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INTRODUCTION

Without doubt the invention of supersymmetry *), a possible fundamental symmetry between fermions and bosons (their contrasting statistics notwithstanding) is one of the most elegant creations in theoretical physics. The recognition of this type of symmetry has enriched the mathematics of relativistic particle theory in a number of basic ways.

1) In supersymmetric theories it is possible to group specified numbers of mesons and fermions (e.g. spins zero and $\frac{1}{2}$, or 0, $\frac{1}{2}$ and 1, or $\frac{3}{2}$ and 2) in supermultiplets of the same mass. The existence of one member of such a supermultiplet must imply the existence of all the other members if supersymmetry is a law of nature. Before the discovery of supersymmetry it had erroneously been assumed that basic tenets of quantum field theory forbade associations of different spins in the same multiplet.

2) One can associate multi-component "superfields" with such supermultiplets and write supersymmetric Lagrangians for these.[2] Such Lagrangians are extremely restrictive in form. In particular, if renormalizability is required then they are very special indeed. The most exciting property of these Lagrangians is their "softness" in that many of the ultraviolet infinities of conventionally renormalizable theories do not appear (as a rule only wave function renormalizations are divergent).

3) Internal symmetries like isospin, SU(3), etc., can be incorporated without any difficulty into the structure of the supermultiplets. These internal symmetries can be gauged, i.e. made into local symmetries. For supersymmetric and gauge invariant Lagrangians the vector gauge bosons are necessarily accompanied by spin-$\frac{1}{2}$ supersymmetry partners.[3] These gauge fermions with their universal gauge coupling are a new phenomenon in particle theory. (The universal couplings manifest themselves not only in the interactions of gauge fermions with their vector partners, the gauge bosons, but also in their interactions with what one may call "matter" supermultiplets.)

4) Supersymmetry can be broken spontaneously in exactly the same way as internal symmetries are broken when the appropriate scalar fields develop a vacuum expectation value.[4] In the lowest approximation this phenomenon is governed by the scalar field potential. In supersymmetric theories these potentials are highly restricted in form, unlike the case of non-supersymmetric Lagrangian theories where there is usually an embarrassment of possibilities.

*) To our knowledge the first appearance of supersymmetry occurs in the work of Gol'fand and Likhtman. It was rediscovered independently by Akulov and Volkov and by Wess and Zumino.[1]

This circumstance is a great but austere virtue - not easily can one set up a model which is flexible enough to meet one's requirements. Among the suprising properties of supersymmetric theories one finds that if the supersymmetry is not broken in zeroth order then it will not break in any finite order and, moreover, the scalar field expectation values - which may describe spontaneously broken internal symmetries - are given exactly by the zeroth order result (i.e. are not renormalized).

5) A basic aspect of supersymmetry - and on which may be associated with developments of physical significance - concerns its relation to the curved spacetime of gravity theory. This would be the generalization of spacetime to a manifold which includes anti-commuting c-numbers θ^{α} (α = 1,2,3,4) along with the familiar x^{μ}. The notion of such an extension of flat spacetime was used to define superfields $\Phi(x,\theta)$. But the curved space generalization in which the metric field $g_{\mu\nu}(x)$ becomes one part of a superfield remains to be made convincingly. The problem of finding a supersymmetric extension of Einstein's gravity theory is being tackled now by many authors and much progress has been made. However, the picture has not yet become quite clear and we shall not attempt to discuss the generalization of Einstein's theory.

It is our purpose in this paper to elucidate the developments (1)-(4) using the superfield notation throughout. Some of the results are published already; others are new. We feel, however, that it is worthwhile to cover the entire development from this unified viewpoint.*) One feature of our presentation is that, from the outset, a fermionic quantum number is defined in the supermultiplets - often at the expense of space reflection symmetry. In the past this has been done more or less as an afterthought. In view of the intrinsic importance of this quantum number in particle physics we believe the present treatment may be more apposite.

One should ask the question: is this very elegant development made use of by nature? At present the answer seems to be in the negative. Now the photon and a neutrino could have shared a supermultiplet - this would be especially plausible after the recent developments concerning unification of weak and electromagnetic interactions - but, as we shall show in Sec.V, it seems that if there is a neutrino accompanying the photon it is probably a Goldstone particle. Such a Goldstone neutrino, arising from the spontaneous breakdown of supersymmetry, is unfortunately subject to low-energy theorems which make its identification with the observed ν_e or ν_{μ} unlikely. The neutrino which partners the photon may yet be found. The spin-$\frac{3}{2}$ object which accompanies the graviton (spin-2) may have a fundamental role in full gravity theory - for example in circumventing singularities of spacetime. On the

*) A number of reviews with extensive lists of references are available.[5]

strong interaction side, the only indication of a symmetry among baryons and mesons appears to be the unversality of Regge slopes but this could, no doubt, be explained in other ways.

SECTION I

(A) SUPERSYMMETRY AS A HIGHER RELATIVISTIC SYMMETRY

The concept of a fundamental symmetry between fermions and bosons is an intriguing one. It belongs to the class of so-called "higher" or relativistic symmetries in that its basic aim is to unite in a single multiplet, particles with different intrinsic spins. In the past such relativistic schemes have been concerned with the problem of uniting, for example, pseudoscalar with vector mesons or spin $\frac{1}{2}$ baryons with spin $\frac{3}{2}$ baryons. Supersymmetry, however, is concerned with the uniting of scalar and vector bosons with spin $\frac{1}{2}$ fermions.

In order to clarify the relationship of the new symmetry to the old proposals for uniting different spins in one multiplet, it is necessary to digress briefly on the structure of the latter. One of the simpler such was arrived at in the attempt to make relativistic the SU(4) and SU(6) symmetries of Wigner, Gürsey, Radicati and Sakita. The SU(6), for example, is a phenomenological attempt to classify the hadronic multiplet structure which could be expected to emerge from the quark model if the dominant forces are independent of spin. Thus the low-lying meson states, pseudoscalar and vector, are expected to fill out a 35-dimensional representation of SU(6) while the low-lying baryons, spins $\frac{1}{2}$ and $\frac{3}{2}$, are expected to fill a 56-fold. In predicting these states as well as in several other respects the symmetry SU(6) is successful. However, and this is the main point for our discussion, the Wigner-Gürsey-Radicati-Sakita symmetry is essentially *non-relativistic*. Its most distinctive feature is the presence among its infinitesimal generators of a set of operators which carry one unit of angular momentum. These are the operators which connect the states, such as π^+ and ρ^+, whose spins differ by one unit. These generators behave in a well defined way under space rotations but <u>not</u> under Lorentz transformations. In a relativistic theory such operators make no sense; if the symmetry is to be taken seriously, they <u>must</u> be supplemented by new operators so as to fill out Lorentz multiplets. A number of attempts were made in this direction. One of the simplest was to embed the 3-vectors of Gürsey, Radicati and Sakita in antisymmetric 4-tensors and this was achieved by enlarging SU(6) to SL(6,ℂ). The latter is in fact the smallest relativistic symmetry which can contain SU(6).

Unfortunately for SL(6,ℂ), it is a non-compact group. This means that its irreducible unitary representations are all infinite dimensional. If the particle spectrum is to be classified in unitary representations of this group, the π and ρ mesons must then be accompanied by an infinite sequence of particles, making the scheme (at the least) unattractive. All attempts at relativistic versions of the spin-containing symmetries met this phenomenon which came to be known as a *no-go theorem*.

Like many no-go theorems in physics, this one apparently was also based on certain hidden assumptions and we shall explain shortly how it is circumvented in the supersymmetric framework. Briefly, the difficulties with SL(6,ℂ) were concerned with extra internal "dimensions" implicit in the symmetry. A rather pictorial understanding of the infinite-dimensionality of SL(6,ℂ)'s unitary representations can be obtained by the method of induced representations. Thus, if SL(6,ℂ) is resolved into cosets with respect to its maximal compact subgroup SU(6) then one obtains a 35-dimensional coset space on which the full group can act. Now what can these 35 real variables signify? The answer is that they must constitute an "internal" space with 35 degrees of freedom. It is these internal degrees of freedom which manifest themselves as an infinite degeneracy in the particle spectrum: they can contribute any amount of spin but no energy. In another version of SL(6,ℂ) symmetry, one of the Lorentz tensors - an SU(3) singlet - among the infinitesimal operators is identified with the generator of Lorentz transformations on space-time itself. For this to make sense it is necessary to embed the space-time co-ordinates, x_μ, in a 72-dimensional manifold. This time the new degrees of freedom contribute to the energy and one finds that, in the rest frame, energy ranges over a continuum. It is difficult to imagine how all the new dimensions could be accommodated in a physically meaningful theory.

The difficulties just outlined can all be avoided by the simple expedient of associating the new dimensions with <u>anticommuting variables</u>. Indeed, it is now rather hard to understand why such a simple solution was not thought of long ago. The idea is to consider fields, $\Phi(x,\theta)$, which are defined over the product of the usual space-time, labelled by four co-ordinates x_μ, with a new space labelled by a set of anticommuting c-numbers, θ_I. In order to see how the new degrees of freedom are controlled one has only to expand the field $\Phi(x,\theta)$ in powers of θ and notice that the series must terminate after a finite number of terms:

$$\Phi(x,\theta) = \phi(x) + \phi_i(x)\,\theta_i + \frac{1}{2}\phi_{ij}(x)\,\theta_i\theta_j + \cdots \frac{1}{N!}\phi_{i_1\ldots i_N}(x)\;\theta_{i_1}\ldots\theta_{i_N}\;, \tag{I.1}$$

where N denotes the number of independent components θ_i. Because of the anticommutativity among the θ_i, the coefficient fields $\phi_{i_1 i_2\ldots}(x)$ must be completely antisymmetric and there can therefore be only a finite number of them.

It remains to be shown, of course, that a consistent group theory can be established on the space of x and θ. Such extensions of the Poincaré group do exist and a minimal example will be discussed in detail in the following sections. At this stage we remark only that the new dimensions must have a spinorial character in keeping with their anticommutativity as demanded by the TCP theorem. Thus, if in the above expansion $\phi(x)$ is a commuting scalar field, we should like the next member, $\phi_i(x)$, to be an anticommuting spinor field. For this reason the new relativistic symmetry must combine bosons with fermions and its infinitesimal generators must include some which carry a half unit of angular momentum.

(B) THE MINIMAL FERMIONIC EXTENSION

The smallest spinorial representation of the homogeneous, proper Lorentz group is the two-component spinor. It follows that the minimal fermionic extension of four-dimensional space-time which can be conceived is made with the two complex (or four real) compoments of such spinors. This is the manifold on which is based the relativistic symmetry known as supersymmetry. In order to establish a notation, we discuss now some of the elementary features of this manifold and of the functions defined over it.

Since the concept of the four-component Dirac spinor and the associated apparatus of Dirac matrices is more familiar to physicists than the two-component (dotted and undotted) spinors and their associated matrices, we choose to work with the former. This is purely a matter of notation: the two-component spinor is identifiable as a chiral projection of the Dirac spinor. Now, a Dirac spinor has two chiral projections (for the definition see Eq.(1.10) and in general these are independent. However, if the Dirac spinor is subject to a reality condition (the Majorana condition) then the two chiral projections are related by complex conjugation. This means, in effect, that the four real components of a complex 2-spinor are identifiable with the four real components of a Majorana spinor. It is possible to set up a one-one correspondence between 2-spinors and Majorana spinors.

Our main conventions are as follows. For the metric of space-time we take

$$\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1) \tag{I.2}$$

and write the scalar product of two 4-vectors

$$A\cdot B = A_\mu B_\mu = A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3\;. \tag{I.3}$$

(There is usually no need for the more exact but cumbersome notation $A\cdot B = \eta^{\mu\nu} A_\mu B_\nu = A_\mu B^\mu = A^\mu B_\mu$. The only point where confusion could arise is in the use of the gradient $\partial_\mu = \partial/\partial x_0\,, -\partial/\partial x_1\,, -\partial/\partial x_2\,, -\partial/\partial x_3$.) The Dirac matrices, γ_μ, satisfy the anticommutation rule

$$\{\gamma_\mu,\gamma_\nu\} = 2\,\eta_{\mu\nu}\;. \tag{I.4}$$

The frequently employed tensor and pseudoscalar combinations are defined by

$$\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu,\gamma_\nu]\;,\qquad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3\;. \tag{I.5}$$

With these definitions it is always possible to choose a representation in which the matrices $\gamma_0, \sigma_{12}, \sigma_{23}, \sigma_{31}$ are hermitian while $\gamma_1, \gamma_2, \gamma_3, \gamma_5$ and $\sigma_{01}, \sigma_{02}, \sigma_{03}$ are antihermitian. Another important matrix in the considerations which follow is the charge conjugation matrix, C, which relates γ_μ to the transposed matrix γ_μ^T,

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T\;. \tag{I.6}$$

The matrix C is necessarily antisymmetric. (It can be fixed, apart from a sign, by imposing the further requirements $C^{-1} = C^\dagger = C$.)

The infinitesimal Lorentz transformations are defined by

$$\delta x_\mu = \omega_{\mu\nu}x_\nu\;, \tag{I.7}$$

where $\omega_{\mu\nu}$ is real and antisymmetric. The 4-vectors A_μ and B_μ are required to transform like x_μ and this ensures the invariance of the product $A_\mu B_\mu$. The Dirac spinor, ψ, is required to transform according to

$$\delta\psi = -\frac{1}{4}\, \omega_{\mu\nu}\, \sigma_{\mu\nu}\, \psi \quad . \tag{I.8}$$

The adjoint spinor $\bar{\psi} = \psi^{\dagger}\gamma_0$ then obeys the transformation rule

$$\delta\bar{\psi} = \frac{1}{4}\, \omega_{\mu\nu}\, \bar{\psi}\, \sigma_{\mu\nu} \quad . \tag{I.9}$$

(This comes about because the matrices $\gamma_0\sigma_{\mu\nu}$, like γ_0, $\gamma_0\gamma_\mu$, $\gamma_0\gamma_5$ and $i\gamma_0\gamma_\mu\gamma_5$, are all hermitian.)

The transformations (I.8) are reducible. From (I.4) and the definitions (I.5) it follows that γ_5 commutes with $\sigma_{\mu\nu}$. The subspaces on which $\gamma_5 = -i$ and $+i$ are invariant and this means that the chiral projections

$$\psi_\pm = \frac{1 \pm i\gamma_5}{2}\, \psi \tag{I.10}$$

separately obey the rule (I.8). These chiral projections are of course the two-component spinors of which the Dirac spinor is comprised.

From (I.9) and the properties of the charge conjugation matrix one can show that the "conjugate" spinor, ψ^c, defined by

$$\psi^c = C\, \bar{\psi}^T \tag{I.11}$$

transforms exactly like ψ itself. The relevant property of C is expressed in the equation $C\sigma^T_{\mu\nu} = -\sigma_{\mu\nu}C$. This identity is one of a number involving C which may be summarized in the statement:

$$\left.\begin{array}{l} \gamma_\mu C \text{ and } \sigma_{\mu\nu}C \text{ are symmetric} \\ \\ C\,,\ \gamma_5 C \text{ and } i\gamma_\mu\gamma_5 C \text{ are antisymmetric} \end{array}\right\} \quad . \tag{I.12}$$

A Majorana spinor is one for which $\psi^c = \psi$ or, in terms of chiral projections,

$$\psi_\mp = C\, \bar{\psi}^T_\pm \quad . \tag{I.13}$$

The Majorana spinor therefore contains the same information as a two-component spinor (and its complex conjugate).

Let ψ_1 and ψ_2 be a pair of anticommuting Majorana spinors,

$$\{\psi_1\,,\, \psi_2\} = 0 \quad .$$

Then the symmetries (I.12) are expressed in a statement about bilinear forms: $\bar{\psi}_1\gamma_\mu\psi_2$ and $\bar{\psi}_1\sigma_{\mu\nu}\psi_2$ are antisymmetric while $\bar{\psi}_1\psi_2$, $\bar{\psi}_1\gamma_5\psi_2$ and $\bar{\psi}_1 i\gamma_\mu\gamma_5\psi_2$ are symmetric under the interchange of ψ_1 and ψ_2. (Note in particular that $\bar{\psi}_1\gamma_\mu\psi_1$ and $\bar{\psi}_1\sigma_{\mu\nu}\psi_1$ vanish identically for Majorana spinors.)

Now consider some of the detailed properties of the expansion (I.1) for the case of a Majorana spinor θ. Because of the anticommutativity among the four components of θ, there are altogether sixteen independent terms in the expansion. These can be conveniently grouped as follows:

$$\begin{aligned}\Phi(x,\theta) = {} & A(x) + \bar{\theta}\psi(x) + \frac{1}{4}\,\bar{\theta}\theta F(x) + \frac{1}{4}\,\bar{\theta}\gamma_5\theta G(x) \\ & + \frac{1}{4}\,\bar{\theta} i\gamma_\nu\gamma_5\theta\, V_\nu(x) + \frac{1}{4}\,\bar{\theta}\theta\bar{\theta}\chi(x) + \frac{1}{32}\,(\bar{\theta}\theta)^2\, D(x) \quad ,\end{aligned} \tag{I.14}$$

where A, F, G, V_ν and D are Bose fields, ψ and χ are Fermi. It is clear that, with respect to the proper Lorentz group, A, F, G and D are scalars, V_ν is a vector while ψ and χ are 4-spinors. We shall defer discussion of the improper transformations.

The product of two such expansions must again have the same form and this can be verified. Thus if

$$\Phi_1(x,\theta)\, \Phi_2(x,\theta) = \Phi_3(x,\theta) \tag{I.15}$$

then the components of Φ_3 are given by

$$\begin{aligned} A_3 &= A_1A_2 \\ \psi_3 &= A_1\psi_2 + \psi_1A_2 \\ F_3 &= A_1F_2 - \overline{\psi_1^c}\psi_2 + F_1A_2 \\ G_3 &= A_1G_2 + \overline{\psi_1^c}\gamma_5\psi_2 + G_1A_2 \\ V_{3\nu} &= A_1V_{2\nu} + \overline{\psi_1^c} i\gamma_\nu\gamma_5\psi_2 + V_{1\nu}A_2 \end{aligned}$$

$$\chi_3 = A_1\chi_2 + \psi_1 F_2 - \gamma_5\psi_1 G_2 + i\gamma_\nu\gamma_5\psi_1 V_{2\nu}$$

$$+ \chi_1 A_2 + F_1\psi_2 - G_1\gamma_5\psi_2 - V_{1\nu} i\gamma_\nu\gamma_5\psi_2$$

$$D_3 = A_1 D_2 + 2F_1F_2 + 2G_1G_2 + 2\, V_{1\nu}\, V_{2\nu} - 2\overline{\psi_1^c}\chi_2 - 2\overline{\chi_1^c}\psi_2 + D_1A_2 \quad . \tag{I.16}$$

In deriving these expressions we have assumed that θ commutes with Bose fields and anticommutes with Fermi fields. A further assumption we shall make is that the order of anticommuting factors is reversed by complex conjugation. With this understanding the bilinears $\bar{\theta}\theta$, $\bar{\theta}\gamma_5\theta$ and $\bar{\theta}i\gamma_\mu\gamma_5\theta$ are "real" while, for example,

$$(\bar{\theta}\psi)^* = \bar{\psi}\theta = \bar{\theta}\psi^c \quad ,$$

and this bilinear is real if ψ is a Majorana spinor.

Derivatives with respect to the Majorana co-ordinates θ are defined by

$$\Phi(x,\theta+\delta\theta) = \Phi(x,\theta) + \delta\bar{\theta}\, \frac{\partial}{\partial\bar{\theta}}\, \Phi(x,\theta) \tag{I.17}$$

to leading order in $\delta\theta$. Since θ and $\delta\theta$ are anticommuting quantities it is important to remark that the infinitesimal $\delta\bar{\theta}$ is placed to the left of $\partial\Phi/\partial\bar{\theta}$ in (I.17). This is a convention we shall adhere to. The derivative defined here has the usual properties of a differential operator with two important exceptions. Firstly, when applied to the product of two functions its effects are distributed according to the rule

$$\frac{\partial}{\partial\bar{\theta}}\,(\Phi_1\Phi_2) = \frac{\partial\Phi_1}{\partial\bar{\theta}}\,\Phi_2 \pm \Phi_1\,\frac{\partial\Phi_2}{\partial\bar{\theta}} \quad , \tag{I.18}$$

where the $+(-)$ sign applies when Φ_1 is bosonic (fermionic), i.e. when $\delta\bar{\theta}$ commutes (anticommutes) with Φ_1 . Secondly, the product of two such differential operators is antisymmetric,

$$\frac{\partial}{\partial\bar{\theta}^\alpha}\,\frac{\partial\Phi}{\partial\bar{\theta}^\beta} = -\,\frac{\partial}{\partial\bar{\theta}^\beta}\,\frac{\partial\Phi}{\partial\bar{\theta}^\alpha} \quad . \tag{I.19}$$

Applied to a general function like (I.14) the derivative takes the explicit form:

$$\frac{\partial\Phi}{\partial\bar{\theta}} = \psi + \frac{1}{2}\,(F + \gamma_5 G + i\gamma_\nu\gamma_5 V_\nu)\theta$$

$$+ \frac{1}{8}\left[\bar{\theta}\theta + \gamma_5\bar{\theta}\gamma_5\theta + i\gamma_\nu\gamma_5\bar{\theta}i\gamma_\nu\gamma_5\theta\right]\chi + \frac{1}{8}\,\bar{\theta}\theta\theta D \quad . \tag{I.20}$$

This formula will prove useful when we come to define the action on component fields of the extended Poincaré group (Sec.(C)).

Finally, it is necessary to make some decisions about the discrete transformations P and CP . There are two possibilities. The first is to refuse to assign any distinctive quantum number to the spinorial components ψ and χ . In this case, if the scalar and vector components of $\Phi(x,\theta)$ are real, then the spinors should also be real (in the Majorana sense), i.e.

$$\Phi(x,\theta) = \Phi(x,\theta)^* \tag{I.21}$$

and there is no concept of fermion number. With such an interpretation one can associate the transformation

$$\theta \rightarrow i\gamma_0\theta \tag{I.22}$$

with space reflection. The component fields A, F and D are scalars, G a pseudoscalar, V_ν an axial vector.

The alternative possibility is that we arrange that the spinor components ψ and χ are distinguished from the Bose components by a fermionic quantum number. There is only one way to do this and it means sacrificing the space reflection. Suppose the positive chirality combination of co-ordinates θ_+ carries unit fermion-number (as well as one-half unit of spin). It follows that θ_- carries minus one unit. Clearly the transformation (I.22) reverses the fermion number,

$$\theta_\pm \rightarrow i\gamma_0\theta_\mp \quad .$$

This transformation can no longer be associated with a simple space reflection. Rather it must be associated with combined space reflection and antiparticle conjugation.

The phase transformations associated with fermion-number,

$$\theta_\pm \rightarrow e^{\pm i\alpha}\,\theta_\pm \qquad \text{or} \qquad \theta \rightarrow e^{-\alpha\gamma_5}\,\theta \quad , \tag{I.23}$$

can now be applied to the function $\Phi(x,\theta)$. Suppose this function transforms according to

$$\Phi(x,\theta) \rightarrow \Phi(x, e^{\alpha\gamma_5}\theta) \quad . \tag{I.24}$$

Then the component fields transform according to

$$\begin{aligned} A &\rightarrow A \\ \psi &\rightarrow e^{\alpha\gamma_5}\psi \\ F \pm iG &\rightarrow e^{\pm 2i\alpha}(F \pm iG) \\ V_\nu &\rightarrow V_\nu \\ \chi &\rightarrow e^{-\alpha\gamma_5}\chi \\ D &\rightarrow D \quad . \end{aligned} \tag{I.25}$$

To summarize, the boson fields A, V_ν and D carry no fermion-number while $F + iG$ carries two units and $F - iG$ carries minus two units; the chiral spinors ψ_- and χ_+ carry one unit while ψ_+ and χ_- carry minus one unit. These assignments are of course compatible with the reality condition (I.21).

We shall return later to the problem of defining both parity as well as fermion-number in this framework.

(C) SUPERTRANSLATIONS

Having established a notation and described some of the properties of the extended space-time and the functions $\Phi(x,\theta)$ defined over it, we are now in a position to discuss the Poincaré group and its extension.

Firstly, the action of an infinitesimal Poincaré transformation on x and θ is given by

$$\left.\begin{aligned} \delta x_\mu &= b_\mu + \omega_{\mu\nu}x_\nu \\ \delta\theta &= -\tfrac{i}{4}\,\omega_{\mu\nu}\,\sigma_{\mu\nu}\theta \end{aligned}\right\} \tag{I.26}$$

where b_μ and $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are infinitesimal real parameters. The new transformations, which we shall call <u>supertranslations</u>, are given by

$$\left.\begin{aligned} \delta x_\mu &= \tfrac{i}{2}\,\bar{\varepsilon}\gamma_\mu\theta \\ \delta\theta &= \varepsilon \end{aligned}\right\} \tag{I.27}$$

where ε is an infinitesimal anticommuting Majorana spinor. We shall assume that the components of ε anticommute among themselves, with θ and with all spinor fields.

The peculiar feature, expressed in (I.27), that a "translation" in the Majorana co-ordinates θ is associated with a θ-dependent translation in the space-time co-ordinate x_μ is highly significant and gives rise to the many unusual characteristics of supersymmetric theory. But first we must verify that the infinitesimal transformations (I.26) and (I.27) form an algebra.

Apply to x and θ two successive infinitesimal transformations characterized, respectively, by the parameters $b^1, \omega^1, \varepsilon^1$ and $b^2, \omega^2, \varepsilon^2$:

$$\begin{aligned} \delta_2\delta_1 x_\mu &= \omega^1_{\mu\nu}\,\delta_2 x_\nu + \tfrac{i}{2}\,\bar{\varepsilon}^1\gamma_\mu\,\delta_2\theta \\ &= \omega^1_{\mu\nu}\left(b^2_\nu + \omega^2_{\nu\lambda}x_\lambda + \tfrac{i}{2}\,\bar{\varepsilon}^2\gamma_\nu\theta\right) \\ &\qquad + \tfrac{i}{2}\,\bar{\varepsilon}^1\gamma_\mu\left(-\tfrac{i}{4}\,\omega^2_{\lambda\rho}\sigma_{\lambda\rho}\theta + \varepsilon^2\right) \\ &= \left(\omega^1_{\mu\nu}b^2_\nu + \tfrac{i}{2}\,\bar{\varepsilon}^1\gamma_\mu\varepsilon^2\right) + \omega^1_{\mu\rho}\omega^2_{\rho\nu}x_\nu \\ &\qquad + \tfrac{i}{2}\left(\omega^1_{\mu\nu}\,\bar{\varepsilon}^2\gamma_\nu - \tfrac{i}{4}\,\bar{\varepsilon}^1\gamma_\mu\omega^2_{\lambda\rho}\sigma_{\lambda\rho}\right)\theta \end{aligned}$$

$$\delta_2\delta_1\theta = -\frac{i}{4}\,\omega^1_{\mu\nu}\,\sigma_{\mu\nu}\,\delta_2\theta$$

$$= -\frac{i}{4}\,\omega^1_{\mu\nu}\,\sigma_{\mu\nu}\left(-\frac{i}{4}\,\omega^2_{\lambda\rho}\,\sigma_{\lambda\rho}\,\theta + \varepsilon^2\right)$$

$$= -\frac{i}{4}\left(-\frac{i}{4}\,\omega^1_{\mu\nu}\,\omega^2_{\lambda\rho}\,\sigma_{\mu\nu}\,\sigma_{\lambda\rho}\right)\theta - \frac{i}{4}\,\omega^1_{\mu\nu}\,\sigma_{\mu\nu}\,\varepsilon^2 .$$

Now apply these variations in the reverse order and subtract to obtain

$$[\delta_2,\delta_1]\,x_\mu = \left(\omega^1_{\mu\nu}\,b^2_\nu - \omega^2_{\mu\nu}\,b^1_\nu + i\,\bar\varepsilon^1\gamma_\mu\varepsilon^2\right) + \left(\omega^1_{\mu\rho}\,\omega^2_{\rho\nu} - \omega^2_{\mu\rho}\,\omega^1_{\rho\nu}\right)x_\nu + \frac{i}{2}\left(\omega^1_{\lambda\rho}\,\bar\varepsilon^2 - \omega^2_{\lambda\rho}\,\bar\varepsilon^1\right)\frac{i}{4}\,\sigma_{\lambda\rho}\,\gamma_\mu\,\theta$$

$$[\delta_2,\delta_1]\,\theta = -\frac{i}{4}\left(\omega^1_{\mu\rho}\,\omega^2_{\rho\nu} - \omega^2_{\mu\rho}\,\omega^1_{\rho\nu}\right)\sigma_{\mu\nu}\,\theta - \frac{i}{4}\,\sigma_{\mu\nu}\left(\omega^1_{\mu\nu}\,\varepsilon^2 - \omega^2_{\mu\nu}\,\varepsilon^1\right),$$

from which it is clear that the commutator $[\delta_2,\delta_1]$ is equivalent to a third infinitesimal transformation δ_3, characterized by the parameters

$$\begin{aligned} b^3_\mu &= \omega^1_{\mu\nu}\,b^2_\nu - \omega^2_{\mu\nu}\,b^1_\nu + i\,\bar\varepsilon^1\gamma_\mu\varepsilon^2 \\ \omega^3_{\mu\nu} &= \omega^1_{\mu\rho}\,\omega^2_{\rho\nu} - \omega^2_{\mu\rho}\,\omega^1_{\rho\nu} \\ \varepsilon^3 &= -\frac{i}{4}\,\sigma_{\mu\nu}(\omega^1_{\mu\nu}\,\varepsilon^2 - \omega^2_{\mu\nu}\,\varepsilon^1) . \end{aligned} \tag{I.28}$$

The infinitesimal transformations do indeed form an algebra and, what is more, they can be integrated to yield finite transformations.

A pure supertranslation integrates quite trivially to the finite form

$$\left.\begin{aligned} x_\mu &\to x_\mu + \frac{i}{2}\,\bar\varepsilon\gamma_\mu\theta \\ \theta &\to \theta + \varepsilon \end{aligned}\right\} . \tag{I.29}$$

It is interesting to note the form taken by the product of two successive supertranslations,

$$\left.\begin{aligned} x_\mu &\to x_\mu + \frac{i}{2}\,\bar\varepsilon^2\gamma_\mu\varepsilon^1 + \frac{i}{2}\,(\bar\varepsilon^1 + \bar\varepsilon^2)\,\gamma_\mu\theta , \\ \theta &\to \theta + \varepsilon^1 + \varepsilon^2 , \end{aligned}\right\} \tag{I.30}$$

which indicates that a space-time translation, $\frac{i}{2}\bar\varepsilon^1\gamma_\mu\varepsilon^2$ emerges when two supertranslations are compounded.

In order to treat the representations of the symmetry defined by (I.26) and (I.27), we shall set up an abstract algebra of the infinitesimal generators. Firstly, notice that the application to x_μ and θ of the first-order differential operator,

$$\delta = b_\kappa\partial_\kappa - \frac{1}{2}\,\omega_{\kappa\lambda}\left(x_\kappa\partial_\lambda - x_\lambda\partial_\kappa + \frac{i}{2}\,\bar\theta\sigma_{\kappa\lambda}\frac{\partial}{\partial\bar\theta}\right) + \bar\varepsilon\left(\frac{\partial}{\partial\bar\theta} + \frac{i}{2}\,\not\partial\theta\right) , \tag{I.31}$$

reproduces the formulae (I.26) and (I.27). This differential operator can be looked upon as a particular realization of the abstract operator,

$$F(\delta) = \frac{1}{i}\left[b_\kappa P_\kappa - \frac{1}{2}\,\omega_{\kappa\lambda}\,J_{\kappa\lambda} + \bar\varepsilon S\right] , \tag{I.32}$$

which defines the infinitesimal generators P, J and S. Moreover, the commutator $[\delta_2,\delta_1] = \delta_3$, derived above, can be looked upon as a particular realization - in terms of differential operators - of the algebraic statement,

$$[F(\delta_2),\,F(\delta_1)] = F(\delta_3) . \tag{I.33}$$

To deduce commutation rules among P, J and S is now straightforward. On the left-hand side of (I.33) substitute for $F(\delta_1)$ and $F(\delta_2)$ the linear expressions (I.32) with parameters b^1, ω^1, ε^1 and b^2, ω^2, ε^2, respectively. On the right-hand side substitute for $F(\delta_3)$ using the values for b^3, ω^3, ε^3 given in (I.28). The resulting formula is an identity in the parameters b^1, ω^1, ε^1 and b^2, ω^2, ε^2. By comparing the coefficients of these parameters one arrives at the following set of rules:

$$
\left.\begin{aligned}
[P_\mu, P_\nu] &= 0 \\
\frac{1}{i}[P_\mu, J_{\nu\lambda}] &= \eta_{\mu\nu} P_\lambda - \eta_{\mu\lambda} P_\nu \\
\frac{1}{i}[J_{\mu\nu}, J_{\kappa\lambda}] &= \eta_{\nu\kappa} J_{\mu\lambda} - \eta_{\nu\lambda} J_{\mu\kappa} + \eta_{\mu\lambda} J_{\nu\kappa} - \eta_{\mu\kappa} J_{\nu\lambda}
\end{aligned}\right\} \quad (I.34)
$$

$$
\left.\begin{aligned}
[S_\alpha, P_\mu] &= 0 \\
[S_\alpha, J_{\mu\nu}] &= \frac{1}{2}(\sigma_{\mu\nu} S)_\alpha
\end{aligned}\right\} \quad (I.35)
$$

$$
\{S_\alpha, S_\beta\} = -(\gamma_\mu C)_{\alpha\beta} P_\mu \quad . \qquad (I.36)
$$

These rules are of course obeyed by the particular realization (I.31), in terms of differential operators, from which we started:

$$
\left.\begin{aligned}
P_\mu &= i\partial_\mu , \\
J_{\mu\nu} &= i(x_\mu \partial_\nu - x_\nu \partial_\mu) - \frac{1}{2}\bar{\theta}\,\sigma_{\mu\nu}\,\frac{\partial}{\partial\bar{\theta}} , \\
S &= i\left(\frac{\partial}{\partial\bar{\theta}} + \frac{i}{2}\,\not{\partial}\theta\right) .
\end{aligned}\right\} \quad (I.37)
$$

The subalgebra (I.34) is, naturally, the familiar set of Poincaré rules. The pair (I.35) merely indicates that the supertranslation generator S transforms as a Dirac spinor under Lorentz transformations and is unaffected by space translations. The anticommutator (I.36) which closes the system reflects the truly novel aspect of this symmetry. That it must be an anticommutator rather than a commutator can be seen in several ways. Notice firstly that the matrix $\gamma_\mu C$ is symmetric. Secondly, the differential operator expression for S involves the anticommuting operator $\partial/\partial\bar{\theta}$. This differential operator anticommutes with ε, which means that, in general, S should anti-commute with ε . Going back to the commutators (I.33) and keeping only ε^1 and ε^2 one finds

$$
[\bar{\varepsilon}^1 S \;,\; \bar{\varepsilon}^2 S] = \bar{\varepsilon}^1 \gamma_\mu \varepsilon^2 P_\mu \qquad (I.38)
$$

and, on using the anticommutativity of ε with S, this yields the anti-commutation rule (I.36).

The appearance of an anticommutator in the algebraic structure is a new phenomenon in particle symmetry. In the mathematical terminology the system (I.34)-(I.36) is called a graded algebra. The linear operators $F(\delta)$ given by (I.32) belong to what is called an "extended Lie algebra" in that some of the co-ordinates ε belong to a Grassmann algebra. (Mathematical precision in such matters will not be emphasised here. Most of the mathematical operations needed here are quite elementary and are familiar to particle theoreticians.)

We conclude with a remark on the fermionic quantum number. Corresponding to the phase transformations (I.23) which serve to assign one unit of fermion-number to the positive chirality component θ_+ , one can define the operator

$$
F = i\bar{\theta}\gamma_5 \frac{\partial}{\partial\bar{\theta}} , \qquad (I.39)
$$

which commutes with P_μ and $J_{\mu\nu}$ and with S gives the commutator

$$
[S \;,\; F] = i\gamma_5 S \quad . \qquad (I.40)
$$

This shows that S_- carries one unit of fermion-number while S_+ carries minus one unit.

(D) UNITARY REPRESENTATIONS

In order to generate a unitary representation, the operators P, J and S must be real. More precisely, the linear form

$$
iF(\delta) = b_\mu P_\mu - \frac{1}{2}\omega_{\mu\nu} J_{\mu\nu} + \bar{\varepsilon} S
$$

must be self-adjoint. Since the parameters b and ω are real numbers while ε is a Majorana spinor, this reality means simply

$$
P_\mu = P_\mu^\dagger \quad , \quad J_{\mu\nu} = J_{\mu\nu}^\dagger \quad , \quad S = C\bar{S}^T \quad . \qquad (I.41)
$$

Now the unitary representations of the Poincaré group are well known, so our problem is to find out how these must be combined so as to form representations of the extended group. In solving this problem the chiral projections $S_\pm$ play a crucial role.

The anticommutator (I.36) resolves into three distinct pieces when chiral projections are taken,

$$\{S_+ , S_+\} = 0 \quad , \quad \{S_- , S_-\} = 0 \quad ,$$

$$\{S_+ , S_-\} = - \frac{1 + i\gamma_5}{2} \gamma_\mu C\, P_\mu \quad . \tag{I.42}$$

That this must happen is clear from the fermion-number assignments: since there are no operators in the algebra which carry two units of fermion-number, the anticommutator $\{S_- , S_-\}$ must equal zero, and likewise for $\{S_+ , S_+\}$.

Since the chiral operator S_- has only two independent components and since these anticommute with each other, it follows that the product of three or more of them must vanish,

$$S_{\alpha -} \; S_{\beta -} \; S_{\gamma -} \cdots = 0 \, . \tag{I.43}$$

The same is true of products with three or more adjacent S_+ factors.

The next important remark is that supertranslations must leave invariant the manifold of states with fixed 4-momentum since $S_\pm$ commute with P_μ . On such a manifold the anticommutator $\{S_+ , S_-\}$ becomes equal to a fixed set of numbers, and the operators S_+ and S_- are seen to generate a Clifford algebra. Since this algebra has just sixteen independent members - the products with up to four factors of S_+ and S_- - its one and only finite-dimensional irreducible representation is in terms of 4×4 matrices. The manifold of states with fixed 4-momentum is therefore reduced by the action of supertranslations into four-dimensional invariant subspaces.

On the other hand, the manifold with fixed 4-momentum is reduced by the Wigner rotations into $(2j + 1)$-dimensional subspaces where j , the intrinsic spin, is integer or half-integer. Taken together, the combined action of Wigner rotations and supertranslations must give rise to invariant subspaces with $4(2j+1)$ dimensions. We now show in detail how this comes about.

Following the method of Wigner we start with a state at rest,

$$P_0 = M \quad , \quad \underline{P} = 0 \quad . \tag{I.44}$$

In this frame the Wigner rotations coincide with the space rotations generated by $\underset{\sim}{J} = (J_{23} , J_{31}, J_{12})$ and one can assume that the chosen state carries a definite angular momentum j and z-component j_z . The basis states for an irreducible representation of the Poincaré group are obtained by applying to the chosen state firstly the space rotations which serve to generate the $2j+1$ values of j_z and, secondly, the Lorentz transformations which generate states with an arbitrary $\underline{P}$ and $P_0 = \sqrt{\underline{P}^2 + M^2}$.

To generate the basis states for an irreducible representation of the extended Poincaré group we can apply the operators S_+ , S_- and their products to the $2j+1$ rest states and then apply Lorentz transformations to the $4(2j+1)$ states which result. We shall suppose that the original $2j+1$ states are annihilated by S_+ . (This is not a restrictive assumption since, if the original states were not annihilated by S_+ , we could have multiplied them either once or twice by S_+ and picked out from among the resulting states a set of $2j'+1$ states which certainly are annihilated by S_+.) Now multiply each of the $2j+1$ states by S_- to generate $2(2j+1)$ new states. Since S_- carries one-half unit of angular momentum, these new states will comprise a pair of multiplets with spins $j-\frac{1}{2}$ and $j+\frac{1}{2}$. Next apply the operator S_-S_- to the original states. Since, in view of the anticommutativity, the product S_-S_- has only one independent component, this operation will generate another multiplet of spin j . The rest frame basis is now complete since products with three or more factors of S_- must vanish. If the original states carried fermion-number f then, since S_- carries one unit of this number, the states generated in this way carry $f+1$ and $f+2$. In summary, the rest frame states have the spin-fermion number content

$$j_f \quad , \quad (j - \tfrac{1}{2})_{f+1} \quad , \quad (j + \tfrac{1}{2})_{f+1} \quad , \quad j_{f+2} \quad . \tag{I.45}$$

(For the case $j = 0$ the multiplet $j-\frac{1}{2}$ is of course absent.)

Starting with the states $|j_z\rangle_f$ which satisfy

$$P_0 |j_z\rangle_f = |j_z\rangle_f \, M$$

$$\underset{\sim}{P} |j_z\rangle_f = 0$$

$$J_{12} |j_z\rangle_f = |j_z\rangle_f \; j_z \quad , \quad j_z = -j \, , -j+1, \ldots, j$$

$$\mathbb{F}|j_z\rangle_f = |j_z\rangle_f\, f \ ,$$

$$S_+|j_z\rangle_f = 0 \qquad . \qquad (I.46)$$

the rest frame basis is completed by adjoining the states $|j_z,\lambda\rangle_{f+1}$, $\lambda = \pm\frac{1}{2}$ and $|j_z\rangle_{f+2}$ defined by

$$\frac{1}{i} S_-|j_z\rangle_f = -\sum_\lambda |j_z\lambda\rangle_{f+1}\, C\, \chi_+^T(\lambda) \ ,$$

$$\bar{S}_+ S_-|j_z\rangle_f = -|j_z\rangle_{f+2}\, 2M \ , \qquad (I.47)$$

where the pair of chiral spinors $\chi_+(\lambda)$ satisfy

$$i\gamma_5\chi_+(\lambda) = \chi_+(\lambda) \ ,$$

$$\frac{1}{2}\sigma_{12}\,\chi_+(\lambda) = \lambda\chi_+(\lambda) \ , \qquad (I.48)$$

and are normalized such that

$$\bar{\chi}_+(\lambda)\,\gamma_0\chi_+(\lambda') = \delta_{\lambda\lambda'}\, M \ . \qquad (I.49)$$

The normalizations are chosen so that all states have the same (positive) norm. The positivity of the norm is basic to the unitarity of the representation and it is a consequence of the reality condition

$$S_+ = C\,\bar{S}_-^T \qquad (I.50)$$

and the anticommutation rules(I.42). The last of these rules reduces on the rest states to the form

$$\{S_+ \, , S_-\} = -\frac{1+i\gamma_5}{2}\,\gamma_0 C\, M \ ,$$

or, on taking account of (I.50)

$$\{S_- \, , \bar{S}_-\} = \frac{1-i\gamma_5}{2}\,\gamma_0\, M \quad .$$

By means of these rules it is straightforward to construct the action of S on the states defined by (I.46) and (I.47). One finds

$$\frac{1}{i} S\,|j_z\rangle_f = -\sum_\lambda |j_z\lambda\rangle_{f+1}\, C\,\bar{\chi}_+^T(\lambda) \ ,$$

$$\frac{1}{i} S\,|j_z\lambda\rangle_{f+1} = |j_z\rangle_f\,\chi_+(\lambda) + |j_z\rangle_{f+2}\,\gamma_0\chi_+(\lambda) \ ,$$

$$\frac{1}{i} S\,|j_z\rangle_{f+2} = \sum_\lambda |j_z\lambda\rangle_{f+1}\,\gamma_0 C\,\bar{\chi}_+^T(\lambda) \ . \qquad (I.51)$$

The states $|j_z,\lambda\rangle$ may be resolved into multiplets with angular momentum $j+\frac{1}{2}$ and $j-\frac{1}{2}$ but this will only complicate the formulae (I.51).

To complete the basis system it remains only to apply the Lorentz boosts. There are many ways to do this,but perhaps the most commonly used conventional scheme is the one due to Jacob and Wick. The rest frame 4-momentum

$$p_\mu = (M, 0, 0, 0)$$

is boosted to the general form

$$p_\mu = M\,(\cosh\alpha,\ \mathrm{sh}\alpha\,\sin\theta\,\cos\phi,\ \mathrm{sh}\alpha\,\sin\theta\,\sin\phi,\ \mathrm{sh}\alpha\,\cos\theta) \qquad (I.52)$$

by the Lorentz transformation

$$U(L_p) = e^{-i\varphi J_{12}}\, e^{-i\theta J_{31}}\, e^{i\varphi J_{12}}\, e^{-i\alpha J_{03}} \ , \qquad (I.53)$$

where the angles α, θ, φ are restricted to the ranges

$$0 \le \alpha < \infty \quad , \quad 0 \le \theta \le \pi \quad , \quad -\pi < \varphi \le \pi \quad .$$

Under this convention the so-called helicity states are defined by

$$|p,j_z\rangle_f = U(L_p)\,|j_z\rangle_f \qquad (I.54)$$

etc. The action of the supertranslation operator on these boosted states is obtained from (I.51) by applying the operator (I.53) to both sides. From the fact that S is a Dirac spinor it follows that

$$U(L_p)\, S\, U(L_p)^{-1} = a(L_p)^{-1}\, S \quad ,$$

where the matrix $a(L_p)$ is given by

$$A(L_p) = e^{-\frac{i}{2}\varphi\sigma_{12}}\, e^{-\frac{i}{2}\theta\sigma_{31}}\, e^{\frac{i}{2}\varphi\sigma_{12}}\, e^{-\frac{i}{2}\alpha\sigma_{03}} \quad . \tag{I.55}$$

On multiplying both sides of the boosted form of (I.51) by the matrix $a(L_p)$ one obtains the formulae:

$$\frac{1}{i} S\, |p\, j_z\rangle_f = -\sum_\lambda |p\, j_z\, \lambda\rangle_{f+1}\, C\, \bar{u}_+^T(p,\lambda) \,,$$

$$\frac{1}{i} S\, |p\, j_z\, \lambda\rangle_{f+1} = |p\, j_z\rangle_f\, u_+(p,\lambda) + |p\, j_z\rangle_{f+2}\, u_-(p,\lambda) \,,$$

$$\frac{1}{i} S\, |p\, j_z\rangle_{f+2} = -\sum_\lambda |p\, j_z\, \lambda\rangle_{f+1}\, C\, \bar{u}_-^T(p,\lambda) \,, \tag{I.56}$$

where the spinor coefficients $u_\pm(p,\lambda)$ are defined,

$$u_\pm(p,\lambda) = \frac{1 \pm i\gamma_5}{2}\, u(p,\lambda)$$

$$= \frac{1 \pm i\gamma_5}{2}\, a(L_p)\, (1 + \gamma_0)\, \chi_+(\lambda) \quad . \tag{I.57}$$

The spinors $u(p,\lambda)$ are positive energy solutions of the Dirac equation, $(\not{p}-M)\, u = 0$, and are normalized by

$$\bar{u}_+(p,\lambda)\, u_-(p,\lambda') = M\, \delta_{\lambda\lambda'} \,,$$

i.e.

$$\bar{u}(p,\lambda)\, u(p,\lambda') = 2M\, \delta_{\lambda\lambda'} \quad .$$

The formulae (I.56) which express the action of S on the states of an irreducible representation of the extended symmetry take the form given regardless of the boosting conventions. Such conventions affect only the functional form of the spinors $u(p,\lambda)$.

This completes the discussion of unitary irreducible representations with a finite mass. These representations are labelled by three parameters, M, j and f, and generally incorporate four distinct irreducible representations of the Poincaré group (as indicated in (I.45)). In the remainder of this paper, which is concerned mainly with renormalizable Lagrangian models, we shall meet only two of these.

(1) The fundamental representation, $f = j = 0$, with Poincaré content

$$(0)_0 \oplus (\tfrac{1}{2})_1 \oplus (0)_2$$

and of course the antiparticles

$$(0)_0 \oplus (\tfrac{1}{2})_{-1} \oplus (0)_{-2} \quad .$$

(2) The "vector" representation, $f = -1$, $j = \frac{1}{2}$, with Poincaré content

$$(\tfrac{1}{2})_{-1} \oplus (0)_0 \oplus (1)_0 \oplus (\tfrac{1}{2})_1 \quad ,$$

which is self-conjugate.

(E) GENERALIZATIONS

To conclude this chapter on the purely group theoretic aspects of the relativistic symmetry we shall consider very briefly some possible generalizations. The fermionic extension of the Poincaré group dealt with above was a minimal one in that it employed a single Majorana spinor. The most significant sacrifice imposed by the minimality requirement – combined with the idea that the new spinor co-ordinates should carry a fermion-number – is the loss of space reflection symmetry. If this is to be restored then a generalization is needed.

The simplest generalization is from Majorana to Dirac spinors which can be subjected to complex supertranslations. The formulae (I.27) are simply replaced by

$$\delta x_\mu = \frac{1}{2}\bar{\varepsilon}\gamma_\mu\theta - \frac{1}{2}\bar{\theta}\gamma_\mu\varepsilon \ ,$$

$$\delta\theta = \varepsilon \ , \tag{I.58}$$

where θ and ε are anticommuting Dirac spinors. It is a simple matter to prove that these form an algebra. The anticommutator (I.36) is here replaced by the set

$$\{S_\alpha , S_\beta\} = 0 \quad , \quad \{\bar{S}^\alpha , \bar{S}^\beta\} = 0 \ ,$$

$$\{S_\alpha , \bar{S}^\beta\} = (\gamma_\mu)_\alpha{}^\beta P_\mu \ , \tag{I.59}$$

where S and $\bar{S}$ are independent generators. (In unitary representations they are related by hermitian conjugation, $\bar{S} = S^\dagger\gamma_0$.)

The complex extension characterized by (I.59) is certainly compatible with reflection symmetry and fermion number. Thus, for example, all four components of S can carry one unit of fermion-number and this is clearly consistent with the parity transformation $S \to \gamma_0 S$.

The fundamental representation has sixteen independent rest frame states. These can be defined by successive applications of S_α to a "lowest" state $|0\rangle_{-2}$ (carrying zero spin and fermion number -2) which is annihilated by $\bar{S}^\alpha$. The successive states (all of the same mass) are:

$$\begin{aligned}
&|0\rangle_{-2} \\
|\alpha\rangle_{-1} &= S_\alpha|0\rangle_{-2} \\
|\alpha\beta\rangle_0 &= S_\alpha S_\beta|0\rangle_{-2} \\
|\alpha\beta\gamma\rangle_1 &= S_\alpha S_\beta S_\gamma|0\rangle_{-2} \\
|\alpha\beta\gamma\delta\rangle_2 &= S_\alpha S_\beta S_\gamma S_\delta|0\rangle_{-2} \ .
\end{aligned} \tag{I.60}$$

They comprise: a scalar difermion and its antiparticle; two spin $\frac{1}{2}$ particles of opposite parities and their anti-particles; two scalars, one pseudoscalar and one polar vector, all fermionically neutral.

In Section IV we shall present a renormalizable Lagrangian model for this multiplet. It is arrived at in a rather indirect way as a special example of a class of models which, although based on the Majorana super-symmetry, are set up in such a way that parity conservation emerges.

Further generalizations of the supertranslations are easily invented. For example,

$$\delta x_\mu = \frac{1}{2}\bar{\varepsilon}_k \gamma_\mu \theta_k \ ,$$

$$\delta\theta_k = \varepsilon_k \ , \tag{I.61}$$

where θ_k and ε_k , $k = 1,\dots,N$, are Majorana spinors. This scheme admits an $O(N)$ symmetry with a consequent enrichment of the multiplet structure. This multiplet appears to exhibit an $SU(2n)$ structure in the rest frame.

(A) LOCAL SUPERFIELDS

In Section I a new symmetry was defined in abstract terms. Through their action on the co-ordinates of an enlarged "space-time" the supertranslations were seen to constitute an extension of the group of inhomogeneous proper Lorentz transformations, the Poincaré group. Consideration of the algebra of infinitesimal transformations led to the formulation of commutation and anticommutation rules among the generators. On the basis of these rules the unitary irreducible representations, or "supermultiplets", were analysed.

By means of such considerations a general understanding of the new symmetry is achieved and it becomes possible to formulate dynamical statements in the form of selection rules, identities among scattering amplitudes, etc. To reach a deeper understanding it is necessary to formulate a local field theory for the supermultiplets and to set up Lagrangian models for its realization. A necessary first step in this programme is the concept of the <u>local superfield</u> which we now consider.

The prototypical local superfield is the real scalar function $\Phi(x,\theta)$, whose expansion in powers of θ was given previously,

$$\Phi(x,\theta) = A(x) + \bar\theta\psi(x) + \frac{1}{4}\,\bar\theta\theta F(x) + \frac{1}{4}\,\bar\theta\gamma_5\theta G(x) + \frac{1}{4}\,\bar\theta i\gamma_\nu\gamma_5\theta\, V_\nu(x) + \frac{1}{4}\,\bar\theta\theta\;\bar\theta\chi(x) + \frac{1}{32}\,(\bar\theta\theta)^2\, D(x) \quad . \qquad \text{(II.1)}$$

This field is local if it commutes with itself at spacelike separations,

$$[\Phi(x,\theta)\ ,\ \Phi(x',\theta')] = 0 \quad , \quad (x-x')^2 < 0 \quad . \qquad \text{(II.2)}$$

This definition of locality is compatible with the usual requirement that ordinary Bose fields commute among themselves and with fermionic fields while fermionic fields anticommute among themselves (always at relatively spacelike points). To ensure this compatibility it is necessary only to require that the spinor co-ordinates θ and θ' anticommute with each other and with the spinor field components $\psi(x)$ and $\chi(x)$, on the one hand, and commute with all boson field components on the other.

The action on $\Phi(x,\theta)$ of an infinitesimal transformation from the algebra of the extended Poincaré group is given, according to (I.31), by

$$\delta\Phi(x,\theta) = \left[b_\mu\partial_\mu - \frac{1}{2}\,\omega_{\mu\nu}\left(x_\mu\partial_\nu - x_\nu\partial_\mu + \frac{i}{2}\,\bar\theta\sigma_{\mu\nu}\,\frac{\partial}{\partial\bar\theta}\right) + \bar\varepsilon\left(\frac{\partial}{\partial\bar\theta} + \frac{i}{2}\,(\not\partial\theta)\right)\right]\Phi(x,\theta) \quad . \qquad \text{(II.3)}$$

This formula defines the "scalar" nature of the superfield $\Phi(x,\theta)$. From it can be deduced (by substituting the expansion (II.1)) the transformation properties of the various component fields. Thus, with respect to the Poincaré group, A, F, G and D are scalars, V_ν is a vector, ψ and χ are spinors. The behaviour of these components under a pure supertranslation is given by:

$$\begin{aligned}
\delta A &= \bar\varepsilon\psi \ , \\
\delta\psi &= \frac{1}{2}\,(F + \gamma_5 G + i\gamma_\mu\gamma_5 V_\mu - i\not\partial A)\,\varepsilon \\
\delta F &= \frac{1}{2}\,\bar\varepsilon\chi - \frac{1}{2}\,\bar\varepsilon i\not\partial\psi \ , \\
\delta G &= \frac{1}{2}\,\bar\varepsilon\gamma_5\chi + \frac{1}{2}\,\bar\varepsilon i\not\partial\gamma_5\psi \ , \\
\delta V_\nu &= \frac{1}{2}\,\bar\varepsilon i\gamma_\nu\gamma_5\chi + \frac{1}{2}\,\bar\varepsilon i\not\partial i\gamma_\nu\gamma_5\psi \ , \\
\delta\chi &= \frac{1}{2}\,(D - i\not\partial F + \gamma_5 i\not\partial G - i\gamma_\nu\gamma_5 i\not\partial V_\nu)\,\varepsilon \ , \\
\delta D &= -\bar\varepsilon i\not\partial\chi \quad .
\end{aligned} \qquad \text{(II.4)}$$

Fermion-number is assigned according to the convention of Section I, i.e. θ_+ carries unit fermion-number, $\mathbb{F} = 1$. If the superfield $\Phi(x,\theta)$ as a whole is required to carry $\mathbb{F} = 0$, then the following distribution of $\mathbb{F}$ values is implied:

$$\begin{aligned}
\mathbb{F} &= 2 &&: \quad F + iG \ , \\
\mathbb{F} &= 1 &&: \quad \psi_- \ , \ \chi_+ \ , \\
\mathbb{F} &= 0 &&: \quad A \ , \ V_\nu , \ D \ , \\
\mathbb{F} &= -1 &&: \quad \psi_+ \ , \ \chi_- \ , \\
\mathbb{F} &= -2 &&: \quad F - iG \quad .
\end{aligned} \qquad \text{(II.5)}$$

These assignments are compatible with the reality condition

$$\Phi(x,\theta)^* = \Phi(x,\theta) \qquad \text{(II.6)}$$

which may be imposed. This condition would imply that Bose components A, F, G, V_ν and D are real, while spinor components are subject to the Majorana condition,

$$\psi = C\bar{\psi}^T \quad , \quad \chi = C\bar{\chi}^T \quad .$$

(In taking the complex or hermitian conjugate of a superfield it is always necessary to reverse the order of anticommuting factors; $(\bar{\theta}\psi)^* = \bar{\psi}\theta = \bar{\theta}C\bar{\psi}^T$, etc.)

In general the superfield $\Phi(x,\theta)$ is locally irreducible. By this we mean that it cannot be decomposed into a sum of distinct, independently transforming components without the intervention of non-local operations. (In this it is analogous to the ordinary vector field $V_\mu(x)$ which can be expressed as the sum of transverse and longitudinal parts, $V_\mu = V^t_\mu + \partial_\mu\phi$, only with the help of the non-local operator $1/\partial^2$, i.e. $\phi = (1/\partial^2)\,\partial_\mu V_\mu$. However, such a non-local decomposition is of considerable help in understanding the structure of the representation (II.4) and its derivation will be our first task.)

An important role is played in the derivation by the operation of <u>covariant differentiation</u>. The covariant derivative of Φ is defined by

$$D\Phi(x,\theta) = \left(\frac{\partial}{\partial\bar{\theta}} - \frac{i}{2}(\not{\partial}\theta)\right)\Phi(x,\theta) \quad . \qquad \text{(II.7)}$$

It is covariant in the sense of transforming like a spinor,

$$[D, J_{\mu\nu}] = \frac{1}{2}\sigma_{\mu\nu} D \; , \qquad \text{(II.8)}$$

with respect to homogeneous Lorentz transformations and an invariant with respect to translations and supertranslations,

$$[D , P_\mu] = 0 \quad ,$$
$$\{D , S\} = 0 \quad . \qquad \text{(II.9)}$$

These results are easily verified with the help of the differential operator expressions for the generators (I.37). From (II.7) it follows that the covariant derivatives generate a "supertranslation" algebra of their own,

$$\{D_\alpha , D_\beta\} = -(\gamma_\mu C)_{\alpha\beta}\, i\partial_\mu \quad . \qquad \text{(II.10)}$$

Again it is useful to distinguish the chiral components $D_\pm$ since, for instance, positive chirality components anticommute and the successive application of three such differentiations must annihilate any superfield. The covariant derivative is basically a Majorana spinor and for this reason it is useful to define the conjugate from $\bar{D}$ by

$$\bar{D}^\alpha = (C^{-1})^{\alpha\beta} D_\beta \quad \text{or} \quad \bar{D}_\pm = -D^T_\mp C^{-1} \quad . \qquad \text{(II.11)}$$

The usefulness of these operators lies in the fact that $\bar{D}_-D_+$ and $\bar{D}_+D_-$ are invariants of the extended group. Applied to one superfield $\Phi(x,\theta)$ they yield another whose components transform in exactly the same way. Out of these operators and their products we can construct **invariant** projections. We begin with some simple identities.

The most useful identities are those which spring immediately from the anticommutativity among components of the same chirality,

$$D_+(\bar{D}_-D_+) = 0 \quad \text{and} \quad D_-(\bar{D}_+D_-) = 0 \quad , \qquad \text{(II.12)}$$

which imply the identities

$$(\bar{D}_+D_-)\, D_\pm(\bar{D}_-D_+) = 0 \quad . \qquad \text{(II.13)}$$

Repeated use of these identities together with the anticommutation rules

$$\{D_\pm , \bar{D}_\pm\} = \frac{1 \pm i\gamma_5}{2}\, i\not{\partial} \quad ,$$

enables one to demonstrate

$$\left\{(\bar{D}_-D_+)(\bar{D}_+D_-)\right\}^2 = -4\partial^2\, (\bar{D}_-D_+)\,(\bar{D}_+D_-) \quad ,$$
$$\left\{(\bar{D}_+D_-)(\bar{D}_-D_+)\right\}^2 = -4\partial^2\, (\bar{D}_+D_-)\,(\bar{D}_-D_+) \quad .$$

These formulae then imply that the non-local operators

$$E_+ = -\frac{1}{4\partial^2}(\bar{D}_+D_-)(\bar{D}_-D_+) ,$$

$$E_- = -\frac{1}{4\partial^2}(\bar{D}_-D_+)(\bar{D}_+D_-) , \qquad (II.14)$$

are idempotent and orthogonal, i.e.

$$E_+^2 = E_+ \quad , \quad E_-^2 = E_- \quad , \quad E_+E_- = 0 \ . \qquad (II.15)$$

The operators (II.14) are two of the projections needed for the decomposition of $\Phi(x,\theta)$. There exists only one more and it is given, of course, by

$$\begin{aligned} E_1 &= 1 - E_+ - E_- , \\ &= 1 + \frac{1}{4\partial^2}\{\bar{D}_-D_+ , \bar{D}_+D_-\} , \\ &= 1 + \frac{1}{4\partial^2}(\bar{D}D)^2 \ . \end{aligned} \qquad (II.16)$$

The decomposition of $\Phi(x,\theta)$ corresponding to this non-local resolution,

$$\begin{aligned} \Phi(x,\theta) &= (E_+ + E_- + E_1)\,\Phi(x,\theta) , \\ &= \Phi_+(x,\theta) + \Phi_-(x,\theta) + \Phi_1(x,\theta) , \end{aligned} \qquad (II.17)$$

is given explicitly by

$$\begin{aligned} A &= A_+ + A_- + A_1 , \\ \psi &= \psi_+ + \psi_- + \psi_1 , \\ F &= F_+ + F_- , \\ G &= iF_+ - iF_- , \\ V_\mu &= i\partial_\mu A_+ - i\partial_\mu A_- + V_{1\mu} , \\ \chi &= -i\partial\!\!\!/\psi_+ - i\partial\!\!\!/\psi_- + i\partial\!\!\!/\psi_1 , \\ D &= -\partial^2 A_+ - \partial^2 A_- + \partial^2 A_1 \ . \end{aligned} \qquad (II.18)$$

where $V_{1\mu}$ is transverse and $\psi_\pm$ are chiral. The expressions (II.18) can be solved for the irreducible components,

$$\left.\begin{aligned} A_\pm &= \frac{1}{4}\left(A - \frac{1}{\partial^2}D\right) \mp \frac{i}{2}\frac{1}{\partial^2}\partial_\mu V_\mu , \\ \psi_\pm &= \frac{1}{2}\,\frac{1 \pm i\gamma_5}{2}\left(\psi - \frac{1}{i\partial\!\!\!/}\chi\right) , \\ F_\pm &= \frac{1}{2}(F \mp iG) , \end{aligned}\right\} \qquad (II.19)$$

$$\left.\begin{aligned} A_1 &= \frac{1}{2}\left(A + \frac{1}{\partial^2}D\right) , \\ \psi_1 &= \frac{1}{2}\left(\psi + \frac{1}{i\partial\!\!\!/}\chi\right) , \\ V_{1\mu} &= \left(\eta_{\mu\nu} - \frac{\partial_\mu\partial_\nu}{\partial^2}\right)V_\nu , \end{aligned}\right\} \qquad (II.20)$$

some of which are seen to be non-local . Only if the components χ and D are themselves first and second derivatives ($\partial\!\!\!/\chi'$ and $\partial^2 D'$ of some local fields χ' and D') can the original superfield be considered locally reducible; otherwise it is not.

The transformation behaviour of the reduced components (II.19) and (II.20) can be deduced from (II.4). One finds,

$$\left.\begin{aligned} \delta A_\pm &= \bar{\varepsilon}_\mp\psi_\pm , \\ \delta\psi_\pm &= F_\pm\varepsilon_\pm - i\partial\!\!\!/A_\pm\varepsilon_\mp , \\ \delta F_\pm &= -\bar{\varepsilon}_\pm i\partial\!\!\!/\psi_\pm \end{aligned}\right\} \qquad (II.21)$$

$$\left.\begin{aligned} \delta A_1 &= \bar{\varepsilon}\psi_1 , \\ \delta\psi_1 &= \frac{1}{2}\left(i\gamma_\mu\gamma_5 V_{1\mu} - i\partial\!\!\!/A_1\right)\varepsilon , \\ \delta V_{1\mu} &= -\bar{\varepsilon}\sigma_{\mu\nu}i\gamma_5 i\partial_\nu\psi_1 \ . \end{aligned}\right\} \qquad (II.22)$$

The so-called chiral scalar representations (II.21) and the transverse vector representation (II.22) are of fundamental importance in the development of renormalizable Lagrangian models. For this reason it is necessary to discuss their properties in some detail and, to gain a deeper understanding, from other points of view. We begin with the chiral scalars.

Since the superfield $\Phi(x,\theta)$ from which Φ_+ and Φ_- are obtained could have been subject to the reality condition (II.6), it is clear that the chiral representations are conjugate, $\Phi_- \sim \Phi_+^*$. We shall therefore restrict our attention to the positive chirality Φ_+ . The existence of this kind of superfield rests on a very simple fact. The behaviour of x and θ under supertranslations is such that the complex 4-vector

$$z_\mu = x_\mu - \frac{1}{4}\bar{\theta}\gamma_\mu\gamma_5\theta \tag{II.23}$$

transforms according to

$$\delta z_\mu = i\bar{\varepsilon}_+\gamma_\mu\theta_+ . \tag{II.24}$$

That is, δz depends on θ_+ (or $\bar{\theta}_-$) but not on θ_- (or $\bar{\theta}_+$) . This means that the space of complex-valued functions, $\varphi(z,\theta_+)$, must be invariant under the action of the extended group. Since θ_+ has only two independent components, the expansion of φ in powers of θ_+ must terminate in the second order rather than the fourth. Indeed, one can write

$$\begin{aligned}\varphi(z,\theta_+) &= A_+(z) + \bar{\theta}_-\psi_+(z) + \frac{1}{2}\bar{\theta}_-\theta_+ F_+(z) , \\ &= e^{-\frac{1}{4}\bar{\theta}\not{\partial}\gamma_5\theta}\left[A_+(x) + \bar{\theta}_-\psi_+(x) + \frac{1}{2}\bar{\theta}_-\theta_+ F_+(x)\right] , \\ &= \Phi_+(x,\theta) \quad ,\end{aligned} \tag{II.25}$$

where, in the second line, we have effected the translation $z \to x$ by means of a displacement operator. If the displacement operator is expanded in powers of θ and the terms in (II.25) are arranged in the standard form (II.1) then one will find precisely the coefficients which are indicated for Φ_+ in the expressions (II.18).

Another way to characterize the chiral superfield Φ_+ is by means of a differential condition. Thus, the covariant derivative of Φ_+ is:

$$\begin{aligned}D\Phi_+ &= \left\{\frac{\partial}{\partial\bar{\theta}} - \frac{i}{2}\not{\partial}\theta\right\} e^{-\frac{1}{4}\bar{\theta}\not{\partial}\gamma_5\theta}\left\{A_+ + \bar{\theta}_-\psi_+ + \frac{1}{2}\bar{\theta}_-\theta_+ F_+\right\} , \\ &= e^{-\frac{1}{4}\bar{\theta}\not{\partial}\gamma_5\theta}\left(\frac{\partial}{\partial\bar{\theta}} - i\not{\partial}\theta_-\right)\left\{A_+ + \bar{\theta}_-\psi_+ + \frac{1}{2}\bar{\theta}_-\theta_+ F_+\right\} , \\ &= \frac{1+i\gamma_5}{2} D\Phi_+ ,\end{aligned}$$

indicating positive chirality with respect to the spinor index. In other words, Φ_+ is annihilated by the negative chiral part of the covariant derivative,

$$D_-\Phi_+ = 0 . \tag{II.26}$$

This set of covariant constraints has as its general solution the expansion (II.25). The positive chiral scalar is therefore defined by (II.26).

A property of the chiral scalars which is of crucial importance in the construction of Lagrangians is their closure with respect to multiplication,

$$\Phi_+(x,\theta)\,\Phi_+'(x,\theta) = \Phi_+''(x,\theta) . \tag{II.27}$$

That the product of two positive chirality scalars (at the same point) should yield a third becomes evident when they are expressed as functions of z and θ_+ . Alternatively, the linearity of the differential operator D_- implies that it will annihilate the product if it annihilates the separate factors. The component fields of Φ_+'' defined by (II.27) are given in terms of the components of Φ_+' and Φ_+ as follows:

$$\begin{aligned}A_+''(x) &= A_+(x)\,A_+'(x) , \\ \psi_+''(x) &= A_+(x)\,\psi_+'(x) + \psi_+(x)\,A_+'(x) , \\ F_+''(x) &= A_+(x)\,F_+'(x) + \psi_+^T(x)\,C^{-1}\,\psi_+'(x) + F_+(x)\,A_+'(x)\end{aligned} \tag{II.28}$$

Generalization of (II.27) and (II.28) to products with any number of factors is clearly possible. Indeed, it is a simple matter to show that the local function $v(\Phi(x,\theta))$ has the component expansion

$$v(\Phi_+) = e^{-\frac{1}{4}\bar{\theta}\not{\partial}\gamma_5\theta}\left[v(A_+) + \bar{\theta}_-\psi_+\, v'(A_+) + \frac{1}{2}\bar{\theta}_-\theta_+\left[F_+\, v'(A_+) + \frac{1}{2}\psi_+^T C^{-1}\psi_+\, v''(A_+)\right]\right] \quad , \qquad (II.29)$$

where v' and v'' denote the first and second derivatives with respect to the complex variable A_+ of the function $v(A_+)$. This function must be analytic in general but, in practice, will be restricted to polynomials.

The above properties of the positive chiral scalar Φ_+ are easily translated for the conjugate representation Φ_- . Thus, the negative chiral scalar Φ_- is expressible as a function of z^* and θ_- and one can write for it the expansion

$$\Phi_-(x,\theta) = e^{+\frac{1}{4}\bar{\theta}\not{\partial}\gamma_5\theta}\left[A_-(x) + \bar{\theta}_+\psi_-(x) + \frac{1}{2}\bar{\theta}_+\theta_-\, F_-(x)\right] . \qquad (II.25')$$

Such superfields are characterized by the differential constraint

$$D_+\Phi_- = 0 \quad . \qquad (II.26')$$

Finally, the product of two negative chiral scalars at the same point yields a third: negative chiral scalars are closed with respect to local multiplication.

The local product of a negative chiral scalar with a positive one does not yield a chiral result. This can be seen from the rather trivial observation that the product of two functions, $\varphi(z,\theta_+)$ and $\varphi'(z^*,\theta_-)$, must depend on both z and z^* as well as θ_+ and θ_- . The result must therefore be a superfield $\Phi''(x,\theta)$ of the general type with which we began this section. Its components are given, in the standard notation, by

$$A'' = A_+A_-' \ ,$$

$$\psi'' = A_+\psi_-' + \psi_+A_-' \ ,$$

$$F'' = A_+F_-' + F_+A_-' \ .$$

$$G'' = -iA_+F_-' + iF_+A_-' \ .$$

$$V_\nu'' = (i\partial_\nu A_+)\, A_-' + A_+(-i\partial_\nu A_-') + \psi_+^T C^{-1}\gamma_\nu\psi_-' \ .$$

$$\chi'' = (i\partial_\nu A_+)\,\gamma_\nu\psi_-' + \gamma_\nu\psi_+(i\partial_\nu A_-') - (i\not{\partial}\psi_+)\, A_-' - A_+(i\not{\partial}\psi_-') + 2\psi_+F_-' + 2F_+\psi_-' \ .$$

$$D'' = (-\partial^2 A_+)\, A_-' + 2(\partial_\nu A_+)\,(\partial_\nu A_-') + A_+(-\partial^2 A_-') + 4F_+F_-' + 2\,\psi_+^T C^{-1}(i\overleftarrow{\not{\partial}} - i\overrightarrow{\not{\partial}})\,\psi_-' \quad . \qquad (II.30)$$

This superfield is clearly not locally reducible.

The fermion-number content of the chiral superfields is easily established. Since θ_+ and θ_- carry $\mathbb{F} = 1$ and $\mathbb{F} = -1$, respectively, it follows from an examination of the expansion (II.25) that A_+ , ψ_+ and F_+ must carry $\mathbb{F} = f$, $f-1$ and $f-2$, respectively. The value f , associated with A_+ is an invariant of the representation. (We shall normally take $f = 0$ so that ψ_+ carries $\mathbb{F} = -1$, i.e. ψ_+ serves to <u>annihilate</u> one fermion and $\bar{\psi}_+$ <u>creates</u> one.) The negative chiral components A_- , ψ_-, F_- carry the values $-f$, $-f+1$, $-f+2$ if Φ_- is identified with Φ_+^* . (As a general rule we shall <u>not</u> identify Φ_- with Φ_+^* but treat it as an independent field with $f = 2$. With this notational convention, ψ_- carries $\mathbb{F} = -1$ and annihilates a fermion. Note that A_- annihilates a difermion.)

New categories of local representations are obtained by generalizing the scalar superfields to spinors and tensors of arbitrary rank. That is, one regards $\Phi(x,\theta)$ not as a single scalar superfield but as a set of superfields belonging to one of the finite-dimensional representations of the proper Lorentz group. With respect to translations and supertranslations, these fields transform as scalars, but under the homogeneous (proper) group they transform according to

$$\Phi(x,\theta) \to \Phi'(x',\theta') = D(\Lambda)\ \Phi(x,\theta) \ , \qquad (II.31)$$

where the matrices $D(\Lambda)$ belong to one of the familiar $(2j_1 + 1)\ (2j_2 + 1)$-dimensional representations, $\mathfrak{D}(j_1,j_2)$ – or a direct sum of such representations. This kind of generalization applies equally to the chiral $(\Phi_\pm)$ or non-chiral (Φ,Φ_1) types. Two examples will illustrate the main features.

The first example employs a chiral spinor superfield, $(\Psi_{\alpha-})_+$, which has negative chirality with respect to its external spinor index,

$$(1 + i\gamma_5)\,\Psi_{-+} = 0 \quad , \qquad \text{(II.32)}$$

and positive chirality with respect to its internal structure,

$$D_-\,\Psi_{-+} = 0 \quad . \qquad \text{(II.33)}$$

The expansion in component fields has the general form (II.25)

$$\Psi_{-+}(x,\theta) = e^{-\frac{1}{4}\bar{\theta}\not\partial\gamma_5\theta}\left(U_-(x) + M_\mu(x)\,\gamma_\mu\theta_+ + \frac{1}{2}\bar{\theta}_-\theta_+\,V_-(x)\right) , \qquad \text{(II.34)}$$

where U_- and V_- are negative chiral spinors and M_μ is a 4-vector. Under an infinitesimal supertranslation these components transform according to

$$\delta U_- = \gamma_\mu\epsilon_+\,M_\mu \quad ,$$
$$\delta M_\mu = -\frac{1}{2}\bar{\epsilon}_-\gamma_\mu V_- - \frac{1}{2}\bar{\epsilon}_+ i\not\partial\gamma_\mu U_- \quad ,$$
$$\delta V_- = \gamma_\mu i\not\partial\epsilon_-\,M_\mu \quad . \qquad \text{(II.35)}$$

This representation is basically complex. Thus, even with the most favourable assignment of fermion-numbers $\mathbb{F} = 1, 0, -1$ to U_- , M_μ and V_- , respectively, it is not possible to treat M_μ as a real vector. However, if certain constraints are applied,this representation will turn out to be equivalent to the real Φ_1 . We shall return to this question after setting out the second example.

The second example employs a chiral spinor, Ψ_{++} ,which has positive chirality both internally and externally, i.e.

$$(1 - i\gamma_5)\,\Psi_{++} = 0 \quad \text{and} \quad D_-\,\Psi_{++} = 0 \quad . \qquad \text{(II.36)}$$

Again the expansion in component fields has the general form (II.25),

$$\Psi_{++}(x,\theta) = e^{-\frac{1}{4}\bar{\theta}\not\partial\gamma_5\theta}\left(U_+(x) + \left(D(x) + \frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}(x)\right)\theta_+ + \frac{1}{2}\bar{\theta}_-\theta_+\,V_+(x)\right) , \qquad \text{(II.37)}$$

where U_+ and V_+ are positive chiral spinors, D is a scalar and $F_{\mu\nu}$ is a self-dual antisymmetric tensor. (It is of course the opposite external chiralities that govern the appearance of a vector M_μ in Ψ_{-+} and of a scalar-tensor combination $D, F_{\mu\nu}$ in Ψ_{++} .) The component fields transform according to

$$\delta U_+ = \left(D + \frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}\right)\epsilon_+ \quad ,$$
$$\delta D = -\frac{1}{2}\bar{\epsilon}_-V_+ - \frac{1}{2}\bar{\epsilon}_+\,i\not\partial U_+ \quad ,$$
$$\delta F_{\mu\nu} = -\frac{1}{2}\bar{\epsilon}_-\sigma_{\mu\nu}V_+ - \frac{1}{2}\bar{\epsilon}_+ i\not\partial\sigma_{\mu\nu}U_+ \quad ,$$
$$\delta V_+ = \left(D + \frac{1}{2}\sigma_{\mu\nu}F_{\mu\nu}\right) i\not\partial\epsilon_- \quad . \qquad \text{(II.38)}$$

This representation also is basically complex but capable of being made real by means of certain constraints.

Now consider the representation (II.35) to see if it can be made self-conjugate. Since U_- and V_- carry $\mathbb{F} = 1$ and -1 , respectively, we should identify the conjugate of one with the other. For the correct balance of dimensions and also to have the correct chiralities, the association must take the form

$$V_- = \omega i\not\partial C\bar{U}_-^T \quad , \qquad \text{(II.39)}$$

where ω is a number to be determined. If the identification (II.39) is to be compatible with supertranslations we must have

$$\delta V_- = \omega i\not\partial C\delta\bar{U}_-^T \quad ,$$

i.e. according to (II.35),

$$\gamma_\mu i\not\partial\epsilon_-M_\mu = \omega i\not\partial C\gamma_\mu^T\bar{\epsilon}_+^T M_\mu^* \quad ,$$
$$= -\omega i\not\partial\gamma_\mu\epsilon_-M_\mu^*$$

for arbitrary ε_- . This gives constraints on M , viz.

$$0 = \partial_\mu (M_\mu + \omega M_\mu^*) ,$$

$$0 = \partial_\mu (M_\nu - \omega M_\nu^*) - \partial_\nu (M_\mu - \omega M_\mu^*) . \qquad \text{(II.40)}$$

Finally, it is necessary to verify the compatibility of (II.39) and (II.40) with the expression (II.35) for δM_μ . This requires $|\omega| = 1$. The phase factor can always be absorbed and so we shall take $\omega = 1$. To summarize, taking

$$M_\mu = \frac{1}{2} \left(V_{1\mu} - i\partial_\mu A_1 \right) ,$$

$$U_- = \psi_{1-} ,$$

$$V_- = i\partial\!\!\!/ C\bar{\psi}^T_{1-} = i\partial\!\!\!/ \psi_{1+} , \qquad \text{(II.41)}$$

where $V_{1\mu}$ is real and transverse, A_1 is real and ψ_1 is Majorana, we find that the transformations (II.35) reduce to the form (II.22), i.e. a <u>real</u> transverse vector representation.

Similar considerations applied to the representation (II.38) show that it too can be real. In this case the reality conditions are

$$V_+ = i\partial\!\!\!/ C\bar{U}^T_+ ,$$

$$2F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + \frac{i}{2} \varepsilon_{\mu\nu\lambda\rho} (\partial_\lambda V_\rho - \partial_\rho V_\lambda) . \qquad \text{(II.42)}$$

with V_μ and D real. The transformation rules (II.38) then take the form

$$\delta U_+ = \left[D + \frac{1}{2} \sigma_{\mu\nu} (\partial_\mu V_\nu - \partial_\nu V_\mu) \right] \varepsilon_+ ,$$

$$\delta V_\mu = \bar{U}_+ \gamma_\mu \varepsilon_+ + \bar{\varepsilon}_+ \gamma_\mu U_+ ,$$

$$\delta D = \frac{1}{2} \bar{U}_+ i\overleftarrow{\partial\!\!\!/} \varepsilon_+ - \frac{1}{2} \bar{\varepsilon}_+ i\partial\!\!\!/ U_+ . \qquad \text{(II.43)}$$

This representation also is related to Φ_1 though it is not locally equivalent. The connection is made through the non-local relations,

$$A_1 = \frac{1}{\partial^2} D ,$$

$$V_{1\mu} = \left(\eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2} \right) V_\nu ,$$

$$\psi_1 = \frac{1}{i\partial\!\!\!/} (U_+ + C\bar{U}^T_+)$$

or, more compactly,

$$\Psi_{++} = i\partial\!\!\!/ D_- \Phi_1 .$$

(B) GREEN'S FUNCTIONS AND INVARIANT AMPLITUDES

One of the main objectives of local field theory is the computation of Green's functions. The structure of these many-point functions is restricted by symmetry considerations and, in the present case, by the requirement of invariance under the action of supertranslations. Such structural constraints are best exhibited by expressing the Green functions in terms of invariant amplitudes and we shall examine here some applications of this idea.

Since it is our purpose to reveal the implications of the new transformations in the extended relativistic symmetry, we shall pay little attention to the subgroup of Poincaré transformations, which can be handled in the usual way. It will therefore be sufficient to deal with a system of fields comprising only chiral scalars $\Phi_\pm$ and their complex conjugates. The appending of Lorentz indices to such fields so as to include non-scalar representations is a relatively trivial modification and would not affect the main argument. (Notice also that the non-chiral real field Φ_1 can always be represented by a chiral spinor, $\Psi_{+-} = D_+\Phi_1$, and so included in the general scheme.)

To begin with we shall ignore the fermionic quantum number which distinguishes Φ_- from Φ_+^* . Insofar as supertranslations only are concerned, these fields are equivalent. It will therefore be sufficient to analyse the structure of many-point Green's functions referring to products of the fields Φ_+ and Φ_- , only. Afterwards we can introduce the fermion-number and quite easily pick out the amplitudes which conserve it.

A considerable simplification in notation is achieved by amputating the displacement operators, $\exp(\mp \frac{1}{4} \bar{\theta}\not{\partial}\gamma_5\theta)$, which always appear as factors in the chiral superfields. Thus, we define the amputated fields,

$$\hat{\Phi}_\pm(x,\theta) = \exp\left(\pm \frac{1}{4} \bar{\theta}\not{\partial}\gamma_5\theta\right) \Phi_\pm(x,\theta) ,$$

$$= A_\pm(x) + \bar{\theta}_\mp\psi_\pm(x) + \frac{1}{2}\bar{\theta}_\mp\theta_\pm F_\pm(x) , \qquad \text{(II.44)}$$

which depend on only one chiral component, either θ_+ or θ_- (and $\bar{\theta}_-$ or $\bar{\theta}_+$), of the spinor co-ordinate. This property is expressed analytically by

$$\frac{\partial\hat{\Phi}_+}{\partial\bar{\theta}_+} = \frac{1 - i\gamma_5}{2} \frac{\partial\hat{\Phi}_+}{\partial\bar{\theta}} = 0 ,$$
$$\frac{\partial\hat{\Phi}_-}{\partial\bar{\theta}_-} = \frac{1 + i\gamma_5}{2} \frac{\partial\hat{\Phi}_-}{\partial\bar{\theta}} = 0 . \qquad \text{(II.45)}$$

Under supertranslations the amputated chiral fields transform according to

$$\delta\hat{\Phi}_\pm = \bar{\varepsilon}\left(\frac{\partial}{\partial\bar{\theta}_\mp} + i\not{\partial}\theta_\pm\right)\hat{\Phi}_\pm ,$$

$$= \bar{\varepsilon}_\mp \frac{\partial\hat{\Phi}_\pm}{\partial\bar{\theta}_\mp} + \bar{\varepsilon}_\pm i\not{\partial}\theta_\pm\hat{\Phi}_\pm . \qquad \text{(II.46)}$$

On occasion in the following we shall use the abbreviated notation $\hat{\Phi}_\pm(1) = \hat{\Phi}_\pm(x_1,\theta_1)$.

The n-point Green's functions $\hat{G}$ are expressed as vacuum expectation values,

$$\hat{G}(1,\ldots,n) = \langle 0|T^*\hat{\Phi}_+(1)\cdots\hat{\Phi}_+(r)\,\hat{\Phi}_-(r+1)\cdots\hat{\Phi}_-(n)|0\rangle , \qquad \text{(II.47)}$$

in which the T*-ordering is used so as to simplify the transformation properties (i.e. avoid the complications due to non-covariant surface-dependent terms). Our main purpose here is to consider the implications of the requirement that $\hat{G}$ be invariant with respect to the transformations (II.46). This requirement is embodied in the single equation

$$0 = \bar{\varepsilon}\left[\sum_1^r \left(\frac{\partial}{\partial\bar{\theta}_{j-}} + i\not{\partial}_j\theta_{j+}\right) + \sum_{r+1}^n \left(\frac{\partial}{\partial\bar{\theta}_{j+}} + i\not{\partial}_j\theta_{j-}\right)\right]\hat{G} \qquad \text{(II.48)}$$

or, equivalently, in the chiral pair,

$$0 = \left(\sum_1^r \frac{\partial}{\partial\bar{\theta}_{j-}} + \sum_{r+1}^n i\not{\partial}_j\theta_{j-}\right)\hat{G} ,$$

$$0 = \left(\sum_{r+1}^n \frac{\partial}{\partial\bar{\theta}_{j+}} + \sum_1^r i\not{\partial}_j\theta_{j+}\right)\hat{G} . \qquad \text{(II.49)}$$

Fortunately it is quite easy to find the general solution to these equations. One starts by looking for solutions of the form

$$\hat{G} = e^{-W} M , \qquad \text{(II.50)}$$

where W is chosen to satisfy the inhomogeneous equations

$$\sum_1^r \frac{\partial W}{\partial\bar{\theta}_{j-}} = \sum_{r+1}^n i\not{\partial}_j\theta_{j-} ,$$

$$\sum_{r+1}^n \frac{\partial W}{\partial\bar{\theta}_{j+}} = \sum_1^r i\not{\partial}_j\theta_{j+} . \qquad \text{(II.51)}$$

There are many ways to satisfy these inhomogeneous equations and we shall return to this question. For the moment assume that one solution has been found. On using this W in (II.50) and substituting it into the conditions (II.49) one finds that the amplitude M must satisfy the simplified equations

$$0 = \sum_1^r \frac{\partial M}{\partial\bar{\theta}_{j-}} ,$$

$$0 = \sum_{r+1}^n \frac{\partial M}{\partial\bar{\theta}_{j+}} . \qquad \text{(II.52)}$$

Thus, quite generally, the amplitude M can depend on the chiral co-ordinates $\theta_{1+},\ldots,\theta_{r+}$ and $\theta_{r+1-},\ldots,\theta_{n-}$ only through their differences. For example,

$$M = M(\theta_{1r+},\ldots,\theta_{r-1r+}\,;\,\theta_{r+1n-},\ldots,\theta_{n-1n-}) \,, \qquad (II.53)$$

where $\theta_{jr+} = \theta_{j+} - \theta_{r+}$ and $\theta_{jn-} = \theta_{j-} - \theta_{n-}$. There are altogether $n-2$ such differences. Further restrictions must arise for disconnected contributions to the Green functions: only differences between co-ordinates referring to a connected piece are admissible. To simplify the discussion we shall tacitly assume that only connected amplitudes are concerned. The expression (II.53) incorporates the supertranslation invariance requirement for all chiral Green's functions with only two exceptions. The exceptional cases involve the products of fields all of which have the same chirality, i.e. $r = 0$ or $r = n$, and these must be treated separately. The reason for this is the impossibility of constructing a W satisfying (II.51) when all θ's have the same chirality. We shall return to the exceptional cases after considering the form of W.

It is a simple matter to verify that the linear operator

$$W_{rn} = \bar\theta_{r-} \sum_{r+1}^{n} i\partial\!\!\!/_j\theta_{j-} + \bar\theta_{n+}\sum_{1}^{r} i\partial\!\!\!/_j\theta_{j+} - \bar\theta_{r-}\Big(\sum_{r+1}^{n} i\partial\!\!\!/_j\Big)\,\theta_{n-} \qquad (II.54)$$

is a solution of (II.51). (One needs only to draw on the fact $\sum_{1}^{n}\partial_j = 0$ resulting from Poincaré invariance.) The expression (II.54) meets all the requirements but it is not very symmetric in appearance. While regretting this asymmetry, we do not feel that it is a very serious shortcoming. We remark only that alternative solutions to (II.51) must take the form

$$W = W_{rn} + W' \quad ,$$

where W' is any linear operator constructed entirely from co-ordinate differences and hence satisfying the homogeneous form of (II.51). By exploiting this arbitrariness one could construct a more symmetrical solution but we shall not pursue the point.

Finally, we must deal with the exceptional, or what might be called "monochiral" cases $r = 0$ and $r = n$. Suppose, then, that $r = n$ and the constraints (II.49) take the form

$$0 = \sum_{1}^{n} \frac{\partial\hat G}{\partial\theta_{j-}} \qquad \text{and} \qquad 0 = \sum_{1}^{n} i\partial\!\!\!/_j\theta_{j+}\hat G \quad . \qquad (II.55)$$

The first of these equations indicates that $\hat G$ must be a function of the $n-1$ co-ordinate differences θ_{jn+}. The second is solved by partitioning the $n-1$ terms of the sum

$$\sum_{1}^{n-1} i\partial_j\theta_{jn+} = \sum_{k} P_{k+} \qquad (II.56)$$

in all possible ways. The partitions involve from one up to $n-1$ pieces,

$$P_{k+} = i\partial_{j_1}\theta_{j_1n+} + \cdots + i\partial_{j_k}\theta_{j_kn+} \quad . \qquad (II.57)$$

For each partition form the monomial

$$\prod_{k} \bar P_{k-} P_{k+} \quad ,$$

which clearly is annihilated by each member of the partition and hence by the sum. The equations (II.55) are therefore satisfied by

$$\hat G = \sum_{\text{partitions}} \prod_{k} \bar P_{k-} P_{k+} M_{\{P\}} \quad , \qquad (II.58)$$

where the sum runs over all independent partitions to each of which there corresponds a scalar amplitude $M_{\{P\}}$.

The other monochiral amplitudes, $r = 0$, are decomposed in the same manner. Some examples of these decompositions are given in Appx. B, where they play an important role in the softening of ultraviolet divergences.

The supertranslation invariance requirement is fully met by the expression (II.58) in the monochiral case and by the expression (II.50) with M and W given by (II.53) and (II.54) in the mixed cases. If we had been dealing with chiral spinor and tensor superfields, these expressions would still be valid, the appropriate Lorentz indices being tacit in the amplitudes M and $M_{\{P\}}$. These amplitudes must of course respect the Poincaré group. Their decomposition into scalar invariants would follow by standard methods which we need not go into. (Even for the case of scalar superfields the decomposition of M into invariant amplitudes, which is just the problem of resolving multispinors of rank $\leqslant n-2$, can be quite complicated.)

Having dealt with the supertranslations it is necessary to consider now the phase transformations generated by fermion-number. On the chiral scalar superfields these transformations take the form

$$U_\alpha \hat{\Phi}_+(x,\theta_+) U_\alpha^{-1} = \hat{\Phi}_+(x,e^{-i\alpha}\theta_+) ,$$
$$U_\alpha \hat{\Phi}_-(x,\theta_-) U_\alpha^{-1} = e^{-2i\alpha} \hat{\Phi}_-(x,e^{i\alpha}\theta_-) , \qquad (II.59)$$

and so $\hat{\Phi}_-$ must be distinguished from $\hat{\Phi}_+^*$. The implications are clear. In the general case the amplitude M must satisfy an identity of the form

$$M\left(e^{-i\alpha}\theta_+ ; e^{i\alpha}\theta_-\right) = e^{2i\alpha N_-} M(\theta_+,\theta_-) , \qquad (II.60)$$

where N_- is an integer defined as the number of times Φ_- occurs in $\hat{G}$ minus the number of times Φ_-^* occurs,

$$N_- = N(\Phi_-) - N(\Phi_-^*) . \qquad (II.61)$$

The condition (II.60) plainly constitutes a fairly strong restriction on the form of the amplitude M.

To illustrate the application of the symmetries discussed here we list a few simple examples. Firstly, from among the 2-point functions we have

$$\langle T^*\hat{\Phi}_+(1)\ \hat{\Phi}_-^*(2)\rangle = \tfrac{1}{2}\,\bar{\theta}_{12-}\ \theta_{12+}\ a ,$$
$$\langle T^*\hat{\Phi}_+(1)\ \hat{\Phi}_+^*(2)\rangle = \exp(-\bar{\theta}_{2+} i\partial_1 \theta_{1+})\ M , \qquad (II.62)$$

where a and M are scalar functions of the invariant $x_{12}^2 = (x_1-x_2)^2$. The only other non-vanishing 2-point functions are those which result from (II.62) by reversing the chirality assignments. We shall have more to say about expressions (II.62) when we come to the insertion of intermediate states.

From among the 3-point functions we have, for example,

$$\langle T^*\hat{\Phi}_-^*(1)\ \hat{\Phi}_-^*(2)\ \hat{\Phi}_+(3)\rangle = \tfrac{1}{4}(\bar{\theta}_{12-}\theta_{13+})(\bar{\theta}_{23-}\theta_{23+})\ M_0 ,$$
$$\langle T^*\hat{\Phi}_+(1)\ \hat{\Phi}_+(2)\ \hat{\Phi}_+^*(3)\rangle = \exp\left[-\bar{\theta}_{3+}(i\not{\partial}_1\theta_{1+} + i\not{\partial}_2\theta_{2+})\right] M_1 ,$$
$$\langle T^*\hat{\Phi}_+(1)\ \hat{\Phi}_-^*(2)\ \hat{\Phi}_+^*(3)\rangle = \exp\left[-\bar{\theta}_{3+}(i\not{\partial}_1\theta_{1+} + i\not{\partial}_2\theta_{2+})\right] \tfrac{1}{2}\,\bar{\theta}_{12-}\theta_{12+}\ M_2 ,$$
$$(II.63)$$

where the scalar amplitudes M_0, M_1 and M_2 depend on the invariants x_{13}^2, x_{23}^2 and $x_{13}.x_{23}$.

The importance of Green's functions in local field theories lies in their close relation to the quantities of more immediate interest, the scattering amplitudes or S-matrix elements. The latter are extracted from the former by means of the limiting procedure known as the LSZ reduction technique. It is not our intention to derive and justify this procedure as applied to superfields but only to point out some of the novel features and exhibit a final formula. To this end we begin with a discussion of the 1-particle contribution to the 2-point functions: the so-called free propagators.

A scalar supermultiplet of mass M comprises the particle states $|p\rangle_0$, $|p,\lambda\rangle_1$, $|p\rangle_2$ defined in Section I. Now we must allow also for the antiparticles $|\bar{p}\rangle_0$, $|\overline{p,\lambda}\rangle_{-1}$, $|\bar{p}\rangle_{-2}$ which belong to a distinct supermultiplet of mass M. The action on these states of the super-translation generator S is given by

$$\tfrac{1}{i} S\,|p\rangle_0 = -\sum_\lambda |p\lambda\rangle_1\ C\bar{u}_+^T(p\lambda) ,$$
$$\tfrac{1}{i} S\,|p\lambda\rangle_1 = |p\rangle_0\ u_+(p\lambda) + |p\rangle_2\ u_-(p\lambda) ,$$
$$\tfrac{1}{i} S\,|p\rangle_2 = -\sum_\lambda |p\lambda\rangle_1\ C\bar{u}_-^T(p\lambda) , \qquad (II.64)$$

$$\tfrac{1}{i} S\,|\bar{p}\rangle_0 = -\sum_\lambda |\overline{p\lambda}\rangle_{-1}\ v_+(p,\lambda) ,$$
$$\tfrac{1}{i} S\,|\overline{p\lambda}\rangle_{-1} = |\bar{p}\rangle_0\ C\bar{v}_+^T(p,\lambda) + |\bar{p}\rangle_{-2}\ C\bar{v}_-^T(p,\lambda) ,$$
$$\tfrac{1}{i} S\,|\bar{p}\rangle_{-2} = -\sum_\lambda |\overline{p\lambda}\rangle_{-1}\ v_-(p,\lambda) , \qquad (II.65)$$

where u and v denote the usual positive and negative energy Dirac spinors. From (II.64) and (II.65) and the field transformation laws (II.46) expressed in the form

$$\frac{1}{i}[\hat{\Phi}_\pm, \bar{\varepsilon} S] = \bar{\varepsilon}\left(\frac{\partial}{\partial\bar{\theta}_\mp} + i\not{\partial}\theta_\pm\right)\hat{\Phi}_\pm \quad ,$$

one can derive a number of relations among the matrix elements of the component fields between the vacuum and the 1-particle states. On suitably normalizing the components A_+ and A_- one finds the following non-vanishing matrix elements:

$$\begin{aligned}
\langle 0|A_+|p\rangle_0 &= 1 \quad , \quad {}_0\langle \bar{p}|A_+|0\rangle = 1 \quad , \\
\langle 0|\psi_+|p\lambda\rangle_1 &= u_+(p\lambda) \quad ; \quad {}_{-1}\langle \overline{p\lambda}|\psi_+|0\rangle = v_+(p,\lambda) \quad , \\
\langle 0|F_+|p\lambda\rangle_2 &= -M \quad , \quad {}_{-2}\langle \bar{p}|F_+|0\rangle = -M \quad ,
\end{aligned} \qquad \text{(II.66)}$$

$$\begin{aligned}
\langle 0|A_-|p\rangle_2 &= 1 \quad , \quad {}_{-2}\langle \bar{p}|A_-|0\rangle = 1 \quad , \\
\langle 0|\psi_-|p\lambda\rangle_1 &= u_-(p\lambda) \quad , \quad {}_{-1}\langle \overline{p\lambda}|\psi_-|0\rangle = v_-(p\lambda) \quad , \\
\langle 0|F_-|p\rangle_0 &= -M \quad , \quad {}_0\langle \bar{p}|F_-|0\rangle = -M \quad .
\end{aligned} \qquad \text{(II.67)}$$

With a standard covariant normalization for these states we can compute their contribution to the absorptive parts of the 2-point functions. Firstly, with $x_{10} > x_{20}$,

$$\langle T^*\hat{\Phi}_+(1)\ \hat{\Phi}^*_-(2)\rangle =$$

$$= \int \frac{dp}{(2\pi)^3}\ \theta(p_0)\ \delta(p^2-M^2)\ \langle 0|\hat{\Phi}_+(1)\left\{|p\rangle_{0\ 0}\langle p| + \sum_\lambda |p\lambda\rangle_{1\ 1}\langle p\lambda| + \right.$$

$$\left. + |p\rangle_{2\ 2}\langle p|\right\}\hat{\Phi}^*_-(2)|0\rangle$$

$$= \int \frac{dp}{(2\pi)^3}\,\theta(p_0)\ \delta(p^2-M^2)\ e^{-ip(x_1-x_2)}\left\{\langle 0|A_+|p\rangle_{0\ 0}\langle p|\tfrac{1}{2}\bar{\theta}_{2-}\theta_{2+}F^*_-|0\rangle + \right.$$

$$\left. + \sum_\lambda \langle 0|\bar{\theta}_+\psi_+|p\lambda\rangle_{1\ 1}\langle p\lambda|\bar{\psi}_-\theta_{2+}|0\rangle + \langle 0|\tfrac{1}{2}\bar{\theta}_{1-}\theta_{1+}F_+|p\rangle_{2\ 2}\langle p|A^*_-|0\rangle\right\}$$

$$= \int \frac{dp}{(2\pi)^3}\,\theta(p_0)\ \delta(p^2-M^2)\ e^{-ip(x_1-x_2)}\left\{-\frac{M}{2}\,\bar{\theta}_{2-}\theta_{2+} + \bar{\theta}_{1-}(\not{p}+M)\theta_{2+} - \frac{M}{2}\,\bar{\theta}_{1-}\theta_{2+}\right\}$$

$$= -M\Delta_+(x_1-x_2\ ;\ M^2)\ \frac{1}{2}\,\bar{\theta}_{12-}\ \theta_{12+} \quad .$$

Similarly, when $x_{10} < x_{20}$ the antiparticle multiplet gives the negative frequency part. Taken together, these expressions clearly take the form (II.62) with the amplitude, a, given by

$$a(x_{12}) = -M\,\Delta_F(x_{12}\ ;\ M^2) \quad . \qquad \text{(II.68)}$$

For the other 2-point function one finds a different complexion. Taking $x_{10} > x_{20}$ one finds

$$\langle T^*\hat{\Phi}_+(1)\ \hat{\Phi}^*_+(2)\rangle = \int \frac{dp}{(2\pi)^3}\,\theta(p_0)\ \delta(p^2-M^2)\ e^{-ip(x_1-x_2)}$$

$$\left\{\langle 0|A_+|p\rangle_{0\ 0}\langle p|A^*_+|0\rangle + \sum_\lambda \langle 0|\bar{\theta}_{1-}\psi_+|p\lambda\rangle_{1\ 1}\langle p\lambda|\bar{\psi}_+\theta_{2-}|0\rangle + \right.$$

$$\left. + \langle 0|\tfrac{1}{2}\bar{\theta}_{1-}\theta_{1+}F_+|p\rangle_{2\ 2}\langle p|\tfrac{1}{2}\bar{\theta}_{2+}\theta_{2-}F^*_+|0\rangle\right\}$$

$$= \int \frac{dp}{(2\pi)^3}\,\theta(p_0)\ \delta(p^2-M^2)\ e^{-ip(x_1-x_2)}\left\{1 + \bar{\theta}_{1-}(\not{p}+M)\theta_{2-} - \frac{M^2}{4}\,\bar{\theta}_{1-}\theta_{1+}\bar{\theta}_{2+}\theta_{1-}\right\}$$

$$= \exp\left[-\bar{\theta}_{2+}i\not{\partial}_1\theta_{1+}\right]\ \Delta_+(x_1-x_2\ ;\ M^2) \quad .$$

Similarly when $x_{10} < x_{20}$. This again conforms with (II.62), the amplitude M being just the Feynman propagator Δ_F. Free propagators for the various component fields are easily picked out from these expressions.

Finally, we give the supersymmetric version of the LSZ reduction formulae. Consider one of the Green functions

$$\langle 0|T\hat{\Phi}_+(x,\theta)\ \Pi(\hat{\Phi})|0\rangle\ , \qquad \text{(II.69)}$$

where $\Pi(\hat{\Phi})$ denotes a product of superfields the details of which do not concern us. According to well-known arguments, the asymptotic behaviour ($x_0 \to \pm\infty$) of (II.69) is dominated by the 1-particle contributions. Thus, formally, as $x_0 \to \infty$,

$$\langle 0|T\hat{\Phi}_+(x,\theta)\Pi|0\rangle \simeq \int \frac{dp}{(2\pi)^4} \frac{i}{p^2-M^2+i\varepsilon} \Big[\langle 0|A_+(x)|p\rangle_{0\ 0}\langle p|\Pi|0\rangle +$$

$$+ \sum_\lambda \langle 0|\bar\theta_-\psi_+(x)|p\lambda\rangle_{1\ 1}\langle p\lambda|\Pi|0\rangle + \langle 0|\tfrac{1}{2}\bar\theta_-\theta_+F_+(x)|p\rangle_{2\ 2}\langle p|\Pi|0\rangle \Big]$$

$$= \int \frac{dp}{(2\pi)^4} \frac{i}{p^2-M^2+i\varepsilon} e^{-ipx} \Big[{}_0\langle p|\Pi|0\rangle +$$

$$+ \sum_\lambda \bar\theta_- u_+(p\lambda)\ {}_1\langle p\lambda|\Pi|0\rangle - \frac{M}{2}\bar\theta_-\theta_+\ {}_2\langle p|\Pi|0\rangle \Big] .$$

More exactly, the 1-particle matrix elements of Π are given by

$${}_0\langle p|\Pi|0\rangle + \sum_\lambda \bar\theta_- u_+(p\lambda)\ {}_1\langle p\lambda|\Pi|0\rangle - \frac{M}{2}\bar\theta_-\theta_+\ {}_2\langle p|\Pi|0\rangle =$$

$$= i\int dx\ e^{ipx} \Big[(\partial^2+M^2)\ \langle 0|T\hat\Phi_+(x,\theta)\Pi|0\rangle\Big] \qquad (II.70)$$

evaluated on the mass shell, $p^2 = M^2$, with $p_0 > 0$. The various matrix elements are associated with distinct θ-coefficients and so can easily be picked out from (II.70). Likewise, the antiparticle matrix elements are given by the integral

$$\langle 0|\Pi|\bar p\rangle_0 + \sum_\lambda \langle 0|\Pi|\overline{p\lambda}\rangle_{-1}\ \bar\theta_- v_+(p\lambda) - \frac{M}{2}\bar\theta_-\theta_+\ \langle 0|\Pi|\bar p\rangle_{-2} =$$

$$= i\int dx\ e^{-ipx} \Big[(\partial^2+M^2)\ \langle 0|T^*\hat\Phi_+(x,\theta)\Pi|0\rangle\Big] . \qquad (II.71)$$

again with $p^2 = M^2$ and $p_0 > 0$. The equations (II.70) and (II.71) constitute the reduction formulae for $\hat\Phi_+$. The same considerations applied to $\hat\Phi_-$ yield the reduction formulae

$${}_2\langle p|\Pi|0\rangle + \sum_\lambda \bar\theta_+ u_-(p\lambda)\ {}_1\langle p\lambda|\Pi|0\rangle - \frac{M}{2}\bar\theta_+\theta_-\ {}_0\langle p|\Pi|0\rangle =$$

$$= i\int dx\ e^{ipx} \Big[(\partial^2+M^2)\ \langle 0|T\hat\Phi_-(x,\theta)\Pi|0\rangle\Big] , \qquad (II.72)$$

$$\langle 0|\Pi|\bar p\rangle_{-2} + \sum_\lambda \langle 0|\Pi|\overline{p\lambda}\rangle_{-1}\ \bar\theta_+ v_-(p\lambda) - \frac{M}{2}\bar\theta_+\theta_-\ \langle 0|\Pi|\bar p\rangle_0 =$$

$$= i\int dx\ e^{-ipx} \Big[(\partial^2+M^2)\ \langle 0|T\hat\Phi_-(x,\theta)\Pi|0\rangle\Big] . \qquad (II.73)$$

It will be remarked that the two formulae (II.70) and (II.72) give precisely the same matrix elements although with different coefficients. This is a reflection of the fact that, in the asymptotic region, the superfields Φ_+ and Φ_- are related in a simple way. They behave like free fields and satisfy a pair of linear differential equations,

$$-\frac{1}{2}\bar D D\Phi_+ + M\Phi_- = 0 ,$$

$$-\frac{1}{2}\bar D D\Phi_- + M\Phi_+ = 0 ,$$

or, in component form,

$$F_\mp + MA_\pm = 0 ,$$

$$i\partial\!\!\!/\psi_\mp - M\psi_\pm = 0 ,$$

$$-\partial^2 A_\mp + MF_\pm = 0 .$$

Such differential equations, and their generalizations to the interacting regions where they become non-linear, are properly the subject of Lagrangian field theory and it is to this subject that the following section is devoted.

III. SUPERSYMMETRIC LAGRANGIANS

(A) KINETIC ENERGY TERMS AND RENORMALIZABLE INTERACTIONS

In this section we shall be concerned with the setting up of Lagrangian models to illustrate various aspects of the theory. Only through a study of such models can one see how the new symmetry interacts with the standard considerations of unitarity, locality, causality and renormalizability.

Now in ordinary Lagrangian field theories the locality requirement is met by starting with a local action which is a space-time integral over a local density function of the fields and their first derivatives, the Lagrangian. We shall adopt this rule also for the superfields. However, with fields which are defined over the eight-dimensional space of x and θ, it is necessary to decide what "density" means. A new concept seems to be needed, that of integration on the anticommuting θ. Such a concept has in fact been described in mathematical literature. Nevertheless we shall avoid any explicit reference to this concept in the following. One reason for avoiding the θ integral is that, so far as we are aware, techniques for dealing with surface terms have not yet been developed and, while no doubt they will be, we suspect that they will turn out to be fairly elaborate. But our main reason for avoiding these integrals, however, is simply that we find we can do without them. All we need to do is make sure that any θ dependence in the action functional is confined to surface terms of the usual (space-time) variety. That is, we must ensure that the θ-dependent terms in the action density always take the form of space-time gradients. Such surface terms do not influence the Euler-Lagrange equations which are governed entirely by the volume part of the action. These equations will be supersymmetric (i.e. covariant with respect to the extended relativistic symmetry) provided the volume part of the action is an invariant. The volume part will, in its turn, be invariant if the Lagrangian is a scalar superfield all of whose θ-dependent terms are space-time gradients. Therefore, the action associated with a region of space-time must take the form

$$S = \int_{\text{volume}} dx\, \mathcal{L}(x,\theta)$$

$$= \int_{\text{volume}} dx\, \mathcal{L}(x,0) + \text{surface integral} \quad ,$$

where $\mathcal{L}$ transforms like a scalar superfield,

$$\delta\mathcal{L} = \bar{\epsilon}\left\{\frac{\partial}{\partial\bar{\theta}} + \frac{1}{2}\not{\partial}\theta\right\}\mathcal{L} \quad .$$

The corresponding δS will clearly be represented by a surface integral.

This asymmetrical treatment of x and θ does not do full justice to the underlying extended space-time structure and our only excuse is that it is expedient.

The θ-independent part of the action density may be characterized very simply. It must transform like a mixture of at most three local representations, <u>viz</u>. the D component of the real (non-chiral) scalar $\Phi(x,\theta)$ and a real combination of the F components of the chiral scalars Φ_- and Φ_-^*. Recall that these components transform under the action of an infinitesimal supertranslation according to

$$\delta D(x) = -\bar{\epsilon}i\not{\partial}\chi(x)$$

$$\delta F_-(x) = -\bar{\epsilon}_-i\not{\partial}\psi_-(x) \quad .$$

Because of the gradients in these formulae it follows that the volume integrals of D and F_- (F_-^*) are invariant. Another candidate for the action density, a real combination of F_+ and F_+^*, must be rejected because of the fermion-number which these components carry. There are no other candidates.

Now the D component of Φ is simply the θ-independent part of $(\bar{D}D)^2\Phi$ and the F component of Φ_- is the θ-independent part of $\bar{D}D\,\Phi_-$. Moreover, the θ-containing parts of these expressions are all space-time gradients. This can be understood by noting that every application of the covariant derivative $D = (\partial/\partial\bar{\theta} - \frac{1}{2}\not{\partial}\theta)$ either removes one power of θ or adds one power along with a space-time derivative. Such are the terms of which the action density is to be made,

$$\mathcal{L} \sim (\bar{D}D)^2\, \Phi \;+\; \bar{D}D\,(\Phi_- + \Phi_-^*) \quad . \qquad \text{(III.1)}$$

Our use of the covariant derivative here rather than the ordinary $\partial/\partial\bar\theta$ has no significance for the equations of motion.

Given a single positive chirality scalar Φ_+ and its conjugate Φ_+^*, what possible Lagrangians can one make? Naturally one must start with bilinears in order to generate linear, free field, equations of motion. It is not difficult to convince oneself that only one bilinear is admissible,

$$\mathcal{L}_0 \;=\; \frac{1}{8}\,(\bar{D}D)^2\, |\Phi_+|^2 \quad . \qquad \text{(III.2)}$$

In terms of component fields the θ-independent part of $\mathcal{L}_0$ is given by

$$\mathcal{L}_0\Big|_{\theta=0} \;=\; -\frac{1}{4}A_+^*\partial^2 A_+ + \frac{1}{2}|\partial A_+|^2 - \frac{1}{4}A_+^*\overleftarrow{\partial}^2 A_+ + \frac{1}{2}\bar\psi_+(i\overrightarrow{\partial\!\!\!/} - i\overleftarrow{\partial\!\!\!/})\psi_+ + |F_+|^2$$

$$= |\partial A_+|^2 + \bar\psi_+ i\partial\!\!\!/\psi_+ + |F_+|^2 + \text{gradient term} \quad . \qquad \text{(III.3)}$$

The gradient term can be gathered into the θ-dependent part and forgotten. The active part of $\mathcal{L}_0$ is just what is shown explicitly in (III.3). It has the form of a **kinetic** term and, taken by itself, would yield the equations of motion

$$\partial^2 A_+ = 0 \quad , \quad i\partial\!\!\!/\psi_+ = 0 \quad , \quad F_+ = 0 \ ,$$

and their complex conjugates. The main interest of the Lagrangian (III.3) is that it determines the dimensionality of the fields. Thus, in natural units,

$$[A_+] \;=\; M \quad , \quad [\psi_+] \;=\; M^{3/2} \ ,$$

which is canonical for scalar and spinor fields with the kinetic term always required to have dimension 4 ,

$$[\mathcal{L}_0] \;=\; M^4 \quad .$$

With Φ_+ having dimension 1 it is clear that no interaction terms can be constructed <u>from this single field</u> without doing violence to the requirement of renormalizability if fermion-number is conserved. To see this, note that simple powers of Φ_+ are forbidden since

$$\bar{D}D(\Phi_+ + \Phi_+^2 + \Phi_+^3 + \cdots)$$

carries two units of fermion-number. The only terms which conserve fermion-number are of the form

$$(\bar{D}D)^2\,(|\Phi_+|^{2n})$$

and these have dimension $2n+2$. On a power-counting basis, these terms conflict with renormalizability for $n > 1$. (Recall that $\bar{D}D$ has dimension 1 and for renormalizability a Lagrangian must not possess dimensions in excess of four.)

Similar arguments applied to the case of a single negative chirality scalar Φ_- and its conjugate Φ_-^* lead to the conclusion that the only possible fermion-number-conserving renormalizable Lagrangian is given by

$$\mathcal{L}_0 \;=\; \frac{1}{8}\,(\bar{D}D)^2\,|\Phi_-|^2 - \frac{1}{2}\,\bar{D}D\,(\mathscr{A}\Phi_- + \mathscr{A}^*\Phi_-^*) \quad , \qquad \text{(III.4)}$$

where $\mathscr{A}$ is a parameter of dimension 2. This expression reduces to

$$\mathcal{L}_0 \;=\; |\partial A_-|^2 + \bar\psi_- i\partial\!\!\!/\psi_- + |F_-|^2 + (\mathscr{A}F_- + \mathscr{A}^*F_-^*) \qquad \text{(III.5)}$$

plus gradient terms. The equations of motion are

$$\partial^2 A_- \;=\; 0 \quad , \quad i\partial\!\!\!/\psi_- \;=\; 0 \quad , \quad F_- \;=\; -\mathscr{A}^*$$

and their complex conjugates.

Thus, for a single chiral field of either type there is no possibility of setting up a non-trivial theory without violating either fermion-number conservation or renormalizability. However, for a system containing two distinct fields Φ_+ and Φ_-, one can construct a non-trivial interaction term. This makes essential use of the closure property discussed earlier, viz. that the product of two scalar superfields of the same chiral type is itself a chiral scalar of that type. Thus, since Φ_-^* and Φ_+ have positive chirality, so too do their products $\Phi_-^*\Phi_+$ and $\Phi_-^*\Phi_+^2$. In both of these products the F components carry zero fermion-number and can be used in the Lagrangian. Thus, for the Φ_+, Φ_- system, the renormalizable fermion-number-conserving Lagrangian takes the form:

$$\mathcal{L} = \frac{1}{8}(\bar{D}D)^2\left[|\Phi_+|^2 + |\Phi_-|^2\right] - \frac{1}{2}\bar{D}D\left[\Phi_-^*(\mathscr{A} + M\Phi_+ + g\Phi_+^2) + \text{h.c.}\right] , \quad \text{(III.6)}$$

where h.c. stands for the (hermitian) conjugate terms. The parameters $\mathscr{A}$, M and g have dimension 2, 1 and 0, respectively. No other terms are admissible. Notice, however, the lack of symmetry of (III.6) between Φ_+ and Φ_- type fields. Later, when we consider the question of parity conservation for the Lagrangian, this asymmetry will be relevant.

Of the three parameters in (III.6) only two independent combinations can appear in physically significant quantities such as scattering amplitudes. This fact results from the invariance of the kinetic terms under the c-number shift,

$$\Phi_+ \to \Phi_+ + c \quad , \qquad \text{(III.7)}$$

where c is independent of x and θ. Such transformations affect only the scalar component A_+ and it is clear from formula (III.3) that they do not alter the volume terms in $(1/8)(\bar{D}D)^2|\Phi_+|^2$. The remaining terms in (III.6), however, change as follows:

$$\begin{aligned} \mathscr{A} &\to \mathscr{A} + Mc + gc^2 \\ M &\to M + 2gc \\ g &\to g \quad . \end{aligned} \qquad \text{(III.8)}$$

Otherwise expressed, the field transformation (III.7) is equivalent to the parameter transformation (III.8). But quantities of physical interest (the

quantum field theoretic analogues of canonical invariants) should not be affected by a field transformation. Therefore, they should not be affected by the parameter change (III.8); they must depend on only two invariants,

$$g \quad \text{and} \quad \mathscr{A} - \frac{M^2}{4g} \quad ,$$

for example. We may conclude that no generality is lost by taking one of the parameters, M or $\mathscr{A}$, equal to zero.

It will be instructive to examine the component structure of the Lagrangian (III.6). By means of the formulae (II.28) and (II.30) one obtains the result

$$\begin{aligned} \mathcal{L} = {} & |\partial A_+|^2 + \bar{\psi}_+ i\partial\!\!\!/\psi_+ + |F_+|^2 + |\partial A_-|^2 + \bar{\psi}_- i\partial\!\!\!/\psi_- + |F_-|^2 \\ & + \Big[F_-^*(\mathscr{A} + MA_+ + gA_+^2) - \bar{\psi}_-(M + 2gA_+)\psi_+ + \\ & + A_-^*\left\{(M + 2gA_+)F_+ + g\psi_+^T C^{-1}\psi_+\right\} + \text{h.c.}\Big] \end{aligned} \qquad \text{(III.9)}$$

plus surface terms which include all the θ dependence. The expression (III.9) is a Lagrangian of the usual sort and one can deal with it by standard methods. First of all one notices that the complex scalars F_+ and F_- are not independent dynamical variables since their derivatives do not appear in $\mathcal{L}$ (i.e. no canonical momenta can be associated with them). These auxiliary variables can therefore be eliminated from the Lagrangian by means of the Euler-Lagrange equations which give

$$\begin{aligned} -F_- &= \mathscr{A} + MA_+ + gA_+^2 \quad , \\ -F_+ &= \quad MA_- + 2gA_+^*A_- \quad . \end{aligned} \qquad \text{(III.10)}$$

Elimination of the auxiliary fields leads to a Lagrangian which describes the interaction of two complex scalars, A_+ and A_- (the latter of which carries fermion-number $F = -2$), and a Dirac fermion $\psi = \psi_+ + \psi_-$,

$$\begin{aligned} \mathcal{L} = {} & |\partial A_+|^2 - |\mathscr{A} + MA_+ + gA_+^2|^2 \\ & + |\partial A_-|^2 - |(M + 2gA_+^*)A_-|^2 \\ & + \bar{\psi}(i\partial\!\!\!/ - M)\psi - \left[2gA_+\bar{\psi}_-\psi_+ - gA_-^*\psi_+^T C^{-1}\psi_+ + \text{h.c.}\right] . \end{aligned} \qquad \text{(III.11)}$$

The behaviour of this Lagrangian with respect to supertranslations has been obscured by the elimination of the auxiliary fields. These transformations are non-linear, viz.

$$\delta A_+ = \bar{\varepsilon}_- \psi \ ,$$

$$\delta\psi = -(M + 2gA_+^*)A_- \varepsilon_+ - (\mathcal{J} + MA_+ + gA_+^2)\varepsilon_- \\ -i\not{\partial} A_+ \varepsilon_- - i\not{\partial} A_- \varepsilon_+ \ ,$$

$$\delta A_- = \bar{\varepsilon}_+ \psi \ ,$$

and the corresponding change in $\mathcal{L}$ is a space-time gradient. However, the physical content of (III.11) is fairly clear. The free field part is most easily read in the parametrization with $\mathcal{J} = 0$ where the bilinear terms take the form

$$\mathcal{L}_{(2)} = |\partial A_+|^2 - |MA_+|^2 \\ + \bar{\psi}(i\not{\partial} - M)\psi \\ + |\partial A_-|^2 - |MA_-|^2 \qquad \text{(III.12)}$$

Thus we are clearly dealing with a supermultiplet of particles of mass M (and, of course, the corresponding supermultiplet of antiparticles). The renormalizable interactions of these supermultiplets are characterized by the dimensionless coupling constant g.

The generalization of (III.6) to systems containing several chiral scalars is immediate. With the set of fields Φ_{j+}, $j = 1,2,\ldots,m$, and Φ_{a-}, $a = 1,2,\ldots,n$ (where m and n are not necessarily equal), the Lagrangian must be

$$\mathcal{L} = \frac{1}{8}(\bar{D}D)^2 \left[\sum_j |\Phi_{j+}|^2 + \sum_a |\Phi_{a-}|^2\right] \\ - \frac{1}{2}\bar{D}D\left[\sum_a \Phi_{a-}^* \left(\mathcal{J}_a + \sum_j M_{aj}\Phi_{j+} + \frac{1}{2}\sum_{jk} g_{ajk}\Phi_{j+}\Phi_{k+}\right) + \text{h.c.}\right], \qquad \text{(III.13)}$$

where $\mathcal{J}_a$, M_{aj} and $g_{ajk} = g_{akj}$ are parameters of dimension 2,1 and 0, respectively. No generality is lost by assuming a diagonal form for the kinetic terms since it can always be arranged by choosing appropriate linear combinations. Again it will be possible to eliminate some of the parameters by making an appropriate shift $\Phi_{j+} \to \Phi_{j+} + C_j$. However, there can now be symmetry or algebraic considerations which restrict this freedom. To formulate a general rule concerning this would be complicated and, in the long run, futile since it is always simpler to deal with specific cases as they arise.

An obvious but nevertheless important remark concerns the field count. If the numbers, m and n, of positive and negative chirality fermions, ψ_{j+} and ψ_{a-}, are not equal then one of two things must happen. Either some particles remain without mass, or else there must occur a spontaneous violation of fermion-number conservation (reflected, for example, by $\langle A_- \rangle \neq 0$). We shall postpone further a discussion on Lagrangians of the general form (III.13) to Sec.IV. For the present we shall consider the simpler form (III.6).

First of all, what are the equations of motion for the superfields Φ_+ and Φ_-? A direct if not very illuminating answer to this question could be obtained by first setting up Euler-Lagrange equations of the usual sort for the component fields. One can deduce these equations from the expression (III.9). Then, knowing that the component equations must be compatible with supersymmetry, one should be able to arrange them in such a way that the supersymmetry is manifest, i.e. convert them back into the notation of superfields. However, such an approach to the problem is not very satisfying. One is left with the uneasy feeling that an unnecessary detour has been made - that the underlying geometrical structure has not been allowed its due role. A more aesthetically pleasing formulation should be possible, and this is indeed the case. It is based on the notion of variational derivatives with respect to superfields.

Consider the question of functional derivatives. Suppose we have a functional M which depends on the chiral fields Φ_+ and Φ_+^*. This is equivalent to saying that it depends on the component fields A_+, ψ_+, F_+ and their conjugates. Derivatives with respect to these components can be defined in the usual way by expressing the infinitesimal δM as a linear form in the component variations,

$$\delta M = \int dx \left[\frac{\delta M}{\delta A_+(x)} \delta A_+(x) + \frac{\delta M}{\delta \psi_+(x)} \delta\psi_+(x) + \frac{\delta M}{\delta F_+(x)} \delta F_+(x) + \delta A_+^*(x) \frac{\delta M}{\delta A_+^*(x)} + \delta\bar\psi_+(x) \frac{\delta M}{\delta\bar\psi_+(x)} + \delta F_+^*(x) \frac{\delta M}{\delta F_+^*(x)} \right] . \tag{III.14}$$

Likewise, we must be able to express δM as a linear form in the superfield variations $\delta\Phi_+(x,\theta)$ and $\delta\Phi_+^*(x,\theta)$. A convenient way to write this is

$$\delta M = \int dx \left\{-\frac{1}{2}\bar{D}D\right\} \left[\frac{\delta M}{\delta\Phi_+(x,\theta)} \delta\Phi_+(x,\theta) + \delta\Phi_+^*(x,\theta) \frac{\delta M}{\delta\Phi_+^*(x,\theta)}\right] . \tag{III.15}$$

The coefficients $\delta M/\delta\Phi_+(x,\theta)$, $\delta M/\delta\Phi_+^*(x,\theta)$ are the required functional derivatives. They are superfields in their own right and their component structure is determined by comparing (III.15) with (III.14). One finds that the linear forms are identical if the superfield derivatives are given by

$$\frac{\delta M}{\delta\Phi_+(x,\theta)} = \exp\left(-\frac{1}{4}\bar\theta\not\partial\gamma_5\theta\right)\left[\frac{\delta M}{\delta F_+(x)} - \frac{\delta M}{\delta\psi_+(x)}\theta_+ + \frac{\delta M}{\delta A_+(x)}\frac{1}{2}\bar\theta_-\theta_+\right]$$

$$\frac{\delta M}{\delta\Phi_+^*(x,\theta)} = \exp\left(\frac{1}{4}\bar\theta\not\partial\gamma_5\theta\right)\left[\frac{\delta M}{\delta F_+^*(x)} - \bar\theta_+\frac{\delta M}{\delta\bar\psi_+(x)} + \frac{1}{2}\bar\theta_+\theta_-\frac{\delta M}{\delta A_+^*(x)}\right] . \tag{III.16}$$

The superfield derivatives are thus seen to have the structure of chiral superfields with components given by ordinary functional derivatives. For example, on substituting $M = \Phi(x',\theta')$ in (III.16), one finds

$$\frac{\delta\Phi_+(x',\theta')}{\delta\Phi_+(x,\theta)} = \exp\left(-\frac{1}{4}\bar\theta\not\partial\gamma_5\theta\right)\left[\frac{\delta\Phi_+(x',\theta')}{\delta F_+(x)} - \frac{\delta\Phi_+(x',\theta')}{\delta\psi_+(x)}\theta_+ + \frac{\delta\Phi_+(x',\theta')}{\delta A_+(x)}\frac{1}{2}\bar\theta_-\theta_+\right]$$

$$= \exp\left(-\frac{1}{4}\bar\theta\not\partial\gamma_5\theta - \frac{1}{4}\bar\theta'\not\partial'\gamma_5\theta'\right)\left[\frac{1}{2}\bar\theta_-'\theta_+' - \bar\theta_-'\theta_+ + \frac{1}{2}\bar\theta_-\theta_+\right] \cdot \delta_4(x-x') =$$

$$= \exp\left(-\frac{1}{4}\bar\theta\not\partial\gamma_5\theta - \frac{1}{4}\bar\theta'\not\partial'\gamma_5\theta'\right) \cdot \frac{1}{2}(\bar\theta_- - \bar\theta_-')(\theta_+ - \theta_+')\ \delta_4(x-x')$$

$$\equiv \delta_+(x,\theta\,;\,x',\theta') , \tag{III.17}$$

which defines a supersymmetric analogue of the Dirac delta function. It is characterized by the integral property

$$\int dx\left(-\frac{1}{2}\bar{D}D\right)\left(\Phi_+(x,\theta)\ \delta_+(x,\theta;x',\theta')\right) = \Phi_+(x',\theta') \tag{III.18}$$

for an arbitrary positive chirality superfield Φ_+ . Similarly, the negative chirality delta function is defined by

$$\frac{\delta\Phi_+^*(x',\theta')}{\delta\Phi_+^*(x,\theta)} = \exp\left(\frac{1}{4}\bar\theta\not\partial\gamma_5\theta + \frac{1}{4}\bar\theta'\not\partial'\gamma_5\theta'\right)\frac{1}{2}(\bar\theta_+ - \bar\theta_+')(\theta_- - \theta_-')\ \delta_4(x-x')$$

$$\equiv \delta_-(x,\theta\,;\,x',\theta') . \tag{III.19}$$

Let us now compute the derivatives of the action functional corresponding to the Lagrangian (III.6),

$$S = \int dx\left[\frac{1}{8}(\bar{D}D)^2\left(|\Phi_+|^2 + |\Phi_-|^2\right) - \frac{1}{2}\bar{D}D\left\{\Phi_-^*(\mathscr{J} + M\Phi_+ + g\Phi_+^2) + \text{h.c.}\right\}\right] . \tag{III.20}$$

Only the kinetic term causes any difficulty and to deal with this we need to express the operator $(\bar{D}D)^2$ in a different form. Two identities are needed for this,

$$(\bar{D}D)^2 = (\bar{D}_-D_+ + \bar{D}_+D_-)^2$$
$$= \{\bar{D}_-D_+ , \bar{D}_+D_-\}$$

and
$$0 = [\bar{D}_-D_+ , \bar{D}_+D_-] - 2\bar{D}\not{\partial}\gamma_5 D$$

whence

$$\frac{1}{2}(\bar{D}D)^2 = \bar{D}_-D_+\,\bar{D}_+D_- - \bar{D}\not{\partial}\gamma_5 D$$
$$= \bar{D}_+D_-\,\bar{D}_-D_+ + \bar{D}\not{\partial}\gamma_5 D \quad . \tag{III.21}$$

Using (III.21) together with the chiral property

$$D_-\Phi_+ = 0 \quad , \quad D_+\Phi_+^* = 0 \ ,$$

one can put the variation of the kinetic term into the following form:

$$\delta\,\frac{1}{8}(\bar{D}D)^2\left\{|\Phi_+|^2 + |\Phi_-|^2\right\} = \frac{1}{8}(\bar{D}D)^2\left\{\delta\Phi_+\Phi_+^* + \delta\Phi_-^*\Phi_- + \text{h.c.}\right\}$$

$$= \frac{1}{4}\left\{\bar{D}_-D_+\,\bar{D}_+D_- - \bar{D}\not{\partial}\gamma_5 D\right\}\left\{\delta\Phi_+\Phi_+^* + \delta\Phi_-^*\Phi_-\right\} +$$
$$+ \frac{1}{4}\left\{\bar{D}_+D_-\,\bar{D}_-D_+ + \bar{D}\not{\partial}\gamma_5 D\right\}\left\{\delta\Phi_+^*\Phi_+ + \delta\Phi_-\Phi_-^*\right\}$$

$$= \frac{1}{4}\,\bar{D}_-D_+\left\{\delta\Phi_+\,\bar{D}_+D_-\Phi_+^* + \delta\Phi_-^*\,\bar{D}_+D_-\Phi_-\right\} +$$
$$+ \frac{1}{4}\,\bar{D}_+D_-\left\{\delta\Phi_+^*\,\bar{D}_-D_+\Phi_+ + \delta\Phi_-\,\bar{D}_-D_+\Phi_-^*\right\} +$$
$$+ \frac{1}{4}\,\bar{D}\not{\partial}\gamma_5 D\left\{\delta\Phi_+^*\Phi_+ - \Phi_+^*\delta\Phi_+ - \delta\Phi_-^*\Phi_- + \Phi_-^*\delta\Phi_-\right\}$$

$$= \frac{1}{4}\,\bar{D}D\left\{\delta\Phi_+^*\,\bar{D}D\,\Phi_+ + \delta\Phi_-^*\,\bar{D}D\,\Phi_- + \text{h.c.}\right\} +$$
$$+ \frac{1}{4}\,\bar{D}\not{\partial}\gamma_5 D\left\{\delta\Phi_+^*\Phi_+ - \delta\Phi_-^*\Phi_- - \text{h.c.}\right\} \quad . \tag{III.22}$$

With the help of this formula we can put the variation of the action into the form:

$$\delta S = \int dx\left(-\frac{1}{2}\bar{D}D\right)\left[\delta\Phi_+^*\left\{-\frac{1}{2}\bar{D}D\Phi_+ + (M + 2g\Phi_+^*)\Phi_-\right\} + \right.$$
$$\left. + \delta\Phi_-^*\left\{-\frac{1}{2}\bar{D}D\Phi_- + (\mathcal{J} + M\Phi_+ + g\Phi_+^2)\right\} + \text{h.c.}\right]$$
$$+ \oint d\sigma_\mu\ \ \frac{1}{4}\,\bar{D}\gamma_\mu\gamma_5 D\left[\delta\Phi_+^*\Phi_+ - \delta\Phi_-^*\Phi_- - \text{h.c.}\right] \quad . \tag{III.23}$$

Thus the variation δS reduces to a surface term when the functional derivatives of S are set equal to zero.

$$\frac{\delta S}{\delta\Phi_+^*} = -\frac{1}{2}\,\bar{D}D\Phi_+ + (M + 2g\Phi_+^*)\Phi_- = 0 \quad .$$
$$\frac{\delta S}{\delta\Phi_-^*} = -\frac{1}{2}\,\bar{D}D\Phi_- + (\mathcal{J} + M\Phi_+ + g\Phi_+^2) = 0 \quad . \tag{III.24}$$

These are the equations of motion. They are manifestly supersymmetric.

(B) LOCAL SYMMETRY AND GAUGE LAGRANGIANS

We have just seen in Section III(A) that the inclusion of global internal symmetries into a supersymmetric scheme gives rise to no difficulty. Supermultiplets are combined into sets which support representations of the internal symmetry and this is effected by arranging the corresponding superfields into multiplets of the internal symmetry. Thus, expressing the positive and negative chiral fields as two independent columns, $\Phi_+ = \{\Phi_{i+}\}$ and $\Phi_- = \{\Phi_{a-}\}$, the action of a global transformation is given by

$$\Phi_\pm \rightarrow \exp(i\Lambda^k Q_\pm^k)\ \Phi_\pm \quad , \tag{III.25}$$

where the parameters Λ^k are real numbers and the matrices Q_+ and Q_- are hermitian. (In a later section expressions for the corresponding conserved currents,

$$J^k = \Phi_+^\dagger Q_+^k \Phi_+ - \Phi_-^\dagger Q_-^k \Phi_- \ , \tag{III.26}$$

will be derived.)

Generalization of the transformations (III.25) to local form is straightforward. Indeed, if the parameters Λ^k are allowed to vary with x_μ then they must necessarily depend on θ as well and in such a way as to preserve the chiral character of the transformed fields. In view of the closure property of chiral products noted before, it is clear that chirality preservation requires that the local parameters $\Lambda^k(x,\theta)$ be themselves chiral, viz.

$$\Phi_+(x,\theta) \to \exp\left\{i\Lambda_+^k(x,\theta)\, Q_+^k\right\} \Phi_+(x,\theta) \quad ,$$
$$\Phi_-(x,\theta) \to \exp\left\{i\Lambda_+^k(x,\theta)^*\, Q_-^k\right\} \Phi_-(x,\theta) \quad . \tag{III.27}$$

These transformations have an unusual structure. As with any chiral field, these matrices can be expanded in powers of θ,

$$\exp\left\{i\Lambda_+^k(x,\theta)\, Q_+^k\right\} = \exp\left(-\tfrac{1}{4}\,\bar\theta \not{\partial}\gamma_5\theta\right)\left\{U_+(x) + \bar\theta_- V_+(x) + \tfrac{1}{2}\,\bar\theta_-\theta_+\, W_+(x)\right\} \;, \tag{III.28}$$

where, of course, $V_+(x)$ carries a spinor index. The product of two transformations yields a third with

$$U_{3+} = U_{1+}\, U_{2+} \;,$$
$$V_{3+} = U_{1+}\, V_{2+} + V_{1+}\, U_{2+} \;.$$
$$W_{3+} = U_{1+}\, W_{2+} + V_{1+}^T\, C^{-1}\, V_{2+} + W_{1+}\, U_{2+} \quad . \tag{III.29}$$

(In the last formula V_{1+}^T denotes the transpose with respect to the spinor index of V_{1+}.) The matrices $U_+(x)$ which represent a subgroup are complex but not, in general, unitary. One might expect that such transformations could not be compatible with the usual requirements of a physically acceptable theory. It is remarkable that no conflict arises: all negative metric excitations are suppressed by the gauge mechanism.

As in all gauge theories, the principal vehicle of the local symmetry is a set of potentials. In this case they take the form of real (non-chiral) superfields, $\Psi^k(x,\theta)$. Their role here is to provide a "metric" for the other fields in the system. Thus, the kinetic terms are to be expressed in the form

$$\mathcal{L}_k = \frac{1}{8}\,(\bar D D)^2 \left[\Phi_+^\dagger\, e^{2\Psi^k Q_+^k}\, \Phi_+ + \Phi_-^\dagger\, e^{-2\Psi^k Q_-^k}\, \Phi_-\right] \quad . \tag{III.30}$$

This form is gauge invariant provided that the potentials transform according to

$$e^{2\Psi^k Q^k} \to e^{i\Lambda_+^{k*} Q^k}\; e^{2\Psi^k Q^k}\; e^{-i\Lambda_+^k Q^k} \quad , \tag{III.31}$$

where $Q^k = Q_\pm^k$. (The choice of an exponential parametrization for the metric in (III.30) is merely a convenience. It is conceivable that other types of co-ordinates might prove useful in specific problems.) For some purposes it is useful to represent the potentials Ψ^k in the form of a hermitian matrix. For example, in the case of SU(3) one may write

$$\Psi = \Psi^k\, \frac{\lambda^k}{2} \quad , \quad \Psi^k = \mathrm{Tr}(\lambda^k \Psi) \quad , \tag{III.32}$$

where λ^k, $k = 1,2,\ldots,8$, denote the Gell-Mann matrices.

A superfield analogue of the field strengths can be defined. It takes the form of a chiral spinor

$$\Psi_{\alpha++} = -\frac{i}{2\sqrt{2}}\;\bar D_+ D_-\left(e^{-2\Psi}\, D_{\alpha+}\, e^{2\Psi}\right) \quad , \tag{III.33}$$

where the numerical coefficient is chosen for later convenience. This superfield is chiral with respect to both its θ structure and its spinor index,

$$D_-\Psi_{++} = 0 \quad \text{and} \quad (1-i\gamma_5)\Psi_{++} = 0 \quad ,$$

and so belongs to the transverse vector representations discussed in Sec.II, Eq.(II.36) ff. Under the gauge transformation (III.31) one finds

$$\Psi_{++} \to e^{i\Lambda_+}\,\Psi_{++}\, e^{-i\Lambda_+} \quad . \tag{III.34}$$

The complex conjugate of Ψ_{++} is a negative chirality field Ψ_{--}. Thus

$$\Psi_{--}^k = C\,\bar\Psi_{++}^{kT}$$

or

$$\bar\theta_+\Psi_{--} = (\bar\theta_-\Psi_{++})^\dagger \quad . \tag{III.35}$$

This field can also be defined by a formula analogous to (III.33), viz.

$$\Psi_{\alpha--} = -\frac{i}{2\sqrt{2}}\;\bar D_- D_+\left(e^{2\Psi}\, D_{\alpha-}\, e^{-2\Psi}\right) \quad ; \tag{III.36}$$

It transforms according to

$$\Psi_{--} \to e^{i\Lambda_+^\dagger}\,\Psi_{--}\, e^{-i\Lambda_+^\dagger} \quad . \tag{III.37}$$

A gauge-invariant kinetic term for the potentials is given by

$$\mathcal{L}_g = \frac{1}{8}\bar{D}_- D_+ \left\{\frac{1}{g^2}\Psi^k_{--}\Psi^k_{++}\right\} + \text{h.c.} \tag{III.38}$$

where g is a dimensionless coupling constant. In general there can be one independent constant for each simple component in the local symmetry group. If this symmetry contains a commutative abelian subgroup, i.e. if the infinitesimal algebra contains certain linear combinations, $\xi^k_{(r)} Q^k$, which commute with all Q^ℓ, then the expression

$$\mathcal{L}_\xi = \frac{1}{8}(\bar{D}D)^2 \left(\sum_r \xi^k_{(r)} \Psi^k\right) \tag{III.39}$$

is gauge invariant (up to surface terms) and can be included in the Lagrangian. The parameters ξ have the dimension of $(\text{mass})^2$ and are very important in the models of massive vector supermultiplets.

On combining the expressions (III.30),(III.38) and (III.39) one obtains a gauge-invariant and supersymmetric Lagrangian for the interacting system. The corresponding equations of motion are necessarily degenerate to a degree which reflects the gauge freedom. In other words, those degrees of freedom which are associated with gauge transformations will not be governed by the equations of motion. They must be dealt with by an external mechanism. This problem is familiar from electrodynamics and Yang-Mills theory. Its treatment in the supersymmetry context is of course more complicated in detail than for these well known cases but will be seen to follow essentially the same lines. We begin with an abelian local symmetry.

For local U(1) symmetry there is just one potential, Ψ, a real superfield. The transformation law (III.31) reduces in this case to the linear inhomogeneous form

$$\Psi \to \Psi - \frac{i}{2}\Lambda_+ + \frac{i}{2}\Lambda_+^* , \tag{III.40}$$

and the implications are clear. Thus, since Ψ can always be reduced, i.e. represented uniquely as the sum of chiral and transverse vector pieces,

$$\Psi = N_+ + N_+^* + \Psi_1 , \tag{III.41}$$

it follows that the transverse vector part, Ψ_1, is gauge invariant while the positive chiral part, N_+, transforms according to

$$N_+ \to N_+ - \frac{i}{2}\Lambda_+ . \tag{III.42}$$

(These formulae are analogous to the following, drawn from electrodynamics,

$$A_\mu \to A_\mu + \partial_\mu \Lambda , \tag{III.40'}$$

$$A_\mu = \partial_\mu A^\ell + A^t_\mu , \tag{III.41'}$$

$$A^\ell \to A^\ell + \Lambda , \tag{III.42'}$$

where A^t_μ, the transverse part, satisfies $\partial_\mu A^t_\mu = 0$. The supersymmetric analogue of this transversality condition is, of course, $\bar{D}_+ D_- \Psi_1 = \bar{D}_- D_+ \Psi_1 = 0$.) The chiral part, N_+, is therefore completely arbitrary. It cannot be determined by any gauge covariant equations of motion and so must instead be fixed by convention. At the classical level this convention or supplementary condition may be taken in the convenient form

$$\bar{D}_+ D_- \Psi = B_+ , \qquad \bar{D}_- D_+ \Psi = B_+^* , \tag{III.43}$$

where B_+ is an arbitrarily chosen chiral field.

The supplementary conditions (III.43) are manifestly supersymmetric and so are the most useful when it is desired to set up the Feynman rules in an explicitly supersymmetric way. However, it is sometimes more convenient to sacrifice this visible supersymmetry with its attendant unphysical modes in favour of a description in terms of physical components. (Analogous to the loss of explicit Lorentz invariance in going from, say, the Landau gauge to the Coulomb gauge.) Indeed, one can choose the components of N_+ in such a way as to compensate some of the components of Ψ_1. In this type of gauge the potential Ψ may be presented in the form

$$\Psi(x,\theta) = \frac{1}{4}\bar{\theta} i\gamma_\nu\gamma_5\theta\, V_\nu(x) + \frac{1}{2\sqrt{2}}\bar{\theta}\theta\bar{\theta}\gamma_5\lambda(x) + \frac{1}{16}(\bar{\theta}\theta)^2 D(x), \tag{III.44}$$

where λ and D are unconstrained but V_ν is subject to the supplementary condition

$$\partial_\mu V_\mu(x) = B(x) , \qquad \text{(III.45)}$$

with $B(x)$ an arbitrary real scalar.[*] One very considerable virtue of this gauge is that it implies the reduction of $\exp(2\Psi)$ to polynomial form since $\Psi^n = 0$ for $n \geqslant 3$. The Lagrangian then assumes a manifestly renormalizable form.

Similar considerations apply to the non-abelian symmetries where a set of real potentials Ψ^k are involved. The linear transformation law (III.40) is no longer relevant and the transverse vector pieces Ψ_1^k are therefore not gauge invariant (just as in the ordinary non-abelian gauge theories) but it is still possible to remove the gauge degrees of freedom by means of subsidiary conditions like (III.43),

$$\bar{D}_+ D_- \Psi^k = B_+^k \quad , \quad \bar{D}_- D_+ \Psi^k = B_+^{k*} \qquad \text{(III.46)}$$

or, alternatively, to represent each Ψ^k in the form (III.44) with $\partial_\mu V_\mu^k = B^k$. On quantizing, the usual Faddeev-Popov complications are encountered. These will be discussed below.

In manifestly renormalizable gauges of the type (III.44) the field strengths are given by

$$\Psi_{++} = \exp\left\{-\frac{1}{4}\,\bar{\theta}\nabla\!\!\!/\gamma_5\theta\right\}\left[\lambda_+(x) + \frac{i}{\sqrt{2}}\left\{D(x) + \frac{1}{2}\sigma_{\mu\nu} V_{\mu\nu}(x)\right\}\theta_+ + \frac{1}{2}\,\bar{\theta}_-\theta_+(-i\nabla\!\!\!/\lambda_-)\right] , \qquad \text{(III.47)}$$

where the Yang-Mills covariant derivatives appear,

$$V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i\,[V_\mu, V_\nu] \quad ,$$

$$\nabla_\mu\lambda_- = \partial_\mu\lambda_- - i[V_\mu, \lambda_-] \quad . \qquad \text{(III.48)}$$

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[*] The normalizations in (III.44) are chosen for later convenience. The γ_5 which multiplies $\lambda(x)$ is an unfortunate relic of the early development of the subject when the role of space reflections was not understood. Its removal would probably cause too much confusion at this stage, however.

(Note that λ_+ is related to λ_- by conjugation, $\lambda_+^k = C\bar{\lambda}_-^{kT}$, since Ψ^k is real.) If such a gauge is used, then it is best to discard the superfield notation and express the Lagrangian entirely in terms of component fields. The contributions (III.30), (III.38) and (III.39) then become, respectively,

$$\begin{aligned}
\mathcal{L}_k = {} & \nabla_\mu A_+^\dagger \nabla_\mu A_+ + \bar{\psi}_+ i\nabla\!\!\!/\psi_+ + F_+^\dagger F_+ \\
& + i\sqrt{2}\left\{A_+^\dagger \bar{\lambda}_-^k Q_+^k \psi_+ - \bar{\psi}_+\lambda_-^k Q_+^k A_+\right\} + A_+^\dagger D^k Q_+^k A_+ \\
& + \nabla_\mu A_-^\dagger \nabla_\mu A_- + \bar{\psi}_- i\nabla\!\!\!/\psi_- + F_-^\dagger F_- \\
& + i\sqrt{2}\left\{A_-^\dagger \bar{\lambda}_+^k Q_-^k \psi_- - \bar{\psi}_-\lambda_+^k Q_-^k A_-\right\} - A_-^\dagger D^k Q_-^k A_- \ ,
\end{aligned} \qquad \text{(III.49)}$$

$$\mathcal{L}_g = \frac{1}{g^2}\left\{-\frac{1}{4}\,V_{\mu\nu}^k V_{\mu\nu}^k + \bar{\lambda}_-^k i\nabla\!\!\!/\lambda_-^k + \frac{1}{2}D^k D^k\right\} , \qquad \text{(III.50)}$$

$$\mathcal{L}_\xi = \sum_r \xi_{(r)}^k D^k , \qquad \text{(III.51)}$$

where the covariant derivatives in (III.49) are given by

$$\begin{aligned}
\nabla_\mu A_\pm &= \left\{\partial_\mu - iV_\mu^k Q_\pm^k\right\} A_\pm \ , \\
\nabla_\mu \psi_\pm &= \left\{\partial_\mu - iV_\mu^k Q_\pm^k\right\} \psi_\pm \ .
\end{aligned} \qquad \text{(III.52)}$$

There may be, in addition, a gauge-invariant self-interaction (or generalized mass term) among the matter fields,

$$\begin{aligned}
\mathcal{L}_m &= -\frac{1}{2}\bar{D}_+ D_-\left[\Phi_-^\dagger\left\{\mathcal{S} + M\Phi_+ + h\,\Phi_+\Phi_+\right\}\right] + \text{h.c.} \\
&= F_-^\dagger\left\{\mathcal{S} + MA_+ + h\,A_+A_+\right\} - \bar{\psi}_-\left\{M\psi_+ + h\,A_+\psi_+ + h\,\psi_+A_+\right\} \\
&\quad + A_-^\dagger\left\{MF_+ + h\,A_+F_+ + h\,F_+A_+ + h\,\psi_+^T C^{-1}\psi_+\right\} + \text{h.c.}
\end{aligned} \qquad \text{(III.53)}$$

provided it is compatible with the global symmetry.

The Lagrangian defined by the expressions (III.49)–(III.53) is manifestly renormalizable (apart, possibly, from anomalies) and respects both fermion-number and gauge symmetry. It is not manifestly supersymmetric owing to the choice of gauge (III.44). An important feature is that the left-handed gauge spinor λ_- carries fermion-number $F = -1$, like ψ_+ and ψ_-. The right-handed spinor λ_+ therefore carries $F = +1$ like $\bar{\psi}_+$ and $\bar{\psi}_-$. This fact will prove significant when we come to consider the problem of parity conservation.

Although the gauge (III.44) appears to conflict with the requirements of supersymmetry, the contradiction is a superficial one since only gauge-dependent quantities such as Green's functions are affected. Gauge-invariant quantities such as scattering amplitudes and matrix elements (between physical states) of the energy momentum tensor should not be sensitive to the choice of subsidiary condition and should therefore respect supersymmetry (unless of course it is spontaneously violated). A formal way to see this is by a (non-linear) transformation law for the fields, which reflects the action of super-translations on them and which does leave the Lagrangian invariant. The method, although somewhat tortuous, can be carried through without difficulty. The idea is to follow an infinitesimal supertranslation – which destroys the form (III.44) – by an appropriate gauge transformation (III.31) which restores it. The combined effect will then leave the Lagrangian invariant (up to surface terms).

Firstly, the infinitesimal supertranslation applied to (III.44) gives

$$\delta_s \Psi = \bar{\varepsilon}\left(\frac{\partial}{\partial\bar{\theta}} + \frac{i}{2}\partial\!\!\!/\theta\right)\left(\frac{1}{4}\bar{\theta} i\gamma_\nu\gamma_5\theta V_\nu + \frac{1}{2\sqrt{2}}\bar{\theta}\theta\bar{\theta}\gamma_5\lambda + \frac{1}{16}(\bar{\theta}\theta)^2 D\right)$$

$$= \frac{1}{2}\bar{\theta} i\gamma_\nu\gamma_5\varepsilon V_\nu$$

$$+ \frac{i}{2\sqrt{2}}\bar{\theta}_-\theta_+\bar{\varepsilon}_+\lambda_- - \frac{i}{2\sqrt{2}}\bar{\theta}_+\theta_-\bar{\varepsilon}_-\lambda_+ - \frac{i}{4\sqrt{2}}\bar{\theta} i\gamma_\nu\gamma_5\theta\;\bar{\varepsilon}\gamma_\nu\lambda$$

$$+ \frac{1}{4}\bar{\theta}\theta\bar{\theta}\left(D + \frac{1}{2} i\gamma_\nu\gamma_5\, i\partial\!\!\!/ V_\nu\right)\varepsilon$$

$$+ \frac{1}{32}(\bar{\theta}\theta)^2\left(-\sqrt{2}\,\bar{\varepsilon}\, i\partial\!\!\!/\gamma_5\lambda\right). \tag{III.54}$$

Next, consider an infinitesimal gauge transformation,

$$\delta_\Lambda e^{2\Psi} = i\Lambda_+^\dagger e^{2\Psi} - e^{2\Psi} i\Lambda_+ , \tag{III.55}$$

where Λ_+ has no θ-independent term, i.e.

$$\Lambda_+ = e^{-\frac{1}{4}\bar{\theta}\partial\!\!\!/\gamma_5\theta}\left\{\bar{\theta}_-\zeta_+ + \frac{1}{2}\bar{\theta}_-\theta_+ f_+\right\}$$

$$= \bar{\theta}_-\zeta_+ + \frac{1}{2}\bar{\theta}_-\theta_+ f_+ + \frac{1}{4}\bar{\theta}_+\theta_-\bar{\theta}_+(-i\partial\!\!\!/\zeta_+) . \tag{III.56}$$

With this special form for Λ_+ it is possible to solve (III.55) for $\delta_\Lambda\Psi$ by one iteration,

$$\delta_\Lambda\Psi = -\frac{i}{2}(\Lambda_+ - \Lambda_+^\dagger) + \frac{i}{2}[\Lambda_+ + \Lambda_+^\dagger, \Psi]$$

$$= -\frac{i}{2}\left(\bar{\theta}_-\zeta_+ - \bar{\zeta}_+\theta_- + \frac{1}{2}\bar{\theta}_-\theta_+ f_+ - \frac{1}{2}\bar{\theta}_+\theta_- f_+^\dagger\right)$$

$$+ \frac{1}{4}\bar{\theta}\theta\bar{\theta}\left(-\frac{i}{2}\partial\!\!\!/\gamma_5\zeta - \frac{i}{2} i\gamma_\nu\gamma_5[\zeta, V_\nu]\right)$$

$$+ \frac{i}{4\sqrt{2}}\bar{\theta}\theta\left[\bar{\theta}\zeta, \bar{\theta}\gamma_5\lambda\right], \tag{III.57}$$

where we have defined ζ_- such that $\bar{\theta}_+\zeta_- = \bar{\zeta}_+\theta_-$. The gauge parameters ζ_+ and f_+ must now be chosen such that

$$\delta_S\Psi + \delta_\Lambda\Psi = \frac{1}{4}\bar{\theta}i\gamma_\nu\gamma_5\theta\,\delta V_\nu + \frac{1}{2\sqrt{2}}\bar{\theta}\theta\bar{\theta}\gamma_5\delta\lambda + \frac{1}{16}(\bar{\theta}\theta)^2\,\delta D . \tag{III.58}$$

This is achieved by taking

$$\zeta_+ = 2V_\nu\gamma_\nu\varepsilon_- \quad\text{and}\quad f_+ = \sqrt{2}\,\bar{\varepsilon}_+\lambda_- \tag{III.59}$$

and the resulting variations of V_ν, λ and D are given by

$$\delta V_\nu = -\frac{i}{\sqrt{2}}\bar{\varepsilon}\gamma_\nu\lambda ,$$

$$\delta\lambda = \frac{i}{\sqrt{2}}\left\{i\gamma_5 D + \frac{1}{2}\sigma_{\mu\nu}V_{\mu\nu}\right\}\varepsilon ,$$

$$\delta D = -\frac{i}{\sqrt{2}}\bar{\varepsilon}\partial\!\!\!/\gamma_5\lambda . \tag{III.60}$$

Applied to matter fields $\Phi_\pm$, the transformations $\delta_S\Phi_\pm + \delta_\Lambda\Phi_\pm$ take the form

$$\delta A_\pm = \bar{\epsilon}\psi_\pm ,$$

$$\delta\psi_\pm = F_\pm\epsilon_\pm - i\not{D}A_\pm\epsilon_\mp ,$$

$$\delta F_\pm = \bar{\epsilon}\left\{-i\not{D}\psi_\pm + i\sqrt{2}\,\lambda^k_\mp Q^k_\pm A_\pm\right\} , \qquad \text{(III.61)}$$

where the covariant derivatives are defined by (III.48) and (III.52). The transformations (III.60) and (III.61) necessarily preserve the form of the Lagrangian (III.49)-(III.51).

(C) PARITY CONSERVATION

In this section we wish to prove the important result that a supersymmetric, renormalizable, fermion-number-conserving Lagrangian theory which also preserves parity must be a gauge theory of a very special form.

First consider the fermion-number-conserving, renormalizable non-gauge type Lagrangian (III.13). For a parity operation to be defined, the supermultiplets Φ_+ and Φ_- must be put into correspondence. Firstly the spinor components can transform only according to a rule of the form

$$\psi_+ \to \omega\gamma_0\psi_- , \qquad \text{(III.62)}$$

where ω is some unitary matrix subject to the constraint $\omega^2 = \pm 1$. Secondly, the fermion-number $\mathbb{F} = 2$ spin-zero components A_- must transform among themselves,

$$A_- \to \omega' A_- . \qquad \text{(III.63)}$$

where ω' is another matrix like ω.

Now consider the particular interaction term in (III.13)

$$\tfrac{1}{2} g_{ajk} A^*_{a-} \psi^T_{j+} C^{-1} \psi_{k+} .$$

Application of the transformations (III.62) and (III.63) to this term changes it to a form which is clearly not present in (III.13). This shows that interactions of the form (III.13) cannot support space reflections.
We shall prove now that if a local symmetry is present it is possible to set up a parity-conserving interaction.

For definiteness consider the case of local SU(n) symmetry. Since the gauge field Ψ — an $n \times n$ hermitian traceless matrix — contains a set of negative chirality fermions, λ_-, in the adjoint representation, it is necessary that the matter system should contain at least a set of positive chirality fermions, ζ_+, also in the adjoint representation. These fermions must belong to a (traceless) supermultiplet S_+ which we shall call the "supplementary gauge fields". Under the action of local SU(n) the augmented gauge field system transforms according to

$$e^{2g\Psi} \to e^{i\Lambda^\dagger_+}\, e^{2g\Psi}\, e^{-i\Lambda_+}$$
$$S_+ \to e^{i\Lambda_+}\, S_+\, e^{-i\Lambda_+} , \qquad \text{(III.64)}$$

where Λ_+ is a traceless $n \times n$ matrix of positive chirality and g is a dimensionless coupling constant.

Other matter fields may also be present in the form of supermultiplets Φ_+ and Φ_-. It is necessary that Φ_+ and Φ^*_- should transform contragrediently. For simplicity we take just a pair of n-component columns transforming according to

$$\Phi_+ \to e^{i\Lambda_+}\,\Phi_+$$
$$\Phi_- \to e^{i\Lambda^\dagger_+}\,\Phi_- . \qquad \text{(III.65)}$$

The most general supersymmetric, fermion-number-conserving and renormalizable Lagrangian for this system is given by

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{8}\bar{D}D\left[\tfrac{1}{2}\,\mathrm{Tr}\left(\bar{\Psi}_-\Psi_+ + \bar{\Psi}_+\Psi_-\right)\right] \\ & + \tfrac{1}{8}(\bar{D}D)^2\left[\tfrac{1}{2}\,\mathrm{Tr}\left(S^\dagger_+ e^{2g\Psi} S_+ e^{-2g\Psi}\right)\right] \\ & + \tfrac{1}{8}(\bar{D}D)^2\left[\Phi^\dagger_+ e^{2g\Psi}\Phi_+ + \Phi^\dagger_- e^{-2g\Psi}\Phi_-\right] \\ & - \tfrac{1}{2}\bar{D}D\left[\Phi^\dagger_-(M + h\,S_+)\,\Phi_+ + \text{h.c.}\right] ,\end{aligned} \qquad \text{(III.66)}$$

where h is a dimensionless coupling constant and M is a mass.

In order to deal with the parity question it is necessary to express (III.66) in terms of the component fields. For $\Phi_{\pm}$ and Ψ we adopt the expansions (II.25) and (III.44), respectively, and for S_+ the expansion

$$S_+ = e^{-\frac{1}{4}\bar{\theta}\not{\partial}\gamma_5\theta}\left[a_+(x) + \bar{\theta}\zeta_+(x) + \frac{1}{4}\bar{\theta}(1+i\gamma_5)\theta\, f_+(x)\right] . \qquad \text{(III.67)}$$

For convenience of writing we introduce the negative chirality antifermion ζ_- defined as the conjugate of ζ_+, i.e.

$$\bar{\theta}\zeta_- = (\bar{\theta}\zeta_+)^\dagger = \bar{\zeta}_+\theta \quad . \qquad \text{(III.68)}$$

This means that the sum, $\zeta = \zeta_+ + \zeta_-$, is a Majorana spinor like λ.

Applying the formulae (III.49), (III.50), (III.53) to the Lagrangian (III.66) one finds

$$\begin{aligned}
\mathcal{L} = \frac{1}{2}\,\mathrm{Tr}\Big[&-\frac{1}{4}U_{\mu\nu}^2 + \bar{\lambda}_- i\not{\nabla}\lambda_- + \frac{1}{2}D^2 \\
&+ \nabla_\mu a_+^\dagger\,\nabla_\mu a_+ + \bar{\zeta}_+ i\not{\nabla}\zeta_+ + f_+^\dagger f_+ \\
&+ i\sqrt{2}\,g\left\{[a_+^\dagger,\bar{\lambda}_-]\zeta_+ - \bar{\zeta}_+[\lambda_-,a_+]\right\} + g\,a_+^\dagger[D,a_+]\Big] \\
&+ \nabla_\mu A_+^\dagger\,\nabla_\mu A_+ + \bar{\psi}_+ i\not{\nabla}\psi_+ + F_+^\dagger F_+ \\
&+ i\sqrt{2}\,g\left\{A_+^\dagger\bar{\lambda}_-\psi_+ - \bar{\psi}_+\lambda_-A_+\right\} + g\,A_+^\dagger D A_+ \\
&+ \nabla_\mu A_-^\dagger\,\nabla_\mu A_- + \bar{\psi}_- i\not{\nabla}\psi_- + F_-^\dagger F_- \\
&+ i\sqrt{2}\,g\left\{A_-^\dagger\bar{\lambda}_+\psi_- - \bar{\psi}_-\lambda_+A_-\right\} - g\,A_-^\dagger D A_- \\
&+ M\left(A_-^\dagger F_+ + F_-^\dagger A_+ - \bar{\psi}_-\psi_+ + \text{h.c.}\right) \\
&+ h\Big[F_-^\dagger a_+ A_+ + A_-^\dagger f_+ A_+ + A_-^\dagger a_+ F_+ \\
&\qquad - A_-^\dagger\bar{\zeta}_-\psi_+ - \bar{\psi}_- a_+\psi_+ - \bar{\psi}_-\zeta_+A_+ + \text{h.c.}\Big] \quad ,
\end{aligned} \qquad \text{(III.69)}$$

where the various covariant derivatives are given by

$$\begin{aligned}
U_{\mu\nu} &= \partial_\mu U_\nu - \partial_\nu U_\mu - ig[U_\mu,U_\nu] \\
\nabla_\mu\lambda_- &= \partial_\mu\lambda_- - ig[U_\mu,\lambda_-] \\
\nabla_\mu a_+ &= \partial_\mu a_+ - ig[U_\mu,a_+] \\
\nabla_\mu\zeta_+ &= \partial_\mu\zeta_+ - ig[U_\mu,\zeta_+] \\
\nabla_\mu A_\pm &= \partial_\mu A_\pm - ig\,U_\mu A_\pm \\
\nabla_\mu\psi_\pm &= \partial_\mu\psi_\pm - ig\,U_\mu\psi_\pm \quad .
\end{aligned} \qquad \text{(III.70)}$$

The appearance of (III.69) can be greatly simplified if the coupling parameter h is given the specific value

$$h = g\sqrt{2} \quad . \qquad \text{(III.71)}$$

Indeed, with this restriction the system is parity conserving. This can be seen more easily by eliminating the auxiliary fields, $F_\pm$, f_+ and D, resolving a_+ into hermitian and antihermitian parts,

$$a_+ = \frac{1}{\sqrt{2}}(a + ib) \quad , \qquad \text{(III.72)}$$

and introducing 4-component spinors,

$$\begin{aligned}
\psi &= \psi_+ + \psi_- \quad , \\
\chi &= \zeta_+ + i\lambda_- \quad .
\end{aligned} \qquad \text{(III.73)}$$

The Lagrangian (III.69) then reduces to

$$\begin{aligned}
\mathcal{L} = \frac{1}{2}\,\mathrm{Tr}\Big[&-\frac{1}{4}U_{\mu\nu}^2 + \bar{\chi}i\not{\nabla}\chi + \frac{1}{2}(\nabla_\mu a)^2 + \frac{1}{2}(\nabla_\mu b)^2 \\
&+ g\bar{\chi}\left\{[a,\chi] + [b,\gamma_5\chi]\right\}\Big] \\
&+ \nabla_\mu A_+^\dagger\,\nabla_\mu A_+ + \nabla_\mu A_-^\dagger\,\nabla_\mu A_- + \bar{\psi}(i\not{\nabla} - M)\psi \\
&- g\,\bar{\psi}(a-\gamma_5 b)\psi - g\sqrt{2}\left\{\bar{\psi}\chi A_+ - i\bar{\psi}\gamma_5\chi^c A_- + \text{h.c.}\right\} \\
&- V \quad .
\end{aligned} \qquad \text{(III.74)}$$

where the covariant derivatives are easily deducible from (III.70),(III.72) and (III.73). The conjugate spinor χ^c is defined in the usual fashion

$$\bar{\theta}\chi^c = (\bar{\theta}\chi)^\dagger = \bar{\theta}(\zeta_- - i\lambda_+) \quad . \tag{III.75}$$

The potential V has the form:

$$V = F_+^\dagger F_+ + F_-^\dagger F_- + \frac{1}{2}\,\mathrm{Tr}\left(f_+^\dagger f_+ + \frac{1}{2}D^2\right) \quad . \tag{III.76}$$

where the auxiliary fields are given by

$$\begin{aligned}
F_+ &= -(M + g(a-ib))A_- \\
F_- &= -(M + g(a+ib))A_+ \\
f_+ &= -g\,2\sqrt{2}\left(A_-A_+^\dagger - \frac{1}{n}A_+^\dagger A_-\right) \\
D &= -2g\left(A_+A_+^\dagger - A_-A_-^\dagger - \frac{i}{2}[a,b]\right) + \frac{2g}{n}\left(A_+^\dagger A_+ - A_-^\dagger A_-\right) \quad .
\end{aligned} \tag{III.77}$$

(Recall that f_+ and D are traceless $n \times n$ matrices.) On substituting the expressions (III.77) into (III.76) one finds

$$\begin{aligned}
V = &-\frac{g^2}{4}\,\mathrm{Tr}[a,b]^2 \\
&+ A_+^\dagger\left[(M+ga)^2 + g^2b^2\right]A_+ + A_-^\dagger\left[(M+ga)^2 + g^2b^2\right]A_- \\
&+ g^2\left(1 - \frac{1}{n}\right)\left[(A_+^\dagger A_+)^2 + (A_-^\dagger A_-)^2\right] \\
&- 2g^2\left(1 + \frac{2}{n}\right)(A_+^\dagger A_-)(A_-^\dagger A_+) \\
&+ 2g^2\left(2 + \frac{1}{n}\right)(A_+^\dagger A_+)(A_-^\dagger A_-) \quad .
\end{aligned} \tag{III.78}$$

Parity conservation in the Lagrangian (III.74) with the potential (III.78) is now manifest. For the spinors we take

$$\begin{aligned}
\psi &\to \gamma_0\psi \\
\chi &\to \gamma_0\chi \quad \text{and} \quad \chi^c \to -\gamma_0\chi^c \quad .
\end{aligned} \tag{III.79}$$

Among the bosons: U_μ is a vector; a, A_+ and A_- are scalars; b is a pseudoscalar. (The odd relative parity of χ and χ^c results from the definition (III.75).) Alternative parity assignments are equally feasible. Thus, instead of (III.79) one could take

$$\begin{aligned}
\psi &\to \omega\gamma_0\psi \\
\chi &\to \omega\gamma_0\chi\omega^{-1} \quad \text{and} \quad \chi^c \to -\omega\gamma_0\chi^c\omega^{-1} \;,
\end{aligned} \tag{III.80}$$

where ω is a unitary matrix satisfying $\omega^2 = 1$. The corresponding rules for the bosons are then:

$$\begin{aligned}
U_0 &\to \omega\,U_0\,\omega^{-1} \;, \quad U_i \to -\omega\,U_i\,\omega^{-1} \;, \\
a &\to \omega\,a\,\omega^{-1} \;, \quad b \to -\omega\,b\,\omega^{-1} \;, \\
A_\pm &\to \omega\,A_\pm \quad .
\end{aligned} \tag{III.81}$$

This completes the discussion of the parity-conserving local SU(n) model. Other local symmetries are treated in exactly the same way and any number of matter supermultiplets may be introduced in the form of contragredient pairs Φ_+, Φ_-^*. In every case the couplings are determined entirely by the gauge principle.

(D) CONSERVED CURRENTS

We conclude this section on Lagrangians by considering the local conserved currents which can be constructed corresponding to a given Lagrangian.

A Lagrangian of the general form (III.13) may admit a global symmetry,

$$\delta\Phi_\pm = i\omega Q_\pm \Phi_\pm \quad , \tag{III.82}$$

where the matrices Q_+ and Q_- are hermitian. In such cases one expects to find a conserved 4-vector. This current vector must of course belong to a supermultiplet and one would like to know what significance the other members can have. We now consider this question.

Notice firstly that the Lagrangian can be expressed quite generally in the form

$$\mathcal{L} = -\frac{1}{2}\bar{D}D(\mathcal{L}_+ + \text{h.c.}) \quad , \tag{III.83}$$

where the chiral part $\mathcal{L}_+$ is given by

$$\mathcal{L}_+ = -\frac{1}{4}\bar{D}_+D_-\left[|\Phi_{i+}|^2 + |\Phi_{a-}|^2\right] + \Phi^*_{a-}\left(\lambda_a + M_{ai}\,\Phi_{i+} + g_{aij}\,\Phi_{i+}\,\Phi_{j+}\right) . \tag{III.84}$$

An arbitrary infinitesimal variation of the fields Φ_{i+} and Φ_{a-} gives

$$\begin{aligned}\delta\mathcal{L}_+ = & -\frac{1}{4}\bar{D}_+D_-\left(\Phi_+^\dagger\,\delta\Phi_+ + \Phi_-^\dagger\,\delta\Phi_- + h.c.\right) \\ & + \delta\Phi^*_{a-}\left(\lambda_a + M_{ai}\,\Phi_{i+} + g_{aij}\,\Phi_{i+}\,\Phi_{j+}\right) \\ & + \Phi^*_{a-}\left(M_{ai} + 2g_{aij}\,\Phi_{j+}\right)\delta\Phi_{i+}\end{aligned} \tag{III.85}$$

and this expression can be arranged in a more appropriate form with the help of the variational derivatives of the action functional, S . The derivatives of S are given by formulae like (III.24), viz.

$$\begin{aligned}\frac{\delta S}{\delta\Phi^*_{a-}} &= -\frac{1}{2}\bar{D}_+D_-\,\Phi_{a-} + \left(\lambda_a + M_{ai}\,\Phi_{i+} + g_{aij}\,\Phi_{i+}\,\Phi_{j+}\right) , \\ \frac{\delta S}{\delta\Phi_{i+}} &= -\frac{1}{2}\bar{D}_+D_-\,\Phi^*_{i+} + \Phi^*_{a-}\left(M_{ai} + 2g_{aij}\,\Phi_{j+}\right) .\end{aligned} \tag{III.86}$$

Using these in (III.85) one finds

$$\begin{aligned}\delta\mathcal{L}_+ = & -\frac{1}{4}\bar{D}_+D_-\left(\Phi_+^\dagger\delta\Phi_+ + \Phi_-^\dagger\delta\Phi_- + h.c.\right) \\ & + \delta\Phi_-^\dagger\left(\frac{\delta S}{\delta\Phi_-^\dagger} + \frac{1}{2}\bar{D}_+D_-\,\Phi_-\right) + \left(\frac{\delta S}{\delta\Phi_+} + \frac{1}{2}\bar{D}_+D_-\Phi_+^\dagger\right)\delta\Phi_+ \\ = & \;\frac{1}{4}\bar{D}_+D_-\left(\Phi_+^\dagger\,\delta\Phi_+ - \Phi_-^\dagger\,\delta\Phi_- - h.c.\right) \\ & + \delta\Phi_-^\dagger\,\frac{\delta S}{\delta\Phi_-^\dagger} + \frac{\delta S}{\delta\Phi_+}\,\delta\Phi_+\end{aligned} \tag{III.87}$$

and for variations of the type (III.82) which correspond to a symmetry this must vanish. The identity (III.87) then assumes the form

$$\frac{1}{2}\bar{D}_+D_-\;J(Q) = -\frac{\delta S}{\delta\Phi_+}\,Q_+\Phi_+ + \Phi_-^\dagger Q_-\,\frac{\delta S}{\delta\Phi_-^\dagger} \quad , \tag{III.88}$$

where the "current" $J(Q)$ is a real (non-chiral) superfield defined by

$$J(Q) = \Phi_+^\dagger\;Q_+\;\Phi_+ - \Phi_-^\dagger\;Q_-\;\Phi_- \;. \tag{III.89}$$

When the equations of motion are satisfied,

$$\frac{\delta S}{\delta\Phi_\pm} = 0 \quad , \quad \frac{\delta S}{\delta\Phi_\pm^\dagger} = 0 \quad ,$$

then the right-hand side of Eq.(III.88) vanishes and it becomes a supersymmetric conservation law. More generally, it can be used to generate Ward-Takahashi identities.

The components of the current $J(Q)$ are defined by the usual expansion

$$\begin{aligned}J(Q) = & \;A(Q) + \bar{\theta}\psi(Q) + \frac{1}{4}\bar{\theta}\theta F(Q) + \frac{1}{4}\bar{\theta}\gamma_5\theta G(Q) \\ & + \frac{1}{4}\bar{\theta}i\gamma_\nu\gamma_5\theta V_\nu(Q) + \frac{1}{4}\bar{\theta}\theta\bar{\theta}\chi(Q) + \frac{1}{32}(\bar{\theta}\theta)^2\,D(Q) \quad ,\end{aligned} \tag{III.90}$$

where the bosonic components A, F, G, V and D are real and the fermionic components ψ and χ are Majorana spinors. They are given explicitly by

$$\begin{aligned}A(Q) &= A_+^\dagger Q_+ A_+ - A_-^\dagger Q_- A_- \;, \\ \psi(Q) &= A_+^\dagger Q_+\psi_+ - A_-^\dagger Q_-\psi_- + C\bar{\psi}_+^T Q_+ A_+ - C\bar{\psi}_-^T Q_- A_- \;, \\ F(Q) &= A_+^\dagger Q_+ F_+ - A_-^\dagger Q_- F_- + F_+^\dagger Q_+ A_+ - F_-^\dagger Q_- A_- \;, \\ G(Q) &= iA_+^\dagger Q_+ F_+ + iA_-^\dagger Q_- F_- - iF_+^\dagger Q_+ A_+ - iF_-^\dagger Q_- A_- \;, \\ V_\nu(Q) &= A_+^\dagger i\overleftrightarrow{\partial}_\nu Q_+ A_+ + A_-^\dagger i\overleftrightarrow{\partial}_\nu Q_- A_- + \bar{\psi}_+\gamma_\nu Q_+\psi_+ + \bar{\psi}_-\gamma_\nu Q_-\psi_- \;,\end{aligned}$$

$$\chi(Q) = 2F_+^\dagger Q_+\psi_+ - 2F_-^\dagger Q_-\psi_- + 2C\bar{\psi}_+^T Q_+ F_+ - 2C\bar{\psi}_-^T Q_- F_-$$

$$-A_+^\dagger i\overleftrightarrow{\not\partial} Q_+\psi_+ + A_-^\dagger i\overleftrightarrow{\not\partial} Q_-\psi_- + \gamma_\nu C\bar{\psi}_+^T i\overleftrightarrow{\partial}_\nu Q_+ A_+ - \gamma_\nu C\bar{\psi}_-^T i\overleftrightarrow{\partial}_\nu Q_- A_- \quad ,$$

$$D(Q) = 4F_+^\dagger Q_+ F_+ - 4F_-^\dagger Q_- F_- + 2\bar{\psi}_+ i\overleftrightarrow{\not\partial} Q_+\psi_+ - 2\bar{\psi}_- i\overleftrightarrow{\not\partial} Q_-\psi_-$$

$$-A_+^\dagger \overleftrightarrow{\partial}^2 Q_+ A_+ + A_-^\dagger \overleftrightarrow{\partial}^2 Q_- A_- \quad , \qquad \text{(III.91)}$$

where $\overleftrightarrow{\partial} = \overrightarrow{\partial} - \overleftarrow{\partial}$. The component structure of the "divergence" of $J(Q)$ is given by

$$-\bar{D}_+ D_- J(Q) = \exp\left\{-\tfrac{1}{4}\bar{\theta}\not\partial\gamma_5\theta\right\}\left[(F + iG) + \bar{\theta}_-(\chi_+ - i\not\partial\psi_-) + \tfrac{1}{2}\bar{\theta}_-\theta_+\left\{\tfrac{1}{2}(D - \partial^2 A) - i\partial_\mu V_\mu\right\}\right] \quad . \qquad \text{(III.92)}$$

When the equations of motion are satisfied, the identity (III.88) gives the "conservation law"

$$\bar{D}_+ D_- J(Q) = 0 \qquad \text{(III.93)}$$

and, since $J(Q)$ is real, this implies $\bar{D}_- D_+ J(Q) = 0$ as well. In terms of components,

$$\begin{aligned} F(Q) + iG(Q) &= 0 \quad , \\ \chi_+(Q) - i\not\partial\psi_-(Q) &= 0 \quad , \\ D(Q) - \partial^2 A(Q) &= 0 \quad , \\ \partial_\mu V_\mu(Q) &= 0 \quad , \end{aligned} \qquad \text{(III.94)}$$

of which the last is the familiar conservation law.

In summary, the current 4-vector, $V_\mu(Q)$, belongs to a real non-chiral supermultiplet $J(Q)$ whose components (III.91) are listed. The divergence, $\partial_\mu V_\mu(Q)$, belongs to a chiral supermultiplet $\bar{D}_+ D_- J(Q)$ all of whose components must vanish when the equations of motion are satisfied. The corresponding Ward-Takahashi identities could be set up in a manifestly supersymmetric form with the help of the classical identity (III.88).

Since the chiral part of $J(Q)$ must vanish when the equations of motion are used, the remaining part must take the form of a transverse vector supermultiplet (Φ_1). In these circumstances J reduces to

$$J(Q) = A(Q) + \bar{\theta}\psi(Q) + \tfrac{1}{4}\bar{\theta}i\gamma_\nu\gamma_5\theta\, V_\nu(Q) + \tfrac{1}{4}\bar{\theta}\theta\bar{\theta}i\not\partial\psi(Q) + \tfrac{1}{32}(\bar{\theta}\theta)^2\partial^2 A(Q) \quad . \qquad \text{(III.95)}$$

The matrix elements of $J(Q)$ between physical states belonging to the fundamental representation are characterized by two invariant amplitudes. We shall pursue this exercise in some detail now since it provides a good illustration of the methods developed in this section.

As a first step in treating the matrix elements of $J(Q)$ it is necessary to replace it by an equivalent chiral superfield to which the decompositions of Section II(B) can be applied directly. Therefore we shall deal with the spinor superfield

$$\begin{aligned} \Psi_{-+} &= D_- J(Q) \\ &= \exp\left(-\tfrac{1}{4}\bar{\theta}\not\partial\gamma_5\theta\right)\Big(\psi_-(Q) + \tfrac{1}{2}\big(V_\mu(Q) - i\partial_\mu A(Q)\big)\gamma_\mu\theta_+ + \tfrac{1}{2}\bar{\theta}_-\theta_+ i\not\partial\psi_+(Q)\Big) \\ &\equiv \exp\left(-\tfrac{1}{4}\bar{\theta}\not\partial\gamma_5\theta\right)\hat{\Psi}_{-+} \quad , \end{aligned} \qquad \text{(III.96)}$$

which has positive chirality in its θ-structure and negative chirality in its external spinor index, i.e.

$$D_-\Psi_{-+} = 0 \quad , \quad (1 + i\gamma_5)\Psi_{-+} = 0 \quad ,$$

and it carries one unit of fermion-number.

The required matrix elements can be extracted from various 3-point functions. A convenient one is,

$$i^2 (\partial_1^2+M^2)(\partial_2^2+M^2) \langle T^* \hat{\Phi}_-(1)\, \hat{\Phi}_+^*(2)\, \hat{\Psi}_{-+}(3) \rangle =$$

$$= M\, e^{-\bar{\theta}_{3-}(i\not{\partial}_1 \theta_{1-} + i\not{\partial}_2 \theta_{2-})} \left(a - b [\not{\partial}_1, \not{\partial}_2] \right) \theta_{12-} \; , \qquad \text{(III.97)}$$

where a and b are scalar invariants. This expression, the most general one compatible with supersymmetry and fermion-number conservation, is of the form (II.63). The LSZ reduction formulae (II.70)-(II.72) can be applied to (III.97). Thus

$$i^2 \int dx_1 dx_2\, e^{ip_1x_1 - ip_2x_2} \left[(\partial_1^2+M^2)(\partial_2^2+M^2) \langle T^* \hat{\Phi}_-(1)\, \hat{\Phi}_+^*(2)\, \hat{\Psi}_{-+}(3) \rangle \right] =$$

$$= \left[{}_2\langle p_1| + \sum_{\lambda_1} \bar{\theta}_{1+} u_-(p_1,\lambda_1)\, {}_1\langle p_1 \lambda_1| - \frac{M}{2} \bar{\theta}_{1+}\theta_{1-}\, {}_0\langle p_1| \right] \cdot$$

$$\cdot\, \hat{\Psi}_{-+}(x_3\, \theta_3) \left[|p_2\rangle_0 + \sum_{\lambda_2} |p_2\lambda_2\rangle_1\, \bar{u}_+(p_2,\lambda_2)\, \theta_{2-} - \frac{M}{2} \bar{\theta}_{2+}\theta_{2-} |p_2\rangle_2 \right]$$

$$= M \exp\left[i(p_1-p_2)x_3 - \bar{\theta}_{3-}(\not{p}_1 \theta_{1-} - \not{p}_2 \theta_{2-}) \right] \cdot \left(a + b [\not{p}_1, \not{p}_2] \right) \theta_{12-} \; . \qquad \text{(III.98)}$$

The various matrix elements are obtained by expanding the right-hand side in powers of θ and comparing coefficients. Thus, with θ_3 and x_3 set equal to zero,

$$\sum_{\lambda_2} {}_2\langle p_1| \psi_-(Q) |p_2\lambda_2\rangle_1\, \bar{u}_+(p_2\lambda_2)\, \theta_{2-} + \sum_{\lambda_1} \bar{\theta}_{1+} u_-(p_1\lambda_1)\, {}_1\langle p_1\lambda_1| \psi_-(Q) |p_2\rangle_0 =$$

$$= M \left(a + b [\not{p}_1, \not{p}_2] \right) (\theta_{1-} - \theta_{2-}) \; . \qquad \text{(III.99)}$$

The terms linear in θ_{3+} give

$$\left[{}_2\langle p_1| + \cdots \right] \left(V_\mu(Q) - i\partial_\mu A(Q) \right) \left[|p_2\rangle_0 + \cdots \right] =$$

$$= -M \left(\bar{\theta}_{1+}\not{p}_1 - \bar{\theta}_{2+}\not{p}_2 \right) \gamma_\mu \left(a + b [\not{p}_1, \not{p}_2] \right) (\theta_{1-} - \theta_{2-}) . \qquad \text{(III.100)}$$

The terms quadratic in θ_{3+} give no further information. To extract particular matrix elements from (III.99) and (III.100) is straightforward. One finds,

$$ {}_2\langle p_1| \psi_-(Q) |p_2\lambda_2\rangle_1 = -(a + b[\not{p}_1, \not{p}_2]) u_-(p_1\lambda_2) \; ,$$

$$ {}_1\langle p_1\lambda_1| \psi_-(Q) |p_2\rangle_0 = (a + b[\not{p}_1, \not{p}_2]) C\bar{u}_+^T(p_1\lambda_1) \; ,$$

$$ {}_0\langle p_1| A(Q) |p_2\rangle_0 = a + (4M^2 - t) b \; ,$$

$$ {}_1\langle p_1\lambda_1| A(Q) |p_2\lambda_2\rangle_1 = \bar{u}(p_1\lambda_1)(-2Mb) u(p_2\lambda_2) \; ,$$

$$ {}_2\langle p_1| A(Q) |p_2\rangle_2 = -a + (4M^2 - t) b \; ,$$

$$ {}_0\langle p_1| V_\mu(Q) |p_2\rangle_0 = (a - tb)(p_1 + p_2)_\mu \; ,$$

$$ {}_1\langle p_1\lambda_1| V_\mu(Q) |p_2\lambda_2\rangle_1 = \bar{u}(p_1\lambda_1) \left(a\gamma_\mu - 2ib\, \varepsilon_{\mu\nu\lambda\rho}\, p_{1\nu} p_{2\lambda} \gamma_\rho \right) u(p_2\lambda_2) \; ,$$

$$ {}_2\langle p_1| V_\mu(Q) |p_2\rangle_2 = (a + tb)(p_1 + p_2)_\mu \; , \qquad \text{(III.101)}$$

where a and b are real functions of $t = (p_1 - p_2)^2$. Matrix elements of $\psi_+(Q)$ can be obtained from these by complex conjugation.

The Noether-like technique used here to treat a global symmetry is not readily applicable to the geometrical symmetries: Poincaré and super-translations. For example, although it is possible to construct a current $T_\mu = J(i\partial_\mu)$ corresponding to space-time translations, which satisfies $\bar{D}_+ D_- T_\mu = 0$, it is not real and it does not satisfy $\bar{D}_- D_+ T_\mu = 0$. It does not contain the canonical energy momentum tensor among its components. At present it is not clear whether such currents will be at all useful. To deal with the geometrical symmetries we shall adopt an indirect approach based on the requirement of fermion-number conservation.

The conserved current 4-vector corresponding to the fermion-number symmetry is given by

$$j_\mu(x) = \bar\psi_+\gamma_\mu\psi_+ + \bar\psi_-\gamma_\mu\psi_- + 2\, A_-^\dagger\, i\overleftrightarrow{\partial}_\mu A_- \quad . \tag{III.102}$$

This is the standard Noether current obtainable by the usual considerations applied to a Lagrangian made out of the fields A_+ , ψ_+ , ψ_- , A_- which carry the fermion-numbers 0, -1, -1, -2, respectively. It satisfies the identity

$$\partial_\mu j_\mu(x) = i\,\frac{\delta S}{\delta\psi_+}\,\psi_+ + i\,\frac{\delta S}{\delta\psi_-}\,\psi_- + 2i\,\frac{\delta S}{\delta A_-}\,A_- + \text{h.c.} \quad , \tag{III.103}$$

from which the Ward-Takahashi identities can be derived.

The supersymmetric versions of (III.102) and (III.103) can be obtained directly by application of the "boost" $\exp(i\bar\theta\hat S)$, where $\hat S$ denotes the supertranslation generator (to be distinguished here from the action functional, S). The supertranslations act on the component fields according to the rules,

$$\begin{aligned} \frac{1}{i}\,[A_\pm,\bar\theta\hat S] &= \bar\theta\psi_\pm \;, \\ \frac{1}{i}\,[\psi_\pm,\bar\theta\hat S] &= F_\pm\theta_\pm - i\partial\!\!\!/ A_\pm\theta_\mp \;, \\ \frac{1}{i}\,[F_\pm,\bar\theta\hat S] &= -\bar\theta i\partial\!\!\!/\psi_\pm \;. \end{aligned} \tag{III.104}$$

It is a simple matter to integrate these formulae to obtain

$$\begin{aligned} \Phi_\pm(x,\theta) &= e^{i\bar\theta\hat S}\, A_\pm(x)\, e^{-i\bar\theta\hat S} \;, \\ D\Phi_\pm(x,\theta) &= e^{i\bar\theta\hat S}\, \psi_\pm(x)\, e^{-i\bar\theta\hat S} \;, \\ -\frac{1}{2}\bar D D\,\Phi_\pm(x,\theta) &= e^{i\bar\theta\hat S}\, F_\pm(x)\, e^{-i\bar\theta\hat S} \end{aligned} \tag{III.105}$$

and, in similar fashion,

$$\begin{aligned} \frac{\delta S}{\delta\Phi_\pm} &= e^{i\bar\theta\hat S}\,\frac{\delta S}{\delta F_\pm(x)}\,e^{-i\bar\theta\hat S} \;, \\ -\bar D\,\frac{\delta S}{\delta\Phi_\pm} &= e^{i\bar\theta\hat S}\,\frac{\delta S}{\delta\psi_\pm(x)}\,e^{-i\bar\theta\hat S} \;, \\ -\frac{1}{2}\bar D D\,\frac{\delta S}{\delta\Phi_\pm} &= e^{i\bar\theta\hat S}\,\frac{\delta S}{\delta A_\pm(x)}\,e^{-i\bar\theta\hat S} \;. \end{aligned} \tag{III.106}$$

[Recall that the derivatives $\delta S/\delta\Phi_\pm$ are chiral superfields with the component structure

$$\frac{\delta S}{\delta\Phi_\pm(x,\theta)} = e^{\mp\frac{1}{4}\bar\theta\partial\!\!\!/\gamma_5\theta}\left(\frac{\delta S}{\delta F_\pm(x)} - \frac{\delta S}{\delta\psi_\pm(x)}\,\theta_\pm + \frac{1}{2}\,\bar\theta_\mp\theta_\pm\,\frac{\delta S}{\delta A_\pm(x)}\right) .$$]

Define the supercurrent $J_\mu(x,\theta)$ by applying the boost to the fermion-number current (III.102), i.e.

$$\begin{aligned} J_\mu(x,\theta) &= e^{i\bar\theta\hat S}\, J_\mu(x,0)\, e^{-i\bar\theta\hat S} \;. \\ &= e^{i\bar\theta\hat S}\, j_\mu(x)\, e^{-i\bar\theta\hat S} \;. \end{aligned} \tag{III.107}$$

It is a simple matter to show that this current can be expressed in the manifestly supersymmetric form

$$J_\mu(x,\theta) = \frac{1}{2}\,\bar D\, i\gamma_\mu\gamma_5 D(\Phi_+^\dagger\Phi_+ - \Phi_-^\dagger\Phi_-) - \Phi_+^\dagger\, i\overleftrightarrow{\partial}_\mu\Phi_+ + \Phi_-^\dagger\, i\overleftrightarrow{\partial}_\mu\Phi_- \quad , \tag{III.108}$$

since this expression has no explicit θ-dependence and satisfies the boundary condition $J_\mu(x,0) = j_\mu(x)$. The current (III.108) is a real vector superfield and it is clearly conserved when both fermion-number and supertranslations are valid symmetries. Indeed, the identity which incorporates the conservation law for J_μ is obtained by boosting (III.103),

$$\partial_\mu J_\mu(x,\theta) = \frac{1}{i}\left(\bar D\,\frac{\delta S}{\delta\Phi_+}\right)D\Phi_+ + \frac{1}{i}\left(\bar D\,\frac{\delta S}{\delta\Phi_-}\right)D\Phi_- + \frac{2}{i}\left(\frac{1}{2}\,\bar D D\,\frac{\delta S}{\delta\Phi_-}\right)\Phi_- + \text{h.c.} \tag{III.109}$$

When the equations of motion are used it is possible to show that J_μ reduces (like the global $J(Q)$) to a transverse vector supermultiplet,

$$\bar D_+ D_-\, J_\mu = \bar D_- D_+\, J_\mu = 0 \quad . \tag{III.110}$$

The proof is as follows. Firstly, there is a simple operator identity

$$\bar D_+ D_-\,\bar D\, i\gamma_\mu\,\gamma_5\, D = -2\bar D_+ D_-\, i\partial_\mu \quad ,$$

which implies that

$$\bar{D}_+ D_- J_\mu = \bar{D}_+ D_- \left\{ \tfrac{1}{2} \bar{D} i \gamma_\mu \gamma_5 D \left(\Phi_+^\dagger \Phi_+ - \Phi_-^\dagger \Phi_- \right) - \Phi_+^\dagger i \overleftrightarrow{\partial}_\mu \Phi_+ + \right.$$
$$\left. + \Phi_-^\dagger i \overleftrightarrow{\partial}_\mu \Phi_- \right\}$$
$$= \bar{D}_+ D_- \left\{ - i \partial_\mu \left(\Phi_+^\dagger \Phi_+ - \Phi_-^\dagger \Phi_- \right) - \Phi_+^\dagger i \overleftrightarrow{\partial}_\mu \Phi_+ + \Phi_-^\dagger i \overleftrightarrow{\partial}_\mu \Phi_- \right\}$$
$$= -2 \bar{D}_+ D_- \left\{ \Phi_+^\dagger i \partial_\mu \Phi_+ - \Phi_-^\dagger i \partial_\mu \Phi_- \right\} . \tag{III.111}$$

Next, in order to show that this vanishes, an argument similar to that used in the global case can be used. Thus, corresponding to an infinitesimal translation,

$$\partial_\mu \mathcal{L}_+ = \frac{1}{4} \bar{D}_+ D_- (\Phi_+^\dagger \overleftrightarrow{\partial}_\mu \Phi_+ - \Phi_-^\dagger \overleftrightarrow{\partial}_\mu \Phi_-) , \tag{III.112}$$

when the equations of motion are used. Further, the equations of motion imply that $\mathcal{L}_+$ takes the simple form

$$\mathcal{L}_+ = - \frac{1}{4} \bar{D}_+ D_- (\Phi_+^\dagger \Phi_+ - \Phi_-^\dagger \Phi_-) . \tag{III.113}$$

Subtraction of the gradient of (III.113) from (III.112) gives the desired result,

$$0 = \frac{1}{2} \bar{D}_+ D_- (\Phi_+^\dagger \partial_\mu \Phi_+ - \Phi_-^\dagger \partial_\mu \Phi_-) \tag{III.114}$$

and (III.111) vanishes. Since J_μ is real, it follows that $\bar{D}_- D_+ J_\mu$ also vanishes and the formulae (III.110) are proved.

The transverse vector part of J_μ can be expanded in the usual fashion.

$$\begin{aligned} J_\mu(x,\theta) = {} & j_\mu(x) \\ & + \bar{\theta}(-\gamma_5 S_\mu(x)) \\ & + \frac{1}{4} \bar{\theta} i \gamma_\nu \gamma_5 \theta (-2T_{\mu\nu}(x)) \\ & + \frac{1}{4} \bar{\theta}\theta\bar{\theta} i \not{\partial} (-\gamma_5 S_\mu(x)) \\ & + \frac{1}{32} (\bar{\theta}\theta)^2 \partial^2 j_\mu(x) , \end{aligned} \tag{III.115}$$

where S_μ is a Majorana spinor whose left-handed part is given explicitly by

$$i S_{\mu-}(x) = A_-^\dagger (\gamma_\mu i \overleftarrow{\not{\partial}} - i \overrightarrow{\not{\partial}} \gamma_\mu) \psi_- + \gamma_\mu C \bar{\psi}_-^T F_- + \gamma_\nu \gamma_\mu C \bar{\psi}_+^T i \partial_\nu A_+$$
$$= F_+^\dagger \gamma_\mu \psi_+ + \gamma_\mu C \bar{\psi}_-^T F_- + (i \partial_\nu A_-^\dagger) \gamma_\nu \gamma_\mu \psi_- + \gamma_\nu \gamma_\mu C \bar{\psi}_+^T i \partial_\nu A_+$$
$$+ i \partial_\nu (A_-^\dagger [\gamma_\mu, \gamma_\nu] \psi_-) \tag{III.116}$$

(where equations of motion have been used to effect the rearrangement). This is a conserved current whose time-component S_{0-} is the density of the supertranslation generator S_- .

The tensor $T_{\mu\nu}$ is transverse with respect to both indices,

$$\partial_\mu T_{\mu\nu} = \partial_\nu T_{\mu\nu} = 0 ,$$

although it is not symmetric. Its explicit form is given, with the help of equations of motion, by

$$\begin{aligned} T_{\mu\nu} = {} & \tfrac{1}{2} A_+^\dagger i \overleftrightarrow{\partial}_\mu i \overleftrightarrow{\partial}_\nu A_+ + \tfrac{1}{2} A_-^\dagger i \overleftrightarrow{\partial}_\mu i \overleftrightarrow{\partial}_\nu A_- \\ & \tfrac{1}{2} \bar{\psi}_+ (\gamma_\mu i \overleftrightarrow{\partial}_\nu + \gamma_\nu i \overleftrightarrow{\partial}_\mu) \psi_+ + \tfrac{1}{2} \bar{\psi}_- (\gamma_\mu i \overleftrightarrow{\partial}_\nu + \gamma_\nu i \overleftrightarrow{\partial}_\mu) \psi_- \\ & + \eta_{\mu\nu} \left\{ F_+^\dagger F_+ + F_-^\dagger F_- + \tfrac{1}{2} \left(A_+^\dagger \partial^2 A_+ + A_-^\dagger \partial^2 A_- + h.c. \right) \right\} \\ & - \tfrac{1}{2} (\eta_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu)(A_+^\dagger A_+ - A_-^\dagger A_-) \\ & + \tfrac{1}{2} \varepsilon_{\mu\nu\lambda\rho} \partial_\lambda \left\{ A_+^\dagger i \overleftrightarrow{\partial}_\rho A_+ + A_-^\dagger i \overleftrightarrow{\partial}_\rho A_- + \tfrac{1}{2} \bar{\psi}_+ \gamma_\rho \psi_+ + \tfrac{3}{2} \bar{\psi}_- \gamma_\rho \psi_- \right\} . \end{aligned} \tag{III.117}$$

The antisymmetric part makes no contribution to the integrated 4-momentum operator.

By its construction the current J_μ transforms according to

$$\frac{1}{i} [J_\mu, \bar{\varepsilon}\hat{S}] = \bar{\varepsilon} \left(\frac{\partial}{\partial\bar{\theta}} + \frac{i}{2} \not{\partial}\theta \right) J_\mu$$

or, in terms of components,

$$[j_\mu, \bar{\varepsilon}\hat{S}] = -\bar{\varepsilon}\,\gamma_5\, S_\mu \ ,$$

$$[S_\mu, \bar{\varepsilon}\hat{S}] = \left\{T_{\mu\nu} + \frac{1}{2}\gamma_5\,\partial_\nu\, j_\mu\right\}\gamma_\nu\,\varepsilon \ ,$$

$$[T_{\mu\nu}, \bar{\varepsilon}\hat{S}] = -\frac{1}{2}\bar{\varepsilon}\,\sigma_{\nu\lambda}\,\partial_\lambda\, S_\mu \ . \tag{III.118}$$

The integrated quantities

$$\mathbb{F} = \int d_3\underline{x}\; j_0 \ , \qquad \hat{S} = \int d_3\underline{x}\; S_0 \ ,$$

$$P_\nu = \int d_3\underline{x}\; T_{0\nu} \ , \qquad J_{\mu\nu} = \int d_3\underline{x}\left\{x_\mu\, T_{(0\nu)} - x_\nu\, T_{(0\mu)}\right\} \ , \tag{III.119}$$

can be shown to satisfy the supersymmetry algebra of Section I. (In $J_{\mu\nu}$ the symmetrical part of $T_{\mu\nu}$ is employed.)

The components of J_μ can be arranged into two distinct supermultiplets. Notice firstly that the covariant derivative

$$D_+ J_\mu = \exp[\tfrac{1}{4}\,\bar{\theta}\not\partial\gamma_5\theta]\left[iS_{\mu+} + (T_{\mu\nu} - \tfrac{i}{2}\,\partial_\nu j_\mu)\gamma_\nu\theta_- + \tfrac{1}{2}\,\bar{\theta}_+\theta_-\not\partial S_{\mu-}\right] \ ,$$

is a chiral superfield. Multiply this by γ_μ to obtain

$$\gamma_\mu D_+ J_\mu = \exp[\tfrac{1}{4}\,\bar{\theta}\not\partial\gamma_5\theta]\left[i\gamma_\mu S_{\mu+} + T_{\mu\mu}\theta_- + (T_{[\mu\nu]} - \tfrac{i}{2}\,\partial_\nu j_\mu)\,\tfrac{1}{i}\,\sigma_{\mu\nu}\theta_- - \tfrac{1}{2}\,\bar{\theta}_+\theta_-\not\partial\,\gamma_\mu\, S_{\mu-}\right] \ , \tag{III.120}$$

which shows that the trace, $T_{\mu\mu}$, and the self-dual part of $T_{[\mu\nu]} - \frac{i}{2}\,\partial_\nu J_\mu$ belong with $\gamma_\mu S_\mu$ in one supermultiplet. The remaining components make up another supermultiplet.

Finally, a few words about Ward-Takahashi identities. These are based on the identities (III.88) for $J(Q)$ and (III.109) for J_μ . A typical Green's function involving $J(Q)$ can be represented by a path integral,

$$\langle T\; J(Q)\; \Pi(\Phi)\rangle = \int (d\Phi)\; J(Q)\; \Pi(\Phi)\; \exp[\tfrac{i}{\hbar}\, S(\Phi)] \ ,$$

where $\Pi(\Phi)$ represents a product of superfields. The identity (III.88) can be applied directly,

$$\frac{1}{2}\bar{D}_+ D_- \langle T\, J(Q)\, \Pi(\Phi)\rangle = \int (d\Phi)\left(-\frac{\delta S}{\delta\Phi_+}\,Q_+\Phi_+ + \Phi_-^\dagger Q_-\frac{\delta S}{\delta\Phi_-^\dagger}\right)\Pi(\Phi)\, e^{\frac{i}{\hbar}S}$$

$$= \frac{\hbar}{i}\int (d\Phi)\,\Pi(\Phi)\left(-\Phi_+^T Q_+^T\frac{\delta}{\delta\Phi_+^T} + \Phi_-^\dagger Q_-\frac{\delta}{\delta\Phi_-^\dagger}\right)e^{\frac{i}{\hbar}S}$$

$$= \frac{\hbar}{i}\int (d\Phi)\left(\frac{\delta\Pi}{\delta\Phi_+}\,Q_+\Phi_+ - \Phi_-^\dagger Q_-\frac{\delta\Pi}{\delta\Phi_-^\dagger}\right)e^{\frac{i}{\hbar}S}$$

$$= \frac{\hbar}{i}\left\langle T\left(\frac{\delta\Pi}{\delta\Phi_+}\,Q_+\Phi_+ - \Phi_-^\dagger Q_-\frac{\delta\Pi}{\delta\Phi_-^\dagger}\right)\right\rangle$$

after integrating by parts. (A singular term has been excluded by assuming $\delta_+(1,1) = 0$, see below.) Thus, for example,

$$\frac{1}{2}\bar{D}_+ D_- \langle T\, J(1,Q)\,\Phi_+(2)\,\Phi_-^\dagger(3)\rangle =$$

$$= \frac{\hbar}{i}\,\delta_+(1,2)\; Q_+\langle T\Phi_+(2)\;\Phi_-^\dagger(3)\rangle - \frac{\hbar}{i}\langle T\Phi_+(2)\;\Phi_-^\dagger(3)\rangle\; Q_-\;\delta_+(3,1) \ ,$$

$$\frac{1}{2}\bar{D}_+ D_- \langle T\, J(1,Q)\,\Phi_-(2)\,\Phi_+^\dagger(3)\rangle = 0 \ ,$$

where $\delta_+(1,2)$ denotes the supersymmetric delta function

$$\delta_+(1,2) = \exp\left[-\frac{1}{4}\,\bar{\theta}_1\not\partial_1\gamma_5\theta_1 - \frac{1}{4}\,\bar{\theta}_2\not\partial_2\gamma_5\theta_2\right]\,\delta\,(x_1 - x_2)\,\frac{1}{2}\,\bar{\theta}_{12-}\,\theta_{12+} \ .$$

This delta function has the important feature that it vanishes when $\theta_1 = \theta_2$, a property used in the above derivation.

In similar fashion one can derive Ward-Takahashi identities for the supercurrent J_μ . Only the details differ.

IV. MODEL BUILDING

(A) SPONTANEOUS SYMMETRY BREAKING

So far we have tacitly assumed that supersymmetry is unbroken. The discussion of representations in Section I and invariant amplitudes in Section II, for example, were based on the presumed existence of a supersymmetric vacuum. Our purpose now is to discard this assumption and investigate some of the consequences of a degeneracy in the vacuum. In other words, we shall suppose that the underlying dynamics (as expressed in the Lagrangian) is invariant and that a conserved supercurrent exists, but that the ground state is not invariant. In this case the space-time symmetry is reduced to the Poincaré group and a Goldstone phenomenon occurs. On the one hand, the most distinctive aspect of supersymmetry - the grouping of fermions into supermultiplets with bosons - is lost while, on the other, a new and equally distinctive feature emerges, the Goldstone "neutrino."

The mechanism is analogous to the familiar chiral dynamics of the pion where chiral $SU(2)_L \times SU(2)_R$ symmetry is realized by means of a triplet of massless pseudoscalar Goldstone bosons. When the chiral symmetry breaks spontaneously to $SU(2)$ there necessarily appears a set of massless particles with the quantum numbers of the lost symmetries. The associated currents are then dominated at low energies by the exchange of these massless particles, and various low-energy theorems result. Exactly the same thing happens when supersymmetry is broken spontaneously.

Let the spinors $\psi_\pm(x)$ belong to scalar superfields of the kind introduced in Section II so that, under an infinitesimal supertranslation,

$$\delta\psi_\pm = \frac{1 \pm i\gamma_5}{2} (F_\pm - i\not{\partial} A_\pm)\varepsilon \quad . \tag{IV.1}$$

If the vacuum is at least translation-invariant then $\langle \not{\partial} A \rangle = 0$ and we have

$$\langle \delta\psi_\pm \rangle = \langle F_\pm \rangle \, \varepsilon_\pm \quad . \tag{IV.2}$$

If the vacuum respects fermion-number conservation, then $\langle F_+ \rangle = 0$ but we shall not require this.

If either $\langle \delta\psi_+ \rangle$ or $\langle \delta\psi_- \rangle$ is non-vanishing, then the vacuum cannot be supersymmetric. Formally,

$$\begin{aligned}\delta\psi_\pm(0) &= \frac{1}{i}\,[\psi_\pm(0)\ ,\ \bar{\varepsilon} S] \\ &= \frac{1}{i} \int d_3\underline{x}\,[\psi_\pm(0)\ ,\ \bar{\varepsilon} S_0(x)] \quad ,\end{aligned} \tag{IV.3}$$

where S_0 is the time component of the current S_μ introduced previously. The vacuum expectation value of this equation can be expressed in the compact form

$$\langle \delta\psi_\pm(0) \rangle \, \delta_4(x) = i\partial_\mu \langle T^* \, \bar{\varepsilon} \, S_\mu(x) \psi_\pm(0) \rangle \tag{IV.4}$$

provided that the vacuum is translation-invariant and the current $S_\mu(x)$ is local and conserved. Substitute the expressions (IV.2) for $\delta\psi_\pm$ and remove the parameter ε from both sides of (IV.4). One finds

$$i\partial_\mu \langle T^* \, \psi_\pm(0) \, \bar{S}_\mu(x) \rangle = \frac{1 \pm i\gamma_5}{2} \langle F_\pm \rangle \, \delta_4(x) \quad . \tag{IV.5}$$

In momentum space,

$$\begin{aligned}&\int dx \, e^{ikx} \, \langle T^* \, \psi_\pm(0) \, \bar{S}_\mu(x) \rangle = \\ &\quad = \frac{1 \pm i\gamma_5}{2} \left[M_1^{(\pm)} k_\mu + \left(M_2^{(\pm)} \eta_{\mu\nu} + M_3^{(\pm)} k_\mu k_\nu \right) \gamma_\nu + M_4^{(\pm)} \sigma_{\mu\nu} k_\nu \right] \quad ,\end{aligned} \tag{IV.6}$$

where the amplitudes $M_1^{(\pm)}, \cdots$ depend on k^2 . The identities (IV.5) take the form

$$\begin{aligned} k^2 M_1^{(\pm)} &= \langle F_\pm \rangle \\ M_2^{(\pm)} + k^2 M_3^{(\pm)} &= 0 \quad . \end{aligned} \tag{IV.7}$$

The second of these serves only to eliminate $M_2^{(\pm)}$ from the decomposition (IV.6). The first gives the explicit form of $M_1^{(\pm)}$, <u>viz.</u> a simple zero-mass pole with residue $\langle F_\pm \rangle$. This indicates that the intermediate states which contribute to the two-point function (IV.6) must include a massless particle of spin 1/2: a Goldstone fermion.

A similar argument can be repeated with the spinor components, λ_- , in the gauge fields. According to the formula (III.60),

$$\langle \delta\lambda_- \rangle = -\frac{i}{\sqrt{2}} \langle D \rangle \, \varepsilon_- \quad , \tag{IV.8}$$

which may be non-vanishing if supersymmetry is broken.

The Goldstone spinor will in general turn out to be a mixture of the spinor fields present in the system. The relative proportions in the mixture are governed by the expectation values of the scalar fields, $F_\pm$ and D. Perhaps the simplest way to see this is through the expression for the conserved supercurrent $S_\mu(x)$,

$$iS_{\mu+} = F_-^\dagger \gamma_\mu \psi_- + \gamma_\mu C\overline{\psi}_+^T F_+ + i\nabla_\nu A_+^\dagger \gamma_\nu \gamma_\mu \psi_+ + \gamma_\nu \gamma_\mu C\overline{\psi}_-^T i\nabla_\nu A_- + \frac{i}{\sqrt{2}}(D^k - \frac{1}{2}\sigma_{\kappa\lambda}W^k_{\kappa\lambda})\gamma_\mu \lambda_-^k . \qquad (IV.9)$$

On linearizing this about the vacuum solution, one finds

$$iS_{\mu+} \approx \langle F_-^\dagger \rangle \gamma_\mu \psi_- + \gamma_\mu C\overline{\psi}_+^T \langle F_+ \rangle + \frac{i}{\sqrt{2}} \langle D^k \rangle \gamma_\mu \lambda_-^k + \cdots$$

$$\approx f\, \gamma_\mu\, \nu_- + \cdots \qquad (IV.10)$$

where the Goldstone spinor, ν_-, is defined by

$$f\, \nu_- = \langle F_-^\dagger \rangle \psi_- + C\overline{\psi}_+^T \langle F_+ \rangle + \frac{i}{\sqrt{2}} \langle D^k \rangle \lambda_-^k \qquad (IV.11)$$

and f is a normalizing factor. If fermion-number is conserved, then $\langle F_+ \rangle = 0$ and ν_- is seen to be a pure negative chirality fermion field.

Low-energy theorems can be derived for ν_- by writing

$$iS_{\mu+} = f\, \gamma_\mu\, \nu_- + R_{\mu+} .$$

Current conservation then implies that the neutrino source $\not\partial \nu_- = -(1/f)\partial_\mu R_{\mu+}$ is a soft operator at low energies.

(B) MASS GENERATION

Owing to the very strict controls that supersymmetry imposes on the form of the action it often happens that one or more particles are found to be massless. Of these only the Goldstone spinor, associated with the spontaneous breakdown of supersymmetry can be understood in group-theoretic terms. Other instances of this phenomenon can usually be explained by a particular representation content in a given system, or by the exigencies of renormalizability which severly limit the variety of interactions. Here we review the mass problem in general terms and list the various ways in which mass can be generated. We conclude with two mechanisms due to Slavnov wherein the supersymmetry is broken explicitly albeit softly.

B.1 Mass generation for scalar supermultiplets

To begin, consider the free spin-1/2 particle. It is usual to represent the massive spin-1/2 states by means of a pair of chiral spinors ψ_+ and ψ_- which belong to independent representations of the proper Lorentz group. They satisfy the Dirac equation

$$-i\not\partial\psi_+ + M\psi_- = 0 ,$$
$$-i\not\partial\psi_- + M\psi_+ = 0 , \qquad (IV.12)$$

and, in the Lagrangian, the mass term takes the form

$$-M\overline{\psi}_-\psi_+ + \text{h.c.} \qquad (IV.13)$$

The fields ψ_+ and ψ_- may be related by hermitian conjugation,

$$\psi_- = C\overline{\psi}_+{}^T , \qquad (IV.14)$$

but this is possible only if the states carry no internal quantum number such as baryon or lepton number. Using (IV.14) the mass term (IV.13) takes the form

$$M\psi_+{}^T C^{-1} \psi_+ + \text{h.c.} \qquad (IV.15)$$

and this term clearly violates such quantum numbers. (Of course, if $M = 0$ one can discard either ψ_+ or ψ_- and allow the remaining one to carry the quantum number.)

The massive spin-1/2 fermion is now considered to belong to a massive scalar supermultiplet of the extended Poincaré symmetry. It has scalar partners which carry fermion number $\mathbb{F} = 0$ and $\mathbb{F} = 2$. This multiplet can be represented by a pair of chiral scalar superfields Φ_+ and Φ_-, which are just the supersymmetric extensions of ψ_+ and ψ_-. They satisfy an extended version of (IV.12),

$$-\frac{1}{2}\overline{D}\, D\, \Phi_+ + M\Phi_- = 0 ,$$
$$-\frac{1}{2}\overline{D}\, D\, \Phi_- + M\Phi_+ = 0 , \qquad (IV.16)$$

and, in the Lagrangian density, the mass term takes the form

$$-\frac{1}{2}\overline{D}\, D\, (M\Phi_-^*\Phi_+ + \text{h.c.})$$

$$= M(F_-^*A_+ - \overline{\psi}_-\psi_+ + A_-^*F_+) + \text{h.c.} , \qquad (IV.17)$$

which includes the spinor term (IV.13). The fields Φ_+ and Φ_- may be related by complex conjugation if there is no fermionic quantum number present. The supersymmetric extension of (IV.14) is simply

$$\Phi_- = \Phi_+^* \quad . \tag{IV.18}$$

However, we shall continue to assume that a fermionic number is involved and so will not identify Φ_- with Φ_+^*. To conclude, in general, mass generation for scalar supermultiplets requires the co-operation of multiplets Φ_+ and Φ_- in a Lagrangian term of the type (IV.17). The massive supermultiplet (represented by Φ_+ and Φ_-) has the content: one spin-1/2 massive particle carrying fermion-number $\mathbb{F} = 1$, one spin-zero particle with $\mathbb{F} = 0$ and one spin-zero particle with $\mathbb{F} = 2$. All these particles have the same mass M.

The most general renormalizable and fermion-number-conserving interactions of a massive scalar multiplet are described by the Lagrangian (III.6).

B.2 Mass generation for a vector supermultiplet

So much for the scalar supermultiplet. The vector supermultiplet is much more complicated. First of all, the only known way to set up a renormalizable interaction for massive vector particles is through the agency of a spontaneously broken gauge symmetry. Hence the vector supermultiplet must be associated with a gauge superfield $\Psi(x,\theta)$. For the spontaneous breaking of this symmetry a set of Higgs scalars is needed and they must be carried in scalar superfields Φ_+. The two superfields, Ψ and Φ_+, co-operate to make a vector supermultiplet of massive states - rather as Φ_- and Φ_+ co-operate to make a scalar supermultiplet.

In general, one may consider a non-Abelian local symmetry which involves a number of generators Q^k and associated gauge fields Ψ^k. The symmetries are spontaneously broken if

$$Q^k \langle A_+ \rangle \neq 0 \quad .$$

Now, it is well known that the vector components acquire in this way the mass term,

$$\nabla_\mu \langle A_+^\dagger \rangle \ \nabla_\mu \langle A_+ \rangle = \frac{1}{2} V_\mu^k (M^2)^{k\ell} \ V_\mu^\ell \ , \tag{IV.19}$$

where the mass matrix $(M^2)^{k\ell}$ is given by

$$(M^2)^{k\ell} = A_+^\dagger \{Q^k, Q^\ell\} A_+ \quad , \tag{IV.20}$$

and the gauge coupling strengths are implicit in the charge matrices Q^k. The new feature of the supersymmetric gauge theory is the generation of fermion masses in the fusion of left-handed gauge fermions λ_-^k with right-handed matter fermions ψ_+ in the term

$$i\sqrt{2} \langle A_+^\dagger \rangle \overline{\lambda}_-^k \ Q^k \ \psi_+ + \text{h.c.} \tag{IV.21}$$

There may of course be additional left-handed fermion components, ψ_-, in matter superfields Φ_-. The full mass matrix then takes the form,

$$\begin{aligned} (\overline{\lambda}_-^k \ \overline{\psi}_-) \begin{pmatrix} i\sqrt{e} \langle A_+^\dagger \rangle Q^k \\ m^\dagger \end{pmatrix} \psi_+ + \text{h.c.} \\ = (\overline{\lambda}_- \ \overline{\psi}_-) M\psi_+ + \text{h.c.} \end{aligned} \tag{IV.22}$$

where M is, in general, a rectangular matrix. The non-vanishing fermion masses are included among the eigenvalues of the matrix

$$\begin{aligned} MM^\dagger &= \begin{pmatrix} i\sqrt{2} \langle A_+^\dagger \rangle Q^k \\ m^\dagger \end{pmatrix} (-i\sqrt{2} \ Q^\ell \langle A_+ \rangle \ \ m) \\ &= \begin{pmatrix} 2 \langle A_+^\dagger \rangle Q^k Q^\ell \langle A_+ \rangle & i\sqrt{2} \langle A_+^\dagger \rangle Q^k \ m \\ -i\sqrt{2} \ m^\dagger Q^\ell \langle A_+ \rangle & m^\dagger m \end{pmatrix} . \end{aligned} \tag{IV.23}$$

If the off-diagonal elements $\langle A_+^\dagger \rangle Q^k \ m$ are non-vanishing, then there is mixing between the guage fermions λ_- and the matter fermions ψ_-. The fermion spectrum will then differ substantially from the vector spectrum. (If there is no such mixing then $MM^\dagger$ includes a submatrix which is effectively the same as the vector mass matrix (IV.20).) In other words, this mixing term $\langle A_+^\dagger \rangle Q^k \ m$ is a signal of supersymmetry breaking - although such breaking can happen even when there is no mixing.

B.3 Minimization of effective potential to generate scalar masses

Scalar masses are obtained by expanding the potential

$$V = F_+^\dagger F_+ + F_-^\dagger F_- + \frac{1}{2} (D^k)^2 \tag{IV.24}$$

about its minimum with F_+, F_- and D^k expressed in terms of A_+ and A_- as discussed in Sec.III. If supersymmetry is not broken then all F's and

D's must vanish at the minimum and the masses are obtained by retaining only their linear parts,

$$F_+ \sim \left\langle \frac{\partial F_+}{\partial A_-} \right\rangle A_- \quad , \quad F_- \sim \left\langle \frac{\partial F_-}{\partial A_+} \right\rangle A_+ \quad ,$$

$$D \sim \left\langle \frac{\partial D}{\partial A_+} \right\rangle A_+ + A_+^\dagger \left\langle \frac{\partial D}{\partial A_+^\dagger} \right\rangle \quad , \tag{IV.25}$$

where the notation $\langle \cdots \rangle$ means "evaluated at the minimum", and we have assumed that fermion-number is not violated, i.e. $\langle A_- \rangle = 0$. (If fermion-number is violated, then we should have more terms in the linearization,

$$F_+ \sim \left\langle \frac{\partial F_+}{\partial A_-} \right\rangle A_- + \left\langle \frac{\partial F_+}{\partial A_+} \right\rangle A_+ + A_+^\dagger \left\langle \frac{\partial F_+}{\partial A_+^\dagger} \right\rangle + A_-^\dagger \left\langle \frac{\partial F_+}{\partial A_-^\dagger} \right\rangle \quad ,$$

etc. We shall continue to assume that fermion-number is preserved since it means a great simplification.)

If the supersymmetry is broken spontaneously, then some components among the F's and D's will fail to vanish at the minimum. In these components it will be necessary to retain quadratic terms. Such terms act to separate the degenerate terms of a supermultiplet. For instance, if $\langle D^k \rangle \neq 0$, then the scalar mass terms will contain

$$\langle D^k \rangle \quad (A_+^\dagger Q_+^k A_+ - A_-^\dagger Q_-^k A_-) \quad ,$$

which separates states with $\mathbb{F} = 2$ from those with $\mathbb{F} = 0$ and both from those with $\mathbb{F} = 1$. To find out whether such a term will appear in a given model is a matter of detailed investigation. One must minimize the potential and see if any F or D is non-vanishing there. However, if the model is not a very elaborate one it will usually be apparent when there are no values of A_+ and A_- for which all F's and D's vanish simultaneously.

A problem which often arises is that of degeneracy in the classical minimum. The potential attains its minimum value not at a single point or even on a set of points which are connected by symmetries of the system, but on a larger manifold. This phenomenon indicates the presence of massless (pseudo-Goldstone) scalar states. In ordinary gauge theories when this kind of instability appears in the classical approximation it is removed by the quantum corrections. <u>However, a peculiar feature of supersymmetric theories is that such instabilities can persist to all orders. Indeed, it can be proved that the pseudo-Goldstone particles remain without mass to all orders in perturbation theory if the supersymmetry is not broken in zeroth</u>

<u>order</u>. On the other hand, if the supersymmetry <u>is</u> broken in zeroth order, then either the instability is removed or it is amplified, i.e. the pseudo-Goldstone particle acquires either a real or an imaginary mass. Again, it requires a detailed investigation to decide which alternative is chosen in any given system.

B.4 <u>Slavnov's mass generating terms</u>

The appearance of vacuum instabilities seriously limits the utility of strictly supersymmetric schemes. If such a pathology arises in a particular model one can only conclude that it is not amenable to perturbative treatment. Whether such a model is sick in some even more fundamental way cannot be answered at present. In order to circumvent these difficulties we consider now the introduction of <u>pseudo-explicit</u> symmetry breaking terms which serve to lift the vacuum degeneracy in zeroth order.

Two mechanisms for breaking supersymmetry in a particularly soft way have been invented by Slavnov. The first (Type I) mechanism is applicable to Lagrangians which admit a global U(1) symmetry. This symmetry is then gauged in the usual supersymmetric way and then a singular limit is taken in which the gauge coupling vanishes while the associated auxiliary field D becomes infinitely large. At no stage is the supersymmetry of the extended system broken explicitly and, in fact, a Goldstone spinor is present. However, this Goldstone spinor, along with the U(1) gauge vector, is decoupled in the limit, and the end result is a scalar mass term which, so far as the original system is concerned, acts like an explicit symmetry breaker. The derivation goes as follows.

The supposed global symmetry acts on the matter fields $\Phi_\pm$ through "charge" matrices $Q_\pm$ and it is required that this symmetry remain unbroken,

$$Q_\pm \langle \Phi_\pm \rangle = 0 \quad .$$

It is made local through the introduction of a gauge superfield Ψ_0 which couples to $\Phi_\pm$ through the kinetic terms,

$$\frac{1}{8} (\bar{D}D)^2 \left[\Phi_+^\dagger \, e^{2g_0 \Psi_0 Q_+ + \cdots} \, \Phi_+ + \Phi_-^\dagger \, e^{-2g_0 \Psi_0 Q_- + \cdots} \, \Phi_- \right] \quad , \tag{IV.26}$$

where g_0 denotes the new coupling constant and the dots indicate all other gauge couplings. With a U(1) local symmetry it is possible to introduce a linear term

$$\frac{1}{8}\,(\bar{D}D)^2\,\left\{\frac{\mu^2}{g_0}\,\Psi_0\right\}\;, \tag{IV.27}$$

where μ is a parameter with the dimensions of mass. The new gauge field gives rise to a new term in the classical potential, viz.

$$\frac{1}{2}\,D_0^2 = \frac{1}{2}\left[\frac{\mu^2}{g_0} + g_0\,(A_+^\dagger Q_+ A_+ - A_-^\dagger Q_- A_-)\right]^2 \tag{IV.28}$$

and this is the only term in the entire Lagrangian which becomes singular in the limit $g_0 \to 0$ (keeping μ fixed). In this limit the vector and spinor components of Ψ_0 will decouple and the only relic of this field will be the terms

$$\lim \frac{1}{2}\,D_0^2 = \frac{\mu^4}{2g_0^2} + \mu^2(A_+^\dagger Q_+ A_+ - A_-^\dagger Q_- A_-)\;, \tag{IV.29}$$

of which the first, an infinite c-number, is irrelevant and may be discarded. Only an effective mass term for A_+ and A_- remains.

As remarked above, this mechanism is not an ordinary sort of symmetry breaking. Although the term (IV.29) acts, in effect, as an explicit breaker of supersymmetry, it has a symmetry respecting origin in the gauge field Ψ_0 of which it is a sort of tadpole. In other words, one can look upon the mass insertion, μ^2, as the contribution of a zero-frequency external D_0 line,

$$\mu^2 = g_0\,\langle D_0\rangle\;. \tag{IV.30}$$

<u>In this view the symmetry breaking is spontaneous rather than explicit</u>. This viewpoint is important when one inquires into the matter of counter-terms. In supersymmetric theories only wave-function counter-terms are needed and this means that only <u>one</u> new divergence is introduced into the theory by this mechanism: the renormalization of D_0, which will manifest itself as a <u>logarithmic</u> divergence in μ^2. By virtue of this paucity of independent parameters, Slavnov was able to set up a model which is both ultraviolet free and infra-red stable.

Slavnov's second (Type II) mechanism is applicable to systems where it is possible to construct chiral bilinears which are invariant under any local symmetries of the system. Let $\Phi_+^T\,\eta\,\Phi_+$ be one such invariant, where η is a symmetric numerical matrix. Introduce the pair of auxiliary negative chirality singlets S_- and S_-' whose kinetic terms and interactions with Φ_+ are governed by a new term in the Lagrangian,

$$\frac{1}{8}\,(\bar{D}D)^2(S_-^* S_-' + \text{h.c.}) - \frac{1}{2}\,\bar{D}D(h\;S_-^*\Phi_+^T\,\eta\,\Phi_+ + \frac{\mu^2}{h}\,S_-' + \text{h.c.})\;. \tag{IV.31}$$

The non-positive definite kinetic term here indicates that one of the new states must be associated with a negative metric,

$$S_-^* S_-' + \text{h.c.} = \left|\frac{S_- + S_-'}{\sqrt{2}}\right|^2 - \left|\frac{S_- - S_-'}{\sqrt{2}}\right|^2\;.$$

However, it is not difficult to show that these states do not couple to the physically significant ones. To see this, reduce (IV.31) to component form,

$$\begin{aligned}
&\Big[\partial a_-^*\,\partial a_-' + \bar{\zeta}_-\,i\partial\!\!\!/\zeta_-' + f_-^*\,f_-' \\
&\quad + a_-^*\left(2h\,A_+^T\eta F_+ + h\psi_+^T C^{-1}\eta\psi_+\right) \\
&\quad - \bar{\zeta}_-\left(2hA_+^T\eta\psi_+\right) - f_-^*\left(hA_+^T\eta A_+\right) + \frac{\mu^2}{h}\,f_-'\Big] + \text{h.c.} \\
&= \Big[a_-^*\left(-\partial^2 a_-' + 2hA_+^T\eta F_+ + h\psi_+^T C^{-1}\eta\psi_+\right) \\
&\quad + \bar{\zeta}_-\left(i\partial\!\!\!/\zeta_-' - 2hA_+^T\eta\psi_+\right) \\
&\quad + \left(f_-^* + \frac{\mu^2}{h}\right)\left(f_-' + hA_+^T\eta A_+\right) - \mu^2 A_+^T\eta A_+\Big] + \text{h.c.}
\end{aligned} \tag{IV.32}$$

with the neglect of surface terms. Now make a field transformation, i.e. substitute in (IV.32) the expressions

$$\begin{aligned}
a_-' &= \tilde{a}_- + \frac{1}{\partial^2}\,(2h\,A_+^T\,\eta\,F_+ + h\psi_+^T\,C^{-1}\,\eta\,\psi_+)\;, \\
\zeta_-' &= \tilde{\zeta}_- + \frac{1}{i\partial\!\!\!/}\,(2h\,A_+^T\,\eta\,\psi_+)\;, \\
f_-' &= \tilde{f}_- - h\,A_+^T\,\eta\,A_+\;.
\end{aligned} \tag{IV.33}$$

The resulting Lagrangian,

$$\left[\partial a_-^* \, \partial \tilde{a}_- + \bar{\zeta}_- i \not\partial \, \tilde{\zeta} + (f_-^* + \frac{\mu^2}{h})\tilde{f}_- - \mu^2 A_+^T \eta A_+\right] + \text{h.c.}$$

describes a pair of non-interacting superfields S_- and $\tilde{S}_-$, together with a scalar mass term,

$$-\mu^2 A_+^T \eta A_+ + \text{h.c.} \tag{IV.34}$$

which is relevant to the original system. Again this is, in effect, an explicit supersymmetry breaker although it has a symmetry respecting origin. Just as in (IV.30), one can write

$$\mu^2 = -h \langle f_-^* \rangle \ , \tag{IV.35}$$

and this implies that μ^2 must diverge logarithmically as in the Type I case. (If the physical system contains a singlet Φ_- then the parameter $\mathscr{A}$ - in the term $\mathscr{A} F_-$ - must develop a logarithmic divergence due to contributions from (IV.34).)

(C) MODELS

To illustrate some general features we list three example. The first two are parity-conserving models in which an internal symmetry is spontaneously broken but supersymmetry is preserved. The third is a very simple model with a discrete symmetry wherein the supersymmetry is spontaneously broken and a Goldstone fermion appears.

C.1 Local U(1)

This parity conserving model involves, in addition to the gauge potential Ψ, a neutral singlet S_+ and a pair of charged fields Φ_+ and Φ_-. The Lagrangian is given by

$$\mathcal{L} = \frac{1}{8}(\bar{D}D)^2\left[|\Phi_+|^2 e^{2g\Psi} + |\Phi_-|^2 e^{-2g\Psi} + |S_+|^2\right]$$
$$- \frac{1}{2}\bar{D}D\left[-\frac{1}{4}\bar{\Psi}_-\Psi_+ + \frac{\xi}{8}\bar{D}D\Psi + \sqrt{2}g\,\Phi_-^* S_+ \Phi_+ + \text{h.c.}\right] \ ,$$

which reduces in the Wess-Zumino gauges to

$$\mathcal{L} = -\frac{1}{4}U_{\mu\nu}^2 + \bar{\chi}i\not\partial\chi + \frac{1}{2}(\partial_\mu a)^2 + \frac{1}{2}(\partial_\mu b)^2$$
$$+ |\nabla_\mu A_+|^2 + |\nabla_\mu A_-|^2 + \bar{\psi}i\not\nabla\psi$$
$$- g\bar{\psi}(a-\gamma_5 b)\psi - g\sqrt{2}\left[\bar{\psi}(\chi A_+ - i\gamma_5\chi^c A_-) + \text{h.c.}\right] - V \ , \tag{IV.36}$$

where the potential is given by

$$V = |F_+|^2 + |F_-|^2 + |f_+|^2 + \frac{1}{2}D^2 \ , \tag{IV.37}$$

with

$$F_\pm = -g(a \mp ib)A_\mp \ ,$$
$$f_+ = -g\sqrt{2}A_-A_+^* \ ,$$
$$D = \xi - g(|A_+|^2 - |A_-|^2) \ . \tag{IV.38}$$

On substitution of these expressions into (IV.37) the potential reduces to

$$V = g^2(a^2+b^2)\left(|A_+|^2 + |A_-|^2\right) + 2g^2|A_+|^2|A_-|^2 + \frac{1}{2}g^2\left(\frac{\xi}{g} - |A_+|^2 + |A_-|^2\right)^2 . \tag{IV.39}$$

In (IV.36) the covariant derivatives take the form appropriate to an abelian symmetry, viz.

$$U_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu \ , \quad \nabla_\mu\psi = (\partial_\mu - igU_\mu)\psi \ , \quad \nabla_\mu A_\pm = (\partial_\mu - igU_\mu)A_\pm \ . \tag{IV.40}$$

Minimization of the classical potential (IV.37) is trivial. Indeed one can see that all the auxiliary fields (IV.38) vanish at the point

$$\langle a \rangle = \langle b \rangle = \langle A_- \rangle = 0 \quad , \langle A_+ \rangle = \sqrt{\xi/g} \quad , \qquad \text{(IV.41)}$$

provided $\xi/g > 0$. This point is non-degenerate and, since $\langle V \rangle = 0$, it is clearly an absolute minimum. The vacuum respects supersymmetry as well as parity and fermion-number. The local symmetry is broken, however.

One can show that all particles in this model have a common mass. The free Lagrangian is easily obtained by substituting into (IV.36) & (IV.39) the shifted field

$$A_+ = \sqrt{\frac{\xi}{g}} + \sqrt{\frac{1}{2}}(A_1 + iA_2) \qquad \text{(IV.42)}$$

and retaining only bilinear terms. One finds

$$\begin{aligned}\mathcal{L}_{(2)} = &- \frac{1}{4} U_{\mu\nu}^2 + \frac{1}{2} M^2\left[U_\mu - \frac{1}{M}\partial_\mu A_2\right]^2 \\ &+ \frac{1}{2}(\partial_\mu a)^2 + \frac{1}{2}(\partial_\mu b)^2 + \frac{1}{2}(\partial_\mu A_1)^2 - \frac{1}{2}M^2(a^2 + b^2 + A_1^2) \\ &+ |\partial_\mu A_-|^2 - M^2|A_-|^2 \\ &+ \bar{\chi} i\partial\!\!\!/ \chi + \bar{\psi} i \partial\!\!\!/ \psi - M(\bar{\psi}\chi + \bar{\chi}\psi) \quad ,\end{aligned} \qquad \text{(IV.43)}$$

where the common mass is given by

$$M = \sqrt{2g\xi} \quad . \qquad \text{(IV.44)}$$

The scalar field A_2 can be removed by a gauge transformation. There remain two real scalars, A_1 and a ; a real pseudoscalar, b ; a scalar difermion, A_- ; a real vector, U_μ ; two Dirac spinors, $\sqrt{\frac{1}{2}}(\psi + \chi)$ and $\sqrt{\frac{1}{2}}\gamma_5(\psi - \chi)$, of opposite parity.

Interactions are characterized by a single dimensionless coupling constant g . The model is presumably renormalizable but we have not examined the quantum corrections.

Although this model can have no broader interest we have included it here as the simplest representative of a supersymmetric system in which both parity <u>and</u> fermion-number are conserved. In fact, it describes the renormalizable interactions of a single multiplet of the complex supersymmetry discussed in Section I.

C.2 $U(3)_{\text{Local}} \times SU(3)_{\text{global}}$

A model of the same type but with more structure is constructed from the gauge potentials Ψ^0 , Ψ^k $(k = 1,2,\dots,8)$, their supplementary pieces S_+^0 , S_+^k and a pair of matter fields Φ_+ and Φ_- in the representation $(3,\bar{3})$. The parity-conserving Lagrangian is

$$\begin{aligned}\mathcal{L} = \frac{1}{8}(\bar{D}D)^2\Big[&\mathrm{Tr}\left\{\Phi_+^\dagger \exp(2g_0\Psi^0 + 2g\Psi)\Phi_+ + \Phi_-^\dagger \exp(-2g_0\Psi^0 - 2g\Psi)\Phi_-\right\} \\ &+ |S_+^0|^2 + \frac{1}{2}\mathrm{Tr}\left\{S_+^\dagger \exp(2g\Psi)\, S_+ \exp(-2g\Psi)\right\}\Big] \\ - \frac{1}{2}\bar{D}D\Big[&- \frac{1}{4}\overline{\Psi^0_-}\Psi^0_+ - \frac{1}{4}\overline{\Psi^k_-}\Psi^k_+ + \frac{\xi}{8}\bar{D}D\Psi^0 \\ &+ \sqrt{2}\,\mathrm{Tr}\,\Phi_-^\dagger(g_0 S_+^0 + g\, S_+)\Phi_+ + \text{h.c.}\Big] \quad ,\end{aligned} \qquad \text{(IV.45)}$$

where $\Psi = \Psi^k\lambda^k$ and $S_+ = S_+^k\lambda^k$ are traceless 3×3 matrices. In this case one can show that the potential is minimized at a point which respects both supersymmetry and global SU(3), <u>viz.</u>

$$\begin{aligned}&\langle A_- \rangle = 0 \quad , \quad \langle A_+ \rangle = \sqrt{-\xi/3g_0} \times \text{unit matrix} \quad , \\ &\langle S_+^0 \rangle = \langle S_+ \rangle = 0 \quad .\end{aligned} \qquad \text{(IV.46)}$$

There are no massless particles. It turns out that the system comprises two multiplets of the complex supersymmetry, an SU(3) singlet with mass $\sqrt{-2g_0\xi}$ and an octet with mass $\sqrt{-4g^2\xi/3g_0}$.

C.3 <u>O'Raigeartaigh's model</u>

This is the simplest model in which spontaneous breakdown is realized. It contains three chiral superfields, Φ_{0-} , Φ_{1+} and Φ_{2-} , and admits a discrete symmetry

$$\Phi_{0-} \to \Phi_{0-} \quad , \quad \Phi_{1+} \to -\Phi_{1+} \quad , \quad \Phi_{2-} \to -\Phi_{2-} \quad . \qquad \text{(IV.47)}$$

The Lagrangian is

$$\mathcal{L} = \frac{1}{8}(\bar{D}D)^2\left[|\Phi_{0-}|^2 + |\Phi_{1+}|^2 + |\Phi_{2-}|^2\right] - \frac{1}{2}\bar{D}D\left[\Phi^*_{0-}\left(\lambda + \frac{h}{2}\Phi^2_{1+}\right) + m\,\Phi^*_{2-}\Phi_{1+} + \text{h.c.}\right] \qquad \text{(IV.48)}$$

and the potential is given by

$$V = |F_{0-}|^2 + |F_{1+}|^2 + |F_{2-}|^2 \quad , \qquad \text{(IV.49)}$$

where the auxiliary fields are

$$-F_{0-} = \lambda + \frac{h}{2}A^2_{1+} \quad , \quad -F_{1+} = mA_{2-} + hA^*_{1+}A_{0-} \quad , \quad -F_{2-} = mA_{1+} \quad .$$

Of the three extremal conditions $\partial V/\partial A^*_{0-} = 0$, etc., only two are independent:

$$0 = mA_{2-} + hA^*_{1+}A_{0-} \quad ,$$

$$0 = \left(m^2 + \frac{h^2}{2}|A_{1+}|^2\right)A_{1+} + h\lambda A^*_{1+} \quad .$$

There are two solutions, but one is deeper, viz.

$$\langle A_{1+}\rangle = \left[(-2/h^2)(m^2 + h\lambda)\right]^{1/2} \quad .$$

Since only the ratio A_{2-}/A_{0-} is fixed here the minimum is degenerate. The system contains a massless di-lepton in the zeroth order. To compute $\langle A_{0-}\rangle$ and $\langle A_{2-}\rangle$ one must take account of quantum effects.*) Notice

$$\langle F_{0-}\rangle = \frac{m^2}{h} \quad , \quad \langle F_{1+}\rangle = 0 \quad , \quad \langle F_{2-}\rangle = -m\left[(-2/h^2)(m^2 + h\lambda)\right]^{1/2} \quad ,$$

which certifies the supersymmetry breakdown.

*) There is a standard technique for obtaining the one-loop contribution to the effective potential. Make the c-number shifts

$$A_{0-} \to a_{0-} + A_{0-} \quad , \quad A_{1+} \to a_{1+} + A_{1+} \quad , \quad A_{2-} \to a_{2-} + A_{2-}$$

and pick out the terms which are bilinear in the q-numbers. This shifted free Lagrangian determines a set of masses which are functions of a_{0-} , etc. The one-loop contribution is then given by

(D) UNIFICATION OF WEAK AND ELECTROMAGNETIC INTERACTIONS IN A SUPERSYMMETRIC APPROACH

To conclude this discussion, we turn to the problem of incorporating the weak and electromagnetic interactions of leptons and hadrons in a supersymmetric framework. As yet there exists no satisfactory resolution of this problem and the following paragraphs are intended to point up the problems as they appear to us now.

In the absence of any more compelling alternative, we shall begin with a minimal extension of the $SU(2) \times U(1)$ model of weak and electromagnetic interactions where left-handed (matter) leptons and quarks are classified into doublets and the right-handed are (matter) singlets. The supersymmetric extension involves introducing doublets and singlets of scalars to go with the fermions and a (gauge) triplet plus singlet of fermions to go with the gauge vectors of $SU(2) \times U(1)$. This much is the bare minimum but in practice many more fields are needed.

The new left-handed gauge fermions ($SU(2) \times U(1)$ triplet and singlet) in the supersymmetric approach can have nothing to do with the matter fermions (leptons and quarks) which are supposed to be doublets. They must therefore be very heavy (or, in some other way effectively decoupled from present-day physics). They must therefore be given right-handed partners from among new matter fields. These new matter fields may then be thought of as part of an "extended gauge system". No members of the extended gauge system – apart from the photon – have yet been seen; presumably they are all very heavy.

$$V^{(1)}(a) = \frac{\hbar}{64\pi^2}\sum_{j=0,1/2,1}(-)^{2j}M^4(a)\,\ln M^2(a) + \text{counter-terms.} \qquad \text{(IV.50)}$$

On restricting the variables a_{0-} , a_{1+} and a_{2-} to the subspace on which the classical potential is minimized (keeping only a_{0-} , say, as an independent variable) the counter-terms, like $V^{(0)}$, are constants. One then minimizes $V^{(1)}$ with regard to a_{0-} . It is found that this minimizes at $a_{0-} = a_{2-} = 0$. Lepton-number is conserved. One finds, in addition to the Goldstone lepton (left-handed), a lepton and a di-lepton of mass $M = \sqrt{-(m^2+2\lambda h)}$, a pair of ordinary bosons of mass $\sqrt{M^2 - m^2}$ and $\sqrt{M^2 + m^2}$. Finally, the pseudo-Goldstone di-lepton is found to have the (mass)^2

$$\frac{\hbar}{16\pi^2}\,h^2M^2\left[\left(1 + \frac{m^2}{M^2}\right)^2\ln\left(1 + \frac{m^2}{M^2}\right) - \left(1 - \frac{m^2}{M^2}\right)^2\ln\left(1 - \frac{m^2}{M^2}\right) - \frac{2m^2}{M^2}\right]$$

We shall continue to assume the existence of the fermionic quantum number $\mathbb{F}$ associated with the chiral components of the supertranslations. This number will not be identified with any of the known ones such as electron- or muon-number. If these numbers are all separately conserved, as we shall assume, then there will be no mixing between the gauge fermions and the matter fermions. This is an important requirement if the universality of the weak interaction (equality of weak $(e\nu_e)$ and $(\mu\nu_\mu)$ couplings) is to be maintained. (It follows also that the Goldstone fermion, arising through spontaneous breakdown of supersymmetry, will not be identified with either of the known neutrinos.)

To ensure the universal strength of the charged weak current one assigns left-handed fermions or right-handed antifermions to doublets of $SU(2)_I \times U(1)_Y$, e.g.

$$\begin{pmatrix} \nu \\ e^- \end{pmatrix}_L \quad \text{with} \quad I = 1/2 \quad , \quad Y = -1 \; ,$$

$$\begin{pmatrix} \mu^+ \\ \nu' \end{pmatrix}_R \quad \text{with} \quad I = 1/2 \quad , \quad Y = +1 \; ,$$

while right-handed fermions or left-handed antifermions are singlets,

$$e^-_R \quad \text{with} \quad I = 0 \quad , \quad Y = -2 \; ,$$

$$\mu^+_L \quad \text{with} \quad I = 0 \quad , \quad Y = +2 \; .$$

Since ν_L has no right-handed partner, it must remain massless (at least if electron-number is conserved). Likewise for ν'_R. The quarks are similarly assigned, e.g.

$$\begin{pmatrix} p \\ n \end{pmatrix}_L \quad \text{with} \quad I = 1/2 \quad , \quad Y = 1/3 \; ,$$

$$p_R \qquad I = 0 \quad , \quad Y = 4/3 \; ,$$

$$n_R \qquad I = 0 \quad , \quad Y = -2/3 \; .$$

A major difficulty concerns the scalar partners of these fermions. These scalars tend to be degenerate with the fermions.

Notice firstly that if these scalars carry globally conserved quantum numbers, like electron- or baryon-numbers, this will require their vacuum expectation values to vanish. In turn, this will mean that the masses of these supermultiplets arise entirely from their interactions with

the extended gauge system. This extended gauge system has two parts: the vectors and left-handed fermions contained in guage superfields $\Psi^k(x,\theta)$, and the scalars and right-handed fermions contained in a set of positive chirality scalars Φ_{i+}. It is the scalar components of the latter which will play the role of Higgs fields and generate all masses.

Now there is some latitude in the assignment of the scalars but, unfortunately, all the versions we have examined suffer similar defects. The main defect is the persistence of a mass formula by which the average $(\text{mass})^2$ of fermions in any <u>charged</u> supermultiplet is equal to the average $(\text{mass})^2$ of bosons. This means, for example, that if the electron is assigned to a scalar supermultiplet, then even after the breaking of supersymmetry, its two bosonic partners must have masses $\sim 1/2$ MeV. This rule, for which we have no general proof, persists even when Slavnov's mechanism is used. This will be illustrated in the example which follows.*)

To illustrate the main features of a supersymmetric version of weak and electromagnetic interactions, we present a model based on the local symmetry $SU(2)_I \times U(1)_{Y_1} \times U(1)_{Y_2}$ wherein the electric charge is identified by

$$Q = I_3 + \frac{Y_1}{2} + \frac{Y_2}{2} \; . \qquad \text{(IV.51)}$$

The other four symmetries must be broken spontaneously. The gauge superfields $\underset{\sim}{\Psi}$, Ψ_{01} and Ψ_{02} include three electrically neutral and two charged ones comprising the photon, two neutral vectors, the charged vectors and the accompanying left-handed fermions. Right-handed fermions are contained in a pair of positive chirality doublets Φ_{1+} and Φ_{2+} with the (I,Y_1,Y_2)

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*) An alternative one might consider is the use of vector supermultiplets. These contain two fermions along with a scalar and a vector. One fermion could be light and the other heavy like the bosons. However, such an approach brings its own difficulties. Firstly, if the theory is to be renormalizable then the vector will have to belong to a gauge system and one will not be able to make do with $SU(2) \times U(1)$. Rather, one would have to associate an independent symmetry generator with each lepton and quark. This very large symmetry would then have to break in such a way as to yield one relatively light charged vector $W^{\pm}$ and perhaps a neutral Z^0. The leptons and quarks would belong to the adjoint representation and it is hard to imagine how they could all resemble doublets of some relatively weakly broken SU(2) subgroup. The problem of setting up a universal weak interaction would seem to be insurmountable in such a framework.

assignments

$$\Phi_{1+} \sim (1/2, -1, 0) ,$$
$$\Phi_{2+} \sim (1/2, 0, 1) . \qquad (IV.52)$$

These fields, together with the gauge fields $\underset{\sim}{\Psi}$, Ψ_{01} , Ψ_{02} make up what we designated as the extended gauge system.

Matter leptons will take one of two possible forms, either left-handed doublet paired with right-handed singlet,

$$\Phi_{3-} \sim (1/2, 0, -1) ,$$
$$S_{3+} \sim (0, 0, -2) , \qquad (IV.53)$$

or right-handed doublet paired with left-handed singlet

$$\Phi_{4+} \sim (1/2, 1, 0) ,$$
$$S_{4-} \sim (0, 1, 1) . \qquad (IV.54)$$

Fractionally charged quarks (to which, for simplicity, we shall confine our considerations) are assigned in a similar way except that they carry fractional hypercharges and each doublet is paired with two singlets instead of one so that all quark states are massive. (See Eq.(IV.91) for the assignments of the second singlet.)

The matter interactions are severely limited with these assignments. Only two types of coupling can be made,

$$-\frac{1}{2}\,\bar{D}D(f\Phi^{\dagger}_{3-}\, S_{3+}\, \Phi_{2+} + h\, S^{*}_{4-}\, \Phi^{T}_{4-}\, i\tau_2\, \Phi_{2+} + \text{h.c.}) \qquad (IV.55)$$

Notice the global symmetries,

$$\Phi_{3-} \rightarrow e^{i\alpha}\, \Phi_{3-} \quad , \quad S_{3+} \rightarrow e^{i\alpha}\, S_{3+} ,$$
$$\Phi_{4+} \rightarrow e^{i\beta}\, \Phi_{4+} \quad , \quad S_{4-} \rightarrow e^{i\beta}\, S_{4+} , \qquad (IV.56)$$
$$\Phi_{1,2+} \rightarrow \Phi_{1,2+} ,$$

which may be interpreted as electron and muon number symmetry.

The gauge interactions are given by the kinetic terms,

$$\frac{1}{8}\,(\bar{D}D)^2 \Big[\Phi^{\dagger}_{1+}\, \exp(2g\Psi - 2g_1\Psi_{01})\, \Phi_{1+} + \Phi^{\dagger}_{2+}\, \exp(2g\Psi + 2g_2\Psi_{02})\Phi_{2+}$$
$$+ \Phi^{\dagger}_{3-}\, \exp(-2g\Psi + 2g_2\Psi_{02})\Phi_{3-} + S^{*}_{3+}\, \exp(-4g_2\Psi_{02})\, S_{3+}$$
$$+ \Phi^{\dagger}_{4+}\, \exp(2g\Psi + 2g_1\Psi_{01})\Phi_{4+} + S^{*}_{4-}\, \exp(-2g_1\Psi_{01} - 2g_2\Psi_{02})\, S_{4-}\Big] \qquad (IV.57)$$

plus the supersymmetric analogue of the Yang-Mills Lagrangian (III.38). With two commutative U(1) groups in the local symmetry we can have two dimensional parameters, ξ_1 and ξ_2, in the term

$$\frac{1}{8}\,(\bar{D}D)^2\, (\xi_1\Psi_{01} + \xi_2\Psi_{02}) . \qquad (IV.58)$$

The scalar potential is the standard one,

$$V = F^{\dagger}_{1+}F_{1+} + F^{\dagger}_{2+}F_{2+} + F^{\dagger}_{3-}F_{3-} + |f_{3+}|^2 + F^{\dagger}_{4+}F_{4+} + |f_{4-}|^2 +$$
$$+ \frac{1}{2}\,\underset{\sim}{D}^2 + \frac{1}{2}\,D^2_{01} + \frac{1}{2}\,D^2_{02} , \qquad (IV.59)$$

where the auxiliary fields are given by

$$-F_{1+} = 0 , \qquad -f_{3+} = f\, A^{\dagger}_{2+}\, A_{3-} ,$$
$$-F_{2+} = fa^{*}_{3+}\, A_{3-} - ha_{4-}\, i\tau_2\, A^{*}_{4+} , \qquad -F_{4+} = ha_{4-}\, i\tau_2\, A^{*}_{2+} ,$$
$$-F_{3-} = fa_{3+}\, A_{2+} , \qquad -f_{4-} = h\, A^{T}_{4+}\, i\tau_2\, A_{2+} ,$$
$$-\frac{1}{g}\,\underset{\sim}{D} = A^{\dagger}_{1+}\,\underset{\sim}{I}\, A_{1+} + A^{\dagger}_{2+}\,\underset{\sim}{I}\, A_{2+} - A^{\dagger}_{3-}\,\underset{\sim}{I}\, A_{3-} + A^{\dagger}_{4+}\,\underset{\sim}{I}\, A_{4+} ,$$
$$-\frac{1}{g_1}\, D_{01} = -\frac{\xi_1}{g_1} - A^{\dagger}_{1+}\, A_{1+} + A^{\dagger}_{4+}\, A_{4+} - |a_{4-}|^2 ,$$
$$-\frac{1}{g_2}\, D_{02} = -\frac{\xi_2}{g_2} + A^{\dagger}_{2+}\, A_{2+} + A^{\dagger}_{3-}\, A_{3-} - 2|a_{3+}|^2 - |a_{4-}|^2 , \qquad (IV.60)$$

where the notation for component fields is exactly as in previous sections, viz.

$$\Phi_{\pm} = \exp(\mp\frac{1}{4}\,\bar{\theta}\not{\partial}\gamma_5\theta)(A_{\pm} + \bar{\theta}_{\mp}\psi_{\pm} + \frac{1}{2}\,\bar{\theta}_{\mp}\theta_{\pm}F_{\pm}) ,$$
$$S_{\pm} = \exp(\mp\frac{1}{4}\,\bar{\theta}\not{\partial}\gamma_5\theta)(a_{\pm} + \bar{\theta}_{\mp}\zeta_{\pm} + \frac{1}{2}\,\bar{\theta}_{\mp}\theta_{\pm}f_{\pm}) ,$$
$$\Psi = \frac{1}{4}\,\bar{\theta}i\gamma_{\nu}\gamma_5\theta\, V_{\nu} + \frac{1}{2\sqrt{2}}\,\bar{\theta}\theta\bar{\theta}\gamma_5\lambda + \frac{1}{16}\,(\bar{\theta}\theta)^2 D .$$

If the various quantum numbers, fermion, electron, muon etc., and electric charge are all conserved then, in the vacuum only two fields can develop an expectation value,

$$\langle A^0_{1+}\rangle = v_1 \quad \text{and} \quad \langle A^0_{2+}\rangle = v_2 \;, \tag{IV.61}$$

where the values v_1 and v_2 are fixed by minimizing the potential (IV.59). They can be expressed in terms of parameters in the Lagrangian by solving the equations

$$g_1^2\left(-\frac{\xi_1}{g_1} - v_1^2\right) = g_2^2\left(-\frac{\xi_2}{g_2} + v_2^2\right) = g^2(v_1^2 - v_2^2) \;. \tag{IV.62}$$

On expanding V at the minimum, one finds the following mass terms:

$$\begin{aligned} V \approx\ & 2g^2|v_2 A_1^{-*} + v_1 A_2^+|^2 \\ & + \left(f^2 v_2^2 - 2g^2(v_1^2 - v_2^2)\right)|a_3^-|^2 + \left(f^2 v_2^2 + 2g^2(v_1^2 - v_2^2)\right)|A_3^-|^2 \\ & + \left(h^2 v_2^2 - 2g^2(v_1^2 - v_2^2)\right)|a_4^+|^2 + \left(h^2 v_2^2 + 2g^2(v_1^2 - v_2^2)\right)|A_4^+|^2 \\ & + 2(g^2 + g_1^2)v_1^2(\mathrm{Re}\ A_1^0)^2 - 4g^2 v_1 v_2(\mathrm{Re}\ A_1^0)(\mathrm{Re}\ A_2^0) + 2(g^2 + g_2^2)v_2^2(\mathrm{Re}\ A_2^0)^2 \;, \end{aligned} \tag{IV.63}$$

(where superscripts refer to electric charge). All non-Goldstone components, except for A^0_{3-} and A^0_{4+}, have mass and these masses are real if

$$f^2v_2^2 > 2g^2|v_1^2 - v_2^2| \quad \text{and} \quad h^2v_2^2 > 2g^2|v_1^2 - v_2^2| \;. \tag{IV.64}$$

The pseudo-Goldstone states A^0_{3-} and A^0_{4+} could receive masses from the radiative corrections. They are without mass in the tree approximation because of their association in supermultiplets with the neutrinos ψ^0_{3L}, ψ^0_{4R}. Couplings to the extended gauge system provide only a modestly effective means for breaking the supersymmetry: charged multiplets are split but neutral ones are not. Fortunately, it is not necessary to rely on radiative effects for lifting this degeneracy. Instead we can exploit the Slavnov Type I mechanism. The necessary global U(1) symmetries are simply, electron-number, muon-number, etc., and so we can immediately adjoin to (IV.63) supplementary mass terms of the type (IV.29),

$$\mu_1^2\left(|A_3^0|^2 + |A_3^-|^2 - |a_3^-|^2\right) + \mu_2^2\left(|A_4^0|^2 + |A_4^+|^2 - |a_4^+|^2\right) . \tag{IV.65}$$

For the positivity of a_3^- and a_4^+ masses we must have

$$\begin{aligned} \mu_1^2 &< f^2v_2^2 - 2g^2(v_1^2 - v_2^2) \;, \\ \mu_2^2 &< h^2v_2^2 - 2g^2(v_1^2 - v_2^2) \;. \end{aligned} \tag{IV.66}$$

Finally the masses of the two real neutral components $\mathrm{Re}\ A_1^0$ and $\mathrm{Re}\ A_2^0$ are determined by the matrix

$$\begin{pmatrix} 2(g^2 + g_1^2)v_1^2 & -2g^2 v_1 v_2 \\ -2g^2 v_1 v_2 & 2(g^2 + g_2^2)v_2^2 \end{pmatrix} \tag{IV.67}$$

whose eigenvalues are certainly positive. Thus, subject to the above inequalities, all scalar masses are real and the minimum is established. In particular, the assumed conservation of charge and the various fermionic quantum numbers is justified. Consider now the spinor and vector masses.

The spinor mass terms are contained in the various Yukawa couplings of A_1 and A_2. They are

$$\begin{aligned} & -f\, v_2\, \overline{\psi_3^-}\, \zeta_3^- \quad - h v_2\, \overline{\zeta_{4-}^+}\, \psi_{4-}^+ \;, \\ & + i2g\, v_1\, \overline{\lambda_-^-}\, \psi_{1+}^- + i2g\, v_2\, \overline{\lambda_-^+}\, \psi_{2+}^+ \;, \\ & + i\sqrt{2}\, v_1(g\overline{\lambda}_{3-} - g_1\, \overline{\lambda_{1-}^0})\psi_{1+}^0 - i\sqrt{2}\, v_2(g\,\overline{\lambda}_{3-} - g_2\, \overline{\lambda_{2-}^0})\psi_{2+}^0 + \text{h.c.} \end{aligned} \tag{IV.68}$$

The masses of the charged particles can be read immediately from this. For definiteness, suppose that the electron is contained in Φ_3, S_3, while the muon is in Φ_4, S_4, i.e. that e^- belongs to a left-handed doublet while μ^+ belongs to a right-handed one. (It would be equally possible if μ^- were assigned to a left-handed doublet. There is no problem of electron muon mixing if none is introduced in the bare Lagrangian.) In addition to the two charged leptons

$$\begin{aligned} e^- &= \psi_{3-}^- + \zeta_{3+}^- \;, & m(e^-) &= f v_2 \;, \\ \mu^+ &= \zeta_{4-}^+ + \psi_{4+}^+ \;, & m(\mu^+) &= h v_2 \;, \end{aligned} \tag{IV.69}$$

there are two charged fermions in the gauge system,

$$\begin{aligned} \Omega^+ &= \psi_{2+}^+ + i\lambda_-^+ \;, & m(\Omega^+) &= 2gv_2 \;, \\ \Omega^- &= \psi_{1+}^- + i\lambda_-^- \;, & m(\Omega^-) &= 2gv_1 \;. \end{aligned} \tag{IV.70}$$

Associated with the charged leptons (IV.69) are the two neutrinos,

$$\nu_L = \psi_{3-}^0 \quad \text{and} \quad \nu_R' = \psi_{4+}^0 \;, \tag{IV.71}$$

which are without mass. Finally there is a set of mass terms for the neutral fermions in the extended gauge system. These can be arranged in the form of a rectangular mass matrix,

$$i(\bar{\lambda}_1^0\ \bar{\lambda}_2^0\ \bar{\lambda}_3)_-\ M \begin{pmatrix} \psi_1^0 \\ \psi_2^0 \end{pmatrix}_+ + \text{h.c.}$$

with

$$M = \sqrt{2}\begin{pmatrix} -g_1 v_1 & 0 \\ 0 & g_2 v_2 \\ g v_1 & -g v_2 \end{pmatrix} \quad . \tag{IV.72}$$

The canonical representation of this matrix would take the form

$$M = U^{-1}\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \\ 0 & 0 \end{pmatrix} V \quad , \tag{IV.73}$$

where U and V are orthogonal matrices (3×3 and 2×2 respectively) and the "eigenvalues" m_1 and m_2 are real and positive. These eigenvalues together with the angles in U and V are obtained in a straightforward fashion by diagonalizing the symmetric matrices MM^T and M^TM. (It is worth remarking that M^TM is precisely the scalar mass matrix (IV.67) while, as will be seen, MM^T is the vector mass matrix. This elegant relationship is a manifestation of the underlying supersymmetry: the massive scalars, spinors and vectors belong to a pair of vector supermultiplets). There are four angles involved in the representation (IV.73), three in U and one in V. They can all be expressed in terms of the three independent ratios, g_1/g, g_2/g and v_1/v_2. Let us write

$$U = \begin{pmatrix} C_3 & -S_3 & 0 \\ S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} C_2 & 0 & -S_2 \\ 0 & 1 & 1 \\ S_2 & 0 & C_2 \end{pmatrix}\begin{pmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ,$$

$$V = \begin{pmatrix} C_4 & -S_4 \\ S_4 & C_4 \end{pmatrix} \quad . \tag{IV.74}$$

where $C_1 = \cos\theta_1$, $S_1 = \sin\theta_1$, etc. These angles can be used to define two massive neutral spinors:

$$\Omega_1^0 = (C_4\psi_1^0 - S_4\psi_2^0)_+ + i\Big((C_3C_2C_1 - S_3S_1)\lambda_1^0 - (C_3C_2S_1 + S_3C_1)\lambda_2^0 - C_3S_2\lambda_3\Big)_-$$

$$\Omega_2^0 = (S_4\psi_1^0 + C_4\psi_2^0)_+ + i\Big((S_3C_2C_1 + C_3S_1)\lambda_1^0 - (S_3C_2S_1 - C_3C_1)\lambda_2^0 - S_3S_2\lambda_3\Big)_- \tag{IV.75}$$

and one massless left-handed spinor:

$$\nu_L^\gamma = (S_2C_1\lambda_1^0 - S_2S_1\lambda_2^0 + C_2\lambda_3)_- \quad . \tag{IV.76}$$

The coefficients in (IV.76) can be expressed entirely in terms of coupling constant ratios,

$$\frac{1}{g_1} = \frac{1}{e}C_1S_2 \quad , \quad \frac{1}{g_2} = -\frac{1}{e}S_1S_2 \quad , \quad \frac{1}{g} = \frac{1}{e}C_2 \quad , \tag{IV.77}$$

where e denotes the electromagnetic combination,

$$\frac{1}{e^2} = \frac{1}{g_1^2} + \frac{1}{g_2^2} + \frac{1}{g^2} \quad . \tag{IV.78}$$

The other angles and the masses m_1, m_2 are given by straightforward but rather less interesting formulae. We pass on now to the vector masses.

The vector mass terms are obtained in the usual way from the Higgs fields A_{1+} and A_{2+}. They are

$$2g^2(v_1^2 + v_2^2)|W^+|^2 + v_1^2(gW_3 - g_1V_1^0)^2 + v_2^2(gW_3 - g_2V_2^0)^2 \quad , \tag{IV.79}$$

where (W^+, W_3, W^-) denote the gauge vectors of $SU(2)_I$, and V_1^0, V_2^0 those of $U(1)_{Y_1} \times U(1)_{Y_2}$. The neutral terms can be arranged in the form

$$(V_1^0\ V_2^0\ W_3)\begin{pmatrix} g_1^2v_1^2 & 0 & -gg_1v_1^2 \\ 0 & g_2^2v_2^2 & -gg_2v_2^2 \\ -gg_1v_1^2 & -gg_2v_2^2 & g^2(v_1^2 + v_2^2) \end{pmatrix}\begin{pmatrix} V_1^0 \\ V_2^0 \\ W_3 \end{pmatrix}$$

$$= \frac{1}{2}\ (V_1^0\ V_2^0\ W_3)\ MM^T \begin{pmatrix} V_1^0 \\ V_2^0 \\ W_3 \end{pmatrix} =$$

$$= \frac{1}{2}\,(V_1^0\ V_2^0\ W_3)\quad U^{-1}\begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} U \begin{pmatrix} V_1^0 \\ V_2^0 \\ W_3 \end{pmatrix} \tag{IV.80}$$

with M given by (IV.72) and (IV.73). The two massive vectors and the photon are, therefore,

$$\begin{aligned} Z_1 &= (C_3C_2C_1 - S_3S_1)V_1^0 - (C_3C_2S_1 + S_3C_1)V_2^0 - C_3S_2W_3 \ , \\ Z_2 &= (S_3C_2C_1 + C_3S_1)V_1^0 - (S_3C_2S_1 - C_3C_1)V_2^0 - S_3S_2W_3 \ , \\ A &= S_2C_1V_1^0 - S_2S_1V_2^0 + C_2W_3 \ . \end{aligned} \tag{IV.81}$$

In summary, the system comprises the following states:

(1) A charged vector supermultiplet

$$\begin{aligned} &\text{spin } \tfrac{1}{2} \quad \Omega^+ = \psi_2^+ + i\lambda_-^+ \ , && m(\Omega^+) = 2gv_2 \ , \\ &\text{spin } 1 \quad W_\mu^+ = \frac{1}{\sqrt{2}}(W_1 - iW_2)_\mu \ , && m(W^+) = \sqrt{2g^2(v_1^2 + v_2^2)} \ , \\ &\text{spin } 0 \quad \phi^+ = \frac{v_2A_{1+}^{-*} + v_1A_{2+}^+}{\sqrt{v_1^2 + v_2^2}} \ , && m(\phi^+) = \sqrt{2g^2(v_1^2 + v_2^2)} \ , \\ &\text{spin } \tfrac{1}{2} \quad \overline{\Omega}^- = \overline{\psi}_{1+}^- - i\overline{\lambda}_-^- \ , && m(\Omega^-) = 2gv_1 \ , \end{aligned} \tag{IV.82}$$

and its conjugate.

(2) Two neutral vector supermultiplets

$$\begin{aligned} &\text{spin } \tfrac{1}{2} \quad \Omega_1^0 = (C_4\psi_1^0 - S_4\psi_2^0)_+ + i\Big((C_3C_2C_1 - S_3S_1)\lambda_1^0 - (C_3C_2S_1 + S_3C_1)\lambda_2^0 - C_3S_2\lambda_3\Big)_- , \\ &\text{spin } 1 \quad Z_1^0 = (C_3C_2C_1 - S_3S_1)V_1^0 - (C_3C_2S_1 + S_3C_1)V_2^0 - C_3S_2W_3 \ , \\ &\text{spin } 0 \quad \phi_1^0 = C_4\sqrt{2}\,\mathrm{Re}\,A_1^0 - S_4\sqrt{2}\,\mathrm{Re}\,A_2^0 \ , \\ &\text{spin } \tfrac{1}{2} \quad \overline{\Omega}_1^0 = (C_4\overline{\psi}_1^0 - S_4\overline{\psi}_2^0)_+ - i\Big((C_3C_2C_1 - S_3S_1)\overline{\lambda}_1^0 - (C_3C_2S_1 + S_3C_1)\overline{\lambda}_2^0 - C_3S_2\overline{\lambda}_3\Big)_- \end{aligned} \tag{IV.83}$$

$$\begin{aligned} &\text{spin } \tfrac{1}{2} \quad \Omega_2^0 = (S_4\psi_1^0 + C_4\psi_2^0)_+ + i\Big((S_3C_2C_1 + C_3S_1)\lambda_1^0 - (S_3C_2S_1 - C_3C_1)\lambda_2^0 - S_3S_2\lambda_3\Big)_- \ , \\ &\text{spin } 1 \quad Z_2^0 = (S_3C_2C_1 + C_3S_1)V_1^0 - (S_3C_2S_1 - C_3C_1)V_2^0 - S_3S_2W_3 \ , \\ &\text{spin } 0 \quad \phi_2^0 = S_4\sqrt{2}\,\mathrm{Re}\,A_1^0 + C_4\sqrt{2}\,\mathrm{Re}\,A_2^0 \ , \\ &\text{spin } \tfrac{1}{2} \quad \overline{\Omega}_2^0 = (S_4\overline{\psi}_1^0 + C_4\overline{\psi}_2^0)_+ - i\Big((S_3C_2C_1 + C_3S_1)\overline{\lambda}_1^0 - (S_3C_2S_1 - C_3C_1)\overline{\lambda}_2^0 - S_3S_2\overline{\lambda}_3\Big)_- \ . \end{aligned} \tag{IV.84}$$

These multiplets are unbroken in zeroth order. Their respective masses are given by

$$m_{1,2}^2 = (g^2 + g_1^2)v_1^2 + (g^2 + g_2^2)v_2^2 \mp \sqrt{\left((g^2 + g_1^2)^2v_1^2 - (g^2 + g_2^2)v_2^2\right)^2 + 4g^4v_1^2v_2^2} \ . \tag{IV.85}$$

(3) The photon supermultiplet

$$\begin{aligned} &\text{spin } 1 \quad A = S_2C_1V_1^0 - S_2S_1V_2^0 + C_2W_3 \ , \\ &\text{spin } \tfrac{1}{2} \quad \nu_L^\gamma = S_2C_1\lambda_{1-}^0 - S_2S_1\lambda_{2-}^0 + C_2\lambda_{3-} \quad \text{(the Goldstone neutrino).} \end{aligned} \tag{IV.86}$$

The vector supermultiplets (1)-(3) exhaust the gauge system. Leptonic matter is classified in scalar supermultiplets: the Φ_{3-}, S_{3+} and Φ_{4+}, S_{4-} type combinations of which there may be duplicates. For definiteness we consider one of each and label them "electronic" and "muonic" respectively.

(4) The electron supermultiplets, one charged,

$$\begin{aligned} &\text{spin } 0 \quad a_{3+}^- \ , && m(a_3^-) = \sqrt{f^2v_2^2 - 2g^2(v_1^2 - v_2^2) - \mu_1^2} \ , \\ &\text{spin } \tfrac{1}{2} \quad e^- = \zeta_{3+}^- + \psi_{3-}^- \ , && m(e^-) = fv_2 \ , \\ &\text{spin } 0 \quad A_{3-}^- \ , && m(A_3^-) = \sqrt{f^2v_2^2 + 2g(v_1^2 - v_2^2) + \mu_1^2} \ , \\ &(\text{note } m^2(a_3^-) + m^2(A_3^-) = 2m^2(e^-)) && \end{aligned} \tag{IV.87}$$

and one neutral,

$$\begin{aligned} &\text{spin } 0 \quad A_{3-}^0 \ , && m(A_3^0) = \mu_1 \ , \\ &\text{spin } \tfrac{1}{2} \quad \nu_L = \psi_{3-}^0 \ , && m(\nu_L) = 0 \ . \end{aligned} \tag{IV.88}$$

(5) The muon supermultiplets, charged,

spin 0 A^+_{4+} , $m(A^+_4) = \sqrt{h^2v_2^2 + 2g^2(v_1^2 - v_2^2) + \mu_2^2}$,

spin $\frac{1}{2}$ $\mu^+ = \psi^+_{4+} + \zeta^+_{4-}$, $m(\mu^+) = hv_2$,

spin 0 a^+_{4-} , $m(a^+_4) = \sqrt{h^2v_2^2 - 2g^2(v_1^2 - v_2^2) - \mu_2^2}$,

(again $m^2(a^+_4) + m^2(A^+_4) = 2m^2(\mu^+)$)

(IV.89)

and neutral,

spin 0 A^0_{4+} , $m(A^0_4) = \mu_2$,

spin $\frac{1}{2}$ $\nu'_R = \psi^0_{4+}$, $m(\nu'_R) = 0$.

(IV.90)

Finally, since the baryonic supermultiplets should not contain any neutrinos, it is necessary to adjoin extra singlets, S'_{3+} or S'_{4-} . Moreover, if the baryonic multiplets are fractionally charged, then their hypercharge assignments and, consequently, their masses will be different from (3) and (4) above. Consider the (I,Y_1,Y_2) assignments,

$$\begin{array}{ll} \Phi_{3-} \sim (1/2,\ Y,\ Y-1)\ , & \Phi_{4+} \sim (1/2,\ Y+1,\ Y)\ , \\ S_{3+} \sim (0,\ Y,\ Y-2) & S_{4-} \sim (0,\ Y+1,\ Y+1)\ , \\ S'_{3+} \sim (0,\ Y+1,\ Y-1) & S'_{4-} \sim (0,\ Y,\ Y)\ , \end{array} \qquad \text{(IV.91)}$$

under which the following interaction terms are permitted:

$$-\frac{1}{2}\bar{D}D\left[\Phi^\dagger_{3-}(f''\Phi_{2+}S_{3+} + f'\ \Phi_{1+}S'_{3+}) + (h''\ S^*_{4-}\ \Phi^T_{2+} + h'\ S'^*_{4-}\Phi^T_{1+})i\ \tau_2\ \Phi_{4+} + \text{h.c.}\right] .$$

(IV.92)

One finds the following spectrum of baryonic states.

(6) Supermultiplets with charges $Q = Y+1$, Y and masses given by

A^{Y+1}_{4+} $\sqrt{h''^2v_2^2 + (Y+1)2g^2(v_1^2 - v_2^2)}$,

$(\psi_{4+} + \zeta_{4-})^{Y+1}$ $h''v_2$,

a^{Y+1}_{4-} $\sqrt{h''^2v_2^2 - (Y+1)2g^2(v_1^2 - v_2^2)}$,

A^Y_{4+} $\sqrt{h'^2v_1^2 + Y\ 2g^2(v_1^2 - v_2^2)}$,

$(\psi_{4+} + \zeta_{4-})^Y$ $h'v_1$,

a^Y_{4-} $\sqrt{h'^2v_1^2 - Y\ 2g^2(v_1^2 - v_2^2)}$,

(IV.93)

and similar supermultiplets *) with $Q = Y,\ Y-1$,

a^Y_{3+} $\sqrt{f'^2v_1^2 + Y\ 2g^2(v_1^2 - v_2^2)}$,

$(\zeta_{3+} + \psi_{3-})^Y$ $f'v_1$,

A^Y_{3-} $\sqrt{f'^2v_1^2 - Y\ 2g^2(v_1^2 - v_2^2)}$,

a^{Y-1}_{3+} $\sqrt{f''^2v_2^2 + (Y-1)\ 2g^2(v_1^2 - v_2^2)}$,

$(\zeta_{3+} + \psi_{3-})^{Y-1}$ $f''v_2$,

A^{Y-1}_{3-} $\sqrt{f''^2v_2^2 - (Y-1)\ 2g^2(v_1^2 - v_2^2)}$.

(IV.94)

To summarize, what we have shown above is that the demand of universality of couplings can be met, if the extended gauge sector fermions do not mix with matter (lepton and quark) fermions. (One important consequence of this is that gauge fermions exist which are about as massive as W and Z particles.) The chief drawback of the model presented lies in that there also exist in the model (charged) spin zero particles - companions of electrons and muons - with light masses. The hope is that radiative effects may cure this defect of the model.

To obtain an effective Lagrangian useful for phenomenology at relatively low energies (< 5 GeV) it would be reasonable to integrate out all massive components of the extended gauge system, i.e. the supermultiplets (1) and (2) which are very heavy (> 50 GeV). The photonic supermultiplet (3) must of course be retained in the effective Lagrangian. That part of the interaction Lagrangian which is linear in the gauge field defines a set of currents,

*) As is well known, there are no anomalies in the theory, provided that there are two(integer charged) leptons and four quarks among the matter fields with their conventional assignments of Y quantum numbers. (The extended gauge fermions do not produce anomalies.)

$$\sum_{j=1,2} \left(\overline{\Omega_j^0}\, J(\Omega_j^0) + Z_{j\mu}^0 J(Z_{j\mu}^0) + \phi_j^0\, J(\phi_j^0) + J(\overline{\Omega_j^0})\Omega_j^0\right)$$

$$+ \left(\overline{\Omega^-}\, J(\Omega^-) + W_\mu^+\, J(W_\mu^-) + \phi^+ J(\phi^-) + J(\overline{\Omega^+})\Omega^+ + \text{h.c.}\right) .$$

Elimination of these gauge fields yields the effective interaction Lagrangian,

$$\sum_{j=1,2} \left(\frac{1}{m_j} J(\overline{\Omega_j^0})\, J(\Omega_j^0) + \frac{1}{2m_j^2} J(\phi_j^0)^2 - \frac{1}{2m_j^2} J(Z_{j\mu}^0)^2\right)$$

$$+ \frac{1}{m(\Omega^-)}\, J(\overline{\Omega^-})\, J(\Omega^-) - \frac{1}{m(W)^2} J(W_\mu^+)\, J_\mu(W_\mu^-) +$$

$$+ \frac{1}{m(\phi^+)^2} J(\phi^+)\, J(\phi^-) + \frac{1}{m(\Omega^+)} J(\overline{\Omega^+})\, J(\Omega^+) \quad .$$

We shall not pursue this any further in view of the unrealistic character of this model. All of the currents in this effective Lagrangian receive contributions from lepton- or baryon-number carrying scalar fields which cannot be ignored. Their masses must, according to formulae like (IV.87), (IV.89), (IV.93), (IV.94) be of the same order as those of their fermionic partners. (This being so, there is even a long-range pseudo-electromagnetic interaction between scalars and spinors due to the exchange of the photonic neutrino (IV.86).)

APPENDIX A

FEYNMAN RULES

A number of apparently renormalizable Lagrangian models has been given. In this appendix rules are developed for computing terms in the perturbation series. To take full advantage of the supersymmetry, these rules are presented in a manifestly supersymmetric fashion.

The formal derivation of Feynman rules proceeds very much as in any ordinary quantized field theory. Consider firstly the most general renormalizable system made entirely from scalar superfields (i.e. without any local symmetries). The action is given by

$$S(\Phi) = \int d_4x \left\{\frac{1}{8} (\bar{D}D)^2\left(|\Phi_{i+}|^2 + |\Phi_{a-}|^2\right)\right.$$

$$\left. - \frac{1}{2} \bar{D}D\left[\Phi_{a-}^* \{\mathcal{M}_a + M_{ai}\, \Phi_{i+} + g_{aij}\, \Phi_{i+}\, \Phi_{j+}\} + \text{h.c.}\right]\right\} . \tag{A.1}$$

Introduce external sources J_{a+} , J_{i-} and define the connected vacuum amplitude $Z(J)$ by the usual path integral,

$$\exp \frac{i}{\hbar}\, Z(J) = \int (d\Phi\, d\Phi^*) \exp \frac{i}{\hbar}\left[S(\Phi) + \int d_4x \left(- \frac{1}{2} \bar{D}D\right)\{J_-^\dagger\Phi_+ + \Phi_-^\dagger J_+ + \text{h.c.}\}\right] . \tag{A.2}$$

Connected Green's functions are given by the functional derivatives of Z evaluated at $J = 0$. For example,

$$\langle T\, \Phi_+(1)\, \Phi_-^\dagger(2)\rangle_{\text{con.}} = \frac{\hbar}{i} \left.\frac{\delta^2 Z}{\delta J_-^\dagger(1)\, \delta J_+(2)}\right|_{J=0} .$$

For many purposes it is convenient to have a special notation for the functional derivatives of Z at arbitrary values of J . Thus we write

$$\begin{aligned} \frac{\delta Z(J)}{\delta J_-^\dagger(1)} &= \Phi_+(1) \quad , \\ \frac{\delta Z(J)}{\delta J_+(1)} &= \Phi_-^\dagger(1) \quad , \end{aligned} \tag{A.3}$$

defining the field averages as functionals of the external currents. In principle these expressions can be solved to give the external currents as functionals of the field averages to which they give rise. We denote field averages by the same symbols as are used for the field operators since no confusion can possibly arise. Higher derivatives of the connected vacuum amplitude are denoted quite generally as follows:

$$\frac{\delta}{\delta J_-^\dagger(1)}\,\frac{\delta}{\delta J_-^\dagger(2)}\cdots\frac{\delta}{\delta J_+(n)}\,Z(J) = G\Big(\Phi_+(1)\,\Phi_+(2)\cdots\Phi_-^\dagger(n)\Big) \quad . \tag{A.4}$$

As an alternative to the connected vacuum amplitude $Z(J)$ one may choose to work with the effective action $\Gamma(\Phi)$ defined by

$$Z(J) = \Gamma(\Phi) + \int d_4x\,(-\tfrac{1}{2}\,\overline{D}D)\{J_-^\dagger\Phi_+ + \Phi_-^\dagger J_+ + \text{h.c.}\} \ . \tag{A.5}$$

The effective equations of motion then read

$$\frac{\delta\Gamma(\Phi)}{\delta\Phi_-^\dagger(1)} = -\,J_+(1) \quad , \quad \frac{\delta\Gamma(\Phi)}{\delta\Phi_+(1)} = -\,J_-^\dagger(1) \tag{A.6}$$

and they are solved by the expressions (A.3). The effective action is characterized by one-particle irreducible graphs and it may be obtained, in principle, by solving the Dyson-Schwinger equations:

$$\begin{aligned}\frac{\delta\Gamma}{\delta\Phi^*_{a-}(1)} &= \frac{\delta S}{\delta\Phi^*_{a-}(1)} + g_{aij}\,\frac{\hbar}{i}\,G\Big(\Phi_{i+}(1)\,\Phi_{j+}(1)\Big) \quad ,\\ \frac{\delta\Gamma}{\delta\Phi_{i+}(1)} &= \frac{\delta S}{\delta\Phi_{i+}(1)} + 2g_{aij}\,\frac{\hbar}{i}\,G\Big(\Phi^*_{a-}(1)\,\Phi_{j+}(1)\Big) \ .\end{aligned} \tag{A.7}$$

In these equations the fields Φ are the independent variables and one is to solve for the functional $\Gamma(\Phi)$. In order to do this it is necessary to have the 2-point functions $G(1,2)$, which appear in (A.7) expressed as functionals of Φ rather than J. This is done by defining $G(1,2)$ as a functional inverse of the matrix of second derivatives of $\Gamma(\Phi)$. We shall not pursue this matter here.

Although the perturbation series for $\Gamma(\Phi)$ is simpler than that for $Z(J)$ in that it contains only irreducible graphs, the latter is easier to derive rules for. The derivation proceeds as follows.

Firstly, separate the classical action into two pieces,

$$S(\Phi) = S_0(\Phi) + S_g(\Phi) \quad , \tag{A.8}$$

where S_g contains all the trilinear terms (i.e. S_0 is obtained from (A.1) by setting $g_{aij} = 0$). Then the connected vacuum amplitude is given by

$$\exp\frac{i}{\hbar}\,Z(J) = \exp\frac{i}{\hbar}\,S_g\Big\{\frac{\hbar}{i}\,\frac{\delta}{\delta J}\Big\}\,\exp\frac{i}{\hbar}\,Z_0(J) \quad , \tag{A.9}$$

where Z_0 denotes the amplitude corresponding to S_0. It is easily evaluated.

To evaluate $Z_0(J)$, notice firstly that the corresponding effective action, Γ_0, is just $S_0(\Phi)$. This is implied by Eqs. (A.7). Turn next to Eqs.(A.6). With Γ replaced by S_0 they become linear inhomogeneous equations, viz.

$$\begin{aligned}\frac{\delta S_0}{\delta\Phi^*_{a-}} &= -\,\frac{1}{2}\,\overline{D}D\Phi_{a-} + \mathcal{A}_a + M_{ai}\,\Phi_{i+} = -J_{a+} \quad ,\\ \frac{\delta S_0}{\delta\Phi^*_{i+}} &= -\,\frac{1}{2}\,\overline{D}D\,\Phi_{i+} + \Phi_{a-}\,M^*_{ai} = -J_{i-} \quad ,\end{aligned} \tag{A.10}$$

which can be solved with the help of the identity

$$(\tfrac{1}{2}\,\overline{D}D)^2 = -\partial^2 \quad .$$

One finds,

$$\begin{aligned}\Phi_-(J) &= -(\partial^2 + MM^\dagger)^{-1}\,(\tfrac{1}{2}\,\overline{D}DJ_+ + MJ_-) \quad ,\\ \Phi_+(J) &= -(\partial^2 + M^\dagger M)^{-1}\,(\tfrac{1}{2}\,\overline{D}DJ_- + M^\dagger(J_+ + \mathcal{A})\ .\end{aligned} \tag{A.11}$$

Finally, substituting these solutions into the right-hand sides of Eqs.(A.3), a quadrature gives

$$\begin{aligned}Z_0(J) = \int dx\Big[\frac{1}{8}\,(\overline{D}D)^2\Big(J_+^\dagger(\partial^2 + MM^\dagger)^{-1}\,J_+ + J_-^\dagger(\partial^2 + M^\dagger M)^{-1}\,J_-\Big)\\ -\,\frac{1}{2}\,\overline{D}D\Big(-J_-^\dagger(\partial^2 + M^\dagger M)^{-1}\,M^\dagger(J_+ + \mathcal{A}) + \text{h.c.}\Big)\Big] \ ,\end{aligned} \tag{A.12}$$

where the Feynman inverse is understood. This functional can also be expressed in the somewhat more cumbersome form,

$$Z_0(J) = \int dx_1\,(-\tfrac{1}{2}\,\overline{D}D)_1\,\Big[J_-^\dagger(1)\langle\Phi_+\rangle_0 + \text{h.c.}\Big] + \int dx_1\,dx_2\,(-\tfrac{1}{2}\,\overline{D}D)_1(-\tfrac{1}{2}\,\overline{D}D)_2 \quad ,$$

$$\Big[J_-^\dagger(1)\,G_0\Big(\Phi_+(1)\Phi_+^\dagger(2)\Big)J_-(2) + J_+^\dagger(1)G_0\Big(\Phi_-(1)\Phi_-^\dagger(2)\Big)J_+(2) +$$

$$+ J_-^\dagger(1)G_0\Big(\Phi_+(1)\Phi_-^\dagger(2)\Big)J_+(2) + J_+^\dagger(1)G_0\Big(\Phi_-(1)\Phi_+^\dagger(2)\Big)J_-(2)\Big] \quad , \tag{A.13}$$

which serves to define the zeroth order expectation value,

$$\begin{aligned}\langle\Phi_+\rangle_0 &= -(\partial^2 + M^\dagger M)^{-1}\, M^\dagger \mathcal{J} \\ &= -M^{-1}\mathcal{J}\end{aligned} \tag{A.14}$$

and the zeroth order propagators,

$$\begin{aligned}G_0\Big(\Phi_+(1)\Phi_+^\dagger(2)\Big) &= (\partial_1^2 + M^\dagger M)^{-1}(-\tfrac{1}{2}\,\bar{D}D)_1\ \delta_-(1,2) \quad , \\ G_0\Big(\Phi_-(1)\Phi_-^\dagger(2)\Big) &= (\partial_1^2 + MM^\dagger)^{-1}(-\tfrac{1}{2}\,\bar{D}D)_2\ \delta_+(1,2) \quad , \\ G_0\Big(\Phi_-(1)\Phi_+^\dagger(2)\Big) &= -(\partial_1^2 + MM^\dagger)^{-1}\ M\ \delta_-(1,2) \quad , \\ G_0\Big(\Phi_+(1)\Phi_-^\dagger(2)\Big) &= -(\partial_1^2 + M^\dagger M)^{-1}\ M^\dagger\ \delta_+(1,2) \quad .\end{aligned} \tag{A.15}$$

The chiral delta functions are given by

$$\delta_\pm(1,2) = \exp \tfrac{1}{4}(\mp\bar\theta_1\not\partial_1\gamma_5\theta_1 \mp \bar\theta_2\not\partial_2\gamma_5\theta_2)\ \tfrac{1}{2}\,\bar\theta_{12\mp}\ \theta_{12\pm}\delta(x_1 - x_2) \quad ,$$

where $\theta_{12} = \theta_1 - \theta_2$.

It is practical to work with the truncated fields $\hat\Phi_\pm$ introduced in Sec. II.B ,

$$\hat\Phi_\pm(x,\theta) = \exp(\pm\,\tfrac{1}{4}\,\bar\theta\not\partial\gamma_5\theta)\Phi_\pm(x,\theta) \quad . \tag{A.16}$$

For these fields the propagators are considerably more simple,

$$\begin{aligned}G_0\Big(\hat\Phi_+(1)\hat\Phi_+^\dagger(2)\Big) &= (\partial_1^2 + M^\dagger M)^{-1}\ \exp(\bar\theta_{1-}\ i\not\partial_1\ \theta_{2-})\delta(x_1 - x_2) \quad , \\ G_0\Big(\hat\Phi_-(1)\hat\Phi_-^\dagger(2)\Big) &= (\partial_1^2 + MM^\dagger)^{-1}\ \exp(\bar\theta_{1+}\ i\not\partial_1\ \theta_{2+})\delta(x_1 - x_2) \quad , \\ G_0\Big(\hat\Phi_-(1)\hat\Phi_+^\dagger(2)\Big) &= -(\partial_1^2 + MM^\dagger)^{-1}\ M\ \tfrac{1}{2}\,\bar\theta_{12+}\ \theta_{12-}\ \delta(x_1 - x_2) \quad , \\ G_0\Big(\hat\Phi_+(1)\hat\Phi_-^\dagger(2)\Big) &= -(\partial_1^2 + M^\dagger M)^{-1}\ M^\dagger\ \tfrac{1}{2}\,\bar\theta_{12-}\ \theta_{12+}\ \delta(x_1 - x_2) \quad .\end{aligned} \tag{A.17}$$

The perturbation series for the connected vacuum amplitude $Z(J)$ is now given by substituting the expression (A.13) (or (A.12)) into (A.9) and expanding in powers of S_g ,

$$\exp \frac{i}{\hbar}\, Z(J) = \sum_n \frac{1}{n!}\left[\frac{i}{\hbar}\, S_g\right]^n \exp\frac{i}{\hbar}\, Z_0(J) \quad , \tag{A.18}$$

where

$$S_g = \int dx \left(-\tfrac{1}{2}\,\bar{D}D\right)\left[\left(\frac{\hbar}{i}\frac{\delta}{\delta J_{a+}}\right) g_{aij}\left(\frac{\hbar}{i}\frac{\delta}{\delta J^*_{i-}}\right)\left(\frac{\hbar}{i}\frac{\delta}{\delta J^*_{j-}}\right) + \text{h.c.}\right] \tag{A.19}$$

This prescription may be summarized in a set of momentum space Feynman rules as follows:

(i) Draw diagrams in the usual way with 3-leg vertices corresponding to the interaction terms

$$g_{aij}\ \hat\Phi^*_{a-}\ \hat\Phi_{i+}\ \hat\Phi_{j+} \quad \text{and} \quad g^*_{aij}\ \hat\Phi_{a-}\ \hat\Phi^*_{i+}\ \hat\Phi^*_{j+} \quad .$$

(There may be 1-leg vertices corresponding to the linear term, $\mathcal{J}_a\ \hat\Phi^*_{a-}$ + h.c. , but in practice these can always be eliminated by a suitable shift in the fields Φ_{i+} . Such a shift would modify the mass matrix but not the couplings.) The vertices each carry a definite chirality, positive or negative.

(ii) With each vertex associate the factor

$$(2\pi)^4\ \delta(p_1 + p_2 + p_3)\ \frac{i}{\hbar}\, g_{aij} \quad \text{(positive chirality)}$$

or

$$(2\pi)^4\ \delta(p_1 + p_2 + p_3)\ \frac{i}{\hbar}\, g^*_{aij} \quad \text{(negative chirality)} \ .$$

These factors are represented graphically as in Fig.1

Fig.1:

positive chirality   negative chirality

(iii) With each line joining a pair of positive vertices associate the propagator

$$-\frac{\hbar}{i}\left[(-p^2 + M^\dagger M)^{-1}\ M^\dagger\right]_{ia}\ \tfrac{1}{2}\,\bar\theta_{12-}\ \theta_{12+} \quad .$$

With each line joining negative vertices associate the propagator

$$- \frac{\hbar}{i} \left[(-p^2 + MM^\dagger)^{-1} M \right]_{ai} \frac{1}{2} \bar{\theta}_{12+} \theta_{12-} \quad .$$

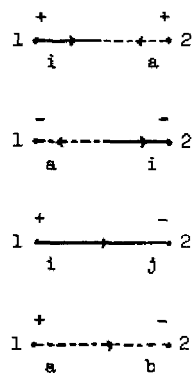
With each line joining vertices of opposite chirality associate as appropriate one of the propagators

$$\frac{\hbar}{i} (-p^2 + M^\dagger M)^{-1}_{ij} \exp(\bar{\theta}_{1-} \not{p} \, \theta_{2-}) \quad ,$$

$$\frac{\hbar}{i} (-p^2 + MM^\dagger)^{-1}_{ab} \exp(\bar{\theta}_{1-} \not{p} \, \theta_{2-}) \quad ,$$

where the 4-momentum is directed from the positive vertex (1) to the negative vertex (2). (Note the asymmetry, $\bar{\theta}_{1-} \not{p} \, \theta_{2-} = -\bar{\theta}_{2+} \not{p} \, \theta_{1+}$.) These propagators are represented graphically in Fig.2,

Fig.2:



(iv) With each external line associate the wave function

$$\hat{\Phi}^{ext}_\pm = A^{ext}_\pm(p) + \bar{\theta}_\mp \psi^{ext}_\pm(p) + \frac{1}{2} \bar{\theta}_\mp \theta_\pm F^{ext}_\pm(p) \quad .$$

(v) At each vertex apply the operator $(-\frac{1}{2}\bar{D}D)$, i.e. extract the coefficient of the term

$$\prod_{\text{vertices}} \frac{1}{2} \bar{\theta}_\mp \theta_\pm$$

and integrate over the momenta,

$$\prod_{\text{lines}} \int \frac{d_4 p}{(2\pi)^4} \quad .$$

(The rule (iv) applies to incoming or outgoing particles. The modifications necessary to characterize lines which terminate on the external currents $J_\pm$ are easily arrived at.)

To conclude this appendix we list the component structure of the propagators (A.17). These are conveniently separated into three sectors according to value of the fermionic number $\mathbb{F}$ in the intermediate states. The non-vanishing components are as follows.

1) <u>$\mathbb{F} = 0$ sector</u>

$$\begin{aligned}
G_0\big(A_+(1)\, A_+^\dagger(2)\big) &= (\partial^2 + M^\dagger M)^{-1}\, \delta(x_1 - x_2) \quad , \\
G_0\big(F_-(1)\, F_-^\dagger(2)\big) &= -\partial^2(\partial^2 + MM^\dagger)^{-1}\, \delta(x_1 - x_2) \quad , \\
G_0\big(F_-(1)\, A_+^\dagger(2)\big) &= -(\partial^2 + MM^\dagger)^{-1}\, M\, \delta(x_1 - x_2) \quad , \\
G_0\big(A_+(1)\, F_-^\dagger(2)\big) &= -(\partial^2 + M^\dagger M)^{-1}\, M^\dagger\, \delta(x_1 - x_2) \quad .
\end{aligned} \tag{A.20}$$

2) <u>$\mathbb{F} = 1$ sector</u>

$$\begin{aligned}
G_0\big(\psi_+(1)\, \bar{\psi}_+(2)\big) &= (\partial^2 + M^\dagger M)^{-1}\, \frac{1 + i\gamma_5}{2}\, i\not{\partial}\, \delta(x_1 - x_2) \quad , \\
G_0\big(\psi_-(1)\, \bar{\psi}_-(2)\big) &= (\partial^2 + M^\dagger M)^{-1}\, \frac{1 - i\gamma_5}{2}\, i\not{\partial}\, \delta(x_1 - x_2) \quad , \\
G_0\big(\psi_-(1)\, \bar{\psi}_+(2)\big) &= (\partial^2 + MM^\dagger)^{-1}\, \frac{1 - i\gamma_5}{2}\, M\, \delta(x_1 - x_2) \quad , \\
G_0\big(\psi_+(1)\, \bar{\psi}_-(2)\big) &= (\partial^2 + M^\dagger M)^{-1}\, \frac{1 + i\gamma_5}{2}\, M^\dagger\, \delta(x_1 - x_2) \quad .
\end{aligned} \tag{A.21}$$

3) <u>$\mathbb{F} = 2$ sector</u>

$$\begin{aligned}
G_0\big(F_+(1)\, F_+^\dagger(2)\big) &= -\partial^2(\partial^2 + M^\dagger M)^{-1}\, \delta(x_1 - x_2) \quad , \\
G_0\big(A_-(1)\, A_-^\dagger(2)\big) &= (\partial^2 + MM^\dagger)^{-1}\, \delta(x_1 - x_2) \quad , \\
G_0\big(A_-(1)\, F_+^\dagger(2)\big) &= -(\partial^2 + MM^\dagger)^{-1}\, M\, \delta(x_1 - x_2) \quad , \\
G_0\big(F_+(1)\, A_-^\dagger(2)\big) &= -(\partial^2 + M^\dagger M)^{-1}\, M^\dagger\, \delta(x_1 - x_2) \quad .
\end{aligned} \tag{A.22}$$

Vertices are governed by the interaction Lagrangian,

$$\mathcal{L}_g = g_{aij} \left[F^*_{a-} A_{i+} A_{j+} - 2\bar{\psi}_{a-} A_{i+} \psi_{j+} + A^*_{a-} \left\{ 2 A_{i+} F_{j+} + \psi^T_{i+} C^{-1} \psi_{j+} \right\} \right] + \text{h.c.} \tag{A.23}$$

The formulae (A.20)-(A.23) give a good indication of the complexity of the model described by the Lagrangian (A.1) whose Feynman rules have been listed above. However, they give no hint of the remarkable cancellations coming from the supersymmetry. To see these effects it is definitely worthwhile to employ the superfield notation and associated Feynman rules.

APPENDIX B

REGULARIZATION AND RENORMALIZATION

In order to regularize the model of Appendix A in a manifestly supersymmetric fashion it is sufficient to replace the kinetic terms by the expression

$$\frac{1}{8}\,(\bar{D}D)^2\left\{\Phi_+^\dagger\, R_+(\partial)\, \Phi_+ + \Phi_-^\dagger\, R_-(\partial)\, \Phi_-\right\} \quad , \qquad (B.1)$$

where $R_\pm(\partial)$ are polynomials in $\partial/\partial x$,

$$R_\pm(\partial) = 1 + \xi_\pm\, \partial^2 + \eta_\pm(\partial^2)^2 + \cdots \qquad (B.2)$$

The exact form of these operators is not important. They need only be of sufficiently high order to dampen the short distance behaviour of propagators so as to make finite the radiative corrections to the Green's functions. The parameters $\xi, \eta, \ldots$ are to be taken to zero at the end of any computation. (In terms of the familiar momentum space cut-off, Λ , we have $\xi \sim 1/\Lambda^2$, $\eta \sim 1/\Lambda^4$, ...)

Since the regularizing factors $R_\pm$ are introduced only in the bilinear terms they will cause changes in the propagators but not in the vertices. It is a simple matter to construct the regularized propagators. One has only to modify the equations (A.10) by introducing the factors R_+ and R_- . They become

$$\begin{aligned} -\frac{1}{2}\bar{D}D\, R_-(\partial)\, \Phi_- + \mathcal{A} + M\, \Phi_+ &= -J_+ \quad , \\ -\frac{1}{2}\bar{D}D\, R_+(\partial)\, \Phi_+ + M^\dagger\Phi_- &= -J_- \quad , \end{aligned} \qquad (B.3)$$

and they are solved by

$$\begin{aligned} \Phi_+ &= -\left\{R_+(\partial)\partial^2 + M^\dagger R_-^{-1}(\partial) M\right\}^{-1} \left\{\frac{1}{2}\bar{D}D\, J_- + M^\dagger R_-^{-1}(\partial)\, (J_+ + \mathcal{A})\right\} \quad , \\ \Phi_- &= -\left\{R_-(\partial)\partial^2 + M\, R_+^{-1}(\partial) M^\dagger\right\}^{-1} \left\{\frac{1}{2}\bar{D}D\, J_+ + M\, R_+^{-1}(\partial)\, J_-\right\} \quad . \end{aligned} \qquad (B.4)$$

Differentiation with respect to $J_\pm$ gives now the regularized propagators,

$$G_0\left(\hat{\Phi}_+(1)\hat{\Phi}_+^\dagger(2)\right)_{reg} = (R_+\partial^2 + M^\dagger R_-^{-1} M)^{-1} \exp(\bar{\theta}_{1-}\, i\not\partial\, \theta_{2-})\, \delta(x_{12}) \quad ,$$

$$G_0\left(\hat{\Phi}_+(1)\hat{\Phi}_-^\dagger(2)\right)_{reg} = -(R_+\partial^2 + M^\dagger R_-^{-1} M)^{-1}\, M^\dagger\, R_-^{-1}\, \frac{1}{2}\bar{\theta}_{12-}\, \theta_{12+}\, \delta(x_{12}) \quad ,$$

$$G_0\left(\hat{\Phi}_-(1)\hat{\Phi}_+^\dagger(2)\right)_{\rm reg} = -(R_-\partial^2 + M\, R_+^{-1}\, M^\dagger)^{-1}\, M\, R_+^{-1}\ \tfrac{1}{2}\,\bar\theta_{12+}\,\theta_{12-}\,\delta(x_{12}) \ ,$$

$$G_0\left(\hat{\Phi}_-(1)\hat{\Phi}_-^\dagger(2)\right)_{\rm reg} = (R_-\partial^2 + M\, R_+^{-1}\, M^\dagger)^{-1}\, \exp(\bar\theta_{1+}\, i\not\partial\, \theta_{2+})\,\delta(x_{12}) \ , \tag{B.5}$$

where $\partial = \partial/\partial x_1$. It is clear that the light-cone behaviour of these propagators can be made as soft as required by choosing the polynomials $R_\pm(\partial)$ of sufficiently high order.

Since the regularized kinetic terms (B.1) are manifestly supersymmetric, it follows that the regularized amplitudes constructed from the propagators (B.5) will be compatible with supersymmetry as well as finite. This is important for the arguments which follow.

The renormalization of supersymmetric models is greatly simplified by the occurrence of zeros in certain amplitudes whose canonical dimensions would, on the face of it, seem to indicate divergence. A good analogy is given by the electrodynamic vacuum polarization,

$$\Pi_{\mu\nu}(x) = (\eta_{\mu\nu}\,\partial^2 - \partial_\mu\partial_\nu)\ \Pi(x^2) \ .$$

The kinematical factor reduces the "effective" dimensionality of $\tilde\Pi_{\mu\nu}(k)$ from 2 to 0 and so the divergence is logarithmic rather than quadratic. This analogy is not perfect, however, in that the softening is due entirely to current conservation - i.e. gauge symmetry - whereas, in the supersymmetric system there is, as we shall see, a subtle cooperation between the symmetry and the constraints due to renormalizability.

In order to reduce the discussion to essentials, we shall henceforth ignore the internal degrees of freedom (including the fermionic number, Σ) and deal with the prototypical system comprised of a single scalar superfield, Φ_+ , in self-interaction,

$$\mathcal{L} = \tfrac{1}{8}\,(\bar{D}D)^2|\Phi_+|^2 - \tfrac{1}{2}\,\bar{D}D\left[\mathscr{J}\,\Phi_+ + \tfrac{M}{2}\,\Phi_+^2 + \tfrac{g}{3}\,\Phi_+^3 + \text{h.c.}\right] \ . \tag{B.6}$$

The simplification is only notational. It will become apparent as the argument proceeds that it is applicable to the general case as well.

Assume now that a supersymmetric propagator regularization of the kind discussed above has been made and consider what sorts of divergence might be expected in the limit when the regularization is removed. The usual dimensional argument would indicate the presence of divergences in the following derivatives of the effective action,

a) $$\frac{\delta\Gamma}{\delta\hat\Phi_+(1)} = \frac{\delta\Gamma}{\delta F_+(x_1)} \ ,$$

b) $$\frac{\delta^2\Gamma}{\delta\hat\Phi_+(1)\ \delta\hat\Phi_+(2)} = \frac{\delta^2\Gamma}{\delta A_+(x_1)\ \delta F_+(x_2)}\,\tfrac{1}{2}\,\bar\theta_{12-}\,\theta_{12+} \ ,$$

c) $$\frac{\delta^3\Gamma}{\delta\hat\Phi_+(1)\ \delta\hat\Phi_+(2)\ \delta\hat\Phi_+(3)} = \frac{\delta^3\Gamma}{\delta A_+(x_1)\ \delta A_+(x_2)\ \delta F_+(x_3)}\ \tfrac{1}{2}\,\bar\theta_{13-}\,\theta_{13+}\,\tfrac{1}{2}\,\bar\theta_{23-}\,\theta_{23+} + \cdots$$

d) $$\frac{\delta^2\Gamma}{\delta\hat\Phi_+(1)\ \delta\hat\Phi_+^*(2)} = \exp(\bar\theta_{1-}\, i\not\partial_1\, \theta_{2-})\,\frac{\delta^2\Gamma}{\delta F_+(x_1)\ \delta F_+^*(x_2)} \ ,$$

evaluated at $\Phi_+ = \Phi_+^* = 0$. Typical components are shown on the right-hand sides in order to reveal the canonical dimensions in a familiar way. Thus the quantity (a) has dimension 2, and one is therefore prepared to find that it diverges quadratically. The amplitudes (b), (c) and (d) have dimensions 1, 0 and 0 respectively, and could therefore diverge logarithmically. These are the only quantities which even potentially diverge *) and, as it happens, only (d) actually diverges. Fortunately there is a simple explanation for this suppression of divergences and there is no need to enquire too deeply into the graphical structure.

Of particular importance are the formulae (II.58) which express supersymmetry for the <u>monochiral</u> Green's functions,

$$\hat{G}_\pm(1,\ldots,n) = \langle T\ \hat\Phi_\pm(1)\ \cdots\ \hat\Phi_\pm(n)\rangle \ . \tag{B.7}$$

The amplitude $\hat{G}_+$ is a function of $\theta_{1+},\theta_{2+},\ldots,\theta_{n+}$, which must satisfy the constraints

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*) The absence of a quadratic divergence from the mass term (b) is to be expected since the supermultiplet includes a fermion whose mass can, at worst, diverge logarithmically. Quadratic divergences in the scalar masses must therefore cancel. A third derivative, like (c) but involving one differentiation with respect to $\hat\Phi_+^*$, need not be considered: its finiteness is assured by supersymmetry since there is no corresponding term, $(1/8)(\bar{D}D)^2$ $(\Phi_+^*\,\Phi_+^2)$ in a renormalizable Lagrangian.

$$\sum_{1}^{n} \frac{\partial \hat{G}_+}{\partial \theta_{j+}} = 0 \ ,$$
$$\Big(\sum_{1}^{n} \not{p}_j \ \theta_{j+}\Big)\hat{G}_+ = 0 \ , \tag{B.8}$$

and can therefore depend only on the $n-1$ differences θ_{jn+} and in a very restricted way, moreover. Only those combinations of the θ_{jn+} and momenta which are annihilated by the sum, $\sum_{1}^{n-1} p_j \ \theta_{jn+}$, can appear in $\hat{G}_+$. For example,

$$\hat{G}_+(1,2) = \theta_{12}^2 \ G_1 \ ,$$

$$\hat{G}_+(1,2,3) = \theta_{13}^2 \ \theta_{23}^2 \ G_1 + (p_1 \ \theta_{13} + p_2 \ \theta_{23})^2 \ G_2 \ ,$$

$$\begin{aligned}\hat{G}_+(1,2,3,4) = {} & \theta_{14}^2 \ \theta_{24}^2 \ \theta_{34}^2 \ G_1 \\ & + \theta_{14}^2 \ (p_2 \ \theta_{24} + p_3 \ \theta_{34})^2 \ G_2 + \theta_{24}^2 (p_3 \ \theta_{34} + p_1 \ \theta_{14})^2 \ G_3 \\ & + \theta_{34}^2 \ (p_1 \ \theta_{14} + p_2 \ \theta_{24})^2 \ G_4 \\ & + (p_1 \ \theta_{14} + p_2 \ \theta_{24} + p_3 \ \theta_{34})^2 \ G_5 \ ,\end{aligned} \tag{B.9}$$

where, to save space, we use the abbreviated notation,

$$\theta^2 = \frac{1}{2}\,\overline{\theta}_-\theta_+ \ .$$

The scalar amplitudes G_1, G_2, ... depend only on scalar combinations of the momenta.

We shall now prove that there can be no radiative corrections to the monochiral vertex functions (a), (b) and (c) evaluated at the origin of momentum space. All we need in addition to the group theoretical results indicated in the structure (B.9) is the remark that the required amplitudes (a), (b) and (c) can be obtained as coincidence limits on amplitudes like $\hat{G}_+(1,2)$, $\hat{G}_+(1,2,3)$ and $\hat{G}_+(1,2,3,4)$, respectively. Thus, for example,

$$\begin{aligned}\frac{\delta\Gamma}{\delta\hat{\Phi}_+(x,\theta)} &= \mathcal{A} + g \left\langle T\hat{\Phi}_+(x,\theta) \ \hat{\Phi}_+(x,\theta)\right\rangle \ , \\ &= \mathcal{A} + g \ \frac{\hbar}{i} \ \hat{G}_+(1,2)\Big|_{\theta_1=\theta_2} \ , \\ &= \mathcal{A} \ ,\end{aligned} \tag{B.10}$$

since, according to (B.9), $\hat{G}_+(1,2)$ must vanish in the coincidence limit $\theta_{12} = 0$.

The second formula of (B.9) gives

$$\hat{G}_+(1,2,3)\Big|_{\theta_1=\theta_2} = (p_1 + p_2)^2 \ \theta_{13}^2 \ G_2 \ , \tag{B.11}$$

which vanishes at $p_1 + p_2 = 0$. If we understand this monochiral amplitude not as the 3-point Green's function but, rather, as the hybrid quantity

$$\frac{\delta}{\delta\hat{\Phi}_+(3)} \ G\big(\hat{\Phi}_+(1) \ \hat{\Phi}_+(2)\big) \ ,$$

then the limit (B.11) representes the self-energy operator, i.e. the radiative corrections to the amplitude (b) (see Fig.3)

<u>Fig.3</u>

self-energy operator

The crucial result here is the emergence of the kinematical factor $(p_1 + p_2)^2$ which causes the self-energy to vanish at the origin of momentum space. Equivalently,

$$\frac{\delta^2\Gamma}{\delta A_+(x) \ \delta F_+(0)} = M \ \delta_4(x) + \partial^2\Pi(x^2) \ . \tag{B.12}$$

Since the potential divergence was only logarithmic we may conclude that Π is strictly finite.

An entirely analogous argument is applied to the monochiral function

$$\hat{G}(1,2,3,4) = \frac{\delta^2}{\delta\hat{\Phi}_+(3)\delta\hat{\Phi}_+(4)} \quad G\big(\hat{\Phi}_+(1) \ \hat{\Phi}_+(2)\big) \ .$$

Its structure is exhibited in (B.9), which shows the vanishing of this quantity in the limit

$$\theta_{12} = 0 \quad , \quad p_1 + p_2 = p_3 = p_4 = 0 \quad .$$

This means the absence of radiative corrections to the coupling constant. The amplitude (c) is finite.

Evidently the arguments given here will go through in exactly the same way when fermion number and other internal degrees of freedom are restored. Only the trilinearity and monochirality of the interaction was used (along with the supersymmetry displayed in the formulae (B.9)) and these features are common to all the renormalizable models made from chiral scalars.

Finally we may mention that the amplitude (d) is truly divergent and must be compensated by a wave function counter-term of the form

$$(Z - 1) \, \frac{1}{8} \, (\bar{D}D)^2 \, |\Phi_+|^2 \quad . \tag{B.13}$$

This wave function rescaling will of course affect the parameters s , M and g in the usual way

$$\begin{aligned} s \to s_r &= Z^{1/2} s \quad , \\ M \to M_r &= ZM \quad , \\ g \to g_r &= Z^{3/2} g \quad . \end{aligned} \tag{B.14}$$

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