IC/74/140

INTERNATIONAL CENTRE FOR
THEORETICAL PHYSICS

TRANSITION ELECTROMAGNETIC FIELDS IN PARTICLE PHYSICS

Abdus Salam

and

J. Strathdee

INTERNATIONAL
ATOMIC ENERGY
AGENCY

UNITED NATIONS
EDUCATIONAL,
SCIENTIFIC
AND CULTURAL
ORGANIZATION

1974 MIRAMARE - TRIESTE

IC/74/140

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

TRANSITION ELECTROMAGNETIC FIELDS IN PARTICLE PHYSICS *

Abdus Salam

International Centre for Theoretical Physics, Trieste, Italy,
and
Imperial College, London, England,

and

J. Strathdee

International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

We present a computation of one-loop effective potentials for elementary systems placed in a strong magnetic or a laser-produced electromagnetic environment. This permits a determination, in principle, of a hierarchy of transition field strengths, for which the systems concerned may (for appropriate values of the parameters in the theory) make transitions from a spontaneously broken asymmetric phase to one of restored symmetry.

MIRAMARE - TRIESTE
21 November 1974

* To be submitted for publication.

I. INTRODUCTION

In an earlier note [1)] it was suggested that if present notions of spontaneously broken symmetries in particle physics are correct then elementary systems may make transitions from the asymmetric phase to a symmetric one in an environment provided by laser-produced electromagnetic or strong static magnetic fields. An analogy was drawn with the case of superconductors where a strong magnetic field destroys the state of order.[*)] This analogy has as its basis the Ginzburg-Landau equations for the Cooper-pair field in superconductivity theory. These equations also happen to be the prototypes of field equations currently used in spontaneously-broken gauge theories of particle physics [2)].

The main purpose of this paper is to present the formalism for determining transition fields rather than to discuss any specific model. Our approach is similar to that followed by a number of authors [3)] who determine the effects of a (high) temperature environment on particle asymmetries. Like these authors, we compute the effective potential (in a one-loop approximation) but in an electromagnetic environment. We show that the non-zero expectation values of certain (neutral) scalar fields - which in a field-free situation signal spontaneously broken symmetries - may make transitions to zero - signalling symmetries which are restored - when strong external fields [**)] are present. The existence and the numerical values of these fields depend on the values of parameters in the model considered. We show in particular that the case of laser-fields is more directly analogous to the case of a high-temperature environment - in that (like temperature) laser-fields affect the mass parameters for the scalar particles in the theory, while strong magnetic fields, more subtly, affect their kinetic energies.

---

*) By a curious inversion of terminology (which unhappily is the source of a fair amount of confusion to the non-cognoscenti), the "state of order" in condensed matter theory is the state of "spontaneously broken symmetry" in particle physics. A magnetic field which destroys the "ordered-phase" in condensed matter physics terminology, in fact, "restores symmetries" in the language of particle physics!

**) Prof. G. Feldman has suggested that there may be analogous transition effects in strong gravitational environments.

The plan of the paper is as follows. In Sec.II we obtain a completely general expression (2.14) for the effective potential in the presence of a uniform external magnetic field, H . To the leading order in H^2 , this expression can be approximated by:

$$V(\phi) = V_0 + \frac{\hbar}{64\pi^2} \sum_j \sum_n (-)^{2j} (2j+1) M^4_{nj} \, \ell n \, M^2_{nj}$$
$$+ \frac{e^2\hbar}{96\pi^2} H^2 \sum_n{}' \left[-\ell n \, M^2_{n0} - 4 \, \ell n \, M^2_{n\frac{1}{2}} + 21 \, \ell n \, M^2_{n1} \right]$$
$$+ \ldots \tag{1.1}$$

In this formula V_0 is the classical potential expressed as a function of neutral scalar fields ϕ , with charged fields set equal to zero. The functions $M_{nj}(\phi)$ are the mass values associated with particles of spin j in the background of constant neutral scalars ϕ . In particular, the spin-zero masses $M^2_{n0}(\phi)$ are given by the eigenvalues of the matrix of second derivatives of the classical potential V_0 evaluated for arbitrary values of the neutral fields, while the spinor and vector mass functions, $M^2_{n\frac{1}{2}}$, M^2_{n1}, are obtained in the standard manner of spontaneously broken theories from the classical Lagrangian, in the unitary gauge. The one-loop contribution to the potential is interpreted as a shift in the vacuum energy density (a gauge-independent concept) due to zero-point oscillations (see Sec.II for details), then it is reasonable to suppose that only physical excitations should contribute. The first correction term in (1.1) involves a sum over all physical particles (and antiparticles where these are distinct) of spins $j = 0,\frac{1}{2},1$. This is the term in the effective potential first given by Coleman and Weinberg *). The second correction term involves a sum over all charged particles (of one sign) only. (It can be identified with the ϕ-dependent part of the photon wave function renormalization.)

The effect of a laser beam on particle symmetries is also of interest. Insofar as this can be treated by means of an effective potential, the relevant correction takes the form:

$$\Delta V \approx \frac{e^2\hbar}{64\pi^2} (-\mathcal{A}^*_\mu \mathcal{A}_\mu) \sum{}' M^2_n (1 + 2 \, \ell n \, M^2_n) , \tag{1.2}$$

*) For problems connected with the renormalization of M^2_n in $\log M^2_n$, see the discussion given by Coleman and Weinberg [4] or Weinberg [5].

where the sum extends over charged particles and $\mathcal{A}_\mu$ is a (complex) space-like vector which is used to represent the amplitude and phase of a monochromatic laser beam:

$$A_\mu(x) = \mathrm{Re}(\mathcal{A}_\mu \, e^{-ik\cdot x}) ,$$

with $k^2 = 0$ and $k_\mu \mathcal{A}_\mu = 0$. The formula (1.2) is discussed briefly in Sec.II.

We briefly consider a simple example in Sec.III in order to illustrate and make more concrete the formalism developed in Sec.II. This example, with local O(3) symmetry, includes two spin-zero triplets: a scalar $\underset{\sim}{\phi}$ and a pseudoscalar $\underset{\sim}{\chi}$. The classical potential involves six independent parameters and, depending on their values, may give rise to a CP-violating vacuum, $\langle \underset{\sim}{\chi} \rangle \neq 0$. In any case there would be a set of distinct zeroth-order vacua amongst which transitions may be expected when the H-dependence of the effective potential is taken into account. Although this model is not a realistic one, it may serve as a prototype from which to deduce (by seeking for minima of the effective potential) a hierarchy of critical fields [for example H_c and H_c', corresponding to $\langle\phi\rangle \neq 0$, $\langle\chi\rangle = 0$ and $\langle\phi\rangle = 0$, $\langle\chi\rangle \neq 0$] at which the various transitions are brought about. The expressions for these critical fields are displayed in Eqs. (3.23), (3.24) and (3.25).

One may enquire, which of the broken symmetries in particle physics are likely to be affected by the considerations of this paper and for what transition fields (if any)? Clearly this depends on the model considered. However, the following are some of the typical situations:

1) Broken strong interaction symmetries like colour, or SU(4) or SU(3) , with the associated mass differences (for example) between n and λ (quarks), which are commonly generated by postulating non-zero expectation values of certain scalar fields. For field-strengths exceeding the critical value, these mass differences would disappear.

2) Asymmetries of unified weak and electromagnetic interactions, e.g. (μ,ν') or (e,ν) mass difference in the simple SU(2) × U(1) model.

3) Parity-violation in those models where (V+A) currents are introduced symmetrically with (V-A) currents. These include Weinberg's [6] $SU_L(3) \times SU_R(3)$ based on the Konopinski-Mahmoud triplet of leptons and the $SU_L(2) \times SU_R(2) \times SU'(4)$ model of Pati and Salam [7] . For these models, P-asymmetry (and with it C-asymmetry responsible for K_1-K_2 mass difference) would disappear above the appropriate transition field strength.

4) Cabibbo angle (and associated hyperon-decays) which would reduce to zero, for strong fields and was discussed in Ref.1.

5) Milliweak (or superweak) CP-violation (for example in the models of Lee, Pais, Mohapatra and Pati [8]) also considered briefly in Ref.1.

6) Finally, baryon- (and possibly fermion-) number violation[9].

As remarked above, one may expect a whole hierarchy of transition electromagnetic fields associated with the restoration of one or more of the symmetries listed above.

Our considerations in this paper are limited to the one-loop corrections to the effective potential. (Even at the one-loop level we are excluding corrections to the kinetic energy terms) What the effects of two-loop and higher corrections (leading, for example, to dynamical phenomena like the composite Cooper-pair field with its non-vanishing expectation value in superconductor theory) on the transition fields will be - that is, on their existence and their magnitudes - is difficult to conjecture. (As a rule, higher loops contribute terms of the type $M^4(\log M^2)^n$ and $e^2H^2(\log M^2)^n$.) There is also the interesting possibility, studied by a number of authors, [10] where some of the scalar field expectation values are generated by the quantum corrections to the potential. The model of Sec.III is relevant to such situations though we do not discuss the problems raised in complete generality.

II. EFFECTIVE POTENTIAL - GENERAL FORMULAE

It is generally agreed now that the most compact and elegant way to treat the symmetry properties of the vacuum is by means of the so-called "effective potential". [4] Our purpose here is to compute the dependence of this function - in the one-loop approximation - on a uniform magnetic external field. [11] The magnetic perturbation may be expected to cause significant shifts in the extrema of the potential and even, when the field is strong enough, to alter the internal symmetry of the vacuum. (Analogous effects due to an external laser beam will also be discussed.)

The general form of the one-loop correction to the effective potential is easily described. Thus, given the classical action functional $S(\phi,H)$ which governs the system of fields ϕ^i in a magnetic environment, one constructs the propagation functions $G^{ij}(\phi,H)$ according to the definition

$$G^{-1}_{ij} = -\frac{\delta^2 S(\phi,H)}{\delta\phi^i\delta\phi^j}, \qquad (2.1)$$

which must be inverted subject to the usual causal boundary conditions. Then the effective action functional is given by [12]

$$\Gamma(\phi,H) = S(\phi,H) + \frac{\hbar}{2i}\ln \text{Det}\, G(\phi,H) + O(\hbar^2) \quad . \qquad (2.2)$$

The terms of order $\hbar^2$ are represented by the one-particle irreducible graphs with two or more loops in which the lines are associated with the propagator $(\hbar/i)\, G(\phi,H)$ and the vertices with the third- and higher-order functional derivatives of the classical action $(i/\hbar)\, S(\phi,H)$.

The functional $\Gamma(\phi,H)$ is far too complicated to be computed in general. Even with the neglect of $O(\hbar^2)$ terms, the best one can obtain is the value of Γ (or any one of its functional derivatives) for constant, x_μ-independent, ϕ or the first few terms of an expansion in powers of the derivatives of a slowly varying [13] $\phi(x)$:

$$\Gamma(\phi,H) = \int dx \Big[-V(\phi(x),H) + \tfrac{1}{2} Z(\phi(x),H)(\partial_\mu\phi(x))^2 + \cdots \Big], \qquad (2.3)$$

In this note we are concerned with the function $V(\phi,H) = V_0(\phi) + \hbar V_1(\phi,H) + \cdots$ The one-loop contribution to this function is given, according to (2.2), by

$$\frac{\hbar}{2i}\ln \text{Det}\, G(\phi,H) = \frac{\hbar}{2i}\text{Tr}\ln G(\phi,H)$$

$$= \frac{\hbar}{2i}\int dx\ \langle x|\ln G(\phi,H)|x\rangle ,$$

i.e.

$$\hbar V_1(\phi,H) = -\frac{\hbar}{2i}\langle x|\ln G(\phi,H)|x\rangle , \qquad (2.4)$$

where, in a rather obvious notation, the matrix elements of the one-body operator G are defined by (2.1),

$$\langle x|G(\phi,H)^{-1}|x'\rangle = -\frac{\delta^2 S(\phi,H)}{\delta\phi(x)\,\delta\phi(x')} , \tag{2.5}$$

evaluated at constant ϕ and H. To obtain V_1, therefore, we need the diagonal elements of $\ln G$.

The definition (2.5) reads, for the case of a single charged particle acted on by a uniform magnetic field directed along the 3-axis,

$$\langle x|G|x'\rangle = \left[\partial_0^2 - \left(\partial_1 - i\tfrac{e}{2}Hx_2\right)^2 - \left(\partial_2 + i\tfrac{e}{2}Hx_1\right)^2 - \partial_3^2 + M^2(\phi) - i\varepsilon\right]^{-1}\delta(x-x')$$

$$= \langle x|\left[-K_0^2 + \left(K_1 + \frac{eH}{2}X_2\right)^2 + \left(K_2 - \frac{eH}{2}X_1\right)^2 + K_3^2 + M^2(\phi) - i\varepsilon\right]^{-1}|x'\rangle , \tag{2.6}$$

where the mass term, $M^2(\phi)$, is allowed to depend on any neutral (constant) fields which may be present. $M^2(\phi)$ is simply the second derivative with respect to the charged fields of the classical potential V_0 evaluated at the origin $\phi^{\pm} = 0$. (Dependence on the charged fields themselves will not be considered since we shall assume that the environment is electrically neutral, i.e. $\langle\phi^{\pm}\rangle = 0$.)

The operators X_μ, K_μ satisfy the commutation rules

$$[X_\mu, K_\nu] = -i\eta_{\mu\nu}$$

and their respective eigenstates are normalized such that

$$\langle x|x'\rangle = \delta_4(x-x') \quad , \quad \langle k|k'\rangle = (2\pi)^4\,\delta(k-k') \quad , \quad \langle x|k\rangle = e^{-ikx} .$$

The gauge combinations

$$\Pi_1 = K_1 + \frac{eH}{2}X_2 \quad , \quad \Pi_2 = K_2 - \frac{eH}{2}X_1 , \tag{2.7}$$

have the commutator

$$[\Pi_1 , \Pi_2] = ieH . \tag{2.8}$$

A formal device for computing G, $\ln G$, etc., is the integral representation

$$(-iG)^\nu = \frac{1}{\Gamma(\nu)}\int_0^\infty dt\; t^{-1+\nu}\; e^{-itG^{-1}}$$

and, in particular,

$$\langle x|(-iG)^\nu|x\rangle = \frac{1}{\Gamma(\nu)}\int_0^\infty dt\; t^{-1+\nu}\;\langle x|\,e^{-itG^{-1}}|x\rangle$$

$$= \int_{-\infty}^{\infty}\frac{dk_0\,dk_3}{(2\pi)^2}\;\frac{1}{\Gamma(\nu)}\int_0^\infty dt\; t^{-1+\nu}\;\exp\left[-it\left(k_0^2 + k_3^2 + M^2 - i\varepsilon\right)\right]$$

$$\langle x_1x_2|\,\exp\left[-it\left(\Pi_1^2 + \Pi_2^2\right)\right]|x_1x_2\rangle ,$$

where we have used the fact that K_0 and K_3 commute with G^{-1}. After rotating the k_0 and k_3 contours through $+\pi/4$ and $-\pi/4$, respectively, we may reverse the orders of integration and carry out the resulting Gaussian intergals over k_0 and k_3. We then rotate the t contour through $-\pi/2$ and take the limit $\varepsilon \to 0$. The result is

$$\langle x|G^\nu|x\rangle = \frac{1}{4\pi\Gamma(\nu)}\int_0^\infty dt\; t^{-2+\nu}\;\langle x_1x_2|\,\exp\left[-t\left(\Pi_1^2 + \Pi_2^2 + M^2\right)\right]|x_1x_2\rangle , \tag{2.9}$$

which appears to converge for $\mathrm{Re}\,\nu > 1$. The matrix element in the integrand can be evaluated explicitly.

To do this introduce the new states $|Q_1Q_2\rangle$ defined by

$$\langle x_1x_2|Q_1Q_2\rangle = \sqrt{\frac{eH}{2\pi}}\;\delta\left(Q_1 + Q_2 - \frac{x_1}{\sqrt{2}}\right)\exp\left[i\,\frac{eH}{\sqrt{2}}\left(Q_1 - Q_2\right)x_2\right] . \tag{2.10}$$

The new states are orthonormal and they have been chosen such that

$$\langle x_1x_2|\left[\Pi_1^2 + \Pi_2^2\right]|Q_1Q_2\rangle = \frac{1}{2}\left[-\frac{\partial^2}{\partial Q_1^2} + (2eH)^2 Q_1^2\right]\langle x_1x_2|Q_1Q_2\rangle .$$

The eigenvalues of this operator are well known. It is now a simple matter to show that

$$\langle x_1x_2|\exp\left[-t\left[\Pi_1^2 + \Pi_2^2\right]\right]|x_1x_2\rangle = \frac{eH}{2\pi}\sum_n \exp\left[-t(n + \tfrac{1}{2})(2eH)\right]$$

$$= \frac{1}{4\pi}\ \frac{eH}{\sinh(teH)} . \tag{2.11}$$

The diagonal elements of G^ν are therefore represented by the integral

$$\langle x|G^\nu|x\rangle = \frac{i}{(4\pi)^2\Gamma(\nu)}\int_0^\infty dt\ t^{-2+\nu}\ \frac{eH}{\sinh(teH)}\ \exp[-tM^2] , \tag{2.12}$$

which converges for $\text{Re}\nu > 2$. The result can be analytically continued in the complex ν-plane and expressed in terms of the generalized Riemann zeta function[14],

$$\langle x|G^\nu|x\rangle = \frac{i}{(4\pi)^2}\frac{(2eH)^{2-\nu}}{\nu - 1}\ \zeta\left[\nu - 1 , \frac{M^2 + eH}{2eH}\right] . \tag{2.13}$$

The derivative with respect to ν at $\nu = 0$ of this equation gives the diagonal element of $\ell n G$,

$$\langle x|\ell n\ G|x\rangle = -\frac{i}{(4\pi)^2}\left[\frac{1}{2}\left[M^4 - \frac{4}{3}e^2H^2\right]\ell n(2eH) + (2eH)^2\ \zeta'\left[-1 , \frac{M^2 + eH}{2eH}\right]\right] . \tag{2.14}$$

This expression is exact. For practical purposes one needs only the leading terms in an expansion about $H = 0$. After discarding such terms as can be absorbed by a finite renormalization one finds for the contribution of a charged scalar particle and its antiparticle to the effective potential,

$$-\frac{\hbar}{i}\ \langle x|\ell n\ G|x\rangle = \frac{\hbar}{64\pi^2}\left[2M^4\ \ell nM^2 - \frac{2}{3}e^2H^2\ell nM^2 + \cdots\right] . \tag{2.15}$$

The contributions of charged spinor and vector particles can likewise be evaluated but with considerably more difficulty. In fact it is not necessary to have detailed knowledge of the various propagators in order to evaluate their contributions to the effective potential. Only the excitation spectrum is needed. We now give an alternative (and abbreviated) derivation of the formula (2.15) which generalizes more readily to the spinor and vector cases.

Firstly, one must recognize that the one-loop contribution (2.4) is nothing more than the ϕ- and H-dependent part of the shift in vacuum energy density due to the zero-point oscillations of the charged field considered as a set of harmonic oscillators.[15] For example, when $H = 0$ one writes

$$\frac{\hbar}{2i}\ \langle x|\ell n\ G|x\rangle = \frac{\hbar}{2i}\int\frac{d_4k}{(2\pi)^4}\ \ell n(-k^2 + M^2 - i\varepsilon) .$$

Differentiating both sides with respect to M^2 gives

$$\frac{\hbar}{2i}\frac{\partial}{\partial M^2}\langle x|\ell n\ G|x\rangle = \frac{\hbar}{2i}\int\frac{d_4k}{(2\pi)^4}\ \frac{i}{-k^2 + M^2 - i\varepsilon} =$$

$$= \frac{\hbar}{2}\int\frac{d_3\underline{k}}{(2\pi)^3}\ \frac{1}{2\sqrt{\underline{k}^2 + M^2}} ,$$

after integration over k_0. This means,

$$-\frac{\hbar}{2i}\ \langle x|\ell n\ G|x\rangle = \int\frac{d_3\underline{k}}{(2\pi)^3}\ \frac{\hbar}{2}\sqrt{\underline{k}^2 + M^2} + \text{constant} , \tag{2.16}$$

which is manifestly a sum over zero-point energies.

When a magnetic field is directed along the z-axis the excitations are labelled by k_z and the integer n,

$$\omega(k_z,n) = \sqrt{k_z^2 + M^2 + (n + \tfrac{1}{2})\ 2eH} \tag{2.17}$$

and the plane wave density of states factor $d_3k/(2\pi)^3$ is replaced by $(eH/2\pi)\ (dk_z/2\pi)$. The sum over zero-point energies then takes the form:

$$\int_{-\infty}^{\infty} \frac{dk_z}{2\pi}\ \frac{eH}{2\pi} \sum_n \frac{\hbar}{2}\, \omega(k_z,n) \quad . \tag{2.18}$$

To evaluate this sum we use the representation

$$\omega^{1-2\nu} = \frac{1}{\Gamma(\nu-\frac{1}{2})} \int_0^{\infty} dt\ t^{\nu-\frac{3}{2}}\, e^{-t\omega^2}$$

and replace (2.17) by the expression

$$\int \frac{dk_z}{2\pi}\ \frac{eH}{2\pi} \sum_n \frac{\hbar}{2}\ \frac{1}{\Gamma(\nu-\frac{1}{2})} \int_0^{\infty} dt\ t^{\nu-\frac{3}{2}}\ e^{-t\omega^2} =$$

$$= \frac{2\pi^{\frac{1}{2}}\Gamma(\nu)}{\Gamma(\nu-\frac{1}{2})}\ \frac{\hbar}{(4\pi)^2}\ \frac{(2eH)^{2-\nu}}{\nu-1}\ \zeta\left\{\nu-1,\ \frac{M^2+eH}{2eH}\right\} \ . \tag{2.19}$$

This result has a simple pole at $\nu = 0$,

$$\frac{2\pi^{\frac{1}{2}}\ \Gamma(\nu)}{\Gamma(\nu-\frac{1}{2})} \approx -\ \frac{1}{\nu} \quad ,$$

which reflects the fact that the sum (2.18) is divergent. To extract the "finite part" from (2.19) one should multiply it by ν and compute the derivative with respect to ν at $\nu = 0$.[16] One sees by comparison with (2.13) and (2.14) that this prescription for the finite part of the vacuum energy shift agrees (up to an unimportant finite renormalization) with $(-\hbar/2i)\ \langle x|\ln G|x\rangle$.

The generalization of (2.17) to particles which carry spin $\frac{1}{2}$ or 1 has been given by Tsai [17]

$$\omega(k_z,n,S_z) \ = \ \left[k_z^2 + M^2 + (2n+1-2q\ S_z)\ eH\right]^{\frac{1}{2}} \ , \tag{2.20}$$

where $q = \pm 1$ denotes the charge and S_z the z-component of spin. The simple formula (2.20) applies only to spin-$\frac{1}{2}$ particles whose magnetic moment is given by the Dirac value and to spin-1 particles whose magnetic moment is fixed by a non-abelian local symmetry SU(2) containing the electromagnetic gauge group.

It is now straightforward to write down the general expression for the one-loop part of the effective potential,

$$V_{(1)}(\phi,H) \ = \ V_{(1)}(\phi) \ + \ H^2\ V'_{(1)}(\phi) + \cdots \tag{2.21}$$

to leading order in H^2 . The neutral term is given by

$$\hbar\ V_{(1)}(\phi) = \frac{\hbar}{64\pi^2} \sum_j \sum_n \ (-)^{2j}\ (2j+1)\ M_{nj}^4\ \ln M_{nj}^2 \ , \tag{2.22}$$

where the sum extends over all particles (and antiparticles where these are distinct) of spins $j = 0,\frac{1}{2},1$. The first magnetic correction is given *) by

*) Note the sequence of signs for charged scalar and Fermi versus charged gauge fields so that the absolute sign of the coefficient of the H^2 term in the effective potential is model dependent. From Tsai's work, it becomes clear that the reason for the opposite signs for scalar versus the gauge-mesons in (2.23) (in what is essentially a one-loop contribution to the photon wave-function renormalization parameter Z_3) is to be sought in the anomalous magnetic moment carried by the charged Yang-Mills particles which (together with the photon) make up the SU(2) gauge triplet. This presumably is also the reason behind the mysterious circumstance that the function β in the renormalization group formalism is negative for non-abelian gauge theories - a circumstance which plays such a crucial role in these theories being asymptotically free.

$$\hbar V'_{(1)}(\phi) = \frac{e^2\hbar}{96\pi^2} \sum_n \left[-\ln M^2_{n0} - 4 \ln M^2_{n\frac{1}{2}} + 21 \ln M^2_{n1} \right] , \qquad (2.23)$$

where the sum extends over charged particles (of one sign) only. It is perhaps worth pointing out that the H^2-term in the effective potential can be identified with the ϕ-dependent part of the photon wave function renormalization, i.e. $Z_3 = 1 + 2\, V'_{(1)}(\phi)$. This indicates also that in a moving frame one should make the replacement $H^2 \to H^2 - E^2$ since Z_3 is invariant.

In deriving the above expressions for the effective potential we have taken the view that the shift in vacuum energy density is to be identified with a sum over the zero-point energies of a collection of oscillators corresponding to the physical one-particle excitations. In theories with a local symmetry it is of course true that the excitation spectrum is gauge dependent. If the only vector in the problem were the photon, then Z_3 would be gauge independent and there would be no difficulty. However, if the photon is contained in a multiplet of vectors which mediate a non-abelian local symmetry (as is the case for (2.23)) then Z_3 and the effective potential in general do involve gauge parameters. In such models it is not clear how seriously to take the effective potential. To us, the most plausible alternative is to adopt the so-called unitary gauge insofar as this can be done consistently, since in this gauge only physical excitations appear in the intermediate states. A potential difficulty here is the lack of renormalizability of gauge-dependent quantities in the unitary gauge. (However, Dolan and Jackiw [18]) have demonstrated the renormalizability in one-loop approximation of the scalar electrodynamics potential $V(\phi)$ in this gauge. Our considerations above, based on Tsai's formulae with the Yang-Mills value for the magnetic moment of the charged vectors, would appear to bear this out.)

Another kind of electrodynamic environment which may prove to have a significant influence on particle properties is that provided by a laser beam. An interesting effect, discovered by Brown and Kibble [19]), is the purely classical shift in the masses of charged particles moving in the beam. For a monochromatic beam characterized by the (Landau gauge) vector potential

$$A_\mu(x) = \mathrm{Re}(\mathcal{A}_\mu\, e^{-ikx}) , \qquad (2.24)$$

with $k^2 = 0$ and $k_\mu \mathcal{A}_\mu = 0$, the mass shift is given by

$$\Delta M^2 = \frac{e^2}{2} (-\mathcal{A}^*_\mu \mathcal{A}_\mu) \qquad (2.25)$$

from the term $\frac{1}{2} e^2 \phi^*\phi A^2_\mu$ in the scalar-electrodynamics Lagrangian (which is positive since $\mathcal{A}_\mu$ is spacelike). An indication of the magnitude of the influence of this phenomenon on the effective potential is obtained by substituting $M^2 \to M^2 + \Delta M^2$ for the charged particle masses in (2.22). For each charged state one obtains the contribution

$$\Delta V_{(1)} = \frac{e^2\hbar}{64\pi^2} (-\mathcal{A}^*_\mu \mathcal{A}_\mu)\, M^2 (1 + 2 \ln M^2) . \qquad (2.26)$$

The factor $(1 + 2\ln M^2)$ in this formula should presumably be read as "a number of order unity". The factor is certainly not reliable since it indicates a logarithmically divergent coupling of two photons to the neutral fields which is clearly out of place in a gauge-invariant theory. The formula (2.26) is therefore indicative at best.

It is interesting to note the significant difference between the laser and magnetic environments. The one gives a perturbation $\sim e^2 A^2_\mu M^2$ and the other $\sim e^2 H^2 \ln M^2$. This is due to the fact that the magnetic field affects the propagation of charged particles through their kinetic terms while the laser field appears to act through their masses.

III. AN EXAMPLE WITH A HIERARCHY OF TRANSITION FIELDS

It is quite easy to invent models which can incorporate a *hierarchy* of spontaneously broken symmetries. We take a relatively simple example in which the effective potential depends on only two neutral spin-zero fields, a scalar, φ , and a pseudoscalar, χ, and take the discussion far enough to exhibit M^2_{n0} , $M^2_{n\frac{1}{2}}$, M^2_{n1} without actually solving for the numerical values of the critical fields. If there are two vacuum expectation values $\langle\varphi\rangle$, $\langle\chi\rangle$ in the model, these are determined, in principle, by solving the pair of algebraic equations

$$\frac{\partial V}{\partial \varphi} = 0 \quad \text{and} \quad \frac{\partial V}{\partial \chi} = 0 ,$$

where V takes the form

$$V = V_0(\varphi,\chi) + \hbar\, V_1(\varphi,\chi,H) + \cdots$$

with V_1 computed according to the rules set out in Sec.II. One chooses the (renormalized) parameters in V such that, for example, the vacuum values $\langle\varphi\rangle_{H=0} = \varphi_0$, $\langle\chi\rangle_{H=0} = \chi_0$ are non-vanishing. This corresponds to the case of two broken symmetries. One then re-solves the equations with $H \neq 0$ and determines the function $\langle\chi\rangle_H$. A critical field, H_c , for restoration of one of the two symmetries is then obtained by solving the equation

$$\langle\chi\rangle_H = 0 ,$$

the resulting H_c being expressed as a function of the renormalized couplings and masses from the $H = 0$ theory. A second critical field H_c' would correspond to the solution of $\langle\phi\rangle_H = 0$.

To illustrate, consider a simplified version of a CP-violating model (due to Lee [8]). The Lagrangian, which has local O(3) symmetry, involves, in addition to the gauge vectors, a fermion, a scalar and a pseudoscalar: all triplets,

$$\mathcal{L} = -\frac{1}{4} W_{\mu\nu}^2 + \bar{\psi}\cdot(i\not\nabla - m)\psi + \bar{\psi}\cdot(f_1\, \phi \times \psi + f_2\, \chi \times \gamma_5\psi) + \frac{1}{2}(\nabla_\mu\phi)^2 + \frac{1}{2}(\nabla_\mu\chi)^2 - V_{(0)}(\phi,\chi) , \quad (3.1)$$

where the covariant derivatives are given as usual by

$$\nabla_\mu\phi = \partial_\mu\phi + e\, W_\mu \times \phi ,$$

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + e\, W_\mu \times W_\nu ,$$

etc., and the potential by

$$V_{(0)}(\phi,\chi) = -\frac{\mu^2}{2}\phi^2 - \frac{\kappa^2}{2}\chi^2 + \frac{a_1}{4}\phi^2 + \frac{a_2}{4}\chi^2 + \frac{a_3}{2}\phi^2\chi^2 + \frac{a_4}{2}(\phi\cdot\chi)^2 . \quad (3.2)$$

For stability it is necessary to have

$$a_1 > 0 , \quad a_2 > 0 , \quad a_1 a_2 > (a_3 \pm a_4)^2 , \quad (3.3)$$

but the parameters μ^2 and κ^2 may be positive or negative.

The extrema of $V_{(0)}$ are determined by the equations

$$\partial V_{(0)}/\partial\phi_i = \phi_i(-\mu^2 + a_1\phi^2 + a_3\chi^2) + \chi_i(a_4\phi\cdot\chi) = 0 ,$$

$$\partial V_{(0)}/\partial\chi_i = \chi_i(-\kappa^2 + a_2\chi^2 + a_3\phi^2) + \phi_i(a_4\phi\cdot\chi) = 0 . \quad (3.4)$$

It will be assumed in the following that $\mu^2 > 0$ so as to exclude the O(3)-symmetric solution $\langle\phi_i\rangle = \langle\chi_i\rangle = 0$. Another solution, one with no residual symmetry, is excluded by requiring $a_4 < 0$. The possibilities which remain preserve an O(2) symmetry which is to be associated with electromagnetism. Thus, we have

$$\langle\phi_\pm\rangle = \left\langle \frac{\phi_1 \pm i\phi_2}{\sqrt{2}} \right\rangle = 0 \quad \text{and} \quad \langle\chi_\pm\rangle = \left\langle \frac{\chi_1 \pm i\chi_2}{\sqrt{2}} \right\rangle = 0 , \quad (3.5)$$

while $\langle\phi_3\rangle = \varphi_0$ and $\langle\chi_3\rangle = \chi_0$ satisfy the pair of equations

$$\frac{\partial V_{(0)}}{\partial\varphi} = \varphi\left[-\mu^2 + a_1\varphi_0^2 + (a_3 + a_4)\chi_0^2\right] = 0 ,$$

$$\frac{\partial V_{(0)}}{\partial\chi} = \chi\left[-\kappa^2 + a_2\chi_0^2 + (a_3 + a_4)\varphi_0^2\right] = 0 . \quad (3.6)$$

There are essentially two alternatives here:

$$\text{(I)} \quad \varphi_0^2 = \mu^2/a_1 \quad \text{and} \quad \chi_0 = 0 \quad (\text{or } \varphi_0 = 0 , \quad \chi_0^2 = \kappa^2/a_2) , \quad (3.7)$$

$$\text{(II)} \quad \varphi_0^2 = \frac{a_2\mu^2 - (a_3+a_4)\kappa^2}{a_1a_2 - (a_3+a_4)^2} \quad \text{and} \quad \chi_0^2 = \frac{a_1\kappa^2 - (a_3+a_4)\mu^2}{a_1a_2 - (a_3+a_4)^2} . \quad (3.8)$$

Solution (I) respects CP while solution (II) violates it (the order of magnitude of CP violation being $\approx \frac{f_2\,\chi_0}{f_1\,\phi_0}$ which is empirically $\approx 10^{-3}$) . To test the viability of these solutions it is necessary to examine the mass matrix. This matrix decomposes into two pieces, charged and neutral:

$$M_c^2 = \begin{pmatrix} \frac{\partial^2 V}{\partial\phi_+ \partial\phi_-} & \frac{\partial^2 V}{\partial\phi_+ \partial\chi_-} \\ \frac{\partial^2 V}{\partial\chi_+ \partial\phi_-} & \frac{\partial^2 V}{\partial\chi_+ \partial\chi_-} \end{pmatrix}$$

$$= \begin{pmatrix} -\mu^2 + a_1\varphi^2 + a_3\chi^2 & a_4\varphi\chi \\ a_4\varphi\chi & -\kappa^2 + a_2\chi^2 + a_3\varphi^2 \end{pmatrix} \qquad (3.9)$$

$$M_n^2 = \begin{pmatrix} \frac{\partial^2 V}{\partial\phi_3 \partial\phi_3} & \frac{\partial^2 V}{\partial\phi_3 \partial\chi_3} \\ \frac{\partial^2 V}{\partial\chi_3 \partial\phi_3} & \frac{\partial^2 V}{\partial\chi_3 \partial\chi_3} \end{pmatrix}$$

$$= \begin{pmatrix} -\mu^2 + 3a_1\varphi^2 + (a_3+a_4)\chi^2 & 2(a_3+a_4)\varphi\chi \\ 2(a_3+a_4)\varphi\chi & -\kappa^2 + 3a_2\chi^2 + (a_3+a_4)\varphi^2 \end{pmatrix} . \qquad (3.10)$$

By substituting the values (3.7) into (3.9) and (3.10) one sees that solution (I) is viable if

$$\mu^2 > 0 \quad \text{and} \quad a_3+a_4 > a_1 \frac{\kappa^2}{\mu^2} . \qquad (3.11)$$

In this domain solution (II) is not viable (i.e. $\chi^2 < 0$ according to (3.8)). On the other hand, solution (I) is excluded while (II) becomes viable if $\mu^2 > 0$ and either

$$a_2 \frac{\mu^2}{\kappa^2} < a_3 + a_4 < a_1 \frac{\kappa^2}{\mu^2} \qquad \text{if } \kappa^2 < 0 , \qquad (3.12)$$

or

$$a_3 + a_4 < \min \left\{ a_2 \frac{\mu^2}{\kappa^2} , a_1 \frac{\kappa^2}{\mu^2} \right\} \qquad \text{if } \kappa^2 > 0 . \qquad (3.12')$$

For the following discussion we shall assume that one of the conditions (3.12) or (3.12') is satisfied. Substitution of the values (3.8) into the charged mass matrix (3.9) yields the singular form

$$\langle M_c^2 \rangle = -a_4 \begin{pmatrix} \chi_0^2 & -\chi_0\varphi_0 \\ -\varphi_0\chi_0 & \varphi_0^2 \end{pmatrix}$$

$$= -a_4 \begin{pmatrix} \chi_0 \\ -\varphi_0 \end{pmatrix} (\chi_0 \quad -\varphi_0)$$

and this result shows that the field combinations

$$G_\pm = \frac{\varphi_0 \phi_\pm + \chi_0 \chi_\pm}{\sqrt{\varphi_0^2 + \chi_0^2}} \qquad (3.13)$$

are Goldstone modes. At this point we can exercise our option to choose the unitary gauge by imposing the constraint

$$G_\pm = 0 . \qquad (3.14)$$

The remaining charged scalar in the system, the Higgs-Kibble meson,

$$H_\pm = \frac{\varphi_0 \chi_\pm - \chi_0 \phi_\pm}{\sqrt{\varphi_0^2 + \chi_0^2}} ,$$

carries the mass

$$M_c^2(\varphi,\chi) = (-\kappa^2 + a_2\chi^2 + a_3\varphi^2)\cos^2\beta - 2a_4\varphi\chi\cos\beta\sin\beta + $$
$$+ (-\mu^2 + a_1\varphi^2 + a_3\chi^2)\sin^2\beta , \qquad (3.15)$$

where the angle β is given by

$$\tan\beta = \frac{\chi_0}{\varphi_0} = \sqrt{\frac{a_1\kappa^2 - (a_3 + a_4)\mu^2}{a_2\mu^2 - (a_3 + a_4)\kappa^2}} . \qquad (3.16)$$

To list the other states in the model we have, in addition to $H_\pm$,

1) two neutral scalars whose masses are given by the eigenvalues of (3.10),

2) a neutral fermion with mass m ,

3) a pair of charged fermions with mass $\sqrt{m^2 + f_1^2\varphi^2 + f_2^2\chi^2}$,

4) the photon,

5) a charged vector with mass $M_W^2 = e^2(\varphi^2 + \chi^2)$.

Now according to the formulae of Sec.II, the one-loop correction to the potential is given (for H = 0) by *)

$$V_{(1)} = \frac{\hbar}{64\pi^2}\left[2M_H^4\ \ell n\left(\frac{M_c^2(\varphi,\chi)}{M_c^2(\varphi_0,\chi_0)}\right) + \mathrm{Tr}\left\{M_n^4\ \ell n\left(\frac{M_n^2(\varphi,\chi)}{M_n^2(\varphi_0,\chi_0)}\right)\right\} + \right.$$
$$+ 3\,e^4(\varphi^2+\chi^2)^2\ \ell n\left(\frac{\varphi^2+\chi^2}{\varphi_0^2+\chi_0^2}\right)$$
$$\left. - 8\,(m^2 + f_1^2\varphi^2 + f_2^2\chi^2)^2\ \ell n\left(\frac{m^2+f_1^2\varphi^2+f_2^2\chi^2}{m^2+f_1^2\varphi_1^2+f_2^2\chi_0^2}\right)\right] , \qquad (3.17)$$

where M_c^2 is defined by (3.15), (3.16), and M_n^2 by (3.10). The magnetic correction is given by

$$H^2\,V'_{(1)} = \frac{e^2\hbar\,H^2}{96\pi^2}\left[-\ell n\left(\frac{M_c^2(\varphi,\chi)}{M_c^2(\varphi_0,\chi_0)}\right) - 8\ \ell n\left(\frac{m^2+f_1^2\varphi^2+f_2^2\chi^2}{m^2+f_1^2\varphi_0^2+f_2^2\chi_0^2}\right) + 21\ \ell n\left(\frac{\varphi^2+\chi^2}{\varphi_0^2+\chi_0^2}\right)\right] . \qquad (3.18)$$

These expressions must be combined with the classical term,

$$V_{(0)} = -\frac{\mu^2}{2}\varphi^2 - \frac{\kappa^2}{2}\chi^2 + \frac{a_1}{4}\varphi^4 + \frac{a_2}{4}\chi^4 + \frac{a_3+a_4}{2}\varphi^2\chi^2 , \qquad (3.19)$$

in order to find the corrections to $\langle\varphi\rangle$ and $\langle\chi\rangle$ and, ultimately, the two critical fields by using the minimizing procedure outlined in the beginning of this section. (Note that for neither of the critical fields, $H_c'(\langle\phi\rangle_H = 0,\ \langle\chi\rangle_H \neq 0)$, $H_c(\langle\phi\rangle_H \neq 0,\ \langle\chi\rangle_H = 0)$, assuming both exist, would the "mass" of the charged gauge particle m_W vanish, since $m_W^2(\phi,\chi) = e^2(\phi^2+\chi^2)$.) With the

---

*) The reference masses which fix the normalization here are the classical (vacuum) values. For the neutral pair we take the larger of the two eigenvalues of $M_n^2(\varphi_0,\chi_0)$.

potential

$$V = V_{(0)}(\varphi,\chi) + \hbar\,V_{(1)}(\varphi,\chi) + \hbar\,H^2\,V'_{(1)}(\varphi,\chi)$$
$$\equiv \mathcal{V}(\varphi,\chi) + H^2\,\mathcal{V}'(\varphi,\chi) , \qquad (3.20)$$

the minimization problem comprises the pair of simultaneous equations

$$\frac{\partial\mathcal{V}}{\partial\varphi} + H^2\frac{\partial\mathcal{V}'}{\partial\varphi} = 0$$
$$\frac{\partial\mathcal{V}}{\partial\chi} + H^2\frac{\partial\mathcal{V}'}{\partial\chi} = 0 . \qquad (3.21)$$

Elimination of H^2 yields the compatibility condition

$$f(\varphi,\chi) = \det\begin{vmatrix}\partial\mathcal{V}/\partial\varphi & \partial\mathcal{V}'/\partial\varphi\\ \partial\mathcal{V}/\partial\chi & \partial\mathcal{V}'/\partial\chi\end{vmatrix} = 0 . \qquad (3.22)$$

Solution of the equation

$$f(\varphi,0) = 0$$

yields the critical value $\langle\varphi\rangle_c$ whose substitution into one of the pair (3.21) gives the critical field H_c . However, owing to the special structure of the mass functions $M_c^2(\varphi,\chi)$, etc., it is possible to obtain quite directly a simple formula relating $\langle\varphi\rangle_c$ to H_c . This is done as follows.

To find the critical field H_c one solves the equations $\partial V/\partial\varphi = 0$ and $\partial V/\partial\chi = 0$ subject to the condition $\chi = 0$. Then we have

$$\left(\frac{\partial V}{\partial\chi}\right)_{\chi=0} = \frac{\hbar}{64\pi^2}\left(\frac{\partial M_c^2}{\partial\chi}\right)_{\chi=0}\left[2M_c^2(\varphi,0)\left(\ell n\,\frac{M_c^2(\varphi,0)}{M_c^2(\varphi_0,\chi_0)} + \frac{1}{2}\right) - \frac{1}{3}\,\frac{(eH_c)^2}{M_c^2(\varphi,0)}\right]$$
$$= 0 ,$$

which gives the formula

$$(eH_c)^2 = 3\,M_c^4(\varphi,0)\left(2\ell n\,\frac{M_c^2(\varphi,0)}{M_c^2(\varphi_0,\chi_0)} + 1\right) , \qquad (3.23)$$

into which must be substituted that value of φ which satisfies the other equation, $\partial V(\varphi,0)/\partial\varphi = 0$. The simplicity of the result (3.23) is due to the fact that most of the terms in V are <u>even</u> in χ and therefore do not contribute to the derivative at $\chi = 0$. Only the function $M_c^2(\varphi,\chi)$, given by (3.15), contains an odd component (provided $\beta \neq 0, \pi/2$). In more complicated models there could be many such terms. The formula which determines the critical value of $\varphi = \varphi_c$ is given (ignoring fermions) by

$$\mu^2 - a_1\varphi^2 = \frac{\hbar}{32\pi^2}\left[3a_1(-\mu^2 + 3a_1\varphi^2)\left\{2\ \ln\frac{-\mu^2 + 3a_1\varphi^2}{M_{n1}^2} + 1\right\}\right.$$
$$+ (a_3+a_4)\left(-\kappa^2+(a_3+a_4)\varphi^2\right)\left\{2\ \ln\frac{-\kappa^2+(a_3+a_4)\varphi^2}{M_{n1}^2}\right\}$$
$$\left. + 6e^4\varphi^2\left\{2\ \ln\left[\frac{\varphi^2}{\varphi_0^2 + \chi_0^2}\right] + 1\right\} + \frac{14}{\varphi^2}\ M_c^4(\varphi,0)\left\{2\ \ln\frac{M_c^2(\varphi,0)}{M_c^2(\varphi_0,\chi_0)}\right\}\right] . \qquad (3.24)$$

We do not know if this equation has a real solution and, if so, whether it yields a real value for H_c when substituted into (3.23) . Nor have we examined under what restrictions on the parameters μ^2, κ^2, a_1, a_2 , etc. does the potential V possess a true minimum for H_c and φ_c given by (3.23) and (3.24). Since the model is a simple and unrealistic one and since we have taken no account of many-loop contributions to the potential, we do not feel that the very complicated equation (3.24) warrants a detailed analysis. We have exhibited it here merely to illustrate the numerical complexity of the problem. (The formulae for the second critical field H_c' are analogous, except, that roles of φ and χ are interchanged. There may also be the possibility of a third critical field corresponding to ($\phi = \chi = 0$), for a special set of values of the parameters.)

If the choice of parameters μ^2, κ^2, a_1, a_2 , etc. is such that $\chi_0 \ll \phi_0$ (and φ_0 is an approximate solution of (3.24)) one may approximate to (3.23) by the expression:

$$e^2H^2 \approx 3M_c^2(\varphi_0,0)\left[3M_c^2(\varphi_0,0) - 2M_c^2(\varphi_0,\chi_0)\right] .$$

A system of equations similar to (3.23) and (3.24) applies for the case of laser induced transitions. The laser modified terms in the one-loop part of the potential: the charged particle contributions with M^2 replaced by $M^2 + \Delta M^2$ include, in particular, the term

$$\approx \frac{\hbar}{32\pi^2}\quad (M_c^2 + \Delta M^2)^2\ \ln\left(\frac{M_c^2 + \Delta M^2}{M_c^2(\varphi_0,\chi_0)}\right) ,$$

whose derivative survives at $\chi = 0$, unless

$$2\ \left(M_c^2(\varphi,0) + \Delta M^2\right)\ \ln\ \frac{M_c^2(\varphi,0) + \Delta M^2}{M_c^2(\varphi_0,\chi_0)} + 1 \approx 0 ,$$

i.e.

$$\Delta M^2 \equiv \frac{e^2}{2}\ (-\mathcal{A}_\mu^*\mathcal{A}_\mu)$$
$$\approx \frac{M_c^2(\varphi_0,\chi_0)}{\sqrt{e}} - M_c^2(\varphi,0), \qquad (3.25)$$

(where $\sqrt{e}$ is the square root of the base of natural logarithms). Into this formula for the critical laser amplitude must be substituted the value of φ which satisfies $\partial V(\varphi,0)/\partial\varphi = 0$ – an equation similar to (3.24).

It will be recognized that the formulae (3.23) and (3.25) owe their relative simplicity to a very special choice of (unitary gauge *) : one which excludes the Goldstone modes and per force leads to the formula (3.15) for the Higgs mass function. One could argue that another choice of gauge would spoil this and make all masses even functions of χ . At this stage we can only warn that such gauges are likely to amplify the Goldstone contributions and cause severe infra-red complications in the neighbourhood of $\chi = 0$. Our hope is that the unitary gauge formulation presented above will prove – to the approximation considered – at least qualitatively correct.

ACKNOWLEDGMENTS

We are idebted to Professors D.J. Bradley, P. Budini, J. Butterworth, R. Delbourgo, G. Feldman, C. Franzinetti, T.W.B. Kibble, P.T. Matthews, D. Rivier, M. Rosenbluth and J. Ziman for discussions on aspects of strong magnetic field and laser physics.

---

*) Weinberg and Dolan and Jackiw [3], in their parallel studies of critical temperatures advocate the use of renormalizable gauges (with a judicious dropping of gauge-dependent terms) rather than the unitary gauge. Their argument is that recognition of comparable orders of magnitude is performed more reliably in renormalizable gauges. We feel these are basic problems of principle which await a convincing solution for all critical phenomena in field theory.

REFERENCES

1) Abdus Salam and J. Strathdee, ICTP, Trieste, preprint IC/74/133 (to appear in Nature).

2) H.B. Nielsen and P. Olesen, Nucl. Phys. B61, 45 (1973); Y. Nambu, preprint EFI 74/40 (1974).

3) D.A. Kirshnitz and A.D. Linde, Phys. Letters 42B, 471 (1972) and Lebedev Institute preprint No.101 (1974);
S. Weinberg, Phys. Rev. D9, 3357 (1974);
L. Dolan and R. Jackiw, Phys. Rev. D9, 3320 (1974);
C. Bernard, Phys. Rev. D9, 3312 (1974).

4) J. Goldstone, Abdus Salam and S. Weinberg, Phys. Rev. 127, 965 (1962);
G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964);
S. Coleman and E. Weinberg, Phys. Rev. D7, 1888 (1973).

5) S. Weinberg, Phys. Rev. D7, 2887 (1973).

6) S. Weinberg, Phys. Rev. D5, 1962 (1972).

7) J.C. Pati and Abdus Salam, Phys. Rev. D10, 275 (1974).

8) R.N. Mohapatra, Phys. Rev. D6, 2023 (1972);
R.N. Mohapatra and J.C. Pati, University of Maryland Technical Report No.74-085 (1974);
T.D. Lee, Physics Reports 9C, 145 (1974); Phys. Rev. D9, 2291 (1974);
A. Pais, Phys.Rev. D8, 625 (1973).

9) Abdus Salam and J.C. Pati, Phys. Rev. Letters 31, 661 (1973);
Abdus Salam, J.C. Pati and J. Strathdee, ICTP, Trieste, Internal Report IC/74/121.

10) S. Coleman and E. Weinberg, Ref. 4;
A. Zee, Phys. Rev. D9, 1772 (1974);
H. Georgi and A. Pais, "CP-violation as a quantum effect", Report No. COO-2232B-50 (1974).

11) This type of problem was first solved by J. Schwinger, Phys. Rev. 82, 664 (1951).

12) B.S. de Witt, Phys. Rev. 162, 1195 (1967).

13) The exact dependence on $\partial_\mu \phi$ for the case when all higher derivatives vanish has been given by M.R. Brown and M.J. Duff, Oxford preprint, 1974.

14) See, for example, W. Magnus, F. Oberhettinger and R.P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, third edition (Springer-Verlag,1966).

15) This interpretation is exploited by L. Dolan and R. Jackiw, and S. Weinberg, Ref.3.

16) I.M. Gel'fand and G.E. Shilov, Generalized Functions, Vol.I (Academic Press, 1964).

17) Wu-Yang Tsai, Phys. Rev. D7, 1945 (1973). This reference provides a simple derivation of the eigenvalue formulae allowing also for anomalous magnetic moments. References to earlier work can be found here.

18) L. Dolan and R. Jackiw, Phys. Rev. D9, 2904 (1974).

19) L.S. Brown and T.W.B. Kibble, Phys. Rev. 133A, 705 (1964).