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CLASSICAL CONFINING SOLUTIONS OF A TENSOR GAUGE THEORY INCORPORATING COLOUR *

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ABSTRACT

We formulate a mass-modified Einstein-Weyl gauge theory of colour-carrying spin-two mesons. A classical solution is exhibited for the case of internal SU(2) symmetry which may confine quarks in colour singlets.

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I. INTRODUCTION

In a recent note [1] we obtained a classical solution for a (mass-modified) Einstein equation for strong gravity, which appears to exhibit a non-perturbative quark-quark or quark-anti-quark rising "potential" of the type $\kappa^2 r^2$. By solving exactly a Klein-Gordon equation [2] describing the interaction of a quark pair in this potential, we were able to show that this equation gives rise to discrete eigenstates only, without any continuum, so that no ionization is possible. Irrespective of whether one believes that exact quark confinement is relevant to particle physics or not, it is interesting that one can obtain confinement through an exact solution of a fundamental gauge equation like that of Einstein. Insofar as the "potential" we have obtained corresponds to a de-Sitter O(3,2) type of strong gravity metric, with a radius proportional to the strong graviton $(\text{mass})^{-1}$, a pictorial interpretation of our results could be of quarks residing in a closed de-Sitter universe of their own (a bag) embedded within ordinary space-time [3].

Now one of the shortcomings of this theory is the absence of colour. In order to describe the physical situation, we must ensure that while quarks exhibit confining potentials of the type $\kappa^2 r^2$, their physical composites do not. As is well known, this may be guaranteed through a colour sensitivity of the confining potential, such that confinement produces colour singlets, which in their turn do not generate "potentials" of the type $\kappa^2 r^2$. It is the purpose of this note to motivate a simple gauge scheme for the marriage of internal symmetries like SU(2), SU(3),... with the SL(2,C) gauge group of Weyl, describing tensor particles. In particular we shall be concerned with the colour symmetry SU(2), so that the physical spectrum of strong spin-2 gravitons consists of a singlet plus a triplet. We show that with a special choice of mass term (equal masses for the singlet and the triplet) we are able to obtain a classical non-perturbative solution, which can give a colour-dependent potential of the general character $\{a + b \langle \underline{I}_1 \rangle \cdot \langle \underline{I}_2 \rangle\}\kappa^2 r^2$. Here the coefficients a and b are slowly varying functions of r, whose magnitudes and absolute signs depend on the parameters of the theory. One may choose these such that $a = 0$, $b < 0$. In this case, the static force between two quarks is attractive (in proportion to the inter-quark distance) in the singlet, and repulsive in the triplet state, while the force between two colour singlets or a singlet and a quark vanishes.

II. THE LAGRANGIAN

A simple Lagrangian for the tensor fields which is invariant under $SU(2)_{global} \times SL(2,C)_{local}$ as well as general co-ordinate transformations is given by

$$\mathcal{L}_1 = \frac{1}{16} \mathrm{Tr}\, [L^\mu, L^\nu]\, B_{\mu\nu} \quad ,$$

where $B_{\mu\nu}$ is a kind of "field strength" made from the "gauge fields", B_μ,

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + i\, [B_\mu, B_\nu] \quad .$$

The fields B_μ and L^μ are 8×8 matrices which can be expanded as follows:

$$B_\mu = \frac{1}{4} B^\alpha_{\mu ab}\, \sigma_{ab}\, \tau_\alpha \quad ,$$

$$L^\mu = L^{\mu a}_\alpha\, \gamma_a\, \tau_\alpha \quad ,$$

where the Minkowski space-time indices a, b and isotopic indices α take the values 0,1,2,3.[4)]

Under the general co-ordinate transformations, $x^\mu \to \bar{x}^\mu$, we have

$$B_\mu(x) \to \bar{B}_\mu(\bar{x}) = \frac{\partial x^\nu}{\partial \bar{x}^\mu}\, B_\nu(x) \quad ,$$

$$L^\mu(x) \to \bar{L}^\mu(\bar{x}) = \left|\det \frac{\partial x}{\partial \bar{x}}\right|^{1/2} \frac{\partial \bar{x}^\mu}{\partial x^\nu}\, L^\nu(x)$$

so that $\mathcal{L}_1$ is a scalar density. Under $SL(2,C)$, on the other hand,

$$B_\mu(x) \to B'_\mu(x) = \Omega(x)\, B_\mu(x)\, \Omega(x)^{-1} + \frac{1}{i}\, \Omega(x)\, \partial_\mu \Omega(x)^{-1} \quad ,$$

$$L^\mu(x) \to L'^\mu(x) = \Omega(x)\, L^\mu(x)\, \Omega(x)^{-1} \quad ,$$

where $\Omega(x)$ is an $SL(2,C)$ matrix [5)] generated by $\sigma_{ab}\tau_0$.

To the kinetic terms provided by $\mathcal{L}_1$ may be added a mass term, $\mathcal{L}_M$, for the tensor fields, $L^{\mu a}_\alpha$. This term preserves $SU(2) \times SL(2,C)$ but breaks the general co-ordinate invariance unless another field, the gravitational metric $g_{\mu\nu}$ of Einstein, is introduced. However, since the gravitational effects are presumably quite negligible ($G_{Newtonian}/G_{strong} \ll 1$) we shall employ Minkowskian co-ordinates and write [6)] $g_{\mu\nu} = \eta_{\mu\nu}$. The precise form of $\mathcal{L}_M$ is much too complicated to be given explicitly. Since we are interested in setting up an exact classical solution we shall find it necessary to adjust the mass term so that it takes a manageable form in the class of solutions in question.

The equations of motion are

$$[L^\mu, B_{\mu\nu}] + \frac{\partial \mathcal{L}_M}{\partial L^\nu} = T_\nu \quad ,$$

$$\partial_\mu [L^\mu, L^\nu] + i[B_\mu, [L^\mu, L^\nu]] = S^\nu \quad ,$$

where T_ν and S^ν denote the matter contributions to the stress (iso-stress) and torsion (iso-torsion), respectively. We shall be concerned with the case where T and S are concentrated at a point and spherical symmetry is maintained. In this case a particularly simple solution emerges if we assume that the isovector components of B_μ and L^μ are constrained to have a fixed direction in isospace,

$$B^i_{\mu ab} = n^i B^n_{\mu ab} \quad , \quad L^{\mu a}_i = n^i L^{\mu a}_n \quad ,$$

where n^i is independent of x. It is now possible to effect a separation of the equations of motion. Define the mixtures,

$$L^\mu_\pm = L^\mu_0 \pm L^\mu_n \quad , \quad B^\pm_\mu = B^0_\mu \pm B^n_\mu \quad .$$

In terms of these combinations the kinetic term reduces to the form:

$$\mathcal{L}_1 = \frac{1}{4} \mathrm{Tr}[L^\mu_+, L^\nu_+]\, B^+_{\mu\nu} + \frac{1}{4} \mathrm{Tr}[L^\mu_-, L^\nu_-]\, B^-_{\mu\nu} \quad ,$$

where

$$B^{\pm}_{\mu\nu} = \partial_\mu B^{\pm}_\nu - \partial_\nu B^{\pm}_\mu + i[B^{\pm}_\mu, B^{\pm}_\nu] \quad .$$

(The separation of $\mathcal{L}_1$ into two independent pieces occurs for SU(2). For SU(3) the problem would require a more elaborate treatment.)

In order to make use of the classical solution found in Ref.1 we make a further change of variable. Define the "metric" tensors $f_{\pm\mu\nu}$ by

$$\frac{1}{4}\,\mathrm{Tr}(L^\mu_+ L^\nu_+) = \sqrt{-f_+}\; f^{\mu\nu}_+$$

and likewise for $f^{\mu\nu}_-$. (In the usual fashion, the contravariant tensor $f^{\mu\nu}_+$ is defined as the matrix inverse of $f_{+\mu\nu}$ while $\sqrt{-f_+}$ denotes the density $\sqrt{-\det f_{+\mu\nu}}$.) In terms of the tensors $f_{\pm\mu\nu}$ the kinetic term becomes the sum of two Einstein-type Lagrangians,

$$\mathcal{L}_1 = \sqrt{-f_+}\, f^{\mu\nu}_+ R_{\mu\nu}(f_+) + \sqrt{-f_-}\, f^{\mu\nu}_- R_{\mu\nu}(f_-) \quad .$$

We now choose the mass terms so as to recover the problem solved in Ref.1, i.e.

$$\mathcal{L}_M = M^2(-g)^{\frac{1}{2}-c}\,(-f_+)^c\,(f^{\mu\nu}_+ - g^{\mu\nu})\,(f^{\kappa\lambda}_+ - g^{\kappa\lambda})\,(g_{\mu\kappa}g_{\nu\lambda} - g_{\mu\nu}g_{\kappa\lambda})$$

plus an identical term with $f_{+\mu\nu}$ replaced by $f_{-\mu\nu}$. On the face of it, this mass term seems not to be compatible with SU(2) invariance since f_+ and f_- are singlet-triplet mixtures. However, it can be shown that there do exist invariant forms which reduce to this one when $L^\mu_i = n_i L^\mu_n$ provided $\det(L^{\mu a}_0) \neq 0$.

III. A SOLUTION

A spherically-symmetric solution to the equations for $f_{+\mu\nu}$ (or $f_{-\mu\nu}$) in the approximation $g_{\mu\nu} = \eta_{\mu\nu}$,

$$R_{\mu\nu} - \frac{1}{2} f_{\mu\nu} R + \frac{1}{\sqrt{-f}}\,\frac{\partial\mathcal{L}_M}{\partial f^{\mu\nu}} = 0$$

has been previously obtained [1),6)]. The contravariant components are given by

$$f^{00} = \frac{3}{2}(1+\alpha+p) \;, \quad f^{0j} = -\frac{3}{2}\sqrt{p(p+\alpha)}\;\frac{x^j}{r} \;, \quad f^{ij} = -\frac{3}{2}\delta^{ij} + \frac{3}{2}\,p\,\frac{x^i x^j}{r^2} \;,$$

where $i,j = 1,2,3$ and $r^2 = x^j x^j$. The function $p(r)$ is given by

$$p(r) = \frac{1}{2}M^2 r^2 \left\{\frac{81}{16}(1+\alpha)\right\}^{\frac{1}{2}-c} \left\{(1-c)(1+\alpha) + \frac{c}{3}\right\} \;,$$

where M and c are parameters [6)] appearing in the mass term and α denotes an integration constant. It is restricted by the condition $1 + \alpha > 0$.

Two such solutions are found independently for f_+ and f_- involving integration constants α_+ and α_- . The next step is to extract the square roots L^μ_+ and L^μ_- . This can be done in a variety of ways but we shall choose schemes such that $L^{\mu a}_\pm = L^{a\mu}_\pm$. For L_+ one finds,

$$L^0_+ = \sqrt{\frac{2}{3}}\,(1+\alpha_+)^{-1/4}\left[\left(\sqrt{1+\alpha_+} - \left(1 - \sqrt{1+\alpha_+}\right)\frac{p_+}{\alpha_+}\right)\gamma^0 - \right.$$
$$\left. - \left(1 - \sqrt{1+\alpha_+}\right)\sqrt{\frac{p_+}{\alpha_+}\left(1 - \frac{p_+}{\alpha_+}\right)}\;\frac{x^i\gamma^i}{r}\right] \;,$$

$$L^j_+ = \sqrt{\frac{2}{3}}\,(1+\alpha_+)^{-1/4}\left[\gamma^j + \left(1 - \sqrt{1+\alpha_+}\right)\frac{x^j x^k \gamma^k}{r^2} + \right.$$
$$\left. + \left(1 - \sqrt{1+\alpha_+}\right)\sqrt{\frac{p_+}{\alpha_+}\left(1 + \frac{p_+}{\alpha_+}\right)}\;\frac{x^j}{r}\gamma^0\right] \;.$$

Two distinct solutions for L_- may be obtained from these expressions by the replacements $\alpha_+ \to \alpha_-$ and $\sqrt{1+\alpha_+} \to -\sqrt{1+\alpha_-}$ or $\sqrt{1+\alpha_+} \to \sqrt{1+\alpha_-}$.
For later purposes we choose the first alternative and obtain:

$$L^0_- = \sqrt{\frac{2}{3}}\,(1+\alpha_-)^{-1/4}\left[\left(-\sqrt{1+\alpha_-} - \left(1 + \sqrt{1+\alpha_-}\right)\frac{p_-}{\alpha_-}\right)\gamma^0 - \right.$$
$$\left. - \left(1 + \sqrt{1+\alpha_-}\right)\sqrt{\frac{p_-}{\alpha_-}\left(1 + \frac{p_-}{\alpha_-}\right)}\;\frac{x^i\gamma^i}{r}\right] \;,$$

$$L_-^j = \sqrt{\frac{2}{3}}\,(1+\alpha_-)^{-1/4}\left[\gamma^j + \left(1+\sqrt{1+\alpha_-}\right)\frac{x^j x^k \gamma^k}{r^2} + \right.$$

$$\left. + \left(1+\sqrt{1+\alpha_-}\right)\sqrt{\frac{p_-}{\alpha_-}\left(1+\frac{p_-}{\alpha_-}\right)}\,\frac{x^j}{r}\gamma^0\right] .$$

Finally, by taking sums and differences one recovers the singlet and triplet vierbein components L_0^μ and L_n^μ.

If one wishes the singlet component not to exhibit the confining phenomenon of unrestricted growth at large r^2 then it is sufficient to correlate the two integration constants α_+ and α_- such that the r^2 dependence is cancelled from L_0^{00}. Since

$$L_0^{00} = \frac{1}{2}\,(L_+^{00} + L_-^{00})$$

$$= \frac{1}{\sqrt{6}}\left[(1+\alpha_+)^{1/4} - (1+\alpha_-)^{1/4}\right]$$

$$+ \frac{1}{\sqrt{6}}\left[\frac{\sqrt{1+\alpha_+}-1}{(1+\alpha_+)^{1/4}}\,\frac{p_+}{\alpha_+} - \frac{\sqrt{1+\alpha_-}+1}{(1+\alpha_-)^{1/4}}\,\frac{p_-}{\alpha_-}\right]$$

it is necessary to choose α_+ and α_- to satisfy the equality

$$\frac{1}{\alpha_+}\left(\sqrt{1+\alpha_+}-1\right)(1+\alpha_+)^{1/4-c}\left((1-c)\,(1+\alpha_+)+\frac{c}{3}\right) =$$

$$= \frac{1}{\alpha_-}\left(\sqrt{1+\alpha_-}+1\right)(1+\alpha_-)^{1/4-c}\left((1-c)\,(1+\alpha_-)+\frac{c}{3}\right) .$$

Clearly, L_0^{00} reduces to a constant as do the L_0^{ij}. Only L_0^{0i} among the singlet components retains any r^2 dependence and this is of a relatively mild character; for large r^2 these components approach fixed values.

IV. INTERPRETATION

The solution given above is a vacuum (exterior) solution whose isovector component maintains a fixed direction n_i independent of x. We would like to interpret it as an idealized version of the tensor fields generated by a point source (with spherical symmetry) which itself carries isospin. In other words we would like to interpret the vector n_i in terms of the source iso-stress (or iso-torsion since, presumably, these fix the same direction because of the equations of motion). For a quark doublet the iso-stress would be

$$T^j_{\mu a} \sim \bar{q}\,\tau^j\,\gamma_a\,(\partial_\mu + iB_\mu)\,q + \text{h.c.}$$

One may expect that in a quasistatic approximation the terms T^j_{00} factorize like $n^j\,T^n_{00}$ and together with T^0_{00} represent the dominant effect of the quark in generating the tensor components L_0^μ and L_j^μ. Thus, adiabatically one may write

$$L_j^{00} \sim \langle \tau_j \rangle\, T^n_{00}\, r^2 , \quad .$$

where $\langle \tau_j \rangle \approx T^j_{00}/T^n_{00}$. If this is so then the action of the source on a test particle moving in this field would take the form:

$$\langle \underline{\tau}_{source} \rangle \cdot \langle \underline{\tau}_{test} \rangle\, r^2_{12}$$

at least if the particles are moving slowly and the non-relativistic approximation has some meaning. (Clearly, the detailed correlation of the source to the tensor fields needs a solution of the coupled set of equations involving both the sources and the fields.)

To conclude, we have shown in the last section that it appears to be possible, by an adjustment of α_+ and α_- to arrange that the singlet fields, L_0^μ, do not grow with r^2 and hence do not participate in the confinement mechanism. The long-range force is therefore a colour sensitive one. However, the suitability of the assumption n_i = constant remains to be tested (It may turn out, for example, that a 't Hooft monopole-like solution with $n_i \sim x_i/r$ is more favourable.) Thus what we have demonstrated here is a distinct possibility of an oscillator-type, colour sensitive, confining potential which has its origin in the exchange of tensor mesons [7]. There is no reason of course why in addition coloured gauge vector mesons may not also exist. However, their major role may be towards saturation rather than confinement [8].

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REFERENCES AND FOOTNOTES

1) Abdus Salam and J. Strathdee, ICTP, Trieste, preprint IC/76/125 (to appear in Phys. Rev. D).

2) Abdus Salam and J. Strathdee, ICTP, Trieste, preprint IC/77/20 (to appear in Phys. Letters).

3) It is noteworthy that by far the most satisfying formulation of supergravity theory appears to be within the context of a de-Sitter rather than an asymptotically flat space-time (Chamseddine Ali and P. West, Imperial College, preprint (1976), S. MacDowell and F. Mansouri, to be published). Supergravity may thus be more relevant to strong rather than weak gravity.

4) The Pauli matrices τ_1, τ_2, τ_3 are joined here by the 2×2 unit matrix denoted by τ_0. The Dirac matrices γ_a satisfy $\{\gamma_a,\gamma_b\} = 2\eta_{ab} = 2\,\mathrm{diag}(+1,-1,-1,-1)$ and $\sigma_{ab} = (i/2)\,[\gamma_a,\gamma_b]$.

5) More elaborate systems have been considered (C.J. Isham, Abdus Salam and J. Strathdee, Phys. Rev. D9, 1702 (1974)) in which the local SL(2,C) symmetry is extended to SL(4,C). For this it is necessary to enlarge the multiplets, writing

$$B_\mu = \left(B^\alpha_\mu + \frac{1}{2} B^\alpha_{\mu ab}\,\sigma_{ab} + B^\alpha_{\mu 5}\gamma_5\right)\tau_\alpha/2 \quad \text{and} \quad L^\mu = L^{\mu a}_\alpha\gamma_a\tau_\alpha + L^{\mu a5}_\alpha i\gamma_a\gamma_5\tau_\alpha$$

(where B^0_μ and $B^0_{\mu 5}$ vanish identically). A full treatment of these multiplets requires the introduction of a number of auxiliary fields and a corresponding complication of the Lagrangian. In the simpler model considered in this note such problems do not arise, the symmetry being $SL(2,C) \times SU(2)$ rather than $SL(4,C)$. One may think of the present model as relevant to the class of solutions of the SL(4,C) type of models, where the extra components B_μ, $B_{\mu 5}$, $L^{\mu a5}$ all vanish.

6) C.J. Isham and D. Storey (Imperial College, London, preprint) suggest a criterion based on the corrections to the behaviour of the tensor $g_{\mu\nu}$ (in the background of the field $f_{\mu\nu}$) which can yield a relation between the two parameters c and α.

7) The physics and the formalism of exact confinement is a new and as yet a dark subject. It could be that there are pitfalls in the approach we have taken above (for example, in the assumption that classical solutions give interparticle potentials) or alternatively it may be that the physically favoured solution of the gauge tensor equations is only a partially confining one (solutions designated as type II in Ref.1 with $L^{0i} = 0$) where the potential first rises like r^2 and then falls exponentially.

8) In fact in the full SL(4,C) version of the theory (C.J. Isham, Abdus Salam and J. Strathdee, Rabi Festschrift Volume, Ed. Lloyd Motz (1977) to be published) the triplet of components B_μ^α of the B_μ multiplet (Footnote 5) do describe precisely the spin-one Yang-Mills particles.