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NULL SURFACE QUANTIZATION AND QUANTUM FIELD THEORY IN ASYMPTOTICALLY FLAT SPACE-TIME

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ABSTRACT

A quantum theory of the free scalar, electromagnetic and gravitational fields in a curved asymptotically flat space-time is developed. It is shown that the Penrose conformal technique makes it possible to reformulate the null infinity quantization as a problem of the quantization on the proper null surface in the corresponding Penrose space. The Schwinger dynamical principle is exploited to derive the corresponding null surface commutation relations. The general covariant and gauge-independent form of the commutation relations is also given. The existence of the asymptotic symmetry (BMS) group in the asymptotically flat space-time is used to define uniquely the 'in' and 'out' vacuum states. The explicit expressions for the S-matrix operator and for the E-matrix elements in the asymptotically simple space-time are given. The functional integration method is used to find the expression for the density matrix describing the observations at $\text{in}$ in the weakly asymptotically simple space-time when the information loss due to the event horizons or the existence of bare singularities is possible. The application of the developed approach to the problem of quantum evaporation of black holes (Hawking effect) is briefly discussed.

The problem of field quantization in a given (external) metric is of particular interest at present. This interest was greatly increased by Hawking's discovery of the possibility of vacuum instability in the strong gravitational field of black holes. The quantization procedure is known to imply the solution of the following two problems: the construction of the commutation relations and the realization of their representation in some state space. While the solution of the first problem does not meet any principal difficulties, the second problem, which can be reduced to the vacuum state definition problem, is in the general case much more complicated. From the physical point of view, the difficulty in the vacuum state definition reflects an ambiguity in separation of the particles into real and virtual ones in an external (gravitational) field. Formally, this uncertainty is caused by the ambiguity (in space-time without a global timelike Killing vector field) in decomposing the field solutions into positive and negative frequency components.

The problem of the vacuum-state definition can be satisfactorily solved in the physically interesting case of asymptotically flat space-times. In such a space-time the geometry deviation from a flat one at large distances from the source becomes negligibly small, and the existence of the asymptotic symmetry group allows one to define unambiguously the asymptotic energy-momentum observables. The state without incoming or outgoing particles (in- or out-vacuums) can be introduced as the lowest asymptotic energy states.

The conformal transformation techniques and the definition of the asymptotically flat space-time proposed by Penrose gave a very convenient mathematical tool to deal with the radiation problem of the classical massless fields. In this paper the corresponding consideration of the quantum theory of massless fields in asymptotically flat space-time is given (see also [7,8]).

The paper is organized as follows. The next three sections are preliminary and contain the reformulation of the known results on the massless scalar field in a flat space-time quantization in an equivalent form but more convenient for generalizations (in terms of $\mathcal{J}$ quantities). The definition of the asymptotically flat space-time and some properties that we use in the sequel are also given. The second part of the paper is devoted to the problem of the null surface quantization in a curved space-time. We find it convenient to use systematically the Schwinger dynamical principle...
approach (9), which allows us to develop the quantization procedure of the scalar, electromagnetic and gravitational fields in a similar manner. The developed null surface quantization and the obtained null surface commutation relations are not only the first step in solving the problem of the null infinity quantization but have their own importance.

In the third part of the paper, the scattering theory of massless fields in asymptotically flat space-time is constructed. Using the asymptotic symmetry group, we define the asymptotic in- and out-vacuum states and find the explicit expression for the S-matrix operator and S-matrix elements in asymptotically simple space-time. The problems arising when some internal "singularities" (e.g. black holes) are present are discussed and the expression for the corresponding density matrix describing the scattering states in this case is given in the last section. Some useful formulas and some technical details are given in the appendix.

II. PRELIMINARIES

2.1 Massless fields in flat space-time: Asymptotic properties

In this section some known results of interest in the following discussions will be collected. We begin by considering the properties of the solutions of the Klein-Gordon equation,

$$\Box \psi = 0 ,$$

(2.1)
in Minkowskian space. We use the approach which can easily be generalized to the case of asymptotically flat space-time and the consideration of this simple example allows us to demonstrate the main ideas of the method developed later in the paper.

a) Scattering problem

Different statements of the problem for Eq. (2.1) are possible. To define the solution $\psi$ over the total space-time manifold one may give the initial value of the field $\psi$ and its time derivative $\dot{\psi}$ on some space-like Cauchy surface (the Cauchy problem) or specify the value $\psi$ on some null Cauchy surface (the characteristic initial value problem). For our purposes it is more convenient to characterize the field $\psi$ by its asymptotic behaviour at infinity (the scattering problem). To consider this problem in more detail, we introduce new co-ordinates $(u,r,\theta,\phi)$

$$r = \sqrt{x^2 + y^2 + z^2} , \quad u = t - r , \quad r \sin \theta \dot{\psi} = x + iy ,$$

(2.2)

instead of the usual Minkowskian co-ordinates $(t,x,y,z)$. In the new co-ordinates the metric takes the form:

$$ds^2 = du^2 + 2dudr - r^2d\theta^2 ,$$

(2.3)

where $d\theta^2 = d\theta^2 + \sin^2 \theta d\phi^2$. We are interested in the asymptotic behaviour of the solution $\psi(p)$ when $u$ and $x^A = (\theta, \phi)$ are fixed and $r \to \infty$. In this case the point $p$ is said to tend to the future null infinity $\mathcal{J}^+$. If $\psi$ at the initial moment $t = 0$ is localized in space and the energy of the field $\psi$ is finite (a wave packet solution) then it can be shown (see for example [10,7]) that the following limit:

$$\phi^\text{out}_{u,x^A} = \lim_{r \to \infty} (\theta_p(u,r,x^A)) ,$$

(2.4)

exists. The function $\phi^\text{out}_{u,x^A}$ will be called the image of $\psi$ on $\mathcal{J}^+$. The correspondence between the solution $\psi$ and its image $\phi^\text{out}$ on $\mathcal{J}^+$ is one-to-one.

If the advanced time co-ordinate $v = t + r$ is introduced instead of the retarded time $u = t - r$, then the following limit:

$$\phi^\text{in}_{v,x^A} = \lim_{r \to \infty} (\theta_p(v,r,x^A)) ,$$

(2.5)

exists and it is called the image of $\psi$ on $\mathcal{J}^-$ (on the past null infinity). $\phi^\text{in}$ also uniquely defines the solution $\psi$. If we take into account that in the line $u,x^A = \text{const.}$ (the null cone generator) is a null geodesic and $r$ is an affine parameter along it, then we can reformulate the above given propositions in a more geometrical way: there is a one-to-one correspondence between the massless field $\psi$ in the flat space-time and its asymptotics along the null geodesics when the affine parameter $r$ tends to infinity. It can also be shown that for the wave-packet-like solutions the corresponding limits along any timelike or spacelike geodesic are zero.

$\mathcal{J}^+(\mathcal{J}^-)$ may be considered as a set of end-points of the generators of the future (past) directed null cones, the tops of which lie on a line $x = y = z = 0$. $\mathcal{J}^+(\mathcal{J}^-)$ has a three-dimensional manifold structure and its
topology is $\mathbb{R}^1 \times S^2$ (co-ordinate $u$ changes from $-\infty$ to $+\infty$ and $\theta, \varphi$ are co-ordinates on the unit sphere $S^2$). The classical scattering problem for the massless field $\varphi$ can be equivalently reformulated in the following manner. In addition to the Minkowskian space-time $\mathbb{M}$ there are two three-dimensional manifolds $\mathbb{M}^-$ and $\mathbb{M}^+$. Given a function $\psi_{in}$ on $\mathbb{M}^-$ (the incoming wave) one needs to find the function $\psi_{out}$ on $\mathbb{M}^+$ (the outgoing wave) such that a solution $\varphi$ of wave Eq. (2.1) in $\mathbb{M}$ exists which satisfies the conditions (2.4) and (2.5).

b) Penrose space

It is more convenient not to deal with three different objects $\mathbb{M}^-$, $\mathbb{M}^+$ and $\mathbb{M}$ but to have a unique one, $\mathbb{M} = \mathbb{M}^+ \cup \mathbb{M}^-$, and to adjoin to the space-time $\mathbb{M}$ as its (improper) boundaries. The corresponding procedure of the construction of such a "compactified" space consists of two steps. The first is the introduction of new co-ordinates $\mathbb{M}'$ in which the limit $r \to \infty$ corresponds to a finite value of $x^1$. It is evident that the metric coefficients in the new co-ordinates are singular at the points of "infinity". It is an obligatory price to pay for the points at infinity to have finite co-ordinates. The second step is the introduction of a new metric, $ds^2$, which is conformal to $ds$ everywhere over $\mathbb{M}$,

$$ds^2 = \Omega^2 ds^2,$$

the conformal factor $\Omega$ being chosen in such a way that the new metric $ds$ is also regular at the points of "infinity". The possibility of fulfilling both steps in the Minkowskian space can be demonstrated as follows [5]. In the co-ordinates $\psi = \arctg (t^2 + r^2)$, $\xi = \arctg (t^2 - r^2)$ ($\frac{\pi}{2} < \xi < \frac{\pi}{2}$) the flat metric is of the form (the first step):

$$ds^2 = (1 + \xi^2 \xi) (1 + \xi^2 \xi) \left[ \frac{1}{r^2} \sin^2 (\psi - \xi) \right] ds^2.$$  

The future (past) null infinity $\mathbb{M}^+$ ($\mathbb{M}^-$) is described in these co-ordinates by the equation $\psi = \frac{\pi}{2}$ ($\xi = -\frac{\pi}{2}$). As a suitable conformal factor $\Omega$, one can choose

$$\Omega = \left( 1 + \xi^2 \xi \right) \left( 1 + \xi^2 \xi \right)^{-1/2},$$

(the second step). The corresponding Penrose picture in this case is given in Fig. 1.

This shows us that for the given Minkowskian space-time $(\mathbb{M}, g)$ a new (non-physical) space $(\mathbb{M}, g, \Omega)$ can be constructed, which has the following properties: 1) the interior of $\mathbb{M}$ is conformal to $\mathbb{M}$ and $\Omega^2 = r^2 \xi$; 2) the conformal factor $\Omega$ is non-negative everywhere over $\mathbb{M}$ and is equal to zero on $\partial \mathbb{M}$; 3) every null geodesic in $\mathbb{M}$ has two end-points on $\partial \mathbb{M}$; 4) the boundary $\partial \mathbb{M}$ consists of two disconnected parts $\mathbb{M}^+ \cup \mathbb{M}^-$, each having the topology $\mathbb{R}^1 \times S^2$; 5) $\mathbb{M}^+$ and $\mathbb{M}^-$ are null surfaces, i.e. $\partial \mathbb{M}$ and $\partial \mathbb{M}$ are null surfaces, i.e. $\Omega^2 = r^2 \xi = 0$ and $\Omega^2 = r^2 \xi = 0$; 6) $\mathbb{M} = \mathbb{M} \cup \mathbb{M}$ has the topology $\mathbb{R}^1$. We shall call this space $(\mathbb{M}, g, \Omega)$ the Penrose space. It is easy to see that there is an ambiguity in the construction of the Penrose space corresponding to a given space-time. This ambiguity consists of the rescaling transformations

$$\Omega \to \Omega' = \omega \Omega,$$

where $\omega$ is a positive smooth function on $\mathbb{M}$ (including its boundary).

The convenience of the Penrose space approach to the massless field scattering problem becomes evident if we note that the existence of the limits given by Eqs. (2.10) and (2.5) is equivalent to the continuity property of a conformally transformed field $\tilde{\varphi} = \Omega^{-1} \varphi$ at the null infinity boundaries $\mathbb{M}^+$ and $\mathbb{M}^-$ in the Penrose space. Moreover, the boundary value of $\tilde{\varphi}$ on $\mathbb{M}$ coincides with the images of $\varphi$ on $\mathbb{M}^+$, i.e. one has

$$\varphi_{out} = \tilde{\varphi}_{out}, \quad \varphi_{in} = \tilde{\varphi}_{in},$$

and the scattering problem in the physical space-time is reduced to the characteristic initial value problem in the Penrose space [6]. The transformation of the images

$$\varphi = \varphi', \quad \varphi' = \omega^{-1} \varphi,$$

under the change (2.8) corresponds to the possibility of a choice in (2.4) and (2.5) of any other affine parameter $r' = \omega^{-1} r$ along the null cone generators instead of $r$. For the image $\varphi'$ on $\mathbb{M}^+$ defines a solution $\varphi$ unambiguously, every quantity depending on $\varphi$, in particular energy-momentum vector, can be expressed in terms of $\varphi$ or equivalently in terms of the conformal field $\tilde{\varphi}$ value on the null surfaces $\mathbb{M}^-$. 

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c) Poincaré group action on $\mathcal{J}^2$

The energy-momentum vector is closely connected with the space-time isometry group. To find an expression for the energy-momentum vector in terms of the null infinity quantities we note that the action of the Poincaré group in the Minkowskian space-time induces some (non-linear) transformations on $\mathcal{J}^2$ surfaces. This representation is isomorphic to the Poincaré group.

To see how this representation naturally appears, one can consider the Poincaré transformation in Minkowskian space

$$x'=x'+a, \quad t'=t'+v, \quad \theta' = \theta + \omega \theta, \quad \phi' = \phi + \omega \phi,$$

where $x, t, \theta, \phi$ are the space-time coordinates and $a, v, \omega$ are the Poincaré parameters. This representation is isomorphic to the Poincaré group, and one has

$$\lim_{r \to \infty} \left( u + a, r, \theta, \phi \right) = \left( u, 0, \theta, \phi \right).$$

Let $(u, r, \theta, \phi)$ and $(u', r', \theta', \phi')$ be co-ordinates corresponding to the points $x'$ and $x''$. Then Eq. (2.11) allows one to express $u', r', \theta'$, and $\phi'$ as complicated functions of the variables $(u, r, \theta, \phi)$. In the limit $r \to \infty$, this dependence becomes simpler and one has

$$\lim u' = H(\theta, \phi), \quad \lim \phi' = I(\theta, \phi).$$

Here $\lim$ denotes the limit $r \to \infty$ and $u, \theta, \phi = \text{const}$. The functions $H, I$ and $K$ are such that the transformation $\theta' = H(\theta, \phi), \phi' = I(\theta, \phi)$ is a conformal transformation on the sphere

$$d\sigma^2 = d\theta^2 + \sin^2 \theta d\phi^2 = K^2(\theta', \phi') \left( d\theta^2 + \sin^2 \theta' d\phi^2 \right).$$

These conformal transformations can be rewritten in a more convenient form,

$$\zeta' = \frac{a \zeta + b}{c \zeta + d},$$

where $\zeta = \zeta(\theta, \phi) \exp(4ip), \zeta' = \zeta(\theta', \phi') \exp(4ip')$ and $a, b, c, d$ are complex parameters satisfying the condition $ad-bc = 1$.

Another way of describing the action of the Poincaré group on $\mathcal{J}^2$ is to pass to a corresponding Penrose space. The generators $\xi^\mu$ of infinitesimal symmetry transformations in $(M, g)$ satisfy the Killing equation

$$\xi^\mu \xi'_\mu + \xi'^\mu \xi^\mu = 0,$$

where $V^\mu$ is a covariant derivative with respect to the flat metric (in flat co-ordinates $V^\mu = 0$). It can easily be verified that in the conformal space $(\mathcal{M}, \tilde{g})$ these vector fields satisfy the following conformal Killing equation

$$\tilde{g}^{\mu \nu} = \frac{\tilde{\Omega}^2}{\Omega} \delta^{\mu \nu}.$$

So the Poincaré group may be defined as a group of conformal motions in the Penrose space corresponding to a flat space-time. An explicit expression for $\xi^\mu$ in co-ordinates $(u, r, \theta, \phi)$ allows one to make sure that $\xi^\mu$ is finite on $\mathcal{J}^2$ and the right-hand side of (2.13) shows that $\xi^\mu \xi^\mu = 0$ on $\mathcal{J}^2$.

Hence the generators $\xi^\mu$ act on $\mathcal{J}^2$ moving it along itself. The solutions of Eq. (2.13) restricted to $\mathcal{J}^2$ give the infinitesimal transformations of the Poincaré group on $\mathcal{J}^2$. In particular, the vector fields

$$\xi^\mu = N^a \partial_a, \quad N^a = (1, \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

correspond to the translations along the $x^a$ axis in flat space-time.

d) Energy-momentum fluxes on $\mathcal{J}^2$

The expression for the energy-momentum tensor $T^{\mu \nu}$

$$T^{\mu \nu} = \beta \Psi \delta^{\mu \nu} + \frac{1}{2} \varepsilon^{\mu \nu \alpha \beta} \partial_{\alpha} \Psi \partial_{\beta} \Phi,$$

corresponding to the field $\Phi$ satisfying (2.1) allows one to calculate the flux of the energy-momentum through a sphere of a large radius $r$ in the limit $r \to \infty$. To obtain this expression we must take into account that Eq. (2.4) provides the following asymptotic behaviour:

$$\frac{\beta \Psi}{r} = \frac{\beta \Psi}{r} \frac{\Phi_{\text{out}}}{r} + O(r^{-2}),$$

where $\Phi_{\text{out}} = \partial_t \Psi$ is a null vector tangent to the light cone $u = \text{const}$. Hence at large distances (in the wave zone)
and the energy-momentum flux may be written in the form:
\[ F_a = \int \sigma \sigma' d^2 \phi \pm \delta \phi \sigma d^2 \phi \pm \delta \phi \sigma \], (2.17)

where \( \sigma = \sin \theta \) is a corresponding Killing vector and the expression for \( N^a = \frac{\partial}{\partial u} \) is given by (2.14). This expression shows that the asymptotic energy-momentum flux can be presented as an integral over \( \delta^+ \) of the form:
\[ F_a = \int \sigma \sigma' d^2 \phi \pm \delta \phi \sigma d^2 \phi \pm \delta \phi \sigma \], (2.18)

where \( \delta \phi \) are vector fields on \( \delta^+ \) corresponding to the translations in space-time and \( d^2 \phi \) is a surface element on \( \delta^3 \).

2.2 Quantum theory in Minkowskian space

Now we want to reformulate the usual quantum theory of massless fields in flat space-time in terms of the Penrose space and show how the quantization procedure (including the vacuum state definition) can be realized in terms of quantities on \( \delta^+ \) (or \( \delta^3 \)) only.

a) Canonical quantization

The usual way to quantize a field is to impose the canonical commutation relations on the field \( \phi \) and its conjugated momentum \( \pi \).

If the Cauchy surface is a hyperplane \( t = 0 \), then \( \pi = \frac{\partial}{\partial t} \) and

\[ \left[ \phi(t,x), \phi(t,y) \right] = \left[ \pi(t,x), \pi(t,y) \right] = 0, \]
\[ \left[ \phi(t,x), \pi(t,y) \right] = i \delta(x-y). \] (2.19)

As in the classical theory case, to pass to the Penrose space picture we need to reformulate the theory in curvilinear co-ordinates (the first step) and after to consider its properties under the conformal transformations. To write down the commutation relations (2.19) in a coordinate-independent way, we note that these commutation relations may be considered as the initial data for the Heisenberg equation:
\[ \Box \phi = 0. \] (2.20)

The solution of this equation allows one to find the commutator of the operators at any different point
\[ \left[ \phi(x), \phi(y) \right] = \frac{1}{2\pi} i \int \epsilon(x-y) \delta'(s^2(x,y)) \], (2.21)

where \( s^2(x,y) = n_{\alpha\beta}(x^\alpha - y^\alpha)(x^\beta - y^\beta) \) is a square of the interval between points \( x \) and \( y \) in Minkowskian space-time. To avoid the difficulties of operating with distributions we can rewrite the commutator (2.21) in an equivalent form:
\[ \left[ \phi(t, x), \phi(t, y) \right] = \left< \delta' \right>, \] (2.22)

where \( \left< f, g \right> \) is a \( \epsilon \)-independent inner product of the complex classical solutions of Eq. (2.20)

\[ \left< f_1, f_2 \right> = \int_{\Sigma} f_1 \overline{f_2} \sigma d\Sigma, \] (2.23)

\[ \overline{f_1} \sigma f_2 = \int_{\Sigma} f_1 \overline{f_2} \sigma d\Sigma. \] (2.24)

and \( d\Sigma \) is a surface element on the Cauchy surface \( \Sigma \). If the surface \( \Sigma \) is determined by the equation \( F(x) = 0 \) and \( x^i (i = 1, 2, 3) \) are inner coordinates on \( \Sigma \) then \( d\Sigma = \sqrt{-g} g^{ij} \partial_i F dx^1 dx^2 dx^3 \). We also use the standard abbreviation for
\[ \phi(t) = \left< f, g \right>. \] (2.25)
b) Conformal transformations

Now we consider the operator algebra properties under the conformal transformations. First of all it should be noted that the inner product (2.23) is conformal invariant. Indeed, let \((M, g = \tilde{g})\) be the space conformal to a given one \((M, g)\) and let us propose that under the conformal transformation an arbitrary scalar field \(f\) changes as follows:

\[
\tilde{f} \rightarrow \tilde{f} = f^{-1} f .
\]

Then noting that \(d\tilde{Z}^\mu = u^{-2} d\tilde{Z}^\mu\) and \(u^{-1} u^{-1} = u^{-2}\), we have

\[
\langle f_2, f_1 \rangle_{(M, g)} = \int \frac{d\tilde{Z}}{2\pi} \delta_{21} \tilde{f}_2^* \tilde{f}_1 \equiv \langle \tilde{f}_2, \tilde{f}_1 \rangle_{(\tilde{M}, \tilde{g})} .
\]

The conformal invariance of the inner product makes it possible to show that the canonical commutation relations in the flat space-time are equivalent to the following commutation relations:

\[
[\Phi(f), \Phi(g)] = \langle \tilde{g}, \tilde{f} \rangle
\]

in the space \((\tilde{M}, \tilde{g})\).

c) Commutation relations on \(\tilde{\mathcal{J}}\)

Now we can pass to the Penrose space \((\tilde{M}, \tilde{g})\) and use (2.23) to establish the commutation relations of the images \(\Phi^\mu (\tilde{f})\) of the operator \(\tilde{f}\) on \(\tilde{\mathcal{J}}\). Because the considerations are very similar for the cases of \(\tilde{\mathcal{J}}^+\) and \(\tilde{\mathcal{J}}^-\), we shall simply omit the superscripts + and - and omitting the corresponding subscripts "out" and "in" for the images of the fields. Eq.(2.28) being considered on the \(\tilde{\mathcal{J}}\) surface in the Penrose space gives

\[
[\tilde{\Phi}(F), \tilde{\Phi}(G)] = \langle \tilde{G}, \tilde{F} \rangle
\]

where the inner product \(\langle \tilde{g}, \tilde{f} \rangle\) on \(\tilde{\mathcal{J}}\) may be written as follows:

\[
\langle \tilde{G}, \tilde{F} \rangle = i \int d\sigma \left( \tilde{G} \partial_u \tilde{F} - \tilde{F} \partial_u \tilde{G} \right) = 2i \int d\sigma \left( \tilde{G} \partial_u \tilde{F} \right).
\]

d) Vacuum state definition

A vacuum, as any other state of a system, must be defined on the total Cauchy surface \(E\), and it will be denoted by \(|0;E\rangle\). In the flat space, without any field sources the vacuum is known to be independent of the \(E\) surface choice and is defined by conditions:

\[
\mathcal{P}_u |0\rangle = 0
\]

where \(\mathcal{P}_u\) are the Poincaré group generators. It is also well known that this definition is equivalent to the fulfillment of the equation:

\[
[\tilde{\Phi}(F), \tilde{\Phi}(G)] = \langle \tilde{G}, \tilde{F} \rangle
\]
for every positive (with respect to the Minkowskian time coordinate \( t \)) frequency solution \( f \) of (2.20)

\[
f(x) = f(u, r, \vec{\sigma}_r) = \int e^{-i(k_r - \vec{\sigma}_r \cdot \vec{k})} \tilde{f}(k) \, dk = \\
= \int e^{-i [k_u + k_r (1 - \vec{\sigma}_r \cdot \vec{k})]} \tilde{f}(k, \vec{\sigma}_r) \, k^2 \, dk \, d\sigma_k ,
\]

where \( \vec{\sigma}_r \) \( (\vec{\sigma}_k) \) is a unit vector in the three-dimensional \( \mathbb{E} \) \( (\mathbb{E}^3) \) space.

To find a description of the vacuum state \(|0>\) in terms of the \( f \) quantities we note that at large distances \( r \to \infty \) \( f(x) \) has the following asymptotic behaviour:

\[
f(u, r, 0) = F_{\text{out}}(u, r) = e^{-2iku} \tilde{f}(k, \vec{\sigma}) \, k \, dk ,
\]

This asymptotic behaviour can be established if we take into account that when the coordinate system is chosen so that the \( z \) axis coincides with the direction of \( \vec{g}_t \), then

\[
f(u, r, 0) = f(u, r, \theta = 0, \varphi = 0) = \int e^{-i(k_u + kr(1-\cos \theta))} \tilde{f}(k, \vec{\sigma}) \, \sin \theta \, d\theta \, d\varphi .
\]

After the integration over the angle variables one has at large distances \( r \),

\[
f(u, r, 0, 0) = -\frac{2\pi}{r} \int_0^\infty e^{-iku} \left[ \tilde{f}(k, 0, 0) - \tilde{f}(k, u, 0) \right] k \, dk + o(r^{-2}) .
\]

The integration of the second term in the square brackets gives the quantity of the order \( O(r^{-1}) \), thus we have

\[
f(u, r, 0, 0) = -\frac{2\pi}{r} \int_0^\infty e^{-iku} \tilde{f}(k, 0, 0) \, k \, dk + O(r^{-2}) .
\]

In the case when the direction of the \( z \) axis does not coincide with the direction of \( g \) this formula transforms into Eq. (2.35). For the inner product is invariant under conformal transformations, the vacuum state \(|0>\) definition (2.24) may be rewritten in a space \((\mathbb{E}^3, g)\) conformal to the flat space-time \((\mathbb{E}^3, \tilde{g})\) as follows:

\[
<\tilde{f}(\vec{g})|0> = 0 .
\]

This makes it possible to define the usual vacuum state \(|0>\) by the following conditions on \( J^+ \) boundary surfaces in the Penrose space \((\mathbb{E}, g)\):

\[
<_{\text{out}}(\vec{f}_{\text{out}})|0> = 0 ,
\]

where

\[
_{\text{out}}(\vec{f}_{\text{out}}) = <\vec{f}_{\text{out}}|_{\text{out}}> = \frac{1}{2\pi} \int du \, d\varphi \, _{\text{out}}(\vec{f}_{\text{out}}) \cdot _{\text{out}}(\vec{f}_{\text{out}}) ,
\]

and \( _{\text{out}}(\vec{f}_{\text{out}}) \) is an arbitrary positive frequency function on \( J^+ \), that is the function which can be presented in the form (2.36).

If we remember that the null generators of the \( J^+ \) coincide with the integral curves of the asymptotic Killing vector field \( \vec{g}_t \) and \( u \) is a "Killing time" parameter along these curves defined by the relation \( d\tilde{u}/du = \tilde{g}^t \), where \( \tilde{g}^t(u) \) is a null generator equation, then we can formulate the final result in the more invariant form: the function \( _{\text{out}}(\vec{f}_{\text{out}}) \) is said to be of positive frequency if its restriction on every null generator of \( J^+ \) contains only positive frequencies with respect to the "Killing time" \( u \).

The vacuum state \(|0>\) is unambiguously defined by Eq. (2.35).

When the field is not free (i.e. when external currents are present and particle creation processes are possible) the vacua on \( J^+ \) and \( J^- \) may not be identical. To distinguish them we shall use the notations \(|0;\text{in}>\) for \( J^- \) and \(|0;\text{out}>\) for \( J^+ \) vacuum state.

If \( _{\text{out}}(\vec{f}_{\text{out}}) \) is a positive frequency normalized function

\[
<_{\text{out}}(\vec{f}_{\text{out}}), _{\text{out}}(\vec{f}_{\text{out}})> = 1
\]

then the negative frequency function \( _{\text{out}}(\vec{f}_{\text{out}}) \) is also normalized

\[
<_{\text{out}}(\vec{f}_{\text{out}}), _{\text{out}}(\vec{f}_{\text{out}})> = -1 ,
\]
and as usual the operators of particle creation and annihilation in a state $a$ can be defined

$$\hat{a}^+_a = \hat{a}^+_{\text{out},a}, \quad \hat{a}_{\text{out},a} = \hat{a}^+_{\text{out},a}.$$  \hfill (2.40)

If $\{\hat{a}^+_{\text{out},a}, \hat{a}^+_{\text{out},a}\}$ is a normalized out basis, i.e. a full system of the functions on $\mathcal{J}^+$ satisfying the conditions

$$\langle \hat{a}^+_{\text{out},a}, \hat{a}^+_{\text{out},a} \rangle = \delta_{ab}, \quad \langle \hat{a}^+_{\text{out},a}, \hat{a}^+_{\text{out},a} \rangle = 0,$$

we then have

$$\hat{a}^+_{\text{out},a} = \sum_a \langle \hat{a}^+_{\text{out},a}, \hat{a}^+_{\text{out},a} \rangle \hat{a}^+_{\text{out},a} \hat{a}^+_{\text{out},a}$$ \hfill (2.41)

and the commutation relations for operators $\hat{a}^+_{\text{out},a}$ and $\hat{a}_{\text{out},a}$:

$$[\hat{a}^+_{\text{out},a}, \hat{a}^+_{\text{out},a}] = \delta_{ab}, \quad [\hat{a}^+_{\text{out},a}, \hat{a}^+_{\text{out},a}] - [\hat{a}^+_{\text{out},a}, \hat{a}^+_{\text{out},a}] = 0,$$

are a simple corollary of Eq. (2.29).

In a similar manner $\hat{a}^+_{\text{in},a}$ and $\hat{a}^+_{\text{in},a}$ operators can be introduced. The Fock space $\mathcal{K}_{\text{out}}$ can be constructed in the usual way and the state in the n-particles section of this space describes the situation when n outgoing (in incoming) particles are present. This allows one to define the S matrix and, after the introduction of the dynamical observables on $\mathcal{J}^+$ and $\mathcal{J}^-$, the reformulation of the quantum scattering theory in terms of $\mathcal{J}^+$ is completed. The above consideration shows the equivalence of the developed approach with the usual one.

2.3 Asymptotically flat space-time: Definitions and properties

If we analyse the question of what properties of the flat space-time and of the solutions of Eq. (2.1) for the scalar field were used when we reformulated the quantization procedure in terms of $\mathcal{J}$ quantities, it appears that they are the following: 1) the possibility of the corresponding Penrose space construction; 2) the universality of the asymptotic field solutions behaviour ($\gamma \sim r^{-2}\phi(n, \theta, \phi) + O(r^{-2})$) along every null geodesic which implies the existence of the images of the field on $\mathcal{J}$; 3) the existence of the symmetry group transformations on $\mathcal{J}$, which makes it possible to define the asymptotic vacuum state unambiguously. There is a wide class of (non-flat) space-times for which the Penrose space construction is possible (so-called asymptotically flat space-times). In these spaces the requirement of the images of the massless field existence on $\mathcal{J}$ appears to be a natural boundary condition for a radiation field (very similar to the Sommerfeld radiation condition). Moreover, the asymptotic symmetry group is inherent in every asymptotically flat space-time and the above-given procedure of $\mathcal{J}$ quantization can be generalized to this class of space-times.

Although the corresponding definitions are well known [5] we find it useful to recall some of them in this section and give here a short review of the asymptotically flat space-time properties which will be used later.

a) Asymptotically simple, weakly asymptotically simple and asymptotically flat space-time

The main idea of Penrose's definition of asymptotically flat space-times is the similarity of the geometrical properties of their null infinities to the geometrical properties of the null infinity in the flat space-time. The space-time $(\mathcal{M}, g)$ is called asymptotically simple if some regular manifold $\mathcal{M}$ with boundary $\partial \mathcal{M}$ and regular metric $ds$ exists, whose interior $\mathcal{M} \setminus \partial \mathcal{M}$ is conformal to $\mathcal{M}$ with $ds = \delta^{3}ds$ and the following properties are satisfied: 1) $\Omega > 0$ is a regular function on $\mathcal{M}$ and $\Omega = 0$ on $\mathcal{J}^-$, while $\Omega / \delta$ on $\mathcal{J}^+$; 2) every null geodesic in $\mathcal{M}$ contains, if maximally extended, two distinct end-points on $\mathcal{J}$. The corresponding Penrose space $(\mathcal{M}, g, \mathcal{J})$ is defined up to a re-scaling transformation

$$(\mathcal{M}, g, \mathcal{J}) \rightarrow (\mathcal{M}, g, \mathcal{J})$$ \hfill (2.43)

where $\omega$ is a regular positive function on $\mathcal{M}$.

If we assume that the metric $g$ satisfies the vacuum Einstein equations in the neighbourhood of $\mathcal{J}$ and some tacit assumptions on time and space orientability and absence of closed timelike curves are made then it is possible to prove [5,6] that: 1) the boundary $\mathcal{J}$ of the Penrose space consists of two disconnected parts $\mathcal{J}^+ = \mathcal{J}^+ \cup \mathcal{J}^-$ each of them having a topology $\mathbb{R}^+ \times S^2$; 2) the boundary surfaces $\mathcal{J}^+$ and $\mathcal{J}^-$ are null surfaces, i.e.
3) the integral curves of the vector field $\xi^\mu = g^{\mu\nu} \nabla_\nu g_{ij}$ on $\mathcal{J}$ are the null generators of $\mathcal{J}$ so that $\mathcal{J}^\pm$ are null hypercylinders ruled by the isotropic lines of their degenerate metrics; 4) the Riemann tensor of the space-time $(\mathcal{M}, g)$ tends to zero along every null geodesic (the space-time is really flat at infinity); 5) the peeling-off property takes place (this property shows in particular how quickly the metric in the asymptotic region tends to a flat one); 6) the topology of $\mathcal{M}\setminus\mathcal{U}$ is $\mathbb{R}^4$.

The last property of the asymptotically simple space-time shows that there is no place in it for any internal singularities (e.g. for black holes). On the other hand, it is evident that the existence of compact objects (or singularities) which are localized in space cannot change the null infinity structure. To take such a possibility into account it is convenient to give the following definition [6]: the space-time $(\mathcal{M}, g)$ is called weakly asymptotically simple if such an asymptotically simple space-time $\mathcal{M}_0$ exists that for some open submanifold $K$ of $\mathcal{M}_0$ the domain $\mathcal{M}_0\setminus K$ is isomorphic to $\mathcal{M}$.

Usually we shall assume that the vacuum Einstein equations are satisfied near $\mathcal{J}$ and we shall refer to the corresponding asymptotically simple space-time as asymptotically flat.

### b) Asymptotically regular fields

Let us consider a field $\varphi_A$ ($A = 1, \ldots, N$) in the space $(\mathcal{M}, g)$ and assume that under the conformal transformations $g \to \omega^2 g$ it transforms as follows:

$$\tilde{\varphi}_A = \tilde{\varphi}_A (\varphi, \omega) .$$

The field $\tilde{\varphi}_A$ in the asymptotically flat space-time is called asymptotically regular if in the Penrose space $(\tilde{\mathcal{M}}, \tilde{g})$ the corresponding conformal field $\tilde{\varphi}_A = r_A (\varphi, \omega)$ is smooth at $\mathcal{J}^+$ and $\mathcal{J}^-$. In the case when some gauge group of transformations exists and the field $\varphi_A$ is defined up to the gauge transformation only, we shall call it asymptotically regular if such a choice of the gauge exists in which the field $\tilde{\varphi}_A = r_A (\varphi, \omega)$ is smooth at $\mathcal{J}^\pm$. This limit of the $\tilde{\varphi}_A$ at $\mathcal{J}$ will be called the image of $\tilde{\varphi}_A$ on $\mathcal{J}$ and will usually be denoted by the capital letter corresponding to the lower case letter describing the field with "in" or "out" subscripts in the case of $\mathcal{J}^+$ or $\mathcal{J}^-$ infinity.

### c) Asymptotic symmetry group

The infinitesimal transformation of the Penrose space corresponding to a given asymptotically flat space-time generated by the vector field $\xi^A$ is called an asymptotic symmetry transformation if on $\mathcal{J}$ it satisfies the equation

$$\left( g^{\nu\mu} \xi^\nu \right) \frac{\partial}{\partial \xi^\mu} = 0 \quad (2.15)$$

(compare with (2.13)). It is evident that this equation does not define the vector field $\xi^A$ in the interior of $\mathcal{M}$. We shall consider the class of solutions of (2.15) coinciding on $\mathcal{J}$ as an infinitesimal element of the asymptotic symmetry group. The remarkable fact is that this group known as the Bondi-Metzner-Sachs (BMS) group does not depend on the concrete choice of the asymptotically flat space-time. The BMS group is an infinite-dimensional group containing the Poincaré group as a subgroup. In Minkowskian space-time this asymptotic symmetry group may be reduced to the Poincaré group if we consider only those solutions of Eq. (2.15) which allow the continuation over the total Penrose space and which are there regular solutions of the conformal Killing Eq. (2.13).

The most important properties of the BMS group are [3]:

1. The BMS group is a semidirect product of the infinite-dimensional abelian normal subgroup $\mathcal{N}$ of supertranslations and the orthochronous homogeneous Lorentz group.

2. There is only one normal four-dimensional subgroup of the BMS group. It is the abelian subgroup of translations.

The explicit form of the asymptotic symmetry transformations in the appropriate coordinate system will be given in Sec. 4.2.

### III. NULL SURFACE QUANTIZATION IN CURVED SPACE-TIME

As a first step in the programme of quantum field theory construction in the asymptotically flat space-time we consider the problem of the null surface quantization. The null infinities of the asymptotically flat space-time are null boundary surfaces in the Penrose space and many features of the null infinity quantization ($\mathcal{J}$-quantization) are a simple reflection of the peculiar geometrical properties of such surfaces. The null surface quantization in the curved space-time is itself of interest. The possibility of the gauge conditions' resolution in the explicit form (in the case of...
the electromagnetic and gravitational fields) allows one to find the description of the field variables through the dynamically independent quantities.

3.1 Null surface geometry

In this section a few remarks about the geometrical structure of a null surface will be given and a convenient co-ordinate system based on a null surface will be described. These co-ordinates will be consistently used in the next sections when the quantization on null surfaces in a curved space-time will be considered.

a) $\mathbb{N}$ co-ordinates

First of all we consider a more technical question regarding the convenient choice of co-ordinates based on the given null surface. Every null surface $N$ in a curved space-time is formed by a two-dimensional family of null geodesics ($N$ generators) \[12\]. Let $S_0$ be a two-dimensional spacelike section of $N$ with $x^A$ ($A = 2,3$) co-ordinates there. These co-ordinates can be continued over the $N$ surface if we define them to be constant along $N$ generators. As a third co-ordinate on $N$ we choose affine parameter $u$ along $N$ generators. If $x^u = x^u(u)$ is the equation of an $N$ generator, then a parameter $u$ is called affine if a tangent vector $n^u = d x^u / d u$ is propagated parallelly along the generator: $n^u n^v = 0$. The co-ordinates $(u,x^A)$ on $N$ are not defined uniquely, the co-ordinate freedom is

$$
\mathcal{A} = \mathcal{A}(x^3) \quad \text{(a change of co-ordinates on } S_0) ,
$$

$$
\mathbb{Z} = \mathbb{A}(x^3) u + \mathbb{B}(x^3) \quad \text{(freedom of choice of the affine parameter).}
$$

(3.1)

The co-ordinate system $(u,x^A)$ on $N$ may be used to construct co-ordinates in some neighbourhood of the $N$ surface. As a first step we continue the co-ordinates $(u,x^A)$. Let $S_u$ be a two-dimensional spacelike section $u =$ const. of $N$, then a two-parameter system of null geodesics orthogonal to $S_u$ and not lying in $N$ generates a null surface which will be denoted by $\Gamma_u$ (Fig.2). We choose the co-ordinates $(u,x^A)$ specifying them to be constant along the generators of $\Gamma_u$. As a fourth co-ordinate $r$ we shall choose an affine parameter along the null generators of $\Gamma_u$. The ambiguity of the affine parameter choice

$$
\mathcal{F} = g(u,x^A) \quad r = \mathcal{A}(u,x^A)
$$

may be used to secure the fulfillment of the following conditions:

1) $\mathcal{F}|_u = 0$ (function $\mathcal{Z}$ specification),
2) $g^{uv}|_u = \delta^{uv}_{\mathcal{A}}$ (function $\mathcal{C}$ specification).

To satisfy the second requirement we must show that such a function $g(u,x^A)$ exists for which the vector field $\mathcal{W} = g^{uv}_\mathcal{A} \mathcal{F}$ obeys the equation

$$
\mathcal{W}_u = 0 .
$$

(3.2)

For this purpose we rewrite the expression $\mathcal{W}_u = 0$ in the form

$$
\mathcal{W}_u = \frac{1}{2} \mathcal{C} (g^{\alpha\beta} r_\alpha r_\beta) + C g^{\alpha\beta} \partial_\alpha C \partial_\beta r = 0 .
$$

For, the vector $g^{\alpha\beta} r_\beta|_u$ is tangent to $N$, we have that the second requirement will be fulfilled if we take the function $C$ as a solution of the first-order partial differential equation

$$
\mathcal{Z}_u = -\mathcal{C}_u (g^{\alpha\beta} r_\alpha r_\beta)
$$

on $N$.

We shall call the so constructed co-ordinates $(u,r,x^2,x^3)$ based on the given null surface $N$, $\mathbb{N}$ co-ordinates. In these co-ordinates the metric $g$ is of the form:

$$
\mathcal{A} = \begin{bmatrix}
0 & 1 & 0 \\
1 & \mathcal{E}^{ll} & \mathcal{E}^{lA} \\
0 & \mathcal{E}^{lA} & \mathcal{E}^{AB}
\end{bmatrix}
$$

with

$$
\mathcal{E}^{uu} = \begin{bmatrix}
\delta^{uu} & 1 & \delta^{uu} \\
1 & 0 & 0 \\
\delta^{uu} & 0 & \delta^{uu}
\end{bmatrix}
$$

(3.3)

where $A,B = 2,3; \mathcal{E}^{ll} = \delta^{ll} + \delta^{AB} = \delta^{ll}$. For, the vector $\mathcal{W} = g^{uv}_\mathcal{A} \mathcal{F}$ is tangent to the null generator of $N$, we have
Moreover, the fulfillment of Eq. (3.2) gives
\[ \partial_{\nu} \xi^1 |_{\mathcal{N}} = \frac{1}{2} \partial_{\rho} (\xi^{1\mu} \xi_{1\mu}) |_{\mathcal{N}} = 0 . \]

Hence on \( \mathcal{N} \) we have
\[ \xi^{1\mu} |_{\mathcal{N}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & q^{AB} \end{bmatrix} , \quad \xi_{1\mu} |_{\mathcal{N}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & q_{AB} \end{bmatrix} , \quad \partial_{\nu} \xi^{1} |_{\mathcal{N}} = \partial_{\rho} \xi_{1\rho} |_{\mathcal{N}} = 0 . \]

The corresponding expression for the Christoffel symbols in \( \mathcal{N} \) co-ordinates is given in the Appendix.

\section*{b) \( \mathcal{N} \) tetrad}

It is also convenient to choose in an appropriate way the null complex tetrad system associated with the given null surface \( \mathcal{N} \). As a first real null vector of this tetrad we take \( n^{\mu} = \xi^{1\mu} |_{\mathcal{N}} \). This null vector field is defined throughout the neighbourhood of \( \mathcal{N} \) where the co-ordinate \( u \) is defined. The null vector field \( n^{\mu} \) is given on \( \mathcal{N} \) by the equation \( n^{\mu} |_{\mathcal{N}} = \xi^{1\mu} |_{\mathcal{N}} \) and off the surface \( \mathcal{N} \) it is considered to be parallelly propagated along the integral lines of the \( \xi^{1\mu} \) vector field (i.e. the \( \Gamma_{\mu} \) generators)
\[ \xi^{1\mu} |_{\mathcal{N}} = 0 . \]

For these integral lines \( x^{\mu}(r) \) of the vector field \( \xi^{\mu} \) are geodesics and \( r \) is an affine parameter along it, i.e.
\[ \xi^{\mu} = \xi^{1\mu} |_{\mathcal{N}} = \frac{dx^{\mu}}{dr} \]  

we have
\[ \xi^{1\mu} |_{\mathcal{N}} = \xi_{1\mu} |_{\mathcal{N}} = 1 . \]  

Two-dimensional sections \( S_{\mu} \) on \( \mathcal{N} \), described by the equation \( u = \text{const} \), are orthogonal to both systems of the null vectors \( n^{\mu} \) and \( m^{\mu} \).

We may complete the tetrad by choosing two additional complex null vectors \( m^{\mu}, \bar{m}^{\mu} \) spanning the tangent space to \( S_{\mu} \). The freedom of choice of these vectors on \( \mathcal{N} \) described by the equation
\[ m^{\mu} |_{\mathcal{N}} \rightarrow \tilde{m}^{\mu} |_{\mathcal{N}} = e^{i\Theta(u,x)} m^{\mu} |_{\mathcal{N}} . \]

This can be used to satisfy the following condition:
\[ (\gamma - \bar{\gamma}) |_{\mathcal{N}} = (\tilde{m}^{\mu} n_{\mu} - m^{\mu} \bar{n}_{\mu}) |_{\mathcal{N}} = 0 . \]

To define \( m^{\mu} \) outside the \( \mathcal{N} \) surface we consider these vectors to be parallelly propagated along the null generators of \( \Gamma_{\mu} \). So constructed tetrad system satisfies the usual normalization conditions
\[ n^{\alpha} n_{\alpha} = -m^{\alpha} m_{\alpha} = 1 , \quad \text{other scalar products are zero} . \]

We call this tetrad system associated with null surface \( \mathcal{N}, \mathcal{N} \) tetrad

Spin coefficients \( \iota_{\mu} |_{\mathcal{N}} \) we have
\[ \kappa = \epsilon = \Pi = 0 \]  

\[ \gamma = 0 \]  

\[ \nu = 0 \]  

\[ \mu = \bar{\Pi} \]  

We call this tetrad system associated with null surface \( \mathcal{N}, \mathcal{N} \) tetrad

Spin coefficients \( \iota_{\mu} |_{\mathcal{N}} \) we have
\[ \iota_{\mu} |_{\mathcal{N}} = -\eta_{\mu} \]  

\[ \lambda = -\eta_{\mu} \]  

\[ \text{and } |\lambda| \]  

where \( \lambda = -\eta_{\mu} \eta^{\mu} \) do not depend on the choice of the tetrad system and are the differential invariants characterizing the internal properties of \( \mathcal{N} \) surface, i.e. its expansion \( \iota_{\mu} \) and shear \( |\lambda| \).
c) Conformal transformations

The last question we are concerned with in this section is conformal properties of the null surface invariants. Let us consider the space $(\mathcal{M}, \mathbb{g} = \delta^2 \mathbb{g})$, which is conformal to the given space $(\mathcal{M}, \mathbb{g})$. If $N$ is a null surface in $(\mathcal{M}, \mathbb{g})$, it will also be a null surface in $(\mathcal{M}, \delta^2 \mathbb{g})$. The $N$ generators are also geodesics in the new metric $\delta^2 \mathbb{g}$ and the new affine parameter $\hat{u}$ is connected with the old $u$ by means of the relation

$$d\hat{u} = \omega^2 du$$

and

$$\hat{u}^\mu = \frac{d\mu}{du} = \omega^{-2} n^\mu.$$ 

The other vectors of the null complex tetrad change as follows:

$$\hat{q}^\mu = q^\mu, \quad \hat{\bar{q}}^\mu = \omega^{-1} n^\mu, \quad \hat{\bar{q}}^\mu = \omega^{-1} n^\mu.$$ 

Simple calculation gives for the conformal transformations of the expansion and shear

$$\hat{\xi} = \omega^{-2} (\xi + \omega^{-1} \eta), \quad \hat{\bar{\eta}} = \omega^{-2} |\eta|.$$ 

3.2 Null surface quantization of scalar massless field: Schwinger approach

In this section we consider the scalar field quantization in a curved space-time. The formal second quantization of the scalar field has been studied elsewhere (e.g., [14-17]). For our purposes it is convenient to use the method of quantization developed by Schwinger [9] which makes it possible to consider not only the quantization on the spacelike surfaces but also the null surface quantization. In the next sections this approach will be used for the electromagnetic and gravitational field quantization.

a) The Schwinger dynamical principle

The action $\mathcal{W}[\varphi]$ for a massless scalar field $\varphi$ in a curved space-time is given by

$$\mathcal{W}[\varphi] = \frac{1}{2} \int \left( K^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + \alpha R \varphi^2 \sqrt{-g} \right) d^4x,$$ (3.12)

where $K^{\mu\nu} = \nabla^\nu \nabla^\mu \varphi$. The corresponding second-order differential equation for $\varphi$ is

$$\frac{D[\varphi]}{\partial R} \varphi = \frac{1}{\sqrt{-g}} \partial_\nu (K^{\mu\nu} \partial_\nu \varphi) - \alpha R \varphi = 0. \quad (3.13)$$

When $\alpha = - \frac{1}{6}$ this equation is conformal invariant. To use the standard Schwinger quantization scheme we introduce new additional variables $\varphi^\mu$ and rewrite the action (3.12) in the first-order form

$$\tilde{\mathcal{W}} [\varphi^\mu, \varphi] = \frac{1}{2} \int \left( K^{\mu\nu} \varphi^\nu \partial_\nu \varphi - K^{\mu\nu} \varphi^\nu \varphi \right. - \left. \varphi \partial_\mu \left( K^{\nu\mu} \varphi^\nu \right) + \alpha R \varphi^2 \sqrt{-g} \right) d^4x.$$ (3.14)

The variation of this action with respect to $\varphi^\mu$ and $\varphi$ gives the following system of first-order equations:

$$\varphi^\nu = \partial_\nu \varphi, \quad \partial_\mu \left( K^{\nu\mu} \varphi^\nu \right) - \alpha R \varphi^2 \sqrt{-g} \varphi = 0,$$ (3.15)

which is equivalent to the second-order Eq.(3.13). The variation $\delta \mathcal{W}[\varphi^\mu, \varphi]$ of the action when the field variable variations $\delta \varphi^\mu$ and $\delta \varphi$ do not vanish on the boundary surfaces $E_0$ and $E_1$ after using the field equations gives

$$\delta \mathcal{W}[\varphi^\mu, \varphi] = \oint_{E_1} - \oint_{E_0},$$

where

$$G_{\Sigma} = \frac{1}{2} \int_{\Sigma} g^{\mu\nu} \left( \partial_\mu \delta \varphi - \varphi \delta \partial_\mu \varphi \right) d\Sigma_\nu.$$ (3.16)

are the generators of infinitesimal transformations of the field variables. Hence the commutation relations on the surface are defined by the following equations:

$$[\varphi^\mu, G_{\Sigma}] = \frac{i}{\hbar} \delta \varphi^\mu, \quad [\varphi^\mu, G_{\Sigma}] = \frac{i}{\hbar} \delta \varphi^\mu.$$ (3.17)
where the fields $\psi$ and $\varphi$, as well as their variations $\delta \psi$ and $\delta \varphi$, are considered on the surface. It should be noted that to deduce the commutation relation one needs to express the variations $\delta \psi$ and $\delta \varphi$ in (3.17) through the independent quantities on $\Sigma$.

b) Spacelike surface quantization

When $\Sigma$ is spacelike, two functions $\psi|_\Sigma$ and $\varphi|_\Sigma$ or, equivalently, $\psi|_\Sigma$, $\varphi|_\Sigma = \chi \psi|_\Sigma$, $\varphi|_\Sigma$ where $\chi$ is a normal to $\Sigma$, may be taken as Cauchy data on $\Sigma$. If we choose a co-ordinate $x^0$ to be constant on $\Sigma$, then $\delta \psi = \delta \varphi = \partial^a \sqrt{-g} \gamma^a$ and the generator $\mathcal{G}_\Sigma$ is given by

$$\mathcal{G}_\Sigma = \frac{1}{2} \int \gamma^a \gamma^b \gamma^c \partial_x \partial_y \partial_z \sqrt{-g} \gamma^a$$

Eqs. (3.17) may now be written in the form:

$$[\psi|_\Sigma, \varphi|_\Sigma] = \frac{i}{2} \delta \psi + \frac{i}{2} \delta \varphi,$$

and taking into account the independence of $\delta \psi$ and $\delta \varphi$ we can derive the following canonical commutation relations:

$$[\psi(x), \varphi(y)]_{x^0 = y^0} = \frac{i}{2} \delta^3(x, y)\gamma^a \gamma^b \gamma^c \partial_x \partial_y \partial_z \sqrt{-g} \gamma^a \gamma^b \gamma^c,$$

$$[\psi(x), \varphi(y)]_{x^0 = y^0} = \frac{i}{2} \delta^3(x, y)\gamma^a \gamma^b \gamma^c \partial_x \partial_y \partial_z \sqrt{-g} \gamma^a \gamma^b \gamma^c,$$

where $\delta^3(x, y)$ is the usual three-dimensional $\delta$ function.

These commutation relations may be written in a more invariant way. For any two complex classical solutions $\psi_1$ and $\psi_2$ their inner product $\langle \psi_1, \psi_2 \rangle$ can be introduced

$$\langle \psi_1, \psi_2 \rangle = \int \sqrt{g} \gamma^a \gamma^b \gamma^c \partial_x \partial_y \partial_z \sqrt{-g} \gamma^a \gamma^b \gamma^c,$$

and the commutation relations (3.18) are equivalent to

$$[\psi_1, \psi_2] = \frac{i}{2} \delta^3(x, y) \gamma^a \gamma^b \gamma^c \partial_x \partial_y \partial_z \sqrt{-g} \gamma^a \gamma^b \gamma^c.$$
where
\[ \tilde{\Psi} = \sqrt{g} \Psi = \sqrt{g} \mathcal{L}, \tag{3.25} \]
and
\[ C_1 = \frac{1}{2} \int \left[ \tilde{\mathcal{L}}(u, X^A) \delta \tilde{\mathcal{L}}(u, X^A) - \mathcal{L}(u, X^A) \delta \mathcal{L}(u, X^A) \right] d^2x \tag{3.26} \]
and \( u_0 = u_0(x^A), u_1 = u_1(x^A) \) are (finite or infinite) values of the affine parameter corresponding to the beginning and the end of the \( N \) surface generators.

Taking variations \( \delta \mathcal{L} \) satisfying the condition \( \mathcal{L} u_0 = \mathcal{L} u_1 = 0 \) and using Eqs.(3.17) we have
\[ \left[ \tilde{\mathcal{L}}(u, X^A), \mathcal{L}(u, X^A) \right] = \frac{i}{2} \delta(u-u') \delta^2(x^A-x'^A), \]
\[ \left[ \mathcal{L}(u, X^A), \mathcal{L}(u, X^A) \right] = -\frac{i}{2} \delta(u-u') \delta^2(x^A-x'^A). \]
The integration of these equations gives
\[ \left[ \tilde{\mathcal{L}}(u, X^A), \mathcal{L}(u, X^A) \right] = -\frac{i}{2} \delta(u-u') \delta^2(x^A-x'^A). \] \( \tag{3.27} \)

Using Eq.(3.27) it is possible to verify that for the variations \( \delta \mathcal{L} \) not vanishing at the end-points one has
\[ \left[ \tilde{\mathcal{L}}(u_0, X^A), \mathcal{L}(u_1, X^A) \right] = 0. \tag{3.28} \]
i.e. \( \tilde{\mathcal{L}}(u_0, X^A) + \mathcal{L}(u_1, X^A) = 0 \) and these variations are not free. From the physical point of view these end-points of the \( N \) generators (if they exist) are the points of the initial input or the final output of the radiation, propagating along the \( N \) surface. In the subsequent consideration we shall assume that these end-points are absent or, if they are present, we shall operate with the quantities like \( \mathcal{L}(x) \), assuming the test functions \( f \) to vanish near the \( N \) surface boundary.

It is an easy exercise to show that Eq.(3.22) is a simple corollary of the \( N \) surface commutation relations (3.28). It is also not difficult to see that the commutation relations (3.22) are conformal invariant in the following sense. Let \((M, \tilde{g})\) be a space conformal to \((M, g)\) and \( \tilde{\mathcal{L}} = \Gamma^{-1} \mathcal{L}, \tilde{\mathcal{L}} = \Gamma^{-1} \mathcal{L} \) be conformal transformed fields in this space, then Eq.(3.22) is equivalent to the following:
\[ \left[ \tilde{\mathcal{L}}(u_0, X^A), \tilde{\mathcal{L}}(u_1, X^A) \right] = \left< \tilde{\mathcal{L}}(u_0, X^A), \tilde{\mathcal{L}}(u_1, X^A) \right>. \tag{3.30} \]
where all inner products of the quantities with a hat are calculated in \((M, \tilde{g})\) space.

### 3.3 Quantization of the electromagnetic field

#### a) First-order formalism for the Maxwell field

To quantize the electromagnetic field in a curved space-time by the Schwinger method we begin with the action \( W[a, \phi] \) in the first-order form
\[ W[a, \phi] = \frac{i}{4} \int \left[ f_{\mu\nu} f^{\mu\nu} - f^{\mu\nu} \left( \partial_\mu a_\nu - \partial_\nu a_\mu \right) \right] + \frac{1}{V} \int V^{-1} \mathcal{L} \left( f_{\mu\nu} f^{\mu\nu} - f^{\mu\nu} \right) d^4x \] \( \tag{3.31} \)
for the field variables \( a_\mu, \phi^{\mu\nu} \). The variations of \( W[a, \phi] \) with respect to \( a_\mu \) and \( \phi^{\mu\nu} \) with the variations \( \delta a_\mu \) and \( \delta \phi^{\mu\nu} \) vanishing on the boundary surfaces \( E_0 \) and \( E_1 \) of the four-dimensional region \( V \) give the usual Maxwell equations.

\(-27-\)
\[
\zeta_{\mu\nu} = \partial_{\mu} \omega_{\nu} - \partial_{\nu} \omega_{\mu}
\]
Using these equations and considering the variations \( \delta \zeta_{\mu\nu} \) and \( \delta \omega_{\mu} \) as not vanishing on the boundary surfaces, we have

\[
\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} \zeta_{\mu\nu}) = 0 \quad (3.32)
\]
where the generator \( G_{\Sigma} \) is

\[
G_{\Sigma} = \frac{1}{2} \sum_{\Sigma} \left( \partial_{\nu} \delta f^{\alpha \mu\nu} - \delta f^{\mu\nu} \partial \alpha \right) d \Sigma_{\mu} \quad (3.34)
\]
and the commutation relations are to be found from the equations

\[
\left[ \partial_{\mu}, \zeta_{\Sigma} \right] = \frac{1}{2} \delta \zeta_{\mu} \quad \left[ \partial_{\mu}, \delta \zeta_{\nu} \right] = \frac{1}{2} \delta \zeta_{\mu} \quad (3.35)
\]

**b) Choice of gauge**

To deduce the explicit form of the commutation relations from Eq.\( (3.35) \) we must express \( \delta \partial_{\mu} \) and \( \delta \zeta_{\mu\nu} \) through the independent quantities. Some difficulty arises from the gauge invariance of the theory, i.e. the invariance of the action \( (3.31) \) under the transformations

\[
\mu_{\nu} \rightarrow \mu_{\nu} + \lambda_{\nu} \quad (3.36)
\]

We shall consider the null surface quantization. In this case it is convenient to impose the following gauge condition:

\[
\chi_{\mu} = 0 \quad (3.37)
\]
where \( \chi_{\mu} \) is a null vector field connected with the given surface \( \Sigma \) in the way described in Sec.3.1. Eq.\( (3.37) \) shows that the vector potential \( \mu_{\nu} \) is of the form:

\[
\mu_{\nu} = \omega_{\nu} + \alpha^{+} \omega_{\nu} + \alpha \lambda_{\nu} \quad (3.38)
\]

The corresponding choice of the gauge we shall call \( \Sigma \) gauge.

**c) Commutation relations**

If we take the null surface \( \Sigma \) as a \( \Sigma \) surface in Eq.\( (3.34) \) and use the \( \Sigma \) co-ordinates and \( \Sigma \) tetrads described in Sec.3.1 we can write \( \delta \zeta_{\mu} \) in the form

\[
\zeta_{\mu} = \frac{1}{2} \sum_{\Sigma} \left( \partial_{\nu} \delta f^{\alpha \mu\nu} - \delta f^{\mu\nu} \partial \alpha \right) d \Sigma_{\mu} \quad (3.35)
\]
where \( \delta f^{\alpha \mu\nu} = \eta_{\mu\nu\alpha} \). On \( \Sigma \) the quantities \( \zeta_{\mu} \) can be expressed through the independent operators \( \omega \) and \( \omega^{+} \) as follows:

\[
f^{\nu} = \Delta \omega \bar{m}^{\nu} + \Delta m^{\nu} \omega^{+} - \Delta^{+} \left( \tau n^{\nu} - \mu m^{\nu} - \lambda \bar{m}^{\nu} \right) - \Delta^{+} \left( \tau n^{\nu} - \mu m^{\nu} - \lambda \bar{m}^{\nu} \right) \quad (3.31)
\]

Here we used the equation

\[
\eta_{\mu\nu} \left( \omega_{\mu} - \omega^{+}_{\nu} \right) = \tau n^{\nu} - \mu m^{\nu} - \lambda \bar{m}^{\nu} \quad (3.32)
\]
which is the consequence of the usual spin coefficient definition and Eq.\( (3.11) \). \( \Delta \) is a standard designation for the operator \( \omega_{\mu} \). In our case \( \Delta = \tau_{\mu} \). When we substitute the expression for \( f^{\nu} \) in \( (3.35) \) all the terms not containing the differential operator \( \Delta \) are cancelled and we have

\[
\zeta_{\mu} = \frac{1}{2} \sum_{\Sigma} \left( \tau n^{\nu} - \mu m^{\nu} - \lambda \bar{m}^{\nu} \right) \quad (3.33)
\]
The integration by parts with respect to the affine parameter \( u \) gives

\[
G_w = \int du \, d^3x \, (\beta_i \, \dot{\alpha}_i + \alpha_i \, \dot{\beta}_i) - \alpha_b ,
\]

where

\[
\tilde{\zeta} = \sqrt{q} \, \zeta = \sqrt{g} \, \zeta , \quad \tilde{\zeta}^+ = \sqrt{g} \, \zeta^+ , \quad \tilde{\zeta}^+ = \sqrt{g} \, \zeta^+ .
\]

(3.45)

\[
G_b = \frac{1}{2} \int \left( \tilde{\zeta}^+ \delta \tilde{\zeta} + \tilde{\zeta} \delta \tilde{\zeta}^+ \right) d^2x .
\]

(3.46)

The quantization conditions (3.35) may now be written through the independent operators \( \tilde{\zeta} \) and \( \tilde{\zeta}^+ \),

\[
[\tilde{\zeta}, \tilde{\zeta}^+] = \frac{i}{2} \, \delta_{\tilde{\zeta}}, \quad [\tilde{\zeta}^+, \tilde{\zeta}] = \frac{i}{2} \, \delta_{\tilde{\zeta}}^+ .
\]

(3.47)

This gives the following commutation relations:

\[
[\tilde{\zeta}(u, x^+), \tilde{\zeta}(u', x'^+)] = [\tilde{\zeta}^+(u, x^+), \tilde{\zeta}^+(u', x'^+)] = 0 ,
\]

\[
[\tilde{\zeta}(u, x^+), \tilde{\zeta}^+(u', x'^+)] = \frac{i}{2} \, \delta^2(x^+ - x'^+) .
\]

(3.48)

\[ \text{d) Inner product} \]

As usual it is convenient to write down the commutation relations (3.48) in a more invariant form. For this purpose we consider complex classical solutions \( a_j \) of the Maxwell Eqs.(3.32), and the inner product of any two such solutions \( a_j = a_j \) and \( a_i = a_i \) can be introduced by:

\[
\langle a_j, a_i \rangle = \int \left( \hat{\beta}_j \, \hat{\alpha}_i - \hat{\alpha}_j \, \hat{\beta}_i \right) d^4 \Sigma .
\]

(3.49)

The \( \Sigma \) independence of this inner product can be established by the following relations:

\[
i \int \left( \int_{\Sigma} \left( \hat{\beta}_j \, \hat{\alpha}_i - \hat{\alpha}_j \, \hat{\beta}_i \right) d^4 \Sigma \right) \, d^4 \Sigma = \int_{\Sigma} \left( \hat{\beta}_j \, \hat{\alpha}_i - \hat{\alpha}_j \, \hat{\beta}_i \right) d^4 \Sigma .
\]

(3.50)

This inner product satisfies the usual conditions (3.20). Stokes theorem implies that the value of the integral (3.49) does not change if we put \( \hat{\beta}_j = \hat{\alpha}_i \) instead of \( \hat{\beta}_j \) provided \( I^{ij} = -I^{ji} \) and \( I^{ij} \) is decreasing rapidly enough at the infinity of \( \Sigma \). This property allows one to prove the gauge independence of the inner product (3.49),

\[
\langle a_j + a_\lambda, a_i \rangle = \langle a_j, a_i + a_\lambda \rangle = \langle a_j, a_i \rangle .
\]

(3.50)

The inner product \( \langle a_j, a_i \rangle \) is also conformal invariant, i.e. if under the conformal transformation \( g_{ij} = \Omega^2 g_{ij} \) the field variables do not change \( a_j = a_j \), then

\[
\langle a_j, a_i \rangle = \langle a_j, a_i \rangle = -i \int \left( \hat{\beta}_j \, \hat{\alpha}_i - \hat{\alpha}_j \, \hat{\beta}_i \right) d^4 \Sigma .
\]

(3.51)

\[ \text{e) Invariant form of commutation relations} \]

Because of the gauge invariance of the inner product (3.49) and its \( \Sigma \) independence, in calculations a null Cauchy surface \( \Sigma \) can be taken,

\[
\langle a_j, a_i \rangle = -i \int_{\Sigma} \left( \hat{\beta}_j \, \hat{\alpha}_i - \hat{\alpha}_j \, \hat{\beta}_i \right) d^4 \Sigma .
\]

(3.51)

where \( \hat{\beta}_j = \hat{\alpha}_i \), and an \( \Sigma \) gauge condition can be invoked

\[
a_j |_{\Sigma} = (a^*_j + a_j) |_{\Sigma} = 0 .
\]

(3.52)

where \( \alpha \) and \( \alpha^* \) are two independent complex functions on \( \Sigma \). A calculation similar to that used in the derivation of Eq.(3.44) gives
<a_1, a_2> = i \int \sqrt{g} d^4x \left( \Delta a_1^\mu a_2^{\mu} - \bar{a}_2^\mu \Delta \bar{a}_1^\mu + \Delta \bar{a}_1^\mu \Delta \bar{a}_2^\mu \right).

This form of the inner product and Eqs. (3.56) allows one to verify that the commutation relations can be written in the form

\[ [a(a_1), a(a_2)] = \langle a_1, a_2 \rangle, \tag{3.54} \]

where, as usual, \[ a(a_1) = \langle a_1, a_2 \rangle \]. Its conformal invariance property now looks as follows:

\[ [\hat{a}(a_1), \hat{a}(a_2)] = \langle \hat{a}(a_1), \hat{a}(a_2) \rangle = [a(a_1), a(a_2)]. \tag{3.55} \]

3.4 Quantization of gravitational perturbations

a) First-order formalism

It is well known (see, for example, [18]) that small perturbations \( h \) of the gravitational field in the empty curved space-time with the background metric \( g \) satisfy the equation

\[ \mathcal{R}_{\mu\nu}[h] = \frac{\lambda}{2g} \mathcal{R}^\rho_{\lambda\mu\nu\rho}h_{\lambda\mu\nu\rho} - \frac{\lambda}{2g} \mathcal{R}_{\mu\nu\rho\lambda}h_{\mu\nu\rho\lambda} = 0. \tag{3.56} \]

This equation is a first-order approximation with respect to \( h_{\mu\nu} \) perturbations of the exact vacuum Einstein equations

\[ R_{\mu\nu}[g - \alpha h] = R_{\mu\nu}[g] + \alpha R_{\mu\nu}[h] + O(h^2) \tag{3.57} \]

the covariant derivatives in (3.56) being understood as the covariant derivatives with respect to the background metric \( g \). If the action \( \mathcal{W} \) for the gravitational field is \( \mathcal{W} = \int \sqrt{-g} d^4x \), then it is convenient to choose \( \alpha = \sqrt{2\lambda \rho} \). In this section the null surface quantization of the gravitational perturbations will be considered.

Let us use a new variable \( k_{\mu\nu} \) connected with \( h_{\mu\nu} \) by the relation

\[ k_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \lambda g_{\mu\nu}, \quad \lambda = e^{\phi} h_{\mu\nu} \tag{3.58} \]

and consider a first-order action \( \mathcal{W}[k, f] \) of the form

\[ \mathcal{W}[k, f] = \int \left[ \mathcal{K}^{\mu\nu}(f^\mu f_{\mu} - f^\mu f_{\mu}) - \mathcal{K}^{\Lambda \mu \nu}(f_{\mu}) - 2 \mathcal{G}^{\mu\nu}(f_{\mu} f_{\nu} - f_{\mu} f_{\nu}) \right] \sqrt{-g} d^4x, \tag{3.59} \]

where \( \mathcal{K}^{\mu\nu} = \mathcal{K}^{\mu\nu} \) and \( \mathcal{G}^{\mu\nu} = \mathcal{G}^{\mu\nu} \) are the field variables. The variation of this action with fixed values of the field variables on the boundary surfaces of the four-dimensional region \( V \) gives

\[ \mathcal{I}_{\mu\nu} \equiv \mathcal{K}^{\rho\sigma}_{\mu\rho\nu\sigma} = 0 \quad (3.60) \]

\[ \mathcal{K}^{\mu\nu}f_{\mu} - \mathcal{K}^{\Lambda \mu \nu}f_{\mu} + 2 \mathcal{G}^{\mu\nu}f_{\mu} - 2 f_{\mu} (\mathcal{G}^{\mu\nu}f_{\nu}) = 0. \tag{3.61} \]

To show that the obtained system of first-order equations is identical to the second-order Eq. (3.56), we note that after putting \( v = \alpha \) one has

\[ \mathcal{E}^{\Lambda \mu} = \mathcal{E}^{\mu\nu} - \frac{1}{2} \lambda g_{\mu\nu} = k_{\mu\nu} \tag{3.62} \]

and the contraction of Eq. (3.61) with \( s_{\mu\nu} \) gives

\[ \mathcal{E}_{\mu\nu} = \frac{1}{2} \lambda s_{\mu\nu} \tag{3.63} \]

Eqs. (3.61) - (3.63) make it possible to express \( k_{\mu\nu} \) in the form

\[ k_{\mu\nu} = \mathcal{E}^{\mu\nu} - \mathcal{E}^{\mu\nu} f_{\mu} f_{\nu} - \mathcal{G}^{\mu\nu} f_{\mu} f_{\nu} = \frac{1}{2} (\mathcal{K}_{\mu\nu} + \mathcal{K}_{\mu\nu} - \mathcal{G}_{\mu\nu} - \mathcal{G}_{\mu\nu} - \mathcal{G}_{\mu\nu}), \tag{3.64} \]
and after substituting this expression for $\alpha$ in (3.60) one reproduces Eq. (3.56).

The variation of $\mathcal{W}(x, f)$ with the unfixed field values on the boundary surfaces $\Sigma_0$ and $\Sigma_1$ gives

$$\mathcal{W}(x, f) = \mathcal{W}_0 - \mathcal{W}_c,$$

where

$$\mathcal{W}_0 = \int \left( \frac{1}{2} \delta \mathcal{F} \cdot \delta \mathcal{F} - \frac{1}{2} \mathcal{F} \cdot \delta \mathcal{F} + \mathcal{F} \cdot \delta \mathcal{F} \right) d \Sigma_v.$$

(3.65)

b) Choice of gauge

The field Eqs. (3.56) are invariant under the gauge transformations

$$\xi_{\mu} = \xi_{\mu} - \xi_{\mu} \xi_{\mu},$$

(3.66)

where $\xi_{\mu}$ is an arbitrary gauge vector field. When the null surface quantization is considered, it is convenient to use the following subsidiary conditions:

$$\xi_{\mu} = 0.$$

(3.67)

These conditions do not fix the gauge fields uniquely. The available transformations $\xi_{\mu}$ are the solutions of the equations

$$\xi_{\mu} \xi_{\nu} + \xi_{\nu} \xi_{\mu} = 0.$$

(3.68)

In the Appendix it is shown that this freedom can be used to achieve the fulfillment of the conditions

$$\xi_{\mu} = 0.$$

(3.69)

i.e. in this gauge we have

$$\xi_{\mu} = 0.$$

(3.70)

We shall call this choice of gauge "$\mathcal{N}$ gauge".

c) Computation relations

Now we represent the generator $\mathcal{G}_N$ through the independent variables on $N$. We use the $N$ gauge, i.e. we have $\xi_{\mu} = 0$ and hence

$$\mathcal{G}_N = \int \left( \xi_{\mu} \xi_{\nu} - \xi_{\nu} \xi_{\mu} \right) d \Sigma_v - \mathcal{G}_0.$$

(3.71)

where $\xi_{\mu} = 0$. The simple calculation gives

$$\xi_{\mu} = 0.$$

(3.72)

This allows us to rewrite (3.71) in the form

$$\xi_{\mu} = \frac{1}{2} \int \mathcal{F} \cdot d \Sigma_v.$$

(3.73)

As usual the integration by parts with respect to $u$ gives

$$\mathcal{G}_N = \int \left( \xi_{\mu} \xi_{\nu} + \xi_{\nu} \xi_{\mu} \right) d \Sigma_v,$$

(3.74)

where

$$\mathcal{G}_N = \int \left( \xi_{\mu} \xi_{\nu} + \xi_{\nu} \xi_{\mu} \right) d \Sigma_v.$$

(3.75)

and

$$\mathcal{G}_N = \frac{1}{2} \int \left( \xi_{\mu} \xi_{\nu} + \xi_{\nu} \xi_{\mu} \right) d \Sigma_v.$$

(3.76)

The expression (3.74) for $\mathcal{G}_N$ makes it possible to obtain the following commutation relations:

$$[\mathcal{F}(u, x), \mathcal{F}(u', x')] = \left[ \mathcal{F}(u, x), \mathcal{F}(u', x') \right] = 0.$$

(3.77)
The inner product for the complex classical solutions of Eq. (3.56) can be introduced as follows:

\[
\langle R_{\alpha}, R_{\beta} \rangle = 2i \int \sum_{\Sigma} \left( \overline{R}^{\alpha}_{\mu \nu} f^{\beta}_{\mu \nu} - \overline{\hat{R}}^{\alpha}_{\mu \nu} f^{\beta}_{\mu \nu} + \frac{f^{\alpha}_{\mu \nu}}{2} \overline{\hat{R}}^{\beta}_{\mu \nu} - \frac{f^{\beta}_{\mu \nu}}{2} \overline{\hat{R}}^{\alpha}_{\mu \nu} \right) d\Sigma_{\alpha} \tag{3.78}
\]

where \( k^{\mu \nu} = h^{\mu \nu} - \frac{1}{2} g^{\mu \nu} \). Its \( \Sigma \) independence is the consequence of the following relations:

\[
\left[ \int_{\Gamma_{\xi}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} = \int_{\Sigma_{0}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} \right] \overline{\eta}^{\Sigma_{0}} \eta^{\Sigma_{0} \Sigma} \tag{3.79}
\]

where \( f^{\alpha}_{\mu \nu} = f^{\beta}_{\mu \nu} \). Its \( \Sigma \) independence is the consequence of the following relations:

\[
\left[ \int_{\Gamma_{\xi}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} = \int_{\Sigma_{0}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} \right] \overline{\eta}^{\Sigma_{0}} \eta^{\Sigma_{0} \Sigma} \tag{3.79}
\]

where \( f^{\alpha}_{\mu \nu} = f^{\beta}_{\mu \nu} \). Its \( \Sigma \) independence is the consequence of the following relations:

\[
\left[ \int_{\Gamma_{\xi}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} = \int_{\Sigma_{0}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} \right] \overline{\eta}^{\Sigma_{0}} \eta^{\Sigma_{0} \Sigma} \tag{3.79}
\]

where \( f^{\alpha}_{\mu \nu} = f^{\beta}_{\mu \nu} \). Its \( \Sigma \) independence is the consequence of the following relations:

\[
\left[ \int_{\Gamma_{\xi}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} = \int_{\Sigma_{0}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} \right] \overline{\eta}^{\Sigma_{0}} \eta^{\Sigma_{0} \Sigma} \tag{3.79}
\]

where \( f^{\alpha}_{\mu \nu} = f^{\beta}_{\mu \nu} \). Its \( \Sigma \) independence is the consequence of the following relations:

\[
\left[ \int_{\Gamma_{\xi}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} = \int_{\Sigma_{0}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} \right] \overline{\eta}^{\Sigma_{0}} \eta^{\Sigma_{0} \Sigma} \tag{3.79}
\]

where \( f^{\alpha}_{\mu \nu} = f^{\beta}_{\mu \nu} \). Its \( \Sigma \) independence is the consequence of the following relations:

\[
\left[ \int_{\Gamma_{\xi}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} = \int_{\Sigma_{0}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} \right] \overline{\eta}^{\Sigma_{0}} \eta^{\Sigma_{0} \Sigma} \tag{3.79}
\]

where \( f^{\alpha}_{\mu \nu} = f^{\beta}_{\mu \nu} \). Its \( \Sigma \) independence is the consequence of the following relations:

\[
\left[ \int_{\Gamma_{\xi}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} = \int_{\Sigma_{0}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} \right] \overline{\eta}^{\Sigma_{0}} \eta^{\Sigma_{0} \Sigma} \tag{3.79}
\]

where \( f^{\alpha}_{\mu \nu} = f^{\beta}_{\mu \nu} \). Its \( \Sigma \) independence is the consequence of the following relations:

\[
\left[ \int_{\Gamma_{\xi}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} = \int_{\Sigma_{0}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} \right] \overline{\eta}^{\Sigma_{0}} \eta^{\Sigma_{0} \Sigma} \tag{3.79}
\]

where \( f^{\alpha}_{\mu \nu} = f^{\beta}_{\mu \nu} \). Its \( \Sigma \) independence is the consequence of the following relations:

\[
\left[ \int_{\Gamma_{\xi}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} = \int_{\Sigma_{0}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} \right] \overline{\eta}^{\Sigma_{0}} \eta^{\Sigma_{0} \Sigma} \tag{3.79}
\]

where \( f^{\alpha}_{\mu \nu} = f^{\beta}_{\mu \nu} \). Its \( \Sigma \) independence is the consequence of the following relations:

\[
\left[ \int_{\Gamma_{\xi}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} = \int_{\Sigma_{0}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} \right] \overline{\eta}^{\Sigma_{0}} \eta^{\Sigma_{0} \Sigma} \tag{3.79}
\]

where \( f^{\alpha}_{\mu \nu} = f^{\beta}_{\mu \nu} \). Its \( \Sigma \) independence is the consequence of the following relations:

\[
\left[ \int_{\Gamma_{\xi}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} = \int_{\Sigma_{0}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} \right] \overline{\eta}^{\Sigma_{0}} \eta^{\Sigma_{0} \Sigma} \tag{3.79}
\]

where \( f^{\alpha}_{\mu \nu} = f^{\beta}_{\mu \nu} \). Its \( \Sigma \) independence is the consequence of the following relations:

\[
\left[ \int_{\Gamma_{\xi}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} = \int_{\Sigma_{0}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} \right] \overline{\eta}^{\Sigma_{0}} \eta^{\Sigma_{0} \Sigma} \tag{3.79}
\]

where \( f^{\alpha}_{\mu \nu} = f^{\beta}_{\mu \nu} \). Its \( \Sigma \) independence is the consequence of the following relations:

\[
\left[ \int_{\Gamma_{\xi}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} = \int_{\Sigma_{0}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} \right] \overline{\eta}^{\Sigma_{0}} \eta^{\Sigma_{0} \Sigma} \tag{3.79}
\]

where \( f^{\alpha}_{\mu \nu} = f^{\beta}_{\mu \nu} \). Its \( \Sigma \) independence is the consequence of the following relations:

\[
\left[ \int_{\Gamma_{\xi}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} = \int_{\Sigma_{0}} \sum_{\Sigma} \langle R_{\alpha}, R_{\beta} \rangle d\Sigma_{\alpha} \right] \overline{\eta}^{\Sigma_{0}} \eta^{\Sigma_{0} \Sigma} \tag{3.79}
\]
and Eqs. (3.80) and (3.86) we obtain

$$\langle \hat{h}_1, \hat{h}_2 \rangle = \langle \hat{h}_1, \hat{h}_2 \rangle + N_f_1 (\hat{h}_1, \hat{h}_2)$$

(3.87)

where $\langle \hat{h}_1, \hat{h}_2 \rangle$ is the inner product of $\hat{h}_1$ and $\hat{h}_2$ in the $(M,g)$ space defined by the expression analogous to Eq. (3.80) and

$$\Pi^{\delta}_{\xi} (\hat{h}_1, \hat{h}_2) = -2i \int d^4 x (\hat{\kappa}_{\xi} - \hat{\kappa}_{\xi}) \hat{\kappa}_{\xi} + \hat{\kappa}_{\xi} \Omega_{\xi} \hat{\kappa}_{\xi} - \hat{\kappa}_{\xi} \Omega_{\xi} \hat{\kappa}_{\xi} - \hat{\kappa}_{\xi} \Omega_{\xi} \hat{\kappa}_{\xi}$$

(3.88)

Eq. (3.87) shows that in the general case the inner product $\langle \hat{h}_1, \hat{h}_2 \rangle$ is not conformally invariant. This property is closely connected with the non-commutability of the gauge and conformal transformations. The conformal invariance property is revived when we restrict ourselves to the consideration of the null surfaces and $N$ gauges only. In this case $N_f_1 (\hat{h}_1, \hat{h}_2)$ vanishes identically. It should also be noted that the notion of the null surface and $N$ gauge connected with this surface has a conformally invariant meaning.

e) Invariant form of commutation relations

Because of the gauge invariance of the inner product and its $\Sigma$ independence, the null Cauchy surface $N$ can be taken for its calculation

$$\langle \hat{h}_1, \hat{h}_2 \rangle = 2i \int d^4 x \sqrt{\gamma} (\hat{\kappa}_{\xi} f_{\xi} - \hat{\kappa}_{\xi} f_{\xi}) +$$

$$+ \hat{\kappa}_{\xi} f_{\xi} - \hat{\kappa}_{\xi} f_{\xi}$$

(3.89)

where $f_{\xi} = \gamma_{\xi}$ and $N$ gauge condition can be assumed

$$\hat{\kappa}_{\xi} f_{\xi} |_{N} = (\eta^+ m_{\nu} m_{\nu} + \eta m_{\nu} m_{\nu}) |_{N}$$

(3.90)

where $\eta^+$ and $\eta$ are two independent complex functions on $N$. After simple calculations the inner product may be expressed in the form

$$\langle \hat{h}_1, \hat{h}_2 \rangle = i \int d^4 x \sqrt{\gamma} \left( \hat{\kappa}_{\xi} \hat{\kappa}_{\xi} - \hat{\kappa}_{\xi} \hat{\kappa}_{\xi} \right)$$

(3.91)

If we denote $\hat{h}_1 (\gamma_{\xi}) = \langle \hat{h}_1, \hat{h}_2 \rangle$ then the commutation relations (3.77) and the explicit form of the inner product (3.91) allows us to verify that

$$[\hat{h}_1 (\gamma_{\xi}), \hat{h}_2 (\gamma_{\xi})] = \langle \hat{h}_2, \hat{h}_1 \rangle$$

(3.92)

This co-ordinate and the gauge-invariant form of commutation relations independent of the choice of Cauchy surface is now proved to be the consequence of the Schwinger quantization procedure.

IV. NULL INFINITY QUANTIZATION AND SCATTERING THEORY IN ASYMPTOTICALLY FLAT SPACE-TIME

A massless particle moving along a null geodesic in asymptotically simple space-time begins its motion at $J^-$ infinity and ends at $J^+$ infinity. It means that at very early or very late times the motion is in the nearly flat space-time, i.e. in the regions where the deflection of the space-time geometry from the flat one is negligibly small. That is why the notion of the incoming or outgoing particles and vacuum states are well defined and the construction of the S matrix is possible. From the mathematical viewpoint the main advantage of the asymptotically simple space-time definition given by Penrose is the possibility of the reformulation of the scattering problem in the physical space-time $(M,g)$ as a characteristic initial value problem in the Penrose space $(\mathbf{H},\mathbf{g},\mathbf{\mathcal{I}})$. It renders possible, in particular, the use of the results on the null surface quantization obtained in the previous sections to derive the commutation relations for 'in' or 'out' operators. The asymptotic 'freedom' of the massless particle behaviour at the infinities (i.e. an almost Minkowskian character of their motion at very early or very late time) is reflected in the existence of the asymptotic symmetry group.

In this part of the paper the quantum scattering problem in the asymptotically flat space-time will be considered.
4.1 **Null infinity quantization**

In this section we derive the commutation relations for the images of the scalar massless, electromagnetic and gravitational fields on $J^\infty$ infinity.

a) **Geometry of null infinity. Conformal Bondi frame**

The null infinities in the asymptotically flat space-time are null surfaces which possess some particular properties. The existence of a conformal factor which compresses the points at infinity into a pair of finite null hypersurfaces $\mathcal{J}^\infty$ and the fulfillment of the vacuum Einstein field equations in the asymptotic region guarantee [5,19] that $\mathcal{J}^\infty$ are non-shearing null hypersurfaces. The ambiguity of the Penrose conformal factor choice can be used to distinguish in the class of Penrose spaces $(\mathcal{M}, g, \Omega)$ corresponding to the given physical space-time, one in which the null surafces $\mathcal{J}$ are also non-expanding and the conformal factor $\Omega$ satisfies the following conditions [5,19,20]

$$\hat{\nabla}_a \hat{\nabla}_b \Omega \big|_{\mathcal{J}} = 0, \quad (\mathcal{O}^{-1} \hat{\nabla}^a \hat{\nabla}_{\rho} \Omega \hat{\nabla}^b \Omega) \big|_{\mathcal{J}} = 0. \quad (4.1)$$

In the Penrose space on the $\mathcal{J}$ surfaces and in some of their neighbourhoods the co-ordinates $(u, x^1, x^2)$ can be introduced in the same manner in Sec.3.1 for an arbitrary null surface $\Sigma$ (see Fig.3). Now it is more convenient to choose $\hat{\rho} = \Omega$ as a fourth co-ordinate instead of the affine parameter $\tau$ along $\Gamma^0$ generators. The metric $\hat{g}$ in these co-ordinates is of the form (a,b=2,3)

$$\hat{g}^{\mu\nu} = \begin{pmatrix} 0 & \hat{g}^{01} & \hat{g}^{02} & \hat{g}^{03} \\ \hat{g}^{10} & 0 & \hat{g}^{12} & \hat{g}^{13} \\ \hat{g}^{20} & \hat{g}^{21} & 0 & \hat{g}^{23} \\ \hat{g}^{30} & \hat{g}^{31} & \hat{g}^{32} & 0 \end{pmatrix}, \quad \hat{g}_{\mu\nu} = \begin{pmatrix} \hat{g}_{00} & \hat{g}_{01} & \hat{g}_{02} & \hat{g}_{03} \\ \hat{g}_{10} & 0 & \hat{g}_{12} & \hat{g}_{13} \\ \hat{g}_{20} & \hat{g}_{21} & 0 & \hat{g}_{23} \\ \hat{g}_{30} & \hat{g}_{31} & \hat{g}_{32} & 0 \end{pmatrix}. \quad (4.2)$$

The careful consideration of the available co-ordinate and conformal choice freedom shows [20,21] that it can be used to provide the fulfillment of the following conditions on $\mathcal{J}$:

$$\delta_{00} = \delta_{0a} = 0, \quad \delta_{01} = 1, \quad \delta_{02} = \delta_{03} = 0; \quad \delta_{ab} = -\delta_{ab}; \quad \delta_{a0} = 0. \quad (4.3)$$

where $\delta_{ab}$ is a metric on the unit sphere. The corresponding co-ordinates are called the conformal Bondi co-ordinates. In the complex co-ordinates $\zeta = x^0 + ix^1 = \exp \Omega$ the metric on the unit sphere can be written in the form

$$d\ell^2 = g_{AB} dx^A dx^B - \frac{d\xi d\tau}{2 \rho^2} - \frac{\dot{\rho}}{\rho} (1 + \frac{\xi}{\rho}). \quad (4.4)$$

The tangent space to this sphere is spanned by the vectors $\hat{m}^\mu$ and $\hat{N}^\mu$.

$$\hat{m}^\mu \partial_\mu = P (\partial_2 + i \partial_3) = -2 \partial_3, \quad \hat{m}^\mu \partial_\mu = -2 \partial_3. \quad \text{The null complex tetrad field} (\hat{e}^\mu_i, \hat{e}^\mu_j, \hat{e}^\mu_k, \hat{e}^\mu_l) \text{can be introduced in the neighborhood of} \mathcal{J} \text{in such a way that} [21]$$

$$\hat{\chi} = \hat{\rho} = \hat{\phi} = \hat{\zeta} = 0, \quad \hat{\xi} = \sigma^0 + O(\mathcal{O}^2), \quad \hat{\mu} = \sigma^0 + \mathcal{O}^2, \quad \hat{\nu} = 0(\mathcal{O}^2),$$

$$\hat{\tau} = - (2 \partial_3 \sigma^0 + 4 \alpha^0 \sigma^0) \Omega + O(\mathcal{O}^2), \quad \hat{\lambda} = \delta^0 \sigma^0 \Omega + O(\mathcal{O}^2), \quad \hat{\gamma} = 4 \partial_3 \sigma^0 \Omega + O(\mathcal{O}^2),$$

$$\hat{\alpha} = \alpha^0 - (2 \partial_3 \sigma^0 + 3 \delta^0 \sigma^0) \Omega + O(\mathcal{O}^2), \quad \hat{\beta} = - \alpha^0 - \alpha^0 \sigma^0 \Omega + O(\mathcal{O}^2),$$

where $\alpha^0 = - \frac{2}{3} \partial_3$ and $\partial_3$ is a tangent vector to the $\mu = \text{const.}$ null surface.
The condition $\xi = \eta = \zeta = 0$ shows that the null complex tetrad is parallelly propagated along the $\mathcal{Y}$ integral curves and the condition $\Phi |_\mathcal{Y} = \Phi' |_\mathcal{Y} = \mathcal{Y}' |_\mathcal{Y}$ implies that this tetrad is also parallelly propagated along the $\mathcal{Y}$ generators.

The admissible combined coordinate and conformal transformations preserving all the conditions given above are

$$u' = K(\xi, \xi) (u + \alpha(\xi, \xi)), \quad r' = K(\xi, \xi) r,$$

$$\xi' = \frac{a\xi + \beta}{c\xi + \delta}, \quad \Omega' = K(\xi, \xi) \Omega,$$  \hspace{1cm} (4.5)

where

$$K(\xi, \xi) = \left( 1 + \frac{\beta}{\delta} \right) \left( 1 + \frac{\xi + \delta}{\xi + a} \right)^{-1}. \hspace{1cm} (4.6)$$

b) Scalar field

Let $(\widetilde{M}, \tilde{g}, \tilde{h})$ be the Penrose space corresponding to a given space-time $(M, g)$ and $\Phi$ be the image of the complex asymptotically regular solution $f$ of the field equation

$$\Box f - m^2 f = 0,$$ \hspace{1cm} (4.7)

i.e. the limit of $\tilde{f} = \Omega^{-1} f$ on $\mathcal{J}$. We shall define the image $\tilde{\Phi}$ of the quantum field $\Phi$ on $\mathcal{J}$ by the condition

$$\tilde{\Phi}(\tilde{F}) = \langle \tilde{F}, \tilde{\Phi} \rangle = \langle a \tilde{f}, \tilde{\Phi} \rangle = \langle \tilde{f}, \tilde{\Phi} \rangle$$ \hspace{1cm} (4.8)

for an arbitrary asymptotically regular function $\tilde{f}$. The commutation relations of the operators $\tilde{\Phi}$ on $\mathcal{J}$ can immediately be obtained from (3.30) in the form

$$[\tilde{\Phi}(\tilde{F}_1), \tilde{\Phi}(\tilde{F}_2)] = \langle \tilde{F}_2, \tilde{\Phi} \rangle - \langle \tilde{F}_1, \tilde{\Phi} \rangle \hspace{1cm} (4.9)$$

In the conformal Bondi co-ordinates

$$\langle \tilde{F}_1, \tilde{\Phi} \rangle = \int du d\omega \tilde{F}_1 e^{2\varphi} \tilde{\Phi}, \hspace{1cm} \langle \tilde{F}_2, \tilde{\Phi} \rangle \hspace{1cm} (4.10)$$

where $d$ is a surface element on a unit sphere and the explicit form of the commutation relations of $\tilde{\Phi}$ on $\mathcal{J}$ in these co-ordinates is

$$[\tilde{\Phi}(u, \sigma), \tilde{\Phi}(u', \sigma')] = -\frac{i}{2} e(u - u') \delta(\sigma - \sigma'), \hspace{1cm} (4.11)$$

where $\delta(\sigma - \sigma')$ is the invariant $\delta$ function on the unit sphere (compare with Eqs.(3.2) and (3.20)).

c) Electromagnetic field

Under the conformal transformations the electromagnetic field potentials $\Phi$, do not change $\tilde{\Phi} = \Phi$ and the condition of asymptotic regularity of this field implies that such a choice of the gauge exists in which the field $\tilde{\Phi}$ is smooth at the $\mathcal{J}$. The available gauge transformations

$$\tilde{\Phi} = \tilde{\Phi} + \left( \frac{\partial}{\partial u} \right) \lambda, \hspace{1cm} (4.12)$$

where $\lambda$ is a smooth finite function on $\mathcal{J}$ can be used to satisfy the conditions (compare (3.39)):

$$\tilde{\Phi}(\mathcal{J}) = 0 \quad \text{on and outside } \mathcal{J}, \hspace{1cm} (4.13)$$

$$\tilde{\Phi}(\mathcal{J}) = \tilde{\Phi}(\mathcal{J}) \quad \text{on } \mathcal{J}. \hspace{1cm} (4.14)$$

If $\Phi$ is a complex classical asymptotically regular solution of the Maxwell equations then $\tilde{\Phi}$ and $\tilde{\Phi}^+$ are two arbitrary complex functions. We denote $\Omega = (\tilde{\Phi}, \tilde{\Phi}^+)$ and call $\Omega$ the image of the field $\Phi$ on $\mathcal{J}$. In the Bondi co-ordinates the inner product $\langle \epsilon_1, \epsilon_2 \rangle$ is given by the following expression:

$$\langle \epsilon_1, \epsilon_2 \rangle = i \int d\sigma d\omega (\tilde{A}_1 \partial_u \tilde{A}_2 - \partial_u \tilde{A}_1 \tilde{A}_2) + (\tilde{A}_1^+ \partial_u \tilde{A}_2^+ - \partial_u \tilde{A}_1^+ \tilde{A}_2^+). \hspace{1cm} (4.15)$$

For the quantum field $\tilde{\Phi}$ we have in the same manner on $\mathcal{J}$

$$\tilde{\Phi}_u = \tilde{\Phi}_u + \tilde{\Phi}_u^+ \tilde{\Phi}_u, \hspace{1cm} (4.16)$$

where $\tilde{\Phi}_u^+$ is the hermitian conjugate to operator $\tilde{\Phi}_u$. Eq.(3.55) gives the following commutation relations for the operators $\tilde{\Phi}_u = (\tilde{A}_u \tilde{A}_u^+)$ on $\mathcal{J}$:

$$\tilde{\Phi}_u = \tilde{\Phi}_u + \tilde{\Phi}_u^+ \tilde{\Phi}_u, \hspace{1cm} (4.17)$$
Gravitational perturbations

The investigation of gravitational perturbations is more complicated.

Let \( h_{ab} \) be the asymptotically regular complex classical solution of Eq. (3.60)

\[
[r_{ab} - h_{ab} = 0],
\]

where

\[
r_{ab} \equiv \nabla_{(a} r_{b)} - \nabla_{(a} r_{b)} = 0.
\]

Eqs. (A.3) and (A.12) of the Appendix allow us to show that the conformally transformed field \( \hat{h}_{ab} = \hat{h}_{ab} \) in the Penrose space (\( \hat{g}_{ab} \)) satisfy the equation

\[
r_{ab} - \hat{h}_{ab} = 0,
\]

where

\[
\hat{r}_{ab} = \hat{\nabla}_{a} \hat{r}_{b} - \hat{\nabla}_{a} \hat{r}_{b},
\]

\[
\hat{r}_{ab} = \frac{1}{2} \hat{g}^{\gamma\lambda}(\hat{\nabla}_{a} \hat{h}_{\lambda b} + \hat{\nabla}_{b} \hat{h}_{\lambda a} - \hat{\nabla}_{\lambda} \hat{h}_{ab}),
\]

\[
\hat{r}_{ab} = \frac{1}{2} \hat{g}^{\gamma\lambda}(\hat{\nabla}_{a} \hat{h}_{\lambda b} + \hat{\nabla}_{b} \hat{h}_{\lambda a} - \hat{\nabla}_{\lambda} \hat{h}_{ab}).
\]

The regularity of the solution shows that near \( \hat{g} \)

\[
p_{ab}^{(\text{reg})} = 0(n),
\]

\[
p_{ab}^{(\text{reg})} = 0(n),
\]

Taking into account Eq. (4.1) we can write (4.26) in the following form:

\[
-\Omega_{a0} \hat{h}_{\gamma b} + \Omega_{0 b} \hat{h}_{\gamma a} + 2 \hat{g}_{ab} \hat{g}^{\gamma \mu} \hat{h}_{\mu} = 0(n).
\]

The contraction of this expression with \( \hat{g}^{ab} \) gives

\[
p_{ab}^{(\text{reg})} \hat{r}_{ab} = 0(n),
\]

hence
and, because $\tilde{R}_\alpha$ does not vanish on $\mathcal{J}$, finally we have that near $\mathcal{J}$

$$\tilde{R}_\alpha^{IA} = O(\eta) .$$

(4.28)

If we denote $f = \lim_{\eta \to 0} \tilde{R}_\alpha^{IA}$ then Eq. (4.27) gives in the same manner that

$$f = \frac{1}{3} (\tilde{\phi} \tilde{\omega} - \tilde{\phi} \omega) .$$

(4.29)

The gauge transformations (3.66) in the Penrose space are of the form

$$\tilde{h}_{\mu \nu} \rightarrow \tilde{h}'_{\mu \nu} = \tilde{h}_{\mu \nu} - \Omega (\tilde{\partial}_\mu \tilde{\xi}_\nu + \tilde{\partial}_\nu \tilde{\xi}_\mu) - \left( \Omega \mu \tilde{\xi}_\nu + \Omega \nu \tilde{\xi}_\mu - \tilde{g}_{\mu \nu} \Omega^\lambda \tilde{\xi}_\lambda \right) ,$$

(4.30)

and if $\tilde{\xi}_\nu$ is a smooth finite vector field at $\mathcal{J}$, this transformation does not violate Eq. (4.28). The gauge freedom (4.30) can be used to satisfy the condition

$$\tilde{h}_{\mu \nu} = 0$$

(4.31)

near and on $\mathcal{J}$ surface. The available gauge transformations $\tilde{\xi}_\nu$ after the fulfillment of these conditions obey the following equation on $\mathcal{J}$:

$$\tilde{\xi}_\nu + \Omega \nu \tilde{E}^\mu \tilde{\xi}_\mu - \tilde{\xi}_\nu \Omega^\mu \tilde{\xi}_\mu = 0 ,$$

i.e.

$$\tilde{\xi}_\nu |_{\mathcal{J}} = \tilde{\xi} (u, x^A) \tilde{E}_\nu .$$

(4.32)

If we now note that the transformation of the trace $\tilde{h}$ of the field $\tilde{h}_{\alpha \beta}$ on $\mathcal{J}$ is

$$\tilde{h} \rightarrow \tilde{h} + 2\alpha \tilde{\xi}_A ,$$

then it is easy to see that the remaining freedom (4.32) can be used to put

$$\tilde{h} |_{\mathcal{J}} = 0 .$$

The above consideration can be summarized as follows. Such a choice of gauge exists in which Eq. (4.31) is satisfied and on $\mathcal{J}$ one has

$$\tilde{h}_\alpha = \tilde{h}_\alpha^\beta \tilde{\xi}_\beta + \tilde{h}_\alpha^\beta \tilde{\xi}_\beta .$$

(4.33)

The pair $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2)$ of complex functions determines the asymptotically regular solution $\tilde{h}_{\alpha \beta}$ unambiguously and is called the image of $\tilde{h}_{\alpha \beta}$ on $\mathcal{J}$. In the conformal Bondi co-ordinates the inner product $\langle \mathcal{H}_1, \mathcal{H}_2 \rangle$ is

$$\langle \mathcal{H}_1, \mathcal{H}_2 \rangle = \int du \, ds \left( \tilde{H}_1 \tilde{H}_2 - \partial_u \tilde{H}_1 \tilde{H}_2 + \tilde{H}_1^+ \partial_u \tilde{H}_2^+ - \partial_u \tilde{H}_1^+ \tilde{H}_2^+ \right) ,$$

(4.34)

If we write for the quantum field $\tilde{h}_{\alpha \beta}$

$$\tilde{\xi}_\alpha \tilde{h}_{\alpha \beta} = \tilde{\xi}_\alpha \tilde{h}_{\alpha \beta} ,$$

(4.35)

then Eq. (3.22) gives

$$[ \mathcal{H} (\mathcal{H}_1), \mathcal{H} (\mathcal{H}_2) ] = \langle \mathcal{H}_2, \mathcal{H}_1 \rangle ,$$

(4.36)

where $\mathcal{H} (\mathcal{H}_1) \equiv \langle \mathcal{H}_1, \mathcal{H}_1 \rangle$. In the conformal Bondi co-ordinates we have

$$[ \mathcal{H} (u, \mathcal{H}_1), \mathcal{H} (u', \mathcal{H}_2) ] = [ \mathcal{H}^+ (u, \mathcal{H}_1), \mathcal{H}^+ (u', \mathcal{H}_2) ] = 0 ,$$

$$[ \mathcal{H} (u, \mathcal{H}_1), \mathcal{H}^+ (u', \mathcal{H}_2) ] = -\frac{i}{2} \left( E(u-u') \delta (u', \mathcal{H}_2) \right) .$$

(4.37)
Asymptotic symmetry group and asymptotic invariants

1) Asymptotic symmetry transformations in the conformal Bondi frame

The investigation [20] of Eq. (2.48) describing the generators \( T^\mu \) of the asymptotic symmetry transformations on \( J \) (asymptotic Killing vector field) shows that its solutions can be written in the conformal Bondi coordinates as follows:

\[
\xi^\mu = 0, \quad \xi^0 = \frac{1}{2} u z^A A + \alpha (x^A), \quad \xi^A = z^A (x^A),
\]

(4.38)

where \( z_{A:B} \) denotes a covariant derivative of \( z_A \) on the unit sphere \( S^2 \) with respect to its metric \( g_{AB} \) and \( z^A \) is a conformal Killing vector field on \( S^2 \).

It is well known that the group of conformal motions on \( S^2 \) is isomorphic to the restricted Lorentz group. The finite transformations on \( J \) generated by the asymptotic Killing vector fields (4.38) are described by (4.5). The transformations satisfying the additional condition \( \xi^0 = 0 \) are known as supertranslations. The subgroup of translations consists of the transformations (4.5) with \( \xi^0 = 0 \) and \( \alpha \) are constants. The asymptotic Killing vector fields generated by the translation subgroup are of the form

\[
\xi^\rho = \frac{1}{2} \xi_{AB} z^C.
\]

(4.41)

The asymptotic Killing Eqs. (2.45) also give that

\[
(\partial^{-1} \xi^\mu \partial_{\mu})|_d = \frac{1}{2} \xi^A A.
\]

(4.42)

b) Asymptotic invariants

Let \( T^\mu = T^\mu + \tilde{T}^\mu \) be any asymptotic Killing vector field, then we define an asymptotic invariant \( P[\xi] \) corresponding to this vector field as follows:

\[
P[\xi] = \int \frac{\zeta^A}{y} e^\mu \delta^\mu y,
\]

(4.43)

where \( \tilde{T}_{\mu \nu} = \gamma^2 T_{\mu \nu} \) and \( T_{\mu \nu} \) is the energy-momentum tensor of the field under consideration. When \( \xi^\mu \) are taken to be generators of the translation subgroup then these quantities describe the incoming (for \( \xi^0 < 0 \)) or outgoing (for \( \xi^0 > 0 \)) fluxes of the energy momentum (compare with (2.18)).

The energy-momentum tensor corresponding to the action (3.12) for the massless scalar field is

\[
T_{\mu \nu} = (2\alpha + 1) \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} (\alpha^2 - \alpha + 1) \partial_\mu \Phi \partial_\nu \Phi + 2\alpha \partial_\mu \Phi \delta^\nu \partial_\nu \Phi + \frac{1}{2} \delta (R_{\mu \nu} - \frac{1}{g} g_{\mu \nu} R) g^2
\]

(4.44)

and the corresponding asymptotic invariants are of the form

\[
P[\xi] = \frac{\sqrt{\gamma} \alpha}{2} \left[ (2\alpha + 1) \partial_\mu \Phi \xi^\mu \partial_\mu \Phi + 2\alpha \Phi \xi^\mu \partial_\mu \Phi \right] d\Sigma,
\]

(4.45)

where the integrand is written in the conformal Bondi co-ordinates and \( \Phi \) is the image on \( J \) of the scalar field \( \Phi \).

In the case of the electromagnetic field (3.32) the energy-momentum tensor is

\[
T_{\mu \nu} = -g^{\nu \rho} \epsilon_{\mu \alpha} \epsilon_{\nu \beta} + \frac{1}{4} g_{\mu \nu} \epsilon_{\alpha \beta} \,
\]

(4.46)

so we have
\[ \hat{T}_{\mu\nu} \equiv \Omega^2 T_{\mu\nu} = -\hat{g}^{\alpha\beta} \hat{f}_{\mu\alpha} \hat{f}_{\nu\beta} + \frac{1}{4} \hat{g}_{\mu\nu} \hat{f}_{\alpha\beta} \hat{f}^{\alpha\beta} \]

and the corresponding asymptotic invariant is
\[ P(\xi) = \int \hat{g}^{\alpha\beta} \hat{f}_{\mu\alpha} \hat{f}_{\nu\beta} d\alpha d\beta, \]

where \( \hat{f} = \hat{g}^{\nu\beta} \hat{f}_{\nu\beta} \). If we take into account that
\[ (\hat{V}_\mu \hat{m}_\nu - \hat{V}_\nu \hat{m}_\mu) |_g = 2 \alpha^o (\hat{m}_\mu \hat{m}_\nu - \hat{m}_\nu \hat{m}_\mu) |_g \]

and use (4.14) we can then find
\[ \hat{p}_\hat{\beta} = \hat{n}^{\alpha\nu} \partial^\alpha A^+ \hat{m}_\beta + \hat{n}^{\alpha\nu} \partial^\alpha A \hat{m}_\beta, \]
\[ \hat{m}^{\alpha} \hat{e}^{\mu} \hat{f}_{\mu\alpha} = -\xi^{\alpha} \partial^\alpha A - \hat{m}^{\alpha} \partial^\alpha A \hat{\xi}^{\mu} \hat{m}_\mu \]
\[ -\hat{m}^{\alpha} \partial^\alpha A \hat{\xi}^{\mu} \hat{m}_\mu + 2 (\alpha^o A^+ - \alpha^o A) \xi^{\mu} \hat{m}_\mu = \]
\[ = -\xi^{\alpha} \hat{n}^{\mu\nu} \partial^\nu A - \eta (\alpha^+ - \alpha^+ \alpha^o). \]

In the last equality we have employed the \( \partial^\alpha \) notation of Newman and Penrose [23]. Let us recall that for a quantity \( z \) of spin weight \( s \)
\[ \partial z = 2 (\partial_+ z + s \partial^o z), \]
\[ \bar{\partial} z = 2 (\partial^- z - s \partial^o z), \]

(4.17)
where $u$ is the Bondi time parameter on $\mathcal{I}^+$ and $F_\omega(x^A)$ is an arbitrary function. The out-vacuum state $|0;\text{out}\rangle$ introduced by Eq.(4.50) satisfies the equations

$$\Psi_{\text{out}}(\Psi_{\text{out}})^*|0;\text{out}\rangle = 0,$$  \hspace{1cm} (4.52)

where $\Psi_{\text{out}}(\Psi_{\text{out}}^*) = \langle \Psi_{\text{out}},\Psi_{\text{out}}^* \rangle$ and $\Psi_{\text{out}}$ is an arbitrary positive frequency function. Eq.(4.52) can be taken as a definition of $|0;\text{out}\rangle$, which is equivalent to definition (4.50). We shall call the set $\{\Psi_{\text{out}},\Psi_{\text{out}}^*\}$ a basis on $\mathcal{I}^+$ if the functions $\Psi_{\text{out}}$ are of positive frequency and the following normalization conditions are satisfied:

$$\langle \Psi_{\text{out}},\Psi_{\text{out}}^* \rangle = \delta_{\omega,\omega}, \hspace{1cm} \langle \Psi_{\text{out}},\Psi_{\text{out}}^* \rangle = -\delta_{\omega,\omega},$$

$$\langle \Psi_{\text{out}},\Psi_{\text{out}}^* \rangle = \langle \Psi_{\text{out}},\Psi_{\text{out}}^* \rangle = 0.$$  \hspace{1cm} (4.53)

We shall also assume this basis to be complete, i.e. every "good" function $Q$ on $\mathcal{I}^+$ allows the expansion

$$Q(u,x^A) = \sum_{\alpha}(a^\alpha \Phi_{\text{out}}(u,x^A) + a_\alpha^\dagger \Phi_{\text{out}}(u,x^A)).$$

For the operator $Q_{\text{out}}$ we can write

$$\Phi_{\text{out}}(u,x^A) = \sum_{\alpha}(F_{\text{out}}(u,x^A)Q_{\text{out}},\alpha(u,x^A) + F_{\text{out}}(u,x^A)Q_{\text{out}},\alpha(u,x^A)^*),$$  \hspace{1cm} (4.54)

where $Q_{\text{out}},\alpha = \langle F_{\text{out}},\alpha,\text{out}\rangle$, $Q_{\text{out}},\alpha^* = -\langle F_{\text{out}},\alpha,\text{out}\rangle$ and the condition (4.52) is equivalent to

$$Q_{\text{out}},\alpha|0;\text{out}\rangle = 0.$$  \hspace{1cm} (4.55)

The commutation relations of the operators $\{Q_{\text{out}},\alpha,\Phi_{\text{out}},\alpha\}$ can be derived from Eq.(4.49)

$$[Q_{\text{out}},\alpha,\Phi_{\text{out}},\beta] = \delta_{\alpha,\beta},$$

$$[Q_{\text{out}},\alpha,\Phi_{\text{out}},\beta] = \delta_{\alpha,\beta},$$

$$[Q_{\text{out}},\alpha,\Phi_{\text{out}},\beta] = [Q_{\text{out}},\alpha,\Phi_{\text{out}},\beta] = 0.$$  \hspace{1cm} (4.56)

$n$-particle states can be introduced in the usual way by the relation

$$|e_1,\ldots,e_n;\text{out}\rangle = \sum_{\alpha_1,\ldots,\alpha_n}^+ \Phi_{\text{out}},\alpha_1,\ldots,\alpha_n^+|0;\text{out}\rangle.$$  \hspace{1cm} (4.57)

These vectors describe the states with $n$ outgoing particles and they form the basis in the Hilbert space $\mathcal{H}_{\text{out}}$ of the out-states.

In a similar manner the Hilbert space $\mathcal{H}_{\text{in}}$ of the in-states can be constructed. The corresponding basis of the positive frequency normalized functions on $\mathcal{I}^-$ are denoted by $\{P_{\text{in}},\alpha,\Phi_{\text{in}},\beta\}$ and the in-particle creation and annihilation operators are given by

$$Q_{\text{in}},\alpha = -\langle P_{\text{in}},\alpha,\Phi_{\text{in}}^* \rangle, \hspace{1cm} Q_{\text{in}},\alpha = \langle P_{\text{in}},\alpha,\Phi_{\text{in}} \rangle.$$  \hspace{1cm} (4.58)

As usual we denote the solution of Eq. (3.13) in the space-time $(M,g)$ by $p_{\alpha}(x)$, the image on $\mathcal{I}^-$ of which is $p_{\alpha}(x^A)$ and we have

$$q_{\alpha} = \sum_{\alpha}(p_{\alpha} Q_{\text{out}},\alpha + \bar{p}_{\alpha} Q_{\text{out}},\alpha) = \sum_{\alpha}(p_{\alpha} Q_{\text{out}},\alpha + \bar{p}_{\alpha} Q_{\text{out}},\alpha).$$  \hspace{1cm} (4.59)

This equation allows us to find the relation between in- and out-operators in the form

$$Q_{\text{in}},\alpha = \sum_{\beta}(A_{\alpha,\beta} Q_{\text{out}},\beta + B_{\alpha,\beta} Q_{\text{out}},\beta),$$  \hspace{1cm} (4.60)

where

$$A_{\alpha,\beta} = \sum_{\gamma}(a_{\alpha,\gamma}^\dagger a_{\gamma,\beta}^\alpha + a_{\alpha,\beta}^\dagger a_{\gamma,\gamma}^\alpha),$$

$$B_{\alpha,\beta} = \sum_{\gamma}(a_{\alpha,\gamma} a_{\gamma,\beta}^\dagger + a_{\alpha,\beta} a_{\gamma,\gamma}^\dagger).$$
\[ A_{\alpha\beta} = \langle \alpha, f_\beta \rangle, \quad B_{\alpha\beta} = \langle \alpha, f_\beta \rangle. \] (4.61)

Because the in-operators \( \mathbf{a}_n \) and \( \mathbf{a}_\alpha \) satisfy the same commutation relations (4.6) as the out-operators \( \mathbf{a}_\alpha' \) and \( \mathbf{a}_n' \), the transformation (4.60) is canonical and the matrices \( A \) and \( B \) satisfy the conditions (4.6):

\[ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \begin{pmatrix} A^\ast & -B^\ast \\ -B^\ast & A^\ast \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \] (4.62)

where the prime denotes the transposition, the bar denotes the complex conjugation and the * star is used for the hermitian conjugation of the matrices.

b) \( S \) matrix

If the sets of operators \( \{ \mathbf{a}_n, \mathbf{a}_\alpha \} \) and \( \{ \mathbf{a}_\alpha', \mathbf{a}_n' \} \) form the full systems and the spaces \( \mathcal{H}_n \) and \( \mathcal{H}_\alpha \) coincide, then the unitary operator \( \mathbf{S} \) (called \( S \) matrix) must exist and possess the following properties:

\[ \mathbf{S}_{\alpha n} = \mathbf{S}_{n \alpha} \mathbf{S}^\ast, \quad \mathbf{S} = \mathbf{S}^{-1}. \] (4.63)

It is evident that

\[ |0;in\rangle = \mathbf{S} |0;out\rangle \] (4.64)

and

\[ \langle \beta_1, \ldots, \beta_n;out | \alpha_1, \ldots, \alpha_m;in\rangle = \mathbf{S}_{\beta_1 \ldots \beta_n \alpha_1 \ldots \alpha_m} = \langle 0;out | \mathbf{a}_n', \mathbf{a}_\alpha | 0;out\rangle. \] (4.65)

The comparison of Eqs. (4.69) and (4.63) allows one to find the following explicit expression for the \( S \) matrix operator through \( A \) and \( B \) matrices [26].

\[ S = \theta [\text{det}(AA^\ast)]^{-\frac{1}{2}} \exp\left(\frac{1}{2} a^\dagger V a^\ast\right). \] (4.66)

where the following abbreviation is used:

\[ \mathbf{a}^\dagger V a^\ast = \sum_{\alpha, \beta} a^\dagger_{\alpha \beta} V(a^\dagger_{\alpha \beta} a^\ast_{\alpha \beta}), \quad \text{etc.,} \] (4.67)

and

\[ V = -A^\ast B, \quad M = A^\ast - I, \quad \Lambda = \overline{B} A^\ast, \]

\[ |\theta| = 1, \quad \Lambda' = \Lambda, \quad V' = V. \] (4.68)

Eqs. (4.64) and (4.66) give

\[ |0;in\rangle = \theta [\text{det}(AA^\ast)]^{-\frac{1}{2}} \exp\left(\frac{1}{2} a^\dagger V a^\ast\right) |0;out\rangle \] (4.69)

and

\[ e^{i W_0} = \langle 0;out | 0;in\rangle = \theta [\text{det}(AA^\ast)]^{-\frac{1}{2}}. \] (4.70)

It can also easily be shown that
In the general case the $S$-matrix elements can be given in the following convenient form [7]:

$$S_{\beta_1 \ldots \beta_n \alpha_1 \ldots \alpha_m} = \frac{\delta^m}{\delta j_{\beta_1} \ldots \delta j_{\beta_n}} \exp \left( i \int \omega(j, j^*) \right) \Bigg|_{j=j^*=0}$$

where

$$\exp \left( i \int \omega(j, j^*) \right) = \exp \left( i \omega_0 \right) \exp \left( \frac{1}{2} \int V j^* j (M+I) j + \frac{1}{2} \lambda j j \right).$$

The simplest way to derive this expression is by using the functional representation of the secondary quantized operators [2]. It should be recalled that one can put in correspondence to every operator $K$ written in the normal form

$$K = \sum_{m,n} K_{mn} (\alpha_1, \ldots, \alpha_m | \beta_1, \ldots, \beta_n) a^*_{\beta_1} \ldots a^*_{\beta_n} a^*_{\alpha_1} \ldots a^*_{\alpha_m}$$

and to every vector $\Psi$

$$\Psi = \sum_{n, (\alpha)} \Psi_n (\alpha_1, \ldots, \alpha_n) a^+_{out, \alpha_1} \ldots a^+_{out, \alpha_n} |0; out>$$

the functionals

$$K(a^* a) = \sum_{m,n} K_{mn} (\alpha_1, \ldots, \alpha_m | \beta_1, \ldots, \beta_n) a^*_{\beta_1} \ldots a^*_{\beta_n} a_{\alpha_1} \ldots a_{\alpha_m}$$

$$\Psi(a^*) = \sum_{n, (\alpha)} \Psi_n (\alpha_1, \ldots, \alpha_n) a^*_{\alpha_1} \ldots a^*_{\alpha_n}$$

so that the following functional corresponds to the result of the action of the operator $K$ on the vector $\Psi$:

$$\Psi'_2 (a^*) = \int K(a^* a) \Psi(b^*) e^{-\int (b^* a^*)} d\beta d b^*$$

and the scalar product of any two vectors $\Psi_1$ and $\Psi_2$ is given by the following functional integral:

$$\langle \Psi_1, \Psi_2 \rangle = \int \Psi_1^*(a^*) \Psi_2(a^*) e^{-\int a^* a} d\alpha d a^*.$$

Here and later on, the matrix designations are used so that, for example, $a^* a$ means

$$a^* a = \sum_{\alpha} a^*_{\alpha} a_{\alpha}.$$
The above construction of the quantum scattering theory in an asymptotically simple space-time including the vacuum state definition, the explicit expression for the S-matrix operator and the functional representation (4.72) for the S-matrix elements allows the evident extension to the case of the electromagnetic and gravitational fields.

4.4 Some remarks on black hole evaporation

a) Scattering theory in weakly asymptotically simple space-time

We now consider the case when the space-time is not asymptotically simple but is only weakly asymptotically simple (for example, it contains one or more black holes or some singularities). We suppose the $f^-$ boundary for example, surface to be the total Cauchy surface for the massless fields, i.e., all the black holes and singularities in the space-time are not external but are created as a result of the matter development. In this case the $f^+$ surface is not the total Cauchy surface and the additional data on the event horizons and near the singularities are needed to specify uniquely the field in the space-time. While the initial vacuum state $|O;\text{in}\rangle$ and the creation and annihilation operators of in-particles on $f^-$ can be defined in the same manner as in the previous section, some difficulties arise when one considers the out-vacuum definition.

Let us suppose that the total future Cauchy surface is

\[ L^+ = J^+ U K^+, \]

and let us introduce the out-basis \{\nu_a, \nu_{\dot{a}}, \bar{\nu}_i, \bar{\nu}_{\dot{i}}\} in the space of the complex solutions of Eq.(3.13) which satisfies the following conditions:

\[ \hat{\nu}_a | \mathcal{K}^+ = 0, \quad \hat{u}_x | \mathcal{K}^+ = 0, \]

\[ \langle \nu^a, \nu_{\dot{b}} \rangle = \delta^a_{\dot{b}}, \quad \langle \nu^a, \bar{\nu}_{\dot{b}} \rangle = 0, \]

\[ \langle u^x, u_y \rangle = \delta^x_y, \quad \langle u^x, \bar{\nu}_{\dot{y}} \rangle = 0. \]
As a consequence of \( (4.82) \) we have
\[
\langle u_\alpha, u_x \rangle = i \int \frac{\partial}{\partial \mu} \hat{u}_x \, d \Sigma_{\mu} = i \int \frac{\partial}{\partial \mu} \hat{u}_x \, d \Sigma_{\mu} = 0.
\]
\( (4.85) \)

The corresponding annihilation and creation operators can be introduced as follows:

\[
\zeta_{\alpha} = \langle u_\alpha, \phi \rangle, \quad \zeta_{\alpha}^+ = -\langle \bar{u}_\alpha, \phi \rangle,
\]
\[
\zeta_x = \langle u_x, \phi \rangle, \quad \zeta_x^+ = -\langle \bar{u}_x, \phi \rangle.
\]
\( (4.86) \)

If we take the functions \( u_\alpha \) to be of positive frequency with respect to the Bondi time coordinate on \( S^+ \), then the operators \( \zeta_{\alpha} \) and \( \zeta_{\alpha}^+ \) are the operators of the physical outgoing particle annihilation and creation.

Any physical observable \( \mathcal{A} \) on \( S^+ \) can be expressed through these operators: \( \mathcal{A} = \mathcal{A} (\zeta_{\alpha}^+ \zeta_{\alpha}) \), and if we consider the calculation of its average in some in-state \( |\psi_{\text{in}}\rangle \), it can be shown that the value of \( \langle \psi_{\text{in}}| \mathcal{A} (\zeta_{\alpha}^+ \zeta_{\alpha}) |\psi_{\text{in}}\rangle \) does not depend on the choice of vacuum state on the \( K^+ \) surface \( [25] \). In fact we can express \( |\psi_{\text{in}}\rangle \) vector through the out-state vectors as follows:

\[
|\psi_{\text{in}}\rangle = \sum_{n,m} \psi_{nm} |n; \mathcal{F}^+ \rangle |m; \mathcal{K}^+ \rangle.
\]
\( (4.87) \)

where \( n = (n_1, \ldots, n_n), = (x_1, \ldots, x_n) \) and

\[
\psi_{nm} = \sum_{n,m} \langle n; \mathcal{F}^+ | \zeta_{\alpha}^+ | m; \mathcal{K}^+ \rangle \psi_{nm} \]

and the vacuum states \( |2; \mathcal{F}^+ \rangle \) and \( |0; \mathcal{K}^+ \rangle \) are defined by conditions

\[
\zeta_{\alpha} |0; \mathcal{F}^+ \rangle = 0, \quad \zeta_x |0; \mathcal{K}^+ \rangle = 0.
\]
\( (4.89) \)

We have
\[
\langle \psi_{\text{in}}| \mathcal{A} (\zeta_{\alpha}^+ \zeta_{\alpha}) |\psi_{\text{in}}\rangle = \sum_{n,m} \overline{\psi}_{nm} \psi_{nm}.
\]
\( \psi_{nm} = \sum_{n,m} \langle n; \mathcal{F}^+ | \zeta_{\alpha}^+ | m; \mathcal{K}^+ \rangle \psi_{nm} \]

where \( R_{nm} = \sum_{n,m} \langle n; \mathcal{F}^+ | \zeta_{\alpha}^+ | m; \mathcal{K}^+ \rangle \). It can easily be seen that \( R_{nm} \) does not change under the unitary transformation in the Hilbert space \( H_{K^+} \) formed by the vectors \( |n; \mathcal{K}^+ \rangle \) and consequently it does not depend on the particular choice of the vacuum state \( |0; \mathcal{K}^+ \rangle \) in \( H_{K^+} \).

If we introduce the density matrix \( \hat{\rho}^+ \) by the equation
\[
\hat{R}_{nm} = \langle n; \mathcal{F}^+ | \hat{\rho}^+ | m; \mathcal{F}^+ \rangle,
\]
we then have
\[
\langle \psi_{\text{in}}| \mathcal{A} (\zeta_{\alpha}^+ \zeta_{\alpha}) |\psi_{\text{in}}\rangle = \hat{\rho}^+ \mathcal{A} (\hat{\rho}^+ \mathcal{A}^+),
\]
\( (4.90) \)
where $S_{p^+}$ denotes that the trace must be taken over the complete set of states on $g^+$. 

b) Density matrix calculation

To find the explicit expression for the density matrix $\rho$, it is convenient to use again the functional representation of the secondary quantized operators. We calculate the $p_0$ operator defined by the equation

$$S_{p^+}(\rho_0\alpha) = \langle 0; in | \alpha (\xi^+, \xi^-) | 0; in \rangle \equiv \langle \alpha \rangle .$$

(4.91)

If we rewrite $\langle \alpha \rangle$ in the form

$$\langle \alpha \rangle = \langle 0; out | S^+ \alpha S^- | 0; out \rangle ,$$

(4.92)

and remember that the functional corresponding to the $S$-matrix operator is given by (4.71), we then have

$$\langle \alpha \rangle = \int \exp \left( \frac{i}{\hbar} \alpha V^* \alpha + \frac{i}{2} a^* a V a + a^* a - a^* \alpha \right) \Omega(\alpha, a) \prod \delta a \delta a^* \delta dx \delta x^* ,$$

(4.93)

where $\Omega(\alpha, a)$ is a functional corresponding to the normal form of the operator $\Omega(\xi^+, \xi^-)$ and $a = (c^a_x, b^a_x), a = (\psi^a_x, b^a_x)$. Now it is convenient to restore the index $a(x)$ corresponding to the particles visible (invisible) at $g^+$ and write down the abbreviated quantities in more detail:

$$a^* a = c^* a c_a + b^* x b_x , \quad \alpha^* a^* V a = c^* a V a, c^* \beta_x^* + c^* a V a^* c_g + b^* x V a^* c_a + b^* y V x a^* b_x^* , \text{ etc.}$$

If we denote $V_{ab} = V_{ab}, V_{ax} = V_{ax}, V_{xy} = V_{xy}$, then

$$V = \begin{pmatrix} U & W \\ W^* & Z \end{pmatrix}$$

(4.94)

and the functional integral (4.93) is equal to

$$\langle \alpha \rangle = \int \exp \left[ \frac{i}{\hbar} \alpha^* G \alpha + \frac{i}{2} \alpha^* H \alpha + \frac{i}{2} \alpha^* \beta \beta \right] \Omega(\alpha^*, \alpha) \prod \delta d \delta b \delta c \delta d \delta b^* \delta \gamma d \delta \gamma^* .$$

(4.95)

After integration over $b, b^*, \beta$ and $\gamma^*$, one can find

$$\langle \alpha \rangle = D \int \exp \left[ \frac{i}{\hbar} \alpha^* G \alpha + \frac{i}{2} \alpha^* L \alpha + \frac{i}{2} \alpha^* H \alpha + \alpha^* \beta \beta \right] \Omega(\alpha^*, \alpha) \prod \delta d \delta b \delta c \delta d \delta b^* \delta \gamma d \delta \gamma^* ,$$

(4.96)

where

$$D = [\det \rho]^{1/2} ,$$

$$(4.96)$$

$$D = U + W P W^* = H = U^* + W P W^* ,$$

$$L = W P W^* , \quad P = (1 - Z Z^*)^{-1} .$$

Comparison of Eq.(4.62) with the functional representation of the $S_{g^+}(\rho_0 \alpha)$

$$S_{g^+}(\rho_0 \alpha) = \int \rho_0 (c^*, \gamma) \Omega(c^*, \gamma) \exp \left[ -(\gamma^* - c^*) \cdot (\gamma - c) \right] \prod \delta c \delta c^* \delta \gamma d \delta \gamma^* ,$$

(4.98)
where $\psi(\epsilon^*, \gamma)$ is a functional corresponding to the normal form of the operator $\xi_0$ gives

$$\rho_0 = D \exp(\frac{i}{\hbar} \xi^* \xi) \exp(\frac{i}{\hbar} L_{\alpha \beta} \xi \xi) \exp(\frac{i}{\hbar} \xi a \xi \xi),$$

(4.99)

c) "Hawking miracle"

There is a special but interesting case when the created pair always consists of one visible ($a$) and one invisible ($b$) particle so that

$$V = \begin{pmatrix} 0 & W \\ \bar{W} & 0 \end{pmatrix}, \quad W_{ax} = -\sum_{\alpha, \beta} (A^{-1})_{\alpha \beta} B_{\alpha x}.$$

(4.100)

where $\alpha$ is an index labelling the $g^-$ states. The corresponding operator $\rho_0$ is of the form

$$\rho_0 = \exp \left( \epsilon^* \left( W W^* - I \right) \epsilon \right).$$

(4.101)

If we use the well-known relation

$$\exp \xi^+ T \xi = \exp \left( \xi^+ \ln \left( I + T \right) \xi \right),$$

(4.102)

we can then get

$$\rho_0 = \exp \left( \xi^+ \ln \left( W W^* \right) \xi \right).$$

(4.103)

Up to this point the results obtained are very general and, for example, Eq. (4.103) can be used in the case when the charged particle production by the electrostatic field is considered and we are interested in the observation of the particles of one sign of charge only.\footnote{Mattis} The most interesting application of the developed formalism is the consideration of massless particle creation by a single spherically symmetric black hole (Hawking effect).\footnote{Hawking}

In this case it is convenient to take $a = (w, l, m), \quad x = (w', l', m')$,

$$V_{out}, \quad a = \frac{e^{-i \omega u}}{\sqrt{4 \omega \pi}} \quad Y_{lm}(\theta, \varphi),$$

(4.104)

$$U_{out}, \quad x \equiv \left. U_x \right|_{H^*} = \frac{e^{-i \omega U}}{\sqrt{4 \omega \pi}} \quad Y_{lm'}(\theta, \varphi),$$

(4.105)

where $u$ is the Bondi time parameter on $g^+$ and $U$ is an affine parameter along the null generators of the event horizon $H^+$. It can be shown\footnote{Kerr} that

$$\left( \kappa^{-1} \right)_{\alpha \beta \gamma} \delta_{\gamma \gamma} = \frac{-\kappa}{\nu} \left( \kappa^{-1} \right)_{\alpha \beta \gamma} \delta_{\gamma \gamma} \quad \left( \kappa c \right)^2$$

(4.106)

where $\kappa = 1/4\hbar c$ is the surface gravity (the velocity of light $c$ is taken to be 1). Thus we have

$$W_{ax} = e^{-\omega \pi / \kappa} \delta (\omega - \omega') \delta_{l l'} \delta_{m m'}$$

(4.107)

and for the density matrix it gives

$$\rho_0 = e^{\left( - \frac{\omega}{\nu} / \kappa \right)} \quad \rho = \exp \left( \frac{-\omega}{\nu} / \kappa \right),$$

(4.108)

where $\rho = e^{\omega/2\nu} \exp \left( \frac{-\omega}{\nu} / \kappa \right)$ is the Hawking-Bekenstein temperature of the black hole. That is, if we neglect the scattering processes of the created particles on the static external gravitational field of the black hole, all the characteristics of the black hole radiation (not only the energy spectrum but also all the correlation functions of the photon number in the different modes) coincide\footnote{Horowitz}.

It should be noted that the general expression for the $\rho$ matrix described by (4.99) or (4.103) in the stationary case can also be presented in the "quasi-thermal" form

$$\rho = \exp \left( - \frac{\omega}{\nu} / \kappa \right),$$

(4.109)
where \( A = \frac{1}{2} T z + \text{const.} \), \( T \) is a hermitian matrix, \( \text{const.} \) and the operators \( z \) and \( z \) are connected with \( z \) and \( z \) by means of some canonical linear transformation [24]. Thus the particle creation processes described by the density matrix \( \rho \) can be considered as the "thermal emission of the quasiparticles". In the case of the black hole evaporation, the intriguing (from the formal viewpoint) result (the "Hawking miracle") is that the corresponding operator \( H \) coincides with the Hamiltonian \( H_0 \) describing the real outgoing particles and hence the thermal structure of the density matrix has a well-defined physical meaning in terms of the observation of the emission of real particles.

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APPENDIX

\( a) \) Conformal transformations

All the quantities in the space \((M, g)\) conformal to a given space \((M, g')\)

\[
\delta_{ab} = \delta_{ab}^1 \delta_{ab}^2 \quad (A.1)
\]

will be denoted by the same symbols as the corresponding quantities in \((M, g)\) with a hat over them. Let us also use the following abbreviations:

\[
\begin{align*}
\hat{\omega} &= \Omega^{-1} \Omega, \\
\hat{\gamma}_{\alpha}^\beta &= \Omega^{-1} \alpha \Omega, \\
\hat{\delta}_{\alpha}^\beta &= \Omega^{-1} \delta_{\alpha}^\beta \Omega,
\end{align*}
\]

we then have

\[
\begin{align*}
\hat{\gamma}_{\alpha}^\beta &= \hat{\gamma}_{\alpha}^\beta + \hat{\gamma}_{\alpha}^\beta, \\
\hat{R}_{\alpha}^\beta &\equiv \Omega^{-2} (\hat{\gamma}_{\alpha}^\beta \hat{\gamma}_{\alpha}^\beta - \delta_{\alpha}^\beta \hat{\omega} - \delta_{\alpha}^\beta \hat{\delta}_{\alpha}^\beta), \\
\hat{R}_{\alpha}^\beta &\equiv \Omega^{-2} \left[ \hat{R}_{\alpha}^\beta - \hat{\gamma}_{\alpha}^\beta \hat{\delta}_{\alpha}^\beta + \hat{\gamma}_{\alpha}^\beta \hat{\omega} \right], \\
\hat{R}_{\alpha}^\beta &\equiv \hat{R}_{\alpha}^\beta - 2 \hat{\delta}_{\alpha}^\beta + (3 \hat{\omega} - \delta) \hat{\gamma}_{\alpha}^\beta, \\
\hat{R} &\equiv \Omega^{-2} \left[ \hat{R} - 6 \hat{\delta} + 12 \hat{\omega} \right], \\
\hat{G}_{\alpha}^\beta &\equiv \hat{R}_{\alpha}^\beta - \frac{1}{2} \hat{g}_{\alpha}^\beta \hat{R} = \hat{G}_{\alpha}^\beta - 2 \hat{\delta}_{\alpha}^\beta + (2 \hat{\omega} - 3 \hat{\delta}) \hat{g}_{\alpha}^\beta,
\end{align*}
\]
\[ C^\alpha_{\beta\gamma\delta} = \hat{C}^\alpha_{\beta\gamma\delta} \quad (A.8) \]

where \( C^\alpha_{\beta\gamma\delta} \) is the Weyl tensor.

For the covariant derivative we have, for example,
\[
\nabla^\mu_f = \hat{\nabla}^\mu_f - \Omega^\mu(\Omega^\mu_{\mu} f + \delta^\mu_{\mu} \nabla f - \nabla f \Omega_\mu). \quad (A.9)\]

If under the conformal transformation (A.1) the field \( h_{\mu\nu} \) transforms as:
\[
\hat{h}_{\mu\nu} = \Omega_{\mu\nu} h_{\mu\nu}, \quad (A.10)\]

we then have
\[
\nabla_\lambda h_{\mu\nu} = \Omega^{-1} \nabla_\lambda \hat{h}_{\mu\nu} + \Omega^{-2} (\Omega_{\alpha\beta} \hat{h}_{\alpha\beta} + \Omega_{\beta\alpha} \hat{h}_{\alpha\beta} + \nabla_\lambda \hat{h}_{\mu\nu} + \nabla_\lambda \hat{h}_{\mu\nu} + \nabla_\lambda \hat{h}_{\mu\nu}) \quad (A.11)\]

\[
f_{\alpha\beta\gamma} = \frac{1}{2} \left( \nabla_\alpha \hat{h}_{\beta\gamma} + \nabla_\beta \hat{h}_{\gamma\alpha} + \nabla_\gamma \hat{h}_{\alpha\beta} - \delta_{\alpha\beta} \nabla_\gamma \hat{h} + \delta_{\alpha\beta} \nabla_\gamma \hat{h} \right) = \Omega^{-2} f_{\alpha\beta\gamma} + \frac{1}{2} \delta^2 (\Omega_{\alpha\beta} \hat{h}_{\gamma\alpha} + \Omega_{\alpha\gamma} \hat{h}_{\beta\alpha} + \Omega_{\beta\gamma} \hat{h}_{\alpha\beta}) - \frac{1}{2} \delta_{\alpha\beta} \delta^2 \hat{h}_{\gamma\alpha} \hat{h}_{\gamma\alpha} \quad (A.12)\]

b) Gravitational perturbations in \( N \) gauge.

The non-vanishing components of the Christoffel symbols
\[
\Gamma_{\mu\nu\lambda} = \frac{1}{2} \left( \partial_\nu g_{\mu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\mu g_{\nu\lambda} \right) \quad (3.6f)\]

on the null surface in \( N \) co-ordinates \((x^0 = u, x^1 = r, x^2, x^3)\) are

\[
\Gamma_{\mu\nu\lambda} = \frac{1}{2} \left( \partial_\nu g_{\mu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\mu g_{\nu\lambda} \right) \quad (3.6f)\]

\[\Gamma_{01,\lambda} = \frac{1}{2} \frac{\partial g_{0\lambda}}{\partial r} \delta^A_{\lambda}, \quad \Gamma_{0A,\lambda} = \frac{1}{2} \left( \frac{\partial g_{0A}}{\partial u} - \frac{\partial g_{0\lambda}}{\partial r} \delta^A_{\lambda} \right), \quad \Gamma_{1A,\lambda} = \frac{1}{2} \frac{\partial g_{1A}}{\partial r} \delta^A_{\lambda} \quad (A.13)\]

We also have
\[
\rho = -\frac{1}{2} \nabla^\mu \ell_{\mu} = \frac{1}{2} g^{\mu\nu} \Gamma_{\mu\nu,1} = -\frac{1}{2} g^{\mu\nu} \Gamma_{\mu\nu1} = \frac{1}{2} g^{AB} \frac{\partial g_{AB}}{\partial r}, \quad \mu = \frac{1}{2} \nabla^\mu \mu_{\nu} = -\frac{1}{2} g^{\mu\nu} \Gamma_{\mu\nu,0} = \frac{1}{2} g^{AB} \frac{\partial g_{AB}}{\partial u} \quad (A.14)\]

Now we shall show that such a choice of gauge exists in which Eq. (3.67) is satisfied in some neighbourhood of the \( N \) surface and the representation (3.70) is valid on \( N \). The fulfillment of (3.67) is a simple consequence of the existence of solutions of the following partial differential equations
\[
\frac{\partial \hat{F}_{\mu\nu}}{\partial r} = -\frac{\partial \hat{F}_{\mu\nu}}{\partial \lambda} + 2 \Gamma_{\lambda,\nu} \hat{F}_{\mu\lambda} + \ell^\mu \hat{h}_{\mu\nu} \quad (A.15)\]

This equation does not fix the value \( \hat{F}_{\mu\nu} \) on \( N \) and the arbitrary functions \( \xi_{\nu}(u, r = 0, x^\lambda) \) can be used for the further simplification of the form of \( h_{\mu\nu} \).

The existence of the solutions of the following equations on \( N \):
\[
\frac{\partial \hat{F}_{00}}{\partial u} = \frac{1}{2} \mu_{00}, \quad \frac{\partial \hat{F}_{0A}}{\partial u} = -\frac{\partial \hat{F}_{0A}}{\partial \lambda} + 2 \Gamma_{0,\lambda} \hat{F}_{0\lambda} + \hat{h}_{0A} \quad (A.16)\]

guarantees that the subsidiary condition
\[
\hat{h}_{\mu\nu}|_{N} = 0 \quad (A.16)\]
can be imposed. The functions $\xi_\alpha(u = 0, r = 0, x^A)$ and $\xi_A(u = 0, r = 0, x^A)$ are arbitrary. It should be noted that $\xi_A^{(1)} = 0$ and the function $\xi_A(u, r = 0, x^A)$ is not determined by (A.16).

We shall now show that it can be chosen to satisfy the fulfillment of the condition

$$h|_N = 0 \quad (A.17)$$

In fact under the gauge transformation, $h$ changes as follows:

$$h \to h' = h - 2\xi h^A \quad (A.18)$$

For $h'$ to be equal to zero, the following equation must be satisfied:

$$2\xi_{u^2} - 2\xi_0 + \xi h^A = \frac{1}{2} h \quad (A.19)$$

It is evident that if $u \neq 0$, then using the available freedom in the choice of $\xi_\alpha$ we can satisfy Eq. (A.17). In the case $u = 0$, one can verify that the field equation $\xi_{u^2} = 0$ does not contain $\partial^2/\partial r^2$ derivatives and we can write it in the form

$$h_{u^2} = h_{u^2} = 0, \quad h_{u^2} + h_{0^2} = h_{0^2} = 0$$

$$2\xi_{u^2} - 2\xi_0 + \xi h^A = \frac{1}{2} h \quad (A.20)$$

If $3\xi_{u^2}/\partial u \neq 0$, we can choose the gauge transformation $\xi_{u^2} = \xi_{u^2}^u |_N$

$$\hat{h}_{ABN} = \hat{h}_{ABN}^u + \frac{\partial \xi_{u^2}}{\partial u}$$

to reduce Eq. (A.20) to the equation

$$\frac{\partial^2 \hat{h}}{\partial u^2} + \frac{\partial \hat{h}}{\partial u} = 0 \quad (A.21)$$

In the opposite case $3\xi_{u^2}/\partial u = 0$, Eqs. (A.20) and (A.21) coincide. We can use the arbitrariness of the $\xi_A(u = 0, r = 0, x^A)$ functions to get

$$h(u = 0, r = 0, x^A) = \frac{2h(u = 0, r = 0, x^A)}{2u^2} = 0,$$

then (A.21) guarantees the fulfillment of (A.17) and the field $h_{u^2}^N$ can be written in the form (3.76).

c) Asymptotic invariants for scalar field

If we take into account that under the conformal transformation the scalar field $\phi$ changes as follows: $\hat{\phi} = \hat{\phi}^{*} \phi^{*}$, we then have

$$2\xi_{u^2} - 2\xi_0 + \xi h^A = \frac{1}{2} h \quad (A.19)$$

$$\frac{\partial^2 \hat{h}}{\partial u^2} + \frac{\partial \hat{h}}{\partial u} = 0 \quad (A.20)$$

These equations and the fulfillment of $\xi$ of the following conditions:

$$\hat{\phi} \hat{\phi} \Omega |_g = 0, (\Omega^{-1} \hat{\phi} \hat{\phi} \Omega \Omega |_g) |_g = 0,$$

$$\Omega_{\alpha} \xi^{\alpha} |_g = 0, (\Omega^{-1} \hat{\phi} \hat{\phi} \Omega \xi^{\alpha} |_g) |_g = (\frac{1}{2} \xi^{\alpha} \Omega \xi^{\alpha} |_g)$$

make it possible to write the expression for the asymptotic invariant (4.13) in the form

$$P[\xi] = \int \frac{d\xi d\eta}{g} \left[ (2\alpha + 1) \partial \phi \hat{\phi} \xi^{\mu} \partial \phi \hat{\phi} + \right.$$  

$$+ 2\alpha \hat{\phi} \left( A \hat{\phi} + \Omega \xi^{\mu} \Omega \hat{\phi} \right) \hat{\phi} \right]$$

where $A = \frac{\partial \xi^{\mu} \Omega \hat{\phi} \hat{\phi}}{\partial \xi^{\mu} \Omega |_g}$.

Eq. (2.45) for the asymptotic Killing vector shows that
and hence

\[ A = \left( \Omega^{-4} \Omega^\nu \nabla_\nu \right) \left( \frac{\partial}{\partial \nu} \left( \omega^\mu \omega_\mu \right) \right) \frac{\partial}{\partial \nu} \left( \frac{\partial}{\partial \nu} \right) \left( \frac{\partial}{\partial \nu} \right) \frac{\partial}{\partial \nu} \left( \frac{\partial}{\partial \nu} \right) \right|_g = 0. \quad (A.26) \]

Using (A.24) we also have

\[ \left( \Omega^{-4} \Omega^\nu \nabla_\nu \phi \right) \left( \frac{\partial}{\partial \nu} \phi \right) \frac{\partial}{\partial \nu} \left( \frac{\partial}{\partial \nu} \phi \right) \frac{\partial}{\partial \nu} \left( \frac{\partial}{\partial \nu} \phi \right) \right|_g = 0. \quad (A.27) \]

Eqs. (A.25)-(A.27) give Eq. (4.45).
1) In order that this operator be hermitian the product of the operators must be interpreted as a symmetrized one. Here and later (for electromagnetic and gravitational fields) for simplicity we do not write this symmetrization explicitly. The corresponding variations of the fields are assumed to possess simple commutation properties with the field operators (see [9]).

2) For another approach to the problem of the vacuum definition in curved space-time based on the affine properties of the null surfaces see [23].

3) It should be noted that these operators, generally speaking, are not well defined. In order to give them a mathematical meaning one must use the same procedure of "subtracting the infinite constant" as in the flat space-time. Later we shall assume this subtraction has been made.

4) References to papers concerned with the problem of quantum particle creation in black holes may be found, for example, in the review articles [27], [28]. Consideration of the analogous problem in the case when the parameters of the black hole are changing is given in [29].

5) For simplicity we do not consider the scattering processes of the created particles on the static gravitational field of the black hole.
Fig. 1. The Penrose picture of Minkowskian space-time.

Fig. 2. The co-ordinates and N tetrad based on the null surface $N$. 
Fig. 3 The conformal Bondi frame.
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