \quad (6.30)
$$

$$
\chi_l = \mathcal{H} \sum_{k} \exp(i\mathbf{p} \cdot \mathbf{p}), \quad (6.30)
$$

and:

$$
\chi_{l+1} = -\chi_l, \quad \chi_{l+2} = \chi_l, \quad \chi_0 = -\chi_0 = 1.
$$

Here $p_0 = \mathcal{H}(1,1)$ is the boundary momentum of the first Brillouin zone, and we explicitly assume the approximation of the nearest neighbour interaction; a parameter $\gamma$ denotes the number of the nearest neighbours of a single site.

From now on we shall restrict our considerations to a special case of spin $s=1/2$ lattice. This restriction is justified both theoretically and experimentally [50,51,9] as a consequence of the fact that the major-
Let us add that if $\xi = \eta$, the coefficient $C(p)$ obviously vanishes, and in that case $E_0$ becomes an energy of the ground state, while $A(p)$ becomes an energy of the ideal gas of Bloch’s magnons.

The general $\xi \neq \eta$ Hamiltonian (6.36) describes the magnetic properties of a few metals, and can also be applied to the quasi-spin formulation of the superfluidity problem for the non-ideal lattice gas and the BCS model in the theory of superconductivity, compare [9, 52-56].

The one-particle dynamics generated by the Hamiltonian (6.36) leads to the following equations of motion:

$$i \dot{b}_p = [b_p, H] = A(p)b_\eta + C(p)b_{p- \eta} + J_p,$$

(6.38)

where

$$J_p = (\xi \eta) (J_p + J_{-p}^\ast) + (\xi \eta)^2 V_{pq} (\xi - \eta) S_q (b_p - b_{p+q}),$$

(6.39)

and a non-symmetric potential $V_{pq}(\xi)$ is given by

$$V_{pq}(\xi) = I(p)(\xi - \eta) S_q + V_{pq}(\xi) - V_{qp}(\xi).$$

(6.40)

In the boson approximation the term $-\xi V_{q \neq p}$ is neglected.

Analogously for $\phi_p$:

$$i \dot{\phi}_p = [\phi_p, H] = (\xi \eta) \sum_{q} V_{pq}(\xi) \phi_q + \sum_{q} (\xi \eta)^2 \sum_{r} W_{pq}^{(1)} (\phi_q - \phi_{q-p}),$$

(6.41)

$$= i \sum_{q} V_{pq}(\xi) \phi_q + \sum_{q} (\xi \eta)^2 \sum_{r} \phi_q (b_{q-p} - b_{q+p}),$$

where

$$W_{pq}^{(1)} = (\xi \eta)^2 V_{pq}(\xi + \eta) \phi_q (\xi + \eta) = \sum_{q} I(0)(\xi \eta) (\phi_q - \phi_{q-p}).$$

(6.42)

In the particular case of the Ising model: $\xi = \eta = 0$, we get

$$V_{pq}(0) = I(0)(\xi + \eta) \phi_q (\xi + \eta) = 0,$$

so that all operators $\phi_p$ are integrals of motion.

Dyson's theory [28, 49] of the isotropic Heisenberg ferromagnet, which was further reproduced in terms of the boson operators [57-64], is commonly treated as the most exact and appears as the standard theory in the low temperature domain. Its generalization onto the anisotropic case is immediate. Let us now reproduce from [9], what happens if the Dyson-Maleev expansions are used. Namely their translation to the momentum space reads:

$$b_p \rightarrow a_p^\ast, \quad b_{-p} \rightarrow a_p - (\xi \eta)^2 \sum_q V_{q-p} a_q,$$

(6.43)

where

$$[a_p, a_{p'}] = \delta_{pp'}, \quad [a_p, V_{pq}] = (\xi \eta)^2 a_{p+q} a_{p'-q},$$

(6.44)

and the subsidiary condition, which reflects the requirement $\phi^2 = 0$,

$$\sum_q a_{p+q} a_{p-q} = 0 \quad \text{for all } p.$$

(6.45)

Then

$$H_B = E_0 + \sum_p [a_p a_{p'} + (\xi \eta)^2 \sum_{q} V_{q-p} a_{p+q} a_{p-q}] -$$

(6.46)

$$- (\xi \eta)^2 I(p) V_{p-p} a_p a_{p'} + (\xi \eta)^2 \sum_{q} I(p) a_{p+q} a_{p-q} a_q +$$

$$+ (\xi \eta)^2 \sum_{q} C(p) V_{q-p} a_{p+q} a_{p-q},$$

In the model $\xi = \eta$, we get $C(\eta) = 0$, and then

$$H_B = E_0 + \sum_p E_p a_p a_{p'},$$

(6.47)

where

$$E_p = E_0 + \sum_{p} E_p a_p^\ast a_p.$$

Hamiltonians (6.46), (6.47) are obviously non-Hermitian and need the introduction of the new topology in the state space. However, if done, the eigenvalues of $H_B$ are allowed to be negative and even complex.

If we make use of (6.30), the Hamiltonian (6.47) can be rewritten in the canonical Dyson form of the symmetrized four-boson interaction, [48]:

-62-
\[ H_b = E_0 + \sum_p \left( \sum_{\alpha, \beta} b_p^* a_{p, \alpha} a_{p, \beta} - \frac{i(\mu V)}{\beta} \sum_{\alpha, \beta} V_{\alpha, \beta}(\xi) a_{p, \alpha} a_{p, \beta} a_{p, \alpha} a_{p, \beta} \right), \]  

(6.48)  

where the Dyson potential \( V_{\alpha, \beta}(\xi) \) is given by

\[ V_{\alpha, \beta}(\xi) = [\chi_{\alpha} + \chi_{\beta} r_{\alpha, \beta} - \frac{1}{\beta} (\chi_{\alpha} r_{\alpha, \beta} + \chi_{\beta} r_{\alpha, \beta})]. I(\alpha). \]  

(6.49)  

In the case of the Ising model \( \xi = 0 \), we would get

\[ V_{\alpha, \beta}(\xi) = \chi_{\alpha} + \chi_{\beta} r_{\alpha, \beta} . \]  

The Hamiltonian (6.40) generates the following equations of motion for the boson operators \( a_{p, \alpha} \), \( a_{p, \beta} \):

\[ i a_{p, \alpha} = \gamma a_{p, \alpha} + E_{p, \alpha} a_{p, \alpha} + \frac{i(\alpha)}{\beta} \sum_{\beta} \left( \chi_{p, \beta} - \chi_{p, \beta} r_{\alpha, \beta} - \frac{1}{\beta} (\chi_{p, \beta} r_{\alpha, \beta} + \chi_{p, \beta} r_{\alpha, \beta}) \right), \]  

(6.50)  

\[ -a_{p, \beta} = \gamma a_{p, \beta} + E_{p, \beta} a_{p, \beta} - \frac{i(\beta)}{\beta} \sum_{\beta} \left( \chi_{p, \beta} - \chi_{p, \beta} r_{\beta, \alpha} - \frac{1}{\beta} (\chi_{p, \beta} r_{\beta, \alpha} + \chi_{p, \beta} r_{\beta, \alpha}) \right). \]  

(6.51)  

The above equations are not mutually adjoint.

By taking into account

\[ \gamma a_{p, \alpha} = \sum_{\beta} \left( \chi_{p, \beta} - \chi_{p, \beta} r_{\alpha, \beta} - \frac{1}{\beta} (\chi_{p, \beta} r_{\alpha, \beta} + \chi_{p, \beta} r_{\alpha, \beta}) \right), \]  

(6.52)  

and combining it with (6.50), one gets one more equation of motion:

\[ (i(\alpha + \beta)) \frac{d}{dt} \left( a_{p, \alpha} - \frac{i(\alpha + \beta)}{\beta} \sum_{\beta} \left( \chi_{p, \beta} - \chi_{p, \beta} r_{\alpha, \beta} - \frac{1}{\beta} (\chi_{p, \beta} r_{\alpha, \beta} + \chi_{p, \beta} r_{\alpha, \beta}) \right) \right) = 0. \]  

(6.53)  

We have thus derived the complete Dyson images of the dynamical equations (6.39), (6.40) in the anisotropic lattice.

The theory of the previous section suggests using the rigorous boson expansions of the spin operators, which in case of lowest spins do not suffer from all peculiarities and difficulties of the Dyson- Maleev approach; note, for example, that the operations of the Hermitian conjugation and the differentiation in time do not commute.

In the rigorous approach, the following substitutions are in order:

\[ b_{\alpha} \rightarrow a_{\alpha}, \quad b_{\beta} \rightarrow a_{\beta}, \quad s_{\alpha} \rightarrow v_{\alpha}, \]  

\[ b_{\alpha} = p_0 a_{\alpha} p_0, \quad b_{\beta} = p_0 a_{\beta} p_0, \quad s_{\alpha} = p_0 s_{\alpha} p_0, \]  

(6.54)  

for all \( \alpha \), and the appropriate projector \( p_0 \) in \( \mathcal{F}_b \).

Note that to make an explicit use of the calculations performed up to now, we have changed the definition of \( s_{\alpha} \) into \( -s_{\alpha} \).

Substituting (6.54) into (6.37), we get

\[ H = p_0 H_b p_0 \]  

(6.55)  

\[ H = E_0 + M \xi \sum_{\alpha, \beta} \left( \alpha_{\alpha} a_{\alpha} - \frac{i(\mu V)}{\beta} \sum_{\alpha, \beta} V_{\alpha, \beta}(\xi) a_{\alpha} a_{\alpha} + a_{\alpha} a_{\alpha} \right) \]  

\[ -\frac{i(\mu V)}{\beta} \sum_{\alpha, \beta} \left( \chi_{p, \beta} - \chi_{p, \beta} r_{\alpha, \beta} - \frac{1}{\beta} (\chi_{p, \beta} r_{\alpha, \beta} + \chi_{p, \beta} r_{\alpha, \beta}) \right) \]  

\[ V_{\alpha, \beta}(\xi) = [\chi_{\alpha} + \chi_{\beta} r_{\alpha, \beta} - \frac{1}{\beta} (\chi_{\alpha} r_{\alpha, \beta} + \chi_{\beta} r_{\alpha, \beta})]. I(\alpha). \]  

(6.56)  

\[ H = E_0 + \sum_{\alpha, \beta} \left( \alpha_{\alpha} a_{\alpha} - \frac{i(\mu V)}{\beta} \sum_{\alpha, \beta} V_{\alpha, \beta}(\xi) a_{\alpha} a_{\alpha} + a_{\alpha} a_{\alpha} \right) \]  

\[ -\frac{i(\mu V)}{\beta} \sum_{\alpha, \beta} \left( \chi_{p, \beta} - \chi_{p, \beta} r_{\alpha, \beta} - \frac{1}{\beta} (\chi_{p, \beta} r_{\alpha, \beta} + \chi_{p, \beta} r_{\alpha, \beta}) \right). \]  

(6.57)  

differing from (6.49) by the absence of the term \( \xi \sum_{\alpha, \beta} \chi_{p, \beta} r_{\alpha, \beta} \) in (6.40).

The canonical form of \( H_b \) is easily achieved,

\[ H_b = E_0 + \sum_{\alpha, \beta} \left( \alpha_{\alpha} a_{\alpha} - \frac{i(\mu V)}{\beta} \sum_{\alpha, \beta} V_{\alpha, \beta}(\xi) a_{\alpha} a_{\alpha} + a_{\alpha} a_{\alpha} \right) \]  

(6.58)  

which implies the following equations of motion for the operators \( a_{\alpha}, a_{\beta} \):

\[ i a_{\alpha} = p_0 i a_{\alpha} p_0, \]  

\[ -i a_{\beta} = p_0 i a_{\beta} p_0, \]  

(6.59)  

and obviously, for \( v_{\alpha} \):

\[ p_0 \gamma v_{\alpha} p_0 = i \gamma v_{\alpha}, \]  

\[ v_{\alpha} = \gamma v_{\alpha} = 0. \]  

(6.60)
One can easily check that Eqs. (6.54) are mutually adjoint with respect to the Hamiltonian constant \( A \).

Here, as in the Ising model (where \( J = 0 \) manifestly), all operators \( \mathcal{V}_p \) are integrals of motion.

In addition, combining the first equation in (6.59) with (6.60), one can derive the following analogue of (6.53):

\[
(\mathbf{i} \mathbf{d} \mathbf{d} \mathbf{t} - \mathbf{E}_p^t)(\mathbf{c}_p - \frac{4}{\mathbf{i} \mathbf{d} \mathbf{t}} \mathbf{V}_p \mathbf{c}_q)(\mathbf{a}_p) = -4 \mathbf{i} \mathbf{d} \mathbf{t} \mathbf{V}_p(\mathbf{c}_q)(\mathbf{a}_p),
\]

where \( \mathbf{V}_{pq}(\mathbf{c}_q) = \mathbf{I}(\mathbf{c}_q) = \mathbf{I}(\mathbf{q}) \).

Let us add that the extensions of the described methods onto the antiferromagnetic case are straightforward.

7. THE ATOMIC NUCLEUS IN THE WEAK EXCITATION LIMIT

In the theory of atomic nuclei, it was noticed that in many cases the spectra of low-lying excited states are similar to those of the excited system of the weakly coupled quadrupole bosons [7,18-26].

Such a boson system can thus be used in the approximate description of the weakly excited atomic nucleus, when the anharmonic corrections, which correspond to the interaction of the quadrupole bosons between themselves, are negligibly small.

On the other hand, it is well known that the Hamiltonians of the microscopic model of the nucleus, if expressed in the representation of the generalized quasi-spin operators, are in close relation with the previously discussed Heisenberg ferromagnet case. This suggests the use of boson expansion methods, to get an approximation of the initial system by the set of collective bosons which are responsible for its low-temperature properties.

In the spherical model of the nucleus, with the \( j-j \) coupling, each \( k \)-th one-particle state is characterized by a collection \( \alpha_k = (n, l, j, m_j) \) of quantum numbers. The magnetic parameter \( m \) plays a distinguished role in further considerations, so we shall use the notation:

\( \alpha_k = (n, l, j, m_j) \) \( k = 1,2, \ldots \),

Let us define the operators of fermion pairs, whose total angular momentum equals \( 1 \), while its projection is:

\[
\begin{align*}
\mathbf{A}_n^m(m_b^b) &= \frac{1}{2} \sum \mathbf{c}_b \mathbf{c}_b^\dagger \mathbf{m}_b \mathbf{m}_b^b \mathbf{M}_b \mathbf{M}_b^b, \\
\mathbf{B}_n^m(m_b^b) &= \frac{1}{2} \sum \mathbf{c}_b \mathbf{c}_b^\dagger \mathbf{m}_b \mathbf{m}_b^b \mathbf{M}_b \mathbf{M}_b^b (-1)^{m_b} \mathbf{b}_b \mathbf{b}_b^b.
\end{align*}
\]

Here \( \mathbf{c}_b \mathbf{c}_b^\dagger \mathbf{m}_b \mathbf{m}_b^b \mathbf{M}_b \mathbf{M}_b^b \) is the Clebsch-Gordan coefficient of the expansion.

Let us introduce the following abbreviations:

\( \mathbf{A}_k = (a_k, m_k) = (a_k, m_k a_k), \quad A_{2j}(a_k, b_k) = A_{a_k} \).

By using the sequence \( k, j, \ldots \) of the operators (7.1) and their adjoints, one can construct the Hamiltonian of the nucleus in the microscopic model, as well as the transition operators between the nuclear states.

The operators (7.1) obey

\[
\begin{align*}
[\mathbf{A}_a, \mathbf{A}_b^\dagger] &= \mathbf{S}_{ab}^{(a)} - 2 \sum (i l)(i + \mathbf{p}_i)(i + \mathbf{p}_i^\dagger) Y(i, k, l) \mathbf{B}_i, \\
[\mathbf{B}_a, \mathbf{A}_b] &= 2 \sum (i l)(i + \mathbf{p}_i)(i + \mathbf{p}_i^\dagger) Y(i, k, l) \mathbf{A}_i^\dagger, \\
[\mathbf{B}_a, \mathbf{B}_b^\dagger] &= -2 \sum (i l)(i - \mathbf{p}_i)(i - \mathbf{p}_i^\dagger) Y(i, k, l) \mathbf{B}_i^\dagger,
\end{align*}
\]

where \( \mathbf{p}_i \) is the permutation operator:

\[
\mathbf{p}_i f(k) = \mathbf{p}_i f(a_k, b_k, j_k, M_k) = -\theta(k, a_k, b_k, j_k, M_k), \quad \theta(a, b, j) = (-1)^{k + \frac{1}{2} \Delta j},
\]

and moreover:

\[
\mathbf{S}_{ab}^{(a)} = (i + \mathbf{p}_i) S_{ab}^{(a)} = (i + \mathbf{p}_i) S_{ab}^{(a)} S_{M_i}^{M_k},
\]

while

\[
\begin{align*}
\mathbf{c}_k &= (i m_k) \mathbf{c}_k \mathbf{c}_k^\dagger \mathbf{m}_k \mathbf{m}_k^b \mathbf{M}_k \mathbf{M}_k^b, \\
\mathbf{Y}(i, k, l) &= \sum (i + \mathbf{p}_i) \mathbf{c}_k \mathbf{c}_k^\dagger \mathbf{m}_k \mathbf{m}_k^b \mathbf{M}_k \mathbf{M}_k^b (-1)^{m_k} \mathbf{b}_k \mathbf{b}_k^b.
\end{align*}
\]
In the particular case of \( J = K = 0 \), the commutation relations (7.2) coincide with the commutation relations for the spin operators. Namely, in that case, for each \( k = 1, 2, \ldots \), we have:

\[
[A_k, A_{k'}^\dagger] = \Lambda - 2 \frac{(2j_k^\pm + 1)}{2j_k^\pm + 1} B_k,
\]

which allows to associate with each \( k \)-th bifermion state a corresponding \( k \)-th infinitesimal generator of the SU(2) group, according to:

\[
S_k^\pm = \frac{1}{\sqrt{2}} (\pm \Lambda + i \Lambda^* \sigma_k^\pm) - \frac{(\pm \Lambda + i \Lambda^* \sigma_k^\pm)}{\sqrt{2}} B_k.
\]

This fact justifies the boson translation of the theory, at least in the simplest case \( J = K = 0 \), but also motivates the attempts to get any extension of the boson expansion methods more complex situations.

Now, as an interlude in the main current of considerations, let us study what happens in the weak coupling limit, also called a harmonic approximation, when higher excitations significantly influence the behaviour of the nucleus. In that case one can imagine the situation, when the compounds of the atomic nucleus though fermions in the basic formalism, can behave like bosons.

Following [7], let us assume we deal with the light nucleus, where the number \( n \) of compounds is relatively small.

We know that the higher the excitation level of the system (high temperature limit), the greater becomes the number \( \sum (2j^l + 1) \) of the one-particle states which are mostly occupied by the single constituents of the nuclear system. Moreover, only the great values \( j_k^l \) become significant then. This corresponds to the weakening of the coupling forces between the nucleons, so that the weak coupling limit of the theory is approached.

Let us note that for \( j_k^l \gg 1 \), we can use instead of \( Y_i^l(k, l) \), the corresponding asymptotic expression:

\[
Y_i^l(k, l) \sim S_{ik}^\pm S_k^\pm \sigma_k^\pm S_{ik}^{-} \sigma_k^{-} \cdot (2j^l + 1)^{-\frac{1}{2}} \cdot \left( \frac{(2j^l + 1)(2j^l + 1)(2j^l + 1)}{(2j^l + 1)(2j^l + 1)(2j^l + 1)} \right),
\]

where \( (2j^l + 1)^3 \equiv (2j^l + 1)(2j^l + 1)(2j^l + 1) \).

By virtue of \( j_k^l \gg 1 \), (7.6) includes a small parameter \( (2j^l + 1)^{-\frac{1}{2}} \). In consequence, the operator part of the right-hand side of the commutator \([A_k, A_{k'}^\dagger]\) can be neglected. Moreover, in the case under consideration, the operators \( B_j^l(a, b) \) can also be neglected. If we denote by \( |m\rangle \) the ground state of our nucleus, and consider the expectation value of \( B_j^l(a, b) \) in this state, we get:

\[
\langle m | B_j^l(a, b) | m \rangle \equiv \langle m | B_j^l(a, b) | m \rangle \equiv \langle m | B_j^l(a, b) | m \rangle.
\]

Hence, in the approximation \( (2j^l + 1)^{-\frac{1}{2}} \), the operators \( A_{j^l}^{-} \) can be considered as the ideal bosons.

\[
A_{j}^{-} = S_{ik}^{-} \sigma_k^{-} \sigma_k^\pm S_{ik}^\pm, \quad \sigma_k^\pm = \frac{1}{\sqrt{2}} (\pm \Lambda^* \sigma_k^\pm).
\]

In consequence, instead of the initial Hamiltonian describing the interaction of Fermion pairs, one can introduce an ideal Boson Hamiltonian, acting in the Hilbert space of the boson states: no Pauli exclusion principle is observed. This is an example of the metamorphosis of Fermions into bosons if higher excitations are essential. In that case, obviously, the bosonic "phase" of the system prevails significantly.

In the above, the Pauli exclusion principle, standing for a defining property of the fermionic behaviour, disappeared. Quite conversely, we shall now prove its appearance. As we consider the weak excitation limit of the appropriately defined boson system, the four-fermion interaction Hamiltonian will now be in order.

In the microscopic theory of the weakly excited atomic nucleus, the collective excitation branches connected with the pairing correlations inside the nucleus, which is assumed to consist of nearly the same number of protons and neutrons. Because the nucleons can occupy the same one-particle levels, differing only by the isospin projection \( \tau \) in addition to the p-p and n-n correlations, the n-p correlations should be
We assume the correlations to be spin independent, so that the quantum numbers of interest are now \( \alpha = (i, n, \ell) \).

Let there be given a four-fermion interaction Hamiltonian [65]:

\[
H = \sum_{\alpha} \left( \varepsilon_{\alpha} - \mu \right) b_{\alpha}^* b_{\alpha} + \frac{\lambda}{16} \sum_{\alpha, \beta} \sum_{n \geq 0} b_{\alpha}^* \left( b_{\beta}^* \right)^n b_{\beta} \left( b_{\alpha} \right)^n W(\alpha, \beta, n) , \tag{7.12}
\]

where \( \mu \) is the chemical potential in the ground state of the system when \( V = 0 \), \( \mu \) equals to the Fermi energy \( E_F \).

We know that in the nucleus, the interaction correlates the antiparallel magnetic moments only, which results in the following form of the Hamiltonian:

\[
H = \sum_{j,m} \left( \varepsilon_j - \mu \right) N_{j,m} + \frac{\lambda}{16} \sum_{j,m,n \geq 0} N_{j,m} N_{j,n} W_{j,m,n} ,
\]

where \( N_{j,m} = \sum_{n \geq 0} \left( b_{j,m}^* b_{j,m+n} + b_{j,m+n}^* b_{j,m} \right) \).

Because terms of different sign give identical counterparts to the total sum \( \sum_{j,m} \left( \varepsilon_j - \mu \right) N_{j,m} \), we can in fact consider \( \sum_{j,m} \left( \varepsilon_j - \mu \right) N_{j,m} = 2 \sum_{j,m \neq n} N_{j,m} N_{j,m} \), compare e.g. [7,65].

In [7] one can find an even more simplified form of the Hamiltonian (7.13):

\[
H = \sum_{j,m} \left( \varepsilon_j - \mu \right) N_{j,m} - G \sum_{j,m,n \geq 0} A_{j,m}^* A_{j,m+n} , \tag{7.14}
\]

where

\[
A_{j,m}^* = (-1)^{m-j} \sum_{\ell = \frac{j-m}{2} \in \mathbb{Z}} c_{m+\ell, j-\ell} b_{j, m+\ell}^* b_{j, m-\ell} .
\]

and \( c_{m+\ell, j-\ell} \) denotes the Clebsch-Gordan coefficient.

To find the spectrum of (7.13) or (7.14), one must perform the appropriate diagonalization of the Hamiltonian, see e.g. [65].

Let us now study the low-temperature behaviour of the nucleus in the microscopic model. The discussion of the traditional Marumori approach can be found in [7], where the following form of the finite Marumori expansions of the biphemion operators was assumed:

\[
(-i)^{m-\ell} A_{j,m}^* \rightarrow c_{m+\ell, j-\ell}^* \left( -\sum_{q \in \mathbb{Z}} c_{m+\ell+2q, j-\ell-2q}^* c_{m+\ell-2q, j-\ell+2q} \right) + \left( (-1)^{m-\ell} \sum_{q \in \mathbb{Z}} \right) c_{m+\ell, j-\ell}^* c_{m+\ell+2q, j-\ell-2q} .
\]

\[
(-i)^{m-\ell} A_{j,m} \rightarrow c_{m+\ell, j-\ell} c_{m+\ell+2q, j-\ell-2q} + \left( (-1)^{m-\ell} \sum_{q \in \mathbb{Z}} \right) c_{m+\ell+2q, j-\ell-2q}^* c_{m+\ell, j-\ell} .
\]

\[
\sum_{q \in \mathbb{Z}} c_{m+\ell+2q, j-\ell-2q} \equiv 0 , \quad \ell = 0, 1, 2.
\]
where $\delta_{ij}$ is the two-point Levi-Civita tensor: $(\delta_{ij}) = (0,1)$. The finite boson expansions read:

\[ p_{\alpha} p_{\beta} = A_{F} A_{\alpha} A_{\beta} A_{F} \delta_{\alpha \beta}, \]
\[ p_{\alpha} p_{\beta} = A_{F} A_{\alpha} A_{\beta} A_{F} \delta_{\alpha \beta}, \]
\[ p_{\alpha} p_{\beta} = A_{F} A_{\alpha} A_{\beta} A_{F} \delta_{\alpha \beta}. \]

(7.19)

Here $\delta_{ij}$ plays the role of the sign factor, while $A_{F}$ restores the Pauli exclusion principle in the pure boson scheme. Identities (7.19) hold only under the sign of the bilinear form, i.e., if multiplied from both sides by an antisymmetric tensor and summed over all the indices. The substitution of (7.19) into (7.14) gives the correct boson expansion of the nuclear Hamiltonian,

\[ H = A_{F} H_{B} A_{F}, \]
\[ H_{B} = \sum_{j,\delta} (\varepsilon_{j} - \mu) \sum_{\gamma} \langle \omega_{\gamma} | C_{\gamma j}^{T} A_{\gamma j} | \omega_{\gamma} \rangle + (4\pi) \sum_{\gamma} \langle \omega_{\gamma} | C_{\gamma j}^{T} A_{\gamma j} | \omega_{\gamma} \rangle. \]

(7.20)

with $\delta_{ij} = \mp 1/2$.

We have thus derived a very simple, purely bosonic Hamiltonian for the microscopic model of the atomic nuclei, which exhibits all the essential properties, as e.g., the Pauli exclusion principle, of the initial one. For this Hamiltonian the diagonalization procedure can be performed on the pure boson level.

The only difficulty here arises if we try to compare the correct expansion formula with that in the Marumori approach. Namely, the bosons we have used to perform the construction are rather unphysical, since carrying the odd isospin quantum numbers. They are the ghost bosons, which in the weak excitation limit behave like the physical fermions. This situation can easily be improved. Let us assume

\[ V(j_{1},m_{1},n_{1},k_{1},c_{1},d_{1}) = (-1)^{m_{1}+n_{1}+k_{1}+d_{1}} \sum_{\gamma} C_{\gamma j_{1}}^{T} C_{\gamma j_{1}}^{T} \bar{C}_{\gamma j_{1}}^{T} \bar{C}_{\gamma j_{1}}^{T}, \]

(7.21)

where $\gamma = 0, \pm 1$, $T = 0, 1$. If we neglect the term with $T = 0$, we get, by using

\[ \sum_{\gamma} C_{\gamma j_{1}}^{T} C_{\gamma j_{1}}^{T} \bar{C}_{\gamma j_{1}}^{T} \bar{C}_{\gamma j_{1}}^{T} = C_{\gamma j_{1}}^{T} C_{\gamma j_{1}}^{T} \bar{C}_{\gamma j_{1}}^{T} \bar{C}_{\gamma j_{1}}^{T}, \]

(7.22)

which, by virtue of the orthogonality and normalizability conditions for the Clebsch-Gordan coefficients, see e.g. [63], implies

\[ [C_{\gamma j_{1}}^{T}, C_{\gamma j_{1}}^{T}] = \delta_{\gamma 0} \]

(7.23)

the following, physical, form of the ghost Hamiltonian:

\[ H_{B} = \sum_{j,\delta} (\varepsilon_{j} - \mu) \sum_{\gamma} \langle \omega_{\gamma} | C_{\gamma j}^{T} A_{\gamma j} | \omega_{\gamma} \rangle + (4\pi) \sum_{\gamma} \langle \omega_{\gamma} | C_{\gamma j}^{T} A_{\gamma j} | \omega_{\gamma} \rangle. \]

(7.24)

where the interaction term now includes only the physical bosons: each one appearing in place of the ghost pair. These are the nuclear analogues of the Cooper bosons, which, by virtue of $A_{F} H_{B} A_{F}$, still obey the Pauli exclusion principle. We see that the explicit use of the orthogonality formulas for the Clebsch-Gordan coefficients, almost indispensable for the four-fermion interaction Hamiltonian in the weak excitation limit, namely, we have here one isospin 1 and isospin 0 "phases" of the weakly excited atomic nucleus. If we restrict considerations to the isospin 1 "phase" only, then an obvious comparison with the Marumori approach is possible. It is quite clear that the Marumori formula, even in the finite case, proposes too many terms in the expansion for $A_{\gamma j}^{T}$ to get a correct equivalence with the starting four-fermion Hamiltonian.

Let us add that in the second quantization approach to the four-fermion interactions, one can also use the supplementary boson field method [66], in which the quanta of the boson field describe a collective motion of a group of fermions. However, in that case, the Hamiltonian is not expressed in terms of bosons only but needs the additional use of the new fermions, which are constructed from the old fermions and the supplementary bosons.
We shall not go beyond the framework of the conventional quantum field theory, and all the considerations are essentially based on its LSZ formulation\[77,78\]. The basic assumption here is that any operator quantifying a given quantum system, the scalar field at the beginning, admits a decomposition into the power series expansions with respect to the normal ordered products of the free asymptotic fields. With the scalar quantum field: \( \Phi(x) \) we associate an algebra of all operators:

\[
\hat{F}(\Phi) = \sum_{\alpha} \left( \hat{a}_{\alpha} \right)^{n} = \sum_{n} \hat{a}_{\alpha_{1}} \ldots \hat{a}_{\alpha_{n}} \Phi(x)_{\alpha_{1}} \ldots \Phi(x)_{\alpha_{n}},
\]

\( \hat{a}_{\alpha} \) \( \hat{a}_{\alpha} \) \( \hat{a}_{\alpha} \) \( \hat{a}_{\alpha} \) \( \hat{a}_{\alpha} \)

\[ \omega = (\alpha_{1}, \ldots, \alpha_{n}), \quad \chi = (\omega, \chi), \quad \chi_{\alpha} \in \mathbb{M}^{4}, \quad (8.1) \]

With the Fock representation of the CCR algebra (the asymptotic condition) we can introduce the coherent state domain for the operator algebra:

\[
K = \mathfrak{F}(\mathbb{R}^{3}) \setminus \mathcal{A}, \quad (\chi, \omega) = \int d^{3}k \left( \omega(k) \right) \mathcal{O}_{\omega}, \quad (8.2)
\]

If \( \alpha, \bar{\alpha} \) are the classical Fourier amplitudes of \( \hat{\Phi}(\omega) \), \( \omega \rightarrow a, a^{*} \Rightarrow \phi(a) \rightarrow \hat{\Phi}(\omega) \) we get:

\[
(\alpha|\hat{a}_{\alpha} | \omega) = \langle \Phi(\omega)| \Phi(\alpha) = \Phi(\omega), \quad (\alpha| : F(\Phi)| : \chi = F(\hat{\Phi}) = \sum_{\alpha} \langle \Phi(\omega)| \Phi(\alpha) \rangle, \quad (a)
\]

which establishes a correspondence between a quantum and the classical level of the scalar field algebra.

To restore correctly the corresponding quantum image, while the classical expressions are given, it is extremely useful to employ the so-called functional representations of the CCR algebras, which arise in the theory of the functional power series, see e.g.\[74-76,27\].

An introduction to the functional methods is given in the Appendix, and here we shall only quote, without any detailed explanations, the basic results.

\[ \text{Namely, if we have an operator expression } (8.4) \text{ and the corresponding classical image, } F(\hat{\Phi}) = \sum_{\alpha} \langle \hat{a}_{\alpha} \rangle \phi_{\alpha}(\omega_{\alpha}) \ldots \phi_{\alpha}(\omega_{\alpha}), \text{ it is sufficient to multiply } \phi_{\alpha}(\omega_{\alpha}) \text{ by the exp}(\alpha, \omega_{\alpha}) \text{ to get so-called functional representation of the operator }: F(\hat{\Phi}) : \text{. The functional power series } F(\hat{\Phi}) \text{ play the role of the operator }: F(\Phi) : \text{ in the Bargmann space, which we denote by } \mathbb{B} = \mathcal{F}(\mathbb{C}^{1}(\mathbb{R}^{3})). \text{ The situation as it stands can be summarized in the correspondence principle, where the correspondence rule reads:}
\]

\[
\{ : F(\Phi) : \} \rightarrow \{ : F(\hat{\Phi}) : \}, \quad (8.4)
\]

while the quantization rule,

\[
\{ F(\hat{\Phi}) : \} \rightarrow \{ : F(\Phi) : \}, \quad (8.5)
\]

\[ F(\hat{\Phi}) = F(\Phi) \exp(\alpha, \omega) = : F(\Phi)(\alpha, \omega) : \]

The pragmatists working in the domain of quantum field theory are strongly convinced that the quite satisfactory classical level for the field algebra associated with any fermion (Dirac say) field is given in the framework of the Grassmann algebra. This algebra is built of the c-number-like but anticommuting objects, which manifestly exhibit the Pauli exclusion principle, whose influence on the initial fermion level is thus taken into account. A complete theory was even founded of the anticommuting numbers in the functional-like integration and differentiation procedures, see e.g.\[72,74\], to justify the use of anticommuting Schwinger sources\[77-81,74\].

On the other hand, we have proved\[27,28\] that one can always associate with any element of the fermion field algebra, the corresponding c-number valued functional power series with respect to the Fourier amplitudes of the Dirac spinors. According to the Klauder prescription\[73\], one can even get the functionals with respect to the free Dirac spinors in case of the quadratic forms at least. So it is rather surprising that no reasonable correspondence was established of the c-number classical level with the prospective fermion level.

The reason is obvious: the Pauli exclusion principle is not yet introduced, because the classical spinors must by no means give account of it. So, the exclusion principle on the classical level is an external and extremely artificial requirement, which should eventually be imposed.
To solve the equation of the c-number classical level for fermion fields, we had previously [27,28] developed the c-number language in the functional formulation of the quantum theory of the Fermi systems: the functional representations of the CAR algebra were invented there. Let us however begin from the chronologically earlier Klauder proposal, [73] whose short presentation will allow a better understanding of our argument.

Let us consider a free scalar time-zero field \( \phi(x) = \frac{1}{\sqrt{2\pi}} \int \frac{dk}{k} e^{ikx} (\phi(k) + e^{i\phi(k)}) \), and the conjugate momentum \( \pi(x) = \frac{-i}{\sqrt{2\pi}} \int \frac{dk}{k} e^{-ikx} (\pi(k) - e^{-i\pi(k)}) \). The coherent states here can be defined by:

\[
|q, p_0\rangle = U(q, p_0)|0\rangle,
\]

where the unitary operators \( U \) are given by

\[
U(q, p_0) = \exp\left\{ ip_0 x - \frac{p_0^2}{2m} \right\} d^3x,
\]

and are labelled by two real, smooth test functions \( q, p \). The overlap of two such states is given by

\[
\langle q', p' | q, p \rangle = \exp\left\{ -im \frac{1}{2} (q'^2 + q^2) - p'^2 p^2 + \frac{2m}{2} \left( \frac{q'^2 - q^2}{2} \right) \right\}.
\]

Here \( \phi \) denotes the Fourier transform of \( \phi \). These states are so overcomplete that diagonal matrix elements (the coherent state expectation values)

\[
\langle q', p' | G | q, p \rangle = G(q, p)
\]

of an operator polynomial uniquely define the operator. One can easily prove that the operator \( G \) is:

\[
G = U(q, p)\phi \psi^\dagger \phi^\dagger
\]

where the \( \phi, \psi \) denote the normal ordering with respect to the creation and annihilation operators \( \phi^\dagger, \psi^\dagger \).

At this point, let us consider a non-relativistic fermion model, with the property that \( \psi(x)|0\rangle = 0 \). It is equivalent to the assumption that \( \psi(x) \) in fact represents the positive frequency portions of both the conventional \( \psi(x) \) and \( \psi^\dagger(x) \).

In any case, let us define the basic states

\[
|x_1, ..., x_n\rangle_A = \psi(x_1) \ldots \psi(x_n)|0\rangle,
\]

which are antisymmetric in the \( x_i \) variables, \( x^2 = (x_1, ..., x_n) \).

Now, following Friedrichs [86], let us introduce an ordering of points, \( x \), in configuration space. We shall say that \( x < y \) if (1) \( x_1 < y_1 \) (these are the first coordinates), or (2) if \( x_1 = y_1 \) and \( x_2 < y_2 \), or (3) if \( x_1 = y_1 \), \( x_2 = y_2 \), and \( x_3 < y_3 \). Next we introduce the ordering sign function, which we call the Friedrichs-Klauder function:

\[
\epsilon(x_1, ..., x_n) = \pm 1,
\]

given by the sign of the permutation \( P \) necessary to bring the arguments of \( \epsilon \) to the "standard" order \( x_1 < x_2 < \ldots < x_n \); if any two \( x_i \) are equal, \( \epsilon \) is defined to be zero.

With the aid of \( \epsilon \) we define the symmetric vectors:

\[
|x_1, ..., x_n\rangle_S = \epsilon(x_1, ..., x_n) \epsilon(x_1, ..., x_n)|0\rangle,
\]

which vanish if any pair of arguments are equal. Note that, by virtue of the considerations of Sec. 3.1, any symmetric fock space vector, admits a decomposition

\[
|x_1, ..., x_n\rangle = \sum_{\sigma} \epsilon(\sigma) |x_{\sigma(1)}, ..., x_{\sigma(n)}\rangle,
\]

where \( \sigma \) is the permutation \( P \) necessary to bring the arguments of \( \epsilon \) to the "standard" order \( x_1 < \ldots < x_n \).

As the special example of the isomorphism \( \lambda \), let us see that the relation

\[
|x_1, ..., x_n\rangle_A = \epsilon(x_1, ..., x_n) |x_1, ..., x_n\rangle_S
\]

also holds, by virtue of \( \epsilon^2 = 1 \).

Armed with the symmetric states (8.13), we form:

\[
|\phi\rangle = N \sum_{\sigma} \frac{1}{\sqrt{\epsilon(\sigma)}} \left( d\bar{x}_n d\bar{x}_{\sigma(n)} \ldots d\bar{x}_{\sigma(1)} \right) \psi(x_1) \ldots \psi(x_n)|0\rangle,
\]

where \( N \) is the suitable normalization factor and \( \psi(x) \) denotes a complex, smooth c-number test function. These states play the role of the fermion "coherent states". Although these states are not eigenstates of the fermion field operator, they exhibit the essential property that

\[
\langle \phi | \psi(x) \psi(x) | \phi \rangle = \langle \phi | \psi(x) \psi(x) | \phi \rangle = \psi(x) \psi(x),
\]

which is a prototypical relation needed to derive classical images for the energy operator, which is a quadratic form.

We do not just wish to tilt at windmills and to advocate any pure c-number point of view, against the conventional Grassmann tools, especially because these latter are widely spread and quite convenient for the explicit calculations (of propagators, for example). We wish,
however, to prove that the correspondence principle of the kind (8.15)
can be established quite generally, by the use of the boson expansion
methods.
To get the Fock representation suitable for the description of a free
Dirac field, we must start from the triples \( \{a^+, a, \Omega \} \) and
\( \{b^+, b, \Omega_B \} \) exhibiting the number four of the internal
degrees of freedom (spin and charge) in the theory. All results on
the boson expansions of the underlying Fermion operators hold here without
any essential changes if compared with Sect. 3.1, see e.g. [28,76].
The standard construction (the analogous formulas for boson operators) allows us to get the quintets
\( \{b^+, b^*, a^+, a, \Omega \} \) and \( \{a^+, a^*, b^+, b, \Omega_B \} \), with

\[
\begin{align*}
[b^+(\phi), b^-(\bar{\phi})]_+ &= \{\phi, \bar{\phi}\} \Delta_F = [b^+(\phi), b^+(\bar{\phi})]_+, \\
[a^+(\phi), a^- (\bar{\phi})]_- &= \{\phi, \bar{\phi}\} \Delta_F = [a^+(\phi), a^+(\bar{\phi})]_-.
\end{align*}
\]

(8.17)

the other \( (anti)\)commutators vanish.
With the Haag-LSZ expansion conjecture extended to the case of Dirac
fields (\( \psi, \bar{\psi} \) are here the asymptotic free limit) we have, for any
element of the field algebra, the following operator expansion:

\[
\begin{align*}
\Omega(\psi, \bar{\psi}) &= \sum_n \frac{1}{n!} (\omega_n, i (\psi^+ \psi^-)^n (\psi^+ \bar{\psi}^-)^n) = \\
&= \sum_n \frac{1}{n!} \sum_{\omega_n} (\omega_n, i (\psi^+ \psi^-)^n (\psi^+ \bar{\psi}^-)^n) = \\
&= \sum_n \frac{1}{n!} \sum_{\omega_n} (\omega_n, i (\psi^+ \psi^-)^n (\psi^+ \bar{\psi}^-)^n).
\end{align*}
\]

(8.18)

\( \phi, \bar{\phi} \) are bispinor indices, and \( \dagger \) denotes the Dirac conjugation
of bispinors.
\( \Omega(\psi, \bar{\psi}) \) can be rewritten in the following form, resulting from
the normal ordering of operators; the total antisymmetry of \( \omega_n \)
in all \( n+m \) variables is essential here:

\[
\begin{align*}
\Omega(\psi, \bar{\psi}) &= \sum_n \frac{1}{n!} \left( \omega_n, i (\psi^+ \psi^-)^n (\psi^+ \bar{\psi}^-)^n \right) = \\
&= \sum_n \frac{1}{n!} \left( \omega_n, i (\psi^+ \psi^-)^n (\psi^+ \bar{\psi}^-)^n \right) = \\
&= \sum_n \frac{1}{n!} \left( \omega_n, i (\psi^+ \psi^-)^n (\psi^+ \bar{\psi}^-)^n \right).
\end{align*}
\]

(8.20)

where the operators \( \psi^+, \psi^- \) depend linearly, through Fourier transformations,
on Fermion creation and annihilation operators \( b^+, b^* \), defined
by (8.17),

\[
\left[ b^+(\phi), b^-(\bar{\phi}) \right]_+ = \delta_{\phi, \bar{\phi}} \cdot e^{\left( -p \right)} A_F = [b^+(\phi), b^+(\bar{\phi})]_+.
\]

(8.21)

The other \( (anti)\)commutators vanish, and the indices \( i, j = 1,2 \) denote the
helicity states while \( \psi, \bar{\psi} \) are the bispinor indices.
In (8.20) we have clearly distinguished the two groups of operators,
\( \psi^+ (\phi) (\psi^- (\bar{\phi}))^* \) and \( \psi^- (\phi) (\psi^+ (\bar{\phi}))^* \), which involve respectively the \( n+m \) point product
of \( b^+ \)'s and the \( k+1 \) point product of \( b^- \)'s.
By virtue of considerations of Sec. 3.1, we have here

\[
\begin{align*}
\psi^+ (\phi) ... \psi^+ (\phi) b^- (\bar{\phi}) ... b^- (\bar{\phi}) &= \delta_{\phi, \bar{\phi}} \cdot e^{\left( -p \right)} A_F = \delta_{\phi, \bar{\phi}} \cdot e^{\left( -p \right)} A_F, \\
\cdots &= \delta_{\phi, \bar{\phi}} \cdot e^{\left( -p \right)} A_F.
\end{align*}
\]

(8.22)

where the superscript \( * \) means that the identity holds true only under
the sign of the bilinear form, i.e., if integrated over all variables
while multiplied from both sides by a suitable, antisymmetric
\( n+m+k+1 \) point function \( \delta_{\phi, \bar{\phi}} \), which means that \( \sim \) reverses the order of variables.

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The operators $\hat{a}_I^\dagger, \hat{a}_I$ stand here for the creation and annihilation operators of the fictitious subsidiary boson field $\hat{\psi}$ whose weak excitation limit exhibits the correct physical properties of the fermion field. This will be the mediating boson level allowing us to get a classic nomenclature for the fermion field algebra.

In this place the fermion Fock space $\mathbb{F}_F$ appears as a subspace $\mathbb{F}_F \subset \mathbb{F}_0$. These are the representation spaces, respectively, for the triples: $\{a, \hat{a}_I^\dagger, \hat{O}_I\}$ and $\{\hat{a}_I^\dagger, \hat{a}_I, \hat{O}_I\}$. For clarity we shall restrict considerations to the two-point product $\psi_+^+(\omega) \psi_+^-(q)$, whereas immediately get

\[
\psi^+_+^+(\omega) \psi^+_+^-(q) = \xi(k(x + \eta y) \cdot \hat{a}_I(\hat{a}_I^\dagger(p) \hat{a}_I^\dagger(p) \hat{a}_I \hat{F}_F . 
\]

(8.23)

Here again $\xi$ means the validity of (8.23) only under the sign of the bilinear form $(\omega, \psi^+ \psi^-)$. To transform a product of the fermion fields into a product of the boson (spinor) fields, it is enough to construct the integral operator $I_2$ with the property

\[
[I_2^{\text{F}}(\omega)]_{\psi_+^+^+(\omega) \psi_+^+^-(q)} \equiv S_{ij} \sum \psi^+_i(\omega) \psi^+_j(\omega) \hat{F}_F . 
\]

(8.24)

With the use of the helicity basis,

\[
\psi^+_i(\omega) \psi^+_j(\omega) \hat{F}_F \equiv \psi^+_i(\omega) \psi^+_j(\omega) \hat{F}_F . 
\]

(8.25)

this purpose is realized if the following integral kernel of $I_2$ is defined:

\[
I_2^{\text{F}}(\omega) = \frac{1}{2\omega} \sum \psi^+_i(\omega) \psi^+_j(\omega) \hat{F}_F . 
\]

(8.26)

where $k, \eta \in \mathbb{N}^3$, $\omega = (\omega', \omega_0)$, and

\[
\frac{\omega_+}{\omega} = \xi \in \mathbb{N}^3 \quad \xi = (\xi', \xi_0) .
\]

The superscript $B$ means that $\hat{\psi}^+ \neq \hat{\psi}^+$ appear as the positive and negative parts, respectively, of the fictitious (as violating the assumptions of the spin-statistics theorem) spinor field, which obeys the Bose-Einstein statistics: Fermi operators are replaced by the boson operators.

The generalization of (8.28) is obvious, and leads to the identity:

\[
\psi_+^+(\omega) \psi_+^-(q) \hat{F}_F \equiv S_{ij} \sum \psi^+_i(\omega) \psi^+_j(\omega) \hat{F}_F . 
\]

(8.29)

The sign $\sim$ in $\hat{\psi}_i^{+\dagger}$ reverses the order of the $k, \eta$ variables.

By virtue of (8.29), the following equivalence formula holds on the fermion Fock space:

\[
\Gamma_{\psi, \psi} : \mathbb{F}_F = \mathbb{F}_F : \Omega(\psi, \psi) ; \mathbb{F}_F = \mathbb{F}_F . 
\]

(8.30)

Here the shorthand notation $\Delta_{mn} = (\omega_{mn} \mathbb{F}_m \mathbb{F}_n)$ is used for the coefficient functions.
We have thus proved that with the fermion field algebra, one can associate a projection of the subsidiary boson field algebra, so that on $\mathcal{F}_F$, both algebras coincide.

The pure boson theory obviously has its own classical image realized either by the use of the coherent state expectation values or by the use of the appropriate functional representations of the CCR algebra:

$$\Omega(\psi, \bar{\phi}) = \Omega(\psi, \bar{\phi}); (\alpha, \bar{\alpha}) \exp(-\bar{\alpha} \cdot \alpha),$$

where the classical fields $\psi, \bar{\phi}$ appear as a result of the replacement of the boson operators $\alpha, \bar{\alpha}$ by the classical Fourier amplitudes $\hat{\alpha}, \bar{\alpha}$, respectively. For more details, see [176].

In consequence of this result, establishing the correspondence rule for fermion fields by the mediation of the subsidiary bosons, one can immediately formulate the quantization prescription for the classical spinors, which completes the derivation of the correspondence rule between spinors and fermions. Namely, if one starts from the set of functionals of free Dirac (c-numbers) fields, then using the functional representation of the CCR algebra,

$$\Omega(\psi, \bar{\phi}) \exp(\alpha, \bar{\alpha}) = \Omega(\psi, \bar{\phi}); (\alpha, \bar{\alpha}) \Rightarrow \Omega(\psi, \bar{\phi});$$

one gets the corresponding set of boson operators. If we build into the boson structure the Pauli exclusion principle through

$$\Omega(\psi, \bar{\phi}) ; 4_F = \Omega(\psi, \bar{\phi}) ; 4_F,$$

we have finished the job by observing that on the subspace $\mathcal{F}_F$ of the boson Fock space $\mathcal{F}_B$, we have the identity

$$\Omega(\psi, \bar{\phi}) ; 4_F = \Omega(\psi, \bar{\phi}) ; \mathcal{F}_F,$$

defining the fermion level of the theory. Note that operators $4_F: \Omega(\psi, \bar{\phi}) ; \mathcal{F}_F$ and $\Omega(\psi, \bar{\phi})$ possess exactly the same matrix elements between states from $\mathcal{F}_F$.

5. Plan Fermion in Quantum Field Theory: Lattice Quantization of the Sine-Gordon System in Two-Dimensional Space-Time

A growing interest in soliton solutions of classical non-linear equations, especially in connection with the Sine-Gordon equation [67, 88], and attempts to [74] what the corresponding quantum Sine-Gordon system is [94, 69, 90, 97], resulted in the rather involved and sophisticated correspondence between the so-called "quantum soliton" of the Sine-Gordon system and the fundamental fermion of the Thirring model. The usual tools at this point were either the canonical quantization procedure or WKB approximation, or perturbation methods in application to the non-linear equations. On the other hand, one a priori states that a correct quantum Sine-Gordon system is that with $\sin \Phi$ on the right-hand side of the equation, which is believed to be a quantum map of the $\sin \Phi$ appearing on the classical level. See in this connection [90, 90, 126], but also [92], where the massive Sine-Gordon quantum system $(\partial_t^2 + \partial_x^2) \Phi(x, t) = \pm \sin(\Phi(x, t))$ was axiomatically studied and, at the price of the imaginary time, its connection with the classical statistical mechanics was established.

However, to the author's knowledge, neither complete nor satisfactory quantization of the classical Sine-Gordon system in two space-time dimensions has so far been proposed. The notion of "quantum soliton" is mostly introduced ad hoc and with no physical justification. The only exception in this connection are the Faddeev papers [89], where a complete characterization of the classical phase space for the Sine-Gordon system was given, together with the semiclassical quantization prescription. We wish to perform here a somewhat naive and intuitive study of the quantization of Sine-Gordon solitons in the lattice approximation of the system. This is a model study, where the peculiarities of the quantum description can be investigated carefully, forming thus the first step in the lattice quantization of the Sine-Gordon equation.

As we know, the linearity of the free field equations allows us to introduce the annihilation and creation operators, by which one can count up the number of particles in a state described by the c-number solution of the same free field equation. One of the most difficult problems in non-linear field theories comes from the situation that it is not so easy to construct a clear-cut relationship between annihilation and creation operators, and the number of particles in the state described by the c-number soliton-type solution.
On the other hand, in the quantum world one can never control the behaviour of any system in terms of the continuous data functions. One should rather imagine a discrete set of the control points (the averaged experimental outcomes), which in the approximated sense can eventually be extrapolated to a continuous control curve.

The most instructive example taking this fact into account is the famous Toda lattice possessing the exactly known solutions, which in the continuous limit goes over to the so-called KdV equation known from hydrodynamics [13].

In the traditional derivation of the quantum field theory of a free scalar field, the Hamiltonian (two space–time dimensions are taken for simplicity)

\[ H = \int dx \left( \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 + \frac{1}{2} \phi^2 \right) \]

(9.1)

can be approximated on the finite linear lattice, by the (rescaled) Hamiltonian:

\[ H = \Lambda \sum_i \left( \frac{1}{2} p_i^2 + \frac{1}{2} \left( \nabla \chi_i \right)^2 + \frac{1}{2} \lambda^2 \chi_i^2 \right) \]

(9.2)

with \( \Lambda \) enumerating the lattice sites.

The omission of the gradient part reduces the problem to its single-site approximation by the linear chain of harmonic oscillators. In this approximation the quantization of the system lies in introducing the quantum oscillators in place of the classical ones. To restore the complete quantum system we must perform here a translation into the quantum language of the neighbour interaction (gradient) term. One can make it according to [123]:

\[ H = \Lambda \sum_i \left( \frac{1}{2} p_i^2 + \frac{1}{2} \left( \nabla \chi_i \right)^2 + \lambda \left( 1 - \cos \phi_i \right) \right) \]

(9.3)

A quite analogous procedure can be repeated in case of the Sine-Gordon system.

The corresponding Hamiltonian,

\[ H = \int dx \left( \frac{1}{2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \left( \frac{\partial \phi}{\partial x} \right)^2 + \lambda \left( 1 - \cos \phi \right) \right) \]

(9.4)

is approximated on the linear lattice by

\[ H = \Lambda \sum_i \left( \frac{1}{2} p_i^2 + \left( \frac{\partial \phi}{\partial x} \right)^2 + \lambda \left( 1 - \cos \phi_i \right) \right) \]

(9.5)

where again the gradient term is in fact the interaction part of the Hamiltonian and carries the nearest neighbour coupling. Its omission leaves us with the linear chain of the independent plane pendula, which was the root for the construction of the Scott mechanical analogue, transmission line[87,88] for the Sine-Gordon pulse. The solitons can be observed experimentally in the chain, though a complete integrability of the lattice Sine-Gordon system was never proved, in contrast to the Toda lattice.

We do not pretend here to get the exact solutions of the discrete problem. We wish rather to find a quantum image of 1-soliton solutions in the single-site approximation, together with the quantum term of the Hamiltonian which takes account of the classically observed long-range correlations between the nearest neighbours in the chain.

The quantization of the Sine-Gordon system in the single-site approximation obviously in the introduction of quantum pendula in place of classical pendula: each one singly occupying a single lattice site.

The quantum pendulum is a solvable problem[91], with a non-degenerate positive set of eigenvalues and Mathieu functions as eigenfunctions. Mathieu functions can be proved to constitute a complete orthonormal set in \( L^2(C) \) which is thus a Hilbert space of pendulany states. The quantum mechanical Hamiltonian though rather not admitting any reasonable number of particles representation, can always be considered in the matrix form:

\[ H = \sum_i E_i f_i \otimes \bar{f}_i \]

(9.6)

where \( E_i \) are energy values while \( f_i \) denotes the \( i \)-th eigenfunction.

We need rather the knowledge of the energy spectrum than of the particular operator (creation and annihilation operators here can easily be constructed in the tensor product form) structure.

Following the preliminary formulation [35], we shall now perform the lattice quantization of the Sine-Gordon 1-solitons. They are the solutions of the equation

\[ \square \phi(x,t) = \left( \frac{\partial^2}{\partial x^2} - \frac{1}{4} \frac{\partial}{\partial x} \right) \phi(x,t) = m^2 \cosh \phi \Phi(x,t) \]

(9.7)

which are of the form:

\[ \Phi(x,t) = 4 \tan^2 \exp \left( \pm m \frac{x - vt}{\sqrt{1 - v^2}} \right) \]

(9.8)

The energy \( E = \frac{8m^2}{1 - v^2} \) and momentum of the soliton can easily be calculated:

\[ p = \frac{8m^2}{1 - v^2} v \]

The approximation of the 1-soliton pulse on the linear lattice is given immediately, if with each site (the spacing is \( \Lambda \)) we associate
a corresponding characteristic function \( \Delta \equiv \chi(\nu) \equiv \{ \nu, x, \Delta \} \), so that:
\[
\Delta \equiv \chi(\nu) = \{ \nu, x, \Delta \} = \{ \nu, x, \Delta \}
\]

so

\[
\phi_2 = \frac{1}{\Delta} \sum \phi(x) \cdot \chi'(x)
\]

\[
\chi'_2 = \frac{1}{\Delta} \sum \phi(x) \cdot \chi'(x)
\]

\[
\gamma(x) = \frac{1}{\Delta} \sum \phi(x) \cdot \chi'(x)
\]

\[
\chi'_2 = \frac{1}{\Delta} \sum \phi(x) \cdot \chi'(x)
\]

\[
\chi'_2 = \frac{1}{\Delta} \sum \phi(x) \cdot \chi'(x)
\]

(9.9)

Having the energy density of the 1-soliton \( \chi(x, t) = \chi(x) \) one can establish the position of the energy centre of the pulse at the initial instant of time \( t = 0 \). Let assume that this particular point \( y \), the collective variable (see, e.g., Christ’s and Gervais’ articles in [126]), belongs to the 0-th site which is identified with the \( s = 0 \) interval: \( y = \Delta \).

In consequence, the 1-soliton is completely described by the following collection of the initial data:

\[
\phi_2 = \frac{1}{\Delta} \sum \phi(x) \cdot \chi'(x)
\]

\[
\chi'_2 = \frac{1}{\Delta} \sum \phi(x) \cdot \chi'(x)
\]

\[
\gamma(x) = \frac{1}{\Delta} \sum \phi(x) \cdot \chi'(x)
\]

\[
\chi'_2 = \frac{1}{\Delta} \sum \phi(x) \cdot \chi'(x)
\]

\[
\chi'_2 = \frac{1}{\Delta} \sum \phi(x) \cdot \chi'(x)
\]

(9.10)

Here \( \phi_2(y) = \phi(y-s\omega) \), \( \pi_2(y) = \pi(y) \), and all \( \phi \)'s and \( \pi \)'s respectively with \( y \rightarrow \infty \) from now on, shall simplify considerations by \( \phi(y) \) in all the formulas, \( \phi(y) = \phi(y) = \pi(y) \), \( E \equiv \pi_2 \).

The uniform motion rule \( \phi(x+vt) = \phi(x,t) \) which holds on the continuous level is now approximated by the following motion rule of the set of the initial data:

\[
\phi_2(t) = \phi_2(y-vt+sa) \rightarrow \phi_2(t) = \phi_2(y-vt+sa)
\]

\[
\pi_2(t) = \pi_2(y-vt+sa)
\]

(9.11)

which simply shifts the data along the chain, following from the influence of the neighbour coupling, implied by the gradient term. We have thus separated, on the classical level, the non-linear geometry (shape) of the solution from the fully linear dynamics, Let us add that a similar procedure can be repeated also in case of the n-soliton solutions where the number \( n \) of collective variables is necessary, for more details see [126].

Let the quantum chain be given now, where in the single-site approximation a sequence of independent quantum pendulums appears, together with a corresponding single-site basis. We shall try to translate the classical data and motion rule to the quantum lattice.

Let us begin from the question of statistics. Because each site of the lattice is occupied by a single quantum pendulum whose spectrum is positive and non-degenerate, if we pretend to describe the line of quantum pendulums, the Pauli exclusion principle should govern its behaviour: the occupation number of each \((s,n)\)-th state of the lattice is either 0 or 1, so we mean that the \( n \)-th energy level of the quantum pendulum is occupied at the \( n \)-th site.

In consequence, the single component Fermions should appear on the quantum level. Because, as we know from the previous section, it is not immediate to have a reasonable correspondence between the classical and Fermion level, we shall formulate all the results for the subsidiary mediating bosons and then, in the sense of the weak excitation limit, the transition to the final Fermion variables will be performed.

Let us denote by \( E_0, E_1 \) the energies of the two lowest stationary levels of the quantum pendulum. We assume to have mapped each plane pendulum, whose energy \( \chi_0 \) does not exceed \( E_0 - E_1 \), into a non-excited - hence occupying the ground state \( \phi_0 \) - quantum pendulum.

This prescription is motivated by the naive hope that such energies cannot be quantized and in the theory play the role of an inessential noise. Now, the question of interest here becomes an energy sharing in-between the quantum pendula of the net energy \( E \) of 1-soliton, which we consider as net in the sense of the renormalization by a subtraction of the ground state energy from the total energy at each site of the lattice. We expect \( E \) to be approximated by the sum of quantized portions \( E/(s,n) \)

\[
\sup_{(s,n)} \sum_{E(s,n)} \{ E(s,n) - E_0 \} \cong E
\]

(9.12)

Because \( E \) is a macroscopic value, the equality in fact holds.

The 1-soliton pulse has a finite energy value, which, if combined with the requirement (9.12), clearly requires at most a finite number of quantum pendulums to be simultaneously excited. Note that the free field techniques, especially the Fock space methods, can be used by virtue of this argument: Fock space vectors are the linear combinations of states which in a finite number of entries (the product states) differ from the vacuum state.
With each single lattice site let us now associate the subsidiary boson field \( \Phi_s \), whose lattice Fourier expansion,

\[
\phi_s = \frac{1}{\sqrt{V}} \sum_k \left( \phi_{k}^+ \exp(\frac{ik}{V} s) + \phi_{k}^- \exp(-\frac{ik}{V} s) \right)
\]

\( \phi_{k}^+ + \phi_{k}^- \),

allows to introduce the corresponding creation and annihilation operators:

\[
[\phi_{k}^+, \phi_{k}^-] = \delta_{kk'}, \quad [\phi_{k}^+, \phi_{k'}^-] = 0, \quad \phi_{k}^+ \phi_{k'}^- = 0 \quad \text{for all } k.
\]  

\( (9.13) \)

In the above, the normalization constant \( V = \text{dim}(s,n) \) for the set of pairs realizing the supremum in \((9.12)\), and \( k \) enumerates the finite set of degrees of freedom (energy levels of pendula reproducing the 1-soliton pulse). The quantum numbers \( k \) are defined by the initial 1-soliton data, if we define the appropriate correspondence rule by using the coherent state methods:

\[
\mathbb{C}^\Omega \left( \gamma \right) = \exp \left\{ \sum_k \left( \phi_{k}^+ \Omega_{k} \cdot \exp(-\frac{i}{V} k \gamma) \right) \right\},
\]

\[
\phi_{k}^+ = \frac{1}{\sqrt{V}} \sum_k \phi_{k}^+ \exp(\frac{ik}{V} s) + \phi_{k}^- \exp(-\frac{ik}{V} s).
\]  

\( (9.15) \)

Let us note that putting \( f_s = \phi_s \), we get

\[
(\phi_s | \phi_s - \phi_s^* \phi_s) = \phi_s.
\]  

\( (9.16) \)

A total energy operator for the 1-soliton then reads:

\[
H = \sum_k \alpha_k \phi_k^- \phi_k^+ \phi_k \exp(-\frac{i}{V} k \gamma) \quad \text{for } k = 1, 2, \ldots
\]

\( (9.18) \)

The solitary evolution rule on the classical level implies:

\[
q_k(t) = \phi_k^+ \exp(-\frac{i}{V} k \gamma) \quad \text{for } k = 1, 2, \ldots
\]

\( (9.19) \)

In consequence, for \( t = \frac{n \alpha}{V} \),

\[
H(t) = \sum_k \alpha_k \phi_k^+ \phi_k \exp(-\frac{i}{V} k \gamma) \quad \text{implies}
\]

\[
(\phi_s | H(t) \phi_s) = \sum_k \alpha_k \phi_k^+ \phi_k \exp(-\frac{i}{V} k \gamma).
\]  

\( (9.20) \)

\( (9.21) \)

On the other hand,

\[
H = \sum K_k H_k
\]

\( (9.22) \)

and \( (\phi_s | H_k \phi_s) = (\phi_s | H_k (t = \frac{n \alpha}{V} \phi_s) \).

If we now require simultaneously

\[
\mathbb{R}_s = \frac{\alpha}{V} \sum_k \mathbb{R}_k \cos \frac{k \alpha}{V}
\]

\( (9.23) \)

then the correspondence rule,

\[
\mathbb{R}_s = (\phi_s | H_s \phi_s),
\]

establishes the following connection between the classical and quantum energy data:

\[
E_k - E_o = \frac{\mathbb{R}_k}{\mathbb{R}_s}
\]  

\( (9.24) \)
To get a quantum image of the 1-soliton evolution, it is useful to know that, if the quantum gradient term is taken in the form [128]:

\[
D_{m}^{2} = D^{2}(n,m) = \frac{1}{2\pi a_{F}^{2}} \sum_{k \in k_{F}} \exp[i(k(m-n))],
\]

(9.25)

then an immediate quantum lattice analogue of the space translation operator can be given,

\[
P = -i \sum_{n} \Pi_{n} \Phi_{m} = \frac{1}{\sqrt{N}} \sum_{k \in k_{F}} \left( \alpha_{n_x} \exp(i k x_{n}) - \alpha_{o} \exp(-i k y_{n}) \right) \hat{b}_{m},
\]

(9.26)

so that

\[
\exp(i P_{n}) \cdot \Phi_{m} \exp(-i P_{n}) = \Phi_{m}.
\]

(9.27)

Obviously, in the sense of the correspondence principle, (9.27) is the quantum image of the 1-soliton evolution rule, which, at first sight, seems to contradict the ordinary expectations that an energy operator \( \mathcal{H} \) should play this role.

Let us at this point prove that the energy operator of the "quantum soliton" just constructed cannot be the correct generator of the solitary time translations. For this purpose it is enough to notice that \( \mathcal{H} \) obeys the restrictions of the Borchers theorem [32]:

\[
\text{Given a one-parameter group } U \text{ defined by } U = \exp(-i \mathcal{H} t) \text{ where } \mathcal{H} \rightarrow -i \mathcal{H} \rightarrow \mathcal{H} \rightarrow U \text{.}
\]

(9.28)

Denote \( \mathcal{F} = U F U^{-1} \) for any operator \( F \). If there is a pair of projectors \( \Pi, \mathcal{F} \) such that for \( |t| < \epsilon \), \( \mathcal{F} = 0 \), then for any \( t \in \mathbb{R}^{+} \), \( \mathcal{F} = 0 \).

Let us remark that the solitary evolution rule:

\[
|t| \quad \text{and} \quad |t| < \epsilon \quad \Rightarrow \quad \Phi_{m}(t) \rightarrow \Phi_{m}(-t) \rightarrow \Phi_{m}.
\]

(9.29)

which realizes the translation of our construction of the "quantum soliton" to the fermion language.

The above considerations suggest that together with the collective shift operator \( \mathcal{P} \), one should introduce a collective velocity operator \( \mathcal{Q} \), which in case of 1-solitons is not proportional to \( \mathcal{P} \), as in that case \( \mathcal{P} \mathcal{Q} \) would be \( \lambda \mathcal{F}^{2} \), and hence positive.

The last step in our considerations is now to make a transition to fermion variables, which should appear by virtue of the built-in Pauli exclusion principle. The most convenient, though obviously not unique, tool here seems to employ the weak excitation limit concept of the previous sections, and then the map,

\[
\Delta_{s} \rightarrow \Delta_{s}^{-} = \Delta_{s} - \Delta_{s}^{-}.
\]

(9.30)

if the positive evolution operator is used. In this connection let us notice that the correct evolution operator - \( \mathcal{P} \mathcal{Q} \) for the quantum image of our 1-soliton is manifestly not positive.

The above arguments justify, in a slightly sophisticated way, the independence of the single sites of the lattice for all times, like that appearing if the gradient term is absent in the Hamiltonian. Does it at all exclude the long range correlations for any class of positive Hamiltonians?

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\[
\mathcal{H}_{F} \mathcal{Q} \mathcal{H}_{F} = \mathcal{H}_{F} \mathcal{Q} \rightarrow \mathcal{H}_{F} \mathcal{Q} = \mathcal{H}_{F} \mathcal{Q},
\]

(9.31)

which realizes the translation of our construction of the "quantum soliton" to the fermion language.

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Here we shall concentrate on a particular domain of applications of the path integral methods in quantum theory, namely on the theory of the so-called functional power series and the theory of functional representations of the canonical algebras arising in it: CCR and CAR respectively. The review of topics is based on the papers [74-76, 27-29], see also [72, 73]. Hilbert spaces of functional power series were introduced into physics by V.Fock, as early as 1934, and have been investigated by many authors since then. Quite a large amount of mathematical literature has also been published on this subject [72].

We are especially interested in their application in connection with Fock space methods, with special account of fermions and bosons. Let us denote by \( \mathcal{K} \) a Hilbert space defined by the scalar product \( \langle \alpha, \beta \rangle \), and let us assume that there exists an involution \( \alpha \rightarrow \bar{\alpha} \) in \( \mathcal{K} \), satisfying \( \bar{\alpha} = \alpha \) and \( \langle \alpha, \bar{\beta} \rangle = \langle \alpha, \beta \rangle \).

We can define in this case a bilinear form in \( \mathcal{K} \) as \( \langle \alpha, \beta \rangle = \langle \alpha, \bar{\beta} \rangle \). The scalar product, bilinear form and involution in \( \mathcal{K} \) induce the corresponding notations in \( \mathcal{K}_n = \mathcal{K}^n \):

\[
\langle f_n, g_n \rangle \equiv \langle f_n, \bar{g}_n \rangle, \quad f_n \rightarrow \bar{f}_n,
\]

\[
(\bar{f}_n, \langle f_n, g_n \rangle = \langle \bar{f}_n, \bar{g}_n \rangle, \quad f_n, g_n \in \mathcal{K}_n).
\] (A.1)

Consider now the space \( \mathcal{F} \) of sequences \( f_n \in \mathcal{K}_n, \ n \in \mathbb{N}, \ n = 0, 1, \ldots \)

\( \mathcal{K}_0 = \mathbb{C} \), satisfying the condition

\[
\| f \| = \left( \sum_{n=0}^{\infty} | f_n |^2 \right)^{1/2} < \infty,
\] (A.2)

where

\[
\mathcal{F} = \{ f = \{ f_n \}_{n=0}^{\infty}, \quad f_n \in \mathcal{K}_n, \quad \| f \| < \infty \}
\] (A.3)

An arbitrary element \( f \) of \( \mathcal{K}^\infty \) can be represented as a sum of
terms which are invariant with respect to the irreducible representations of the symmetry group, acting in the n-th tensor product of Hilbert spaces. With the help of Young's idempotents $Y_n$, we can write:

$$f_n = \sum_Y Y_n f_n,$$

(A.4)

where from the various $Y_n$ of particular interest for us will be the two:

$$S_n = \frac{1}{n!} \sum P_n, \quad A_n = \frac{1}{n!} \sum (-1)^P P_n,$$

(A.5)

being the symmetrizing and antisymmetrizing operators. The sums are extended over all permutations $P_n$ of $n$ elements. Young's operators have their duals $Y^d_n$, among themselves, e.g. $S_n = A_n^d$, $A_n = S_n^d$.

Let us now introduce the inversion operators $P_{ik}$ in $\mathcal{H}_n$, interchanging the indices of the $k$-th and $i$-th element of the tensor product $\otimes^m$.

Further, let $E_n$ be a bounded operator in $\mathcal{H}_n$ satisfying the relations

$$E_n^2 = E_n, \quad E_n^3 = E_n,$$

$$P_{ik} E_n = -E_n P_{ik}.$$  

(A.6)

It follows from (A.6) that $E_n^2$ is a projector, which realizes the following decomposition of $\mathcal{H}_n$:

$$\mathcal{H}_n = \mathcal{H}_n^+ \oplus \mathcal{H}_n^-, \quad \mathcal{H}_n^+ = E_n^+ \mathcal{H}_n, \quad \mathcal{H}_n^- = (I - E_n) \mathcal{H}_n.$$  

(A.7)

If we take into account Young's decompositions,

$$\mathcal{H}_n = \sum_Y Y_n \mathcal{H}_n \oplus \sum Y_n \mathcal{H}_n^-,$$

(A.8)

then by the basic property of $E_n$,

$$Y_n E_n = Y_n E_n^+, \quad Y_n E_n^- = Y_n (I - E_n) \mathcal{H}_n^-,$$

$$E_n^+ Y_n = Y_n E_n^+,$$

(A.9)

$E_n$ is an automorphism of $\mathcal{H}_n$ consisting of the isomorphisms $Y_n^d \mapsto Y_n^d$, which implies:

$$A_n \mathcal{H}_n \mapsto S_n \mathcal{H}_n,$$

$$E_n^1 A_n \mathcal{H}_n = S_n \mathcal{H}_n, \quad E_n^2 A_n \mathcal{H}_n = A_n \mathcal{H}_n.$$

(A.10)

We are basically interested in the $E_n$'s possessing the additional property

$$E_n^2 A_n \mathcal{H}_n = A_n \mathcal{H}_n \Rightarrow A_n^2 \mathcal{H}_n = 0,$$

(A.11)

which allows us to consider the map

$$A_n \mathcal{H}_n \mapsto S_n \mathcal{H}_n, \quad E_n^2 A_n \mathcal{H}_n = S_n \mathcal{H}_n.$$

(A.12)

The simplest examples of $E_n$ are found in [27]. The operator

$$E_n = \sum \epsilon(i) \cdots \epsilon(k) E_{1i} \cdots E_{nk} \cdots E_{kn},$$

(A.13)

is defined by the eigenfunctions $\{ \epsilon(i_1) \cdots \epsilon(k) \}$ and the eigenvalues $\epsilon(i_1) \cdots \epsilon(k)$. $E_n$ is an example if $K$ is separable, and $\{ \epsilon(i_1) \cdots \epsilon(k) \}$ is an orthonormal set in $\mathcal{H}_n$ corresponding to the orthonormal set $\{ \epsilon(i) \}$ in $K$. In addition, we need $E_{1i} \cdots E_{kn}$ to be the totally antisymmetric Levi-Civita tensor with $\epsilon_{i_1 \cdots i_n} = 1$ for $i_1 \neq i_2 \neq \cdots \neq i_n$ and $E_{i_1 \cdots i_n} = 0$ otherwise.

Another example is the operator with the integral kernel:

$$E_n(x, y) = \delta(x - y) \cdots \delta(x - y_n),$$

(A.14)

which in the case of a number greater than one of internal degrees of freedom in theory, goes over to:

$$E_n^1(x, y) = \delta(x - y) \cdots \delta(x - y_n) \cdots \delta(x_n - y_n),$$

(A.15)
In the above $\mathcal{C}(x_1, \ldots, x_n) = \mathcal{C}(x^n)$ is the previously introduced Friedrichs--Klauder sign function.

The study of the symmetry structure of the $n$th tensor product is immediately generalized onto $\mathcal{F}$, so that in the symmetric and antisymmetric cases we have:

\[ \mathcal{F}^s = \bigoplus_{n=0}^{\infty} \mathcal{S}_n^s = \bigoplus_{n=0}^{\infty} \left\{ \sum_{\pi} \sigma_{\pi} S^\pi \mathcal{C} \right\} = \mathcal{F}^s \oplus \mathcal{F}^s, \tag{A.16} \]

\[ \mathcal{F}^a = \bigoplus_{n=0}^{\infty} \mathcal{A}_n^a = \bigoplus_{n=0}^{\infty} \left\{ \sum_{\pi} \sigma_{\pi} A^\pi \mathcal{C} \right\} = \mathcal{F}^a \oplus \mathcal{F}^a. \]

Let us consider now the corresponding Hilbert spaces of functional power series (generating functionals) which are defined as mappings $V: \mathcal{X} \rightarrow \mathcal{Y}(\mathbb{C})$ in the following way:

\[ \mathbb{B}^s(\mathcal{X}) = \mathbb{B}^s = \{ V: \mathcal{F}^s(\mathcal{X}) = \sum_{n=0}^{\infty} \left\langle \sum_{\pi} \sigma_{\pi} S^\pi \mathcal{C}, \mathcal{Y} \right\rangle \mid \| \mathcal{V} \| < \infty \}, \tag{A.17} \]

\[ \mathbb{B}^a(\mathcal{X}) = \mathbb{B}^a = \{ V: \mathcal{F}^a(\mathcal{X}) = \sum_{n=0}^{\infty} \left\langle \sum_{\pi} \sigma_{\pi} A^\pi \mathcal{C}, \mathcal{Y} \right\rangle \mid \| \mathcal{V} \| < \infty \}. \]

The first definition is the conventional one in the symmetric case, the second definition is the generalization to the antisymmetric case.

In this last connection see also [73].

Given a series $V(\mathcal{X}) = \sum_{n=0}^{\infty} \left\langle \mathcal{X}^n, \mathcal{Y} \right\rangle$, we construct the series

\[ (q_m, V(\mathcal{X}))(\mathcal{Z}) = \sum_{n=0}^{\infty} \left\langle \frac{(m+n)!}{n!} \mathcal{X}^m, \mathcal{Y} \right\rangle, \tag{A.18} \]

obtained from $V(\mathcal{X})$ by differentiating each term in this series $m$ times according to the formula

\[ (q_m, \mathcal{Z})V(\mathcal{X}) = \lim_{\varepsilon \to 0} (q_m, \varepsilon^i \{ V(\mathcal{X} + \varepsilon q) - V(\mathcal{X}) \}) \tag{A.19} \]

where $\mathcal{Z}, q \in \mathcal{K}$, $\varepsilon \in \mathbb{R}$.

One can prove [75] that (A.18) is equal to the $m$-th derivative of $V(\mathcal{X})$, so that we can interchange summation and differentiation:

\[ (q_m, \frac{d^m}{d\mathcal{X}^m})V(\mathcal{X}) = (q_m, V^{(m)})(\mathcal{Z}) \tag{A.20} \]

We have here the isomorphism of $\mathbb{B}^s$ and $\mathbb{B}^a$:

\[ \mathbb{B}^s = \{ V^s: \mathcal{F}^s(\mathcal{X}) = \sum_{n=0}^{\infty} \left\langle \sum_{\pi} \sigma_{\pi} S^\pi \mathcal{C}, \mathcal{Y} \right\rangle \mid \| \mathcal{V} \| < \infty \}, \tag{A.21} \]

Thus $\mathbb{B}^s \subset \mathbb{B}^s$ and we can specialize the results proved in the symmetric case to the antisymmetric case by putting everywhere $\mathcal{F}^s = E_n \mathcal{F}^a$.

The operator $E_n$ can be omitted in all scalar products due to the property (A.10), which makes $\langle \mathcal{X}_m, q_n^a \rangle$ invariant with respect to $E_n$ if at least one of the elements in $\langle \mathcal{X}_m, q_n^a \rangle$ belongs to $\mathcal{C}_m$. Indeed,

\[ \langle \mathcal{X}_m, q_n^a \rangle = E_n \langle \mathcal{X}_m, q_n^a \rangle, \quad q_m^a = E_n q_n^a, \quad q_n^a \in \mathcal{C}_m \Rightarrow \langle q_m^a, q_n^a \rangle = \langle q_m^a, E_n q_n^a \rangle = \langle q_m^a, q_n^a \rangle. \]

In particular:

\[ \langle E_m^a \mathcal{X}_m^a, E_n \mathcal{X}_n^a \rangle = \langle E_m^a \mathcal{X}_m^a, q_n^a \rangle, \tag{A.22} \]

\[ \langle E_m^a \mathcal{X}_m^a, E_n^a \mathcal{X}_n^a \rangle = \langle E_m^a \mathcal{X}_m^a, q_n^a \rangle. \]

This shows that the theory of Hilbert spaces of functional power series, which is originally developed for pure symmetric coefficient functions $U$, works also in the antisymmetric case, where all scalar products containing symmetric functions can be replaced by the corresponding scalar products of antisymmetric functions.

The scalar product in the Hilbert space of functional power series can be written formally in several alternative ways, which explicitly exploit path integration methods:
\[ \langle V, \psi \rangle = \sum_{n} \langle \psi_n, \xi_n \rangle = V \left( \frac{d}{dx} \right) V'(\xi) \bigg|_{x=0} = \]
\[ = \sum_{n} \int V(\xi) \psi_n(\xi) \, d\left( \frac{\xi}{12} \right), \]
where
\[ K = \text{the Hilbert space defined at the beginning of this Appendix, and } K_0 = \text{the corresponding real Hilbert space} \]
\[ \langle x, x' \rangle = \frac{1}{\sqrt{2}} (x + i x'), \quad \xi \in K_0, \]
\[ \{ e_i \} = \text{an orthonormal set in } K_0 : \langle e_i, e_j \rangle = \delta_{ij}, \quad \xi \in K_0. \]
In the last expression in (A.23),
\[ \mathcal{Q}(\xi) = \sum_{n} \mathcal{A}(\xi, \psi_n) \exp(-\|\xi\|^2) \, d\left( \frac{\xi}{12} \right), \]
where \( \psi_n \) are the coefficients of the expansion for \( V(\xi) \) (in the same way \( \psi(\xi) \) corresponds to \( G(\xi) \)), \( J(\psi_n, \xi^n) \) are the Hermite functionals defined by
\[ J(\psi_n, \xi^n) = \frac{1}{\sqrt{4!^n}} \exp\left(-\frac{1}{4} \|\xi\|^4\right) \exp\left(-\frac{1}{4} \left( \frac{d^2}{dx^2} + \frac{d}{dx} \right) \right) \left( \psi_n, \xi^n \right), \]
and \( (\frac{d^2}{dx^2} + \frac{d}{dx}) \) is the unit operator in \( K_0 \).
The kernel \( A(\psi_n, \xi^n) \) of the integral in (A.25) is the generalisation of the Bargmann operator [72] to countably dimensional spaces.

The Hilbert space of the functional power series can be considered as a carrier space of the algebra of operators in it, defined by the double functional power series:
\[ A(\xi, \alpha) = \sum_{n,m} \frac{d^n}{dx^n} \left( a_{mn}, \overline{\alpha} \alpha^n \right), \]
whose action as operators from \( K(\xi) \) into \( K(\xi) \) is given in accordance with
\[ (AV)(\xi) = \sum_{n} \frac{d^n}{dx^n} \left( \sum_{m} a_m \overline{\alpha} \alpha^m \right) \]
\[ = A(\xi, \frac{d}{dx}) V(\xi) \bigg|_{x=0} = \sum_{n} \mathcal{A}(\xi, \psi_n) \exp(-\|\xi\|^2) \, d\left( \frac{\xi}{12} \right). \]
To establish a connection of the formalism developed above with the Fock space, let us first notice that (A.29) induces the following multiplication law for operators:
\[ (A \cdot B)(\xi, \alpha) = \sum_{n,m} \frac{d^n}{dx^n} \left( \sum_{m} a_{mn} \overline{\alpha} \alpha^n \right) \]
\[ = A(\xi, \frac{d}{dx}) B(\xi, \alpha) \bigg|_{x=0} = \sum_{n} \mathcal{A}(\xi, \psi_n) \exp(-\|\xi\|^2) \, d\left( \frac{\xi}{12} \right), \]
so that introducing the double series:
\[ A(\xi, \alpha) = \sum_{n,m} \frac{d^n}{dx^n} \left( a_{mn}, \overline{\alpha} \alpha^n \right) \]
connected with (A.28) by the relation
\[ A(\xi, \alpha) = A(\xi, \psi) \exp(\xi, \alpha), \]
we can express the multiplication law (A.30) in terms of \( A(\xi, \alpha) \),
\[ (A \cdot B)(\xi, \alpha) = \exp(\xi, \alpha) \cdot A(\xi, \alpha) \cdot B(\xi, \alpha) \]
\[ = \mathcal{A}(\xi, \psi_n) \cdot \mathcal{A}(\xi, \psi_n) \cdot \exp(\xi, \alpha), \]
where:
\[
A_\alpha(2, \alpha)(\omega) = \Lambda_\alpha(\omega, \alpha + \frac{1}{2} \omega) \Lambda_\alpha(\omega, \alpha) = \Lambda_\alpha(\omega, \alpha) \exp\left(\frac{\omega}{2} \alpha \right) \Lambda_\alpha(\omega, \alpha),
\]
and \( :A(\omega, \alpha) : \) means that in the series (A.31) \( \alpha \) is replaced by \( \frac{d}{d \omega} \) in such a way that \( \frac{d}{d \omega} \) always stands to the right of \( \omega \)
(similarly in \( :A(\omega, \alpha + \frac{1}{2} \omega) : \)).

For \( A(\omega, \alpha) = 1 \), \( A_{00} = 1 \), \( A_{\omega} = 0 \), \( A_{\omega} = 0 = \cdots = 0 \)
we obtain the unit operator \( A_0 \) with the kernel
\[
A_0(\omega, \alpha) = \exp(\omega, \alpha) = \sum_{\alpha = 0}^{\infty} \frac{1}{\omega!} (\omega, \alpha),
\]
\[
A_0' = (\omega, \alpha).
\]
Note that the coefficients in the expression (A.31) for the unit operator are of the form \( \omega \omega \lambda \omega \), \( \omega \omega \lambda \omega \), where \( A_0 = (\omega, \alpha) \).

The next simple operators after unity are the operators represented by double series (A.32) where \( \Lambda(\omega, \alpha) \) is a first-order polynomial.

There are two such independent operators: the annihilation operator \( A(\omega, \alpha) \) and the creation operator \( A(\omega, \alpha) \) given by the kernels
\[
A(\omega, \alpha) = \exp(\omega, \alpha) \cdot (\omega, \alpha),
\]
\[
A(\omega, \alpha) = \exp(\omega, \alpha) \cdot (\omega, \alpha),
\]
of certain operator-valued elements of \( K \), which we shall denote by \( A(\omega, \alpha) \) and \( A(\omega, \alpha) \).

One easily derives the commutation relations
\[
[A(\omega, \alpha), A(\omega, \alpha)] = (\omega, \alpha) \exp(\omega, \alpha) = [A(\omega, \alpha), A(\omega, \alpha)],
\]
\[
[A(\omega, \alpha), A(\omega, \alpha)] = (\omega, \alpha) = 0,
\]
and
\[
[A(\omega, \alpha), A(\omega, \alpha)] = (\omega, \alpha) = 0.
\]

Furthermore,
\[
:A(\omega, \alpha) : (\omega, \alpha) = \exp(\omega, \alpha) \cdot (\omega, \alpha) = A(\omega, \alpha),\]
which enables to express every operator with a kernel \( A(\omega, \alpha) \) of the form (A.31), (A.32) in terms of creation and annihilation operators.

In (A.39), \( :A(\omega, \alpha) : \) is defined by the equation
\[
:A(\omega, \alpha) : = \sum_{\alpha = 0}^{\infty} \frac{1}{\omega!} (\omega, \alpha),
\]
\[
A(\omega, \alpha).
\]

Similarly, one can express every element of \( B(K) \) in terms of creation operators,
\[
V(\omega, \alpha) = \exp(\omega, \alpha) \cdot (\omega, \alpha) = V(\omega).
\]

In particular, if \( \omega = \frac{\alpha}{\omega} \cdot f^2 \), \( f \in K \), we obtain the set of so-called principal vectors or coherent states:
\[
\exp(f, \omega) = \exp(f, \omega) \cdot (\omega, 0).
\]

Evidently \( \omega(\alpha) = 0, \omega \in C \) and \( f_0 \) (if normalized) is to be considered the vacuum state,
\[
V(\omega) = (V(\omega), \omega) = \exp(\omega, \omega) \cdot (\omega, 0),
\]
which shows that each element of \( B(K) \) is obtained from the vacuum by repeated application of creation operators. Together with relations (A.39), they establish the connection between the Hilbert space \( B(K) \).
of functional power series and the Fock space $\mathcal{F}_B$. 

The triple $\{A$, $A^{-1}$, $B\} \in \mathcal{F}_B$ we call a functional representation of the CCR (canonical commutation relations) algebra: $\mathcal{B}(k) = \mathcal{F}_B$.

If we now define more complex operators

$$b(x, \omega^*_{\alpha}) = \sum_{n=0}^{\infty} (\omega E_{n+1}, \omega E_{n+1} \omega^*_{\alpha}),$$

$$b(y, \omega_{\alpha}) = \sum_{n=0}^{\infty} (\omega E_{n+1}, \omega E_{n+1} \omega^*_{\alpha}),$$

one gets the following formulas, see e.g. [25, 26]:

$$\langle \omega_{\alpha}, b^* \rangle (\omega, \omega^*_{\alpha}) = \sum_{n=1}^{\infty} (\omega E_{n+1}, \omega E_{n+1} \omega^*_{\alpha}),$$

$$\langle \omega^*_{\alpha}, b^* \rangle (\omega, \omega_{\alpha}) = \sum_{n=0}^{\infty} (\omega E_{n+1}, \omega E_{n+1} \omega^*_{\alpha}),$$

where $W_{jj}$ denotes the element $W_{jj}$ with inverse order of indices.

The calculation of $b^* b$ gives

$$b(b(x, \omega^*_{\alpha})) = \sum_{n=0}^{\infty} (\omega E_{n+1}, \omega E_{n+1} \omega^*_{\alpha}) = \sum_{n=0}^{\infty} (\omega E_{n+1}, \omega E_{n+1} \omega^*_{\alpha}) = \sum_{n=0}^{\infty} (\omega E_{n+1}, \omega E_{n+1} \omega^*_{\alpha}) = \sum_{n=0}^{\infty} (\omega E_{n+1}, \omega E_{n+1} \omega^*_{\alpha}).$$

so that the canonical anticommutation relations for $b$ and $b^*$ follow at once:

$$\{b(x, \omega^*_{\alpha}), b(y, \omega_{\alpha})\} = \{b(x, \omega^*_{\alpha}), b(y, \omega_{\alpha})\} = 0,$$

$$\{b(x, \omega^*_{\alpha}), b(y, \omega_{\alpha})\} = 0,$$

$$\{b(x, \omega^*_{\alpha}), b(y, \omega_{\alpha})\} = 0,$$

$$\{b(x, \omega^*_{\alpha}), b(y, \omega_{\alpha})\} = 0.$$
The projection theorem:

\[ A_F : \hat{F}(a^*_a, a) \cdot A_F \mathcal{F}_F = : F(b^*_b, b) \cdot \mathcal{F}_F , \]  \hspace{1cm} (A.51)

then needs the proof of the following identity:

\[ \left( : F(b^*_b, b) \cdot \mathcal{F}_F (a^*_a, a) \right)(a) = \Sigma_{\lambda=0}^{\infty} \frac{1}{\lambda!} \mathcal{F}_{\lambda} \hat{F}(\lambda, \lambda) \left( \hat{a}^{(\lambda + 1)} \cdot \hat{a}^{(\lambda + 1)} \right) \mathcal{F}_{\lambda} \mathcal{F}_F \]

\[ \left( A_F : \hat{F}(a^*_a, a) \cdot \mathcal{F}_F (a^*_a, a) \right)(a) = \Sigma_{\lambda=0}^{\infty} \frac{1}{\lambda!} \mathcal{F}_{\lambda} \hat{F}(\lambda, \lambda) \left( \hat{a}^{(\lambda + 1)} \cdot \hat{a}^{(\lambda + 1)} \right) \mathcal{F}_{\lambda} \mathcal{F}_F \]

\[ = \left( A_F : \hat{F}(a^*_a, a) \cdot \mathcal{F}_F (a^*_a, a) \right)(a) = \Sigma_{\lambda=0}^{\infty} \frac{1}{\lambda!} \mathcal{F}_{\lambda} \hat{F}(\lambda, \lambda) \left( \hat{a}^{(\lambda + 1)} \cdot \hat{a}^{(\lambda + 1)} \right) \mathcal{F}_{\lambda} \mathcal{F}_F \]

\[ \left( \hat{a}^{(\lambda + 1)} \cdot \hat{a}^{(\lambda + 1)} \right) \mathcal{F}_{\lambda} \mathcal{F}_F \]

\[ \left( A_F : \hat{F}(a^*_a, a) \cdot \mathcal{F}_F (a^*_a, a) \right)(a) = \Sigma_{\lambda=0}^{\infty} \frac{1}{\lambda!} \mathcal{F}_{\lambda} \hat{F}(\lambda, \lambda) \left( \hat{a}^{(\lambda + 1)} \cdot \hat{a}^{(\lambda + 1)} \right) \mathcal{F}_{\lambda} \mathcal{F}_F \]

At this point one must notice that the integrations symbolized by the sign of the bilinear form \( (\cdot, \cdot) \) include a non-zero counterpart from those symmetry group decomposition terms of functions which are totally symmetric in the group of \( k+n \) variables.

Denoting by \( A(\nu, \omega) \) the anti-symmetrization operator of the product \( \mathcal{F} \cdot \mathcal{F} \) of sign functions, we immediately have:

\[ \mathcal{F}_{\nu} \mathcal{F}_{\omega} \mathcal{F} = \text{Sym} \left\{ \mathcal{F}_{\nu} \mathcal{F}_{\omega} \mathcal{F} \right\} + \text{other decomposition terms} = \mathcal{F}_{\nu} \mathcal{F}_{\omega} \mathcal{F} + \text{c.d.t.} \]  \hspace{1cm} (A.54)

In consequence,

\[ \mathcal{F}_{\nu} \mathcal{F}_{\omega} \mathcal{F} = \left\{ \mathcal{F}_{\nu} \mathcal{F}_{\omega} \mathcal{F} \right\} \mathcal{F} + \text{c.d.t.} \]

Quite analogously,

\[ \mathcal{F}_{\nu} \mathcal{F}_{\omega} \mathcal{F} = \left\{ \mathcal{F}_{\nu} \mathcal{F}_{\omega} \mathcal{F} \right\} \mathcal{F} + \text{c.d.t.} \]

which proves (A.52) and hence (A.51).

As an application of the projection theorem one can prove that operators

\[ \hat{a}^*_a \mathcal{F}_F, \mathcal{F}_F \hat{a}_a^* \mathcal{F}_F \]

satisfy the canonical anticommutation relation in \( \mathcal{F}_F = \mathcal{F}_F \mathcal{F}_F \).

More details can be found in (67).
REFERENCES


F.J. Dyson, Phys. Rev. 102, 1217 (1956).

F.J. Dyson, Phys. Rev. 102, 1230 (1956).


A.B. Harris, Phys. Rev. 172, 674 (1968).


N.P. Dühr, see Ref. 70.


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[122] "Extended systems in field theory", J. Gervais, A. Neveu (Eds.).


[127] J. Cervais, A. Neveu (Eds.).


