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UNITARY REPRESENTATIONS OF SUPER-GAUGE SYMMETRIES

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and

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ADDENDUM

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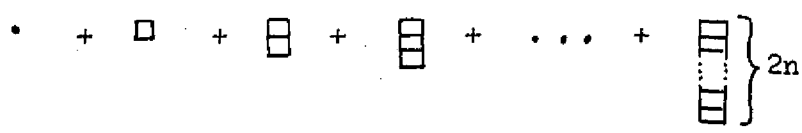
Generalizations to larger algebraic structures could proceed in at least two distinct ways. The more economical one, which we consider first, involves a set of n Majorana spinors $S_{\alpha k}$ transforming as a vector of $O(n)$. The fundamental anticommutator would take the form:

$$\{S_{\alpha k}, S_{\beta \ell}\} = -\delta_{k\ell}\,(\gamma_\mu C)_{\alpha\beta}\,P_\mu \ .$$

The smallest irreducible representation has 2^{2n} components. The rest frame states span a 2^{2n}-dimensional Fock space which is obtained by operating repeatedly with the $2n$ "creation operators" S_{ak}, $a = 1,2$, in the manner explained in the text. These states may be classified according to SU(2n) which is contained in the little algebra. In the notation of Young tableaux one finds the antisymmetric representations



For the case $n = 3$, for example, one finds the SU(6) decomposition

$$2^6 = 1 + 6 + 15 + 20 + \overline{15} + \overline{6} + 1 \qquad .$$

Among these states is an O(3) singlet with spin $\frac{3}{2}$. In general, the maximum spin value in the fundamental representation is $n/2$.

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The other (and less economical) scheme is to require the generators $S_{\alpha p}$, $p = 1,2,\ldots,n$, to transform as an n-fold of SU(n). For $n \geqslant 3$ it is necessary to discard the Majorana constraint and treat the generators $\bar{S}^{\alpha p}$ as independent. The fundamental anticommutators would take the form:

$$\{S_{\alpha p} , S_{\beta q}\} = 0 \quad , \quad \{\bar{S}^{\alpha p} , S^{\beta q}\} = 0 ,$$

$$\{S_{\alpha p} , \bar{S}^{\beta q}\} = \delta_p^{\ q} (\gamma_\mu)_\alpha^{\ \beta} P_\mu .$$

In this case the smallest representation has 2^{4n} components since there are now 4n independent anticommuting creation operators $S_{\alpha p}$. For the case n = 3, the rest frame algebra contains SU(12) which can therefore be used to classify the $2^{12} = 4,096$ states. One finds all the antisymmetric tensors of SU(12). In terms of dimensions the SU(12) decomposition reads:

$$2^{12} = 1 + 12 + 66 + 220 + 495 + 792 + 924 + \overline{792} + \overline{495} + \overline{220} + \overline{66} + \overline{12} + 1 .$$

In this multiplet one finds spins up to the value J = 3.

Although the examples sketched here may not be realistic, they do, at least, show that algebraic generalizations of the Fermi-Bose symmetry are possible. It now becomes important to search out the economical ones.

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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

UNITARY REPRESENTATIONS OF SUPER-GAUGE SYMMETRIES *

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ABSTRACT

A method is given for constructing some of the unitary irreducible representations of the Wess-Zumino super-gauge symmetry. Application of this symmetry to the analysis of S-matrix elements is considered. A new super-gauge symmetry which includes isospin is introduced and some of its representations are constructed.

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The super-gauge operations invented by Wess and Zumino [1] have the remarkable property of transforming bosons into fermions and vice versa. These authors define such transformations on a multiplet of fields which is then built into an invariant Lagrangian. The absence of ghosts (in the perturbative development at least) shows, among other things, that the symmetry is consistent with unitarity. Indeed, it is possible to proceed directly to the construction of unitary representations. This we shall do in the following.

The super-gauge symmetry of Wess and Zumino may be looked upon as the first example of a unitary and relativistic spin-containing theory. Particles with distinct intrinsic spins are here combined into irreducible multiplets. In this letter we present a generalized super-gauge symmetry which contains isospin as well. It is not a relativistic version of Wigner's SU(4), though it resembles it. It appears to be a potent new symmetry, though we do not speculate at present on its usefulness in particle physics.

The full super-gauge symmetry involves dilatations, conformal and γ_5 transformations. In a recent note [2] we considered the more easily manageable subalgebra which is generated by the Poincaré operators $J_{\mu\nu}$, P_μ and the Majorana spinor S_α for which we adopted the following algebra:

$$[S_\alpha , P_\mu] = 0 \quad , \quad [S_\alpha , J_{\mu\nu}] = (1/2)(\sigma_{\mu\nu})_\alpha{}^\beta S_\beta \quad ,$$

$$\{S_\alpha , S_\beta\} = -(\gamma_\mu C)_{\alpha\beta} P_\mu \quad . \tag{1}$$

Here C denotes the charge conjugation matrix *) and the spinor S_α is constrained to be real in the sense,

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*) Our notational conventions are as follows. The Dirac matrices satisfy $(1/2)\{\gamma_\mu, \gamma_\nu\} = \eta_{\mu\nu} = \mathrm{diag}(+---)$. Adjoint spinors are defined by $\bar\psi = \psi^+\gamma_0$. The matrices γ_0 , $\gamma_0\gamma_\mu$, $\gamma_0\sigma_{\mu\nu}$, $\gamma_0 i\gamma_\mu\gamma_5$, $\gamma_0\gamma_5$ are hermitian. The antisymmetric matrix C defined by $C^{-1}\gamma_\mu C = -\gamma_\mu{}^T$ is real.

$$S_\alpha = C_{\alpha\beta}\,\bar{S}^\beta \quad . \tag{2}$$

Under space reflections it transforms according to *)

$$S_\alpha \to i(\gamma_0)_\alpha{}^\beta\, S_\beta \quad . \tag{3}$$

The above-mentioned generalized super-gauge symmetry is obtained by replacing S_α with the isospinor $S_{\alpha i}$ $(i = 1,2)$ for which a modified anti-commutator is postulated,

$$\{S_{\alpha i}\,,\, S_{\beta j}\} = \varepsilon_{ij}(i\gamma_\mu\gamma_5 C)_{\alpha\beta}\, P_\mu \quad . \tag{4}$$

The space reflection rule (3) is unchanged but the Majorana constraint (2) is replaced by the SU(2)-covariant form

$$S_{\alpha i} = i\,\varepsilon_{ij}(\gamma_5 C)_{\alpha\beta}\,\bar{S}^{\beta j} \quad . \tag{5}$$

The construction of unitary representations begins with the observation that super-gauge transformations must leave invariant the manifold of states with fixed 4-momentum since S_α commutes with P_μ . On this manifold the anticommutator $\{S_\alpha\,,\, S_\beta\}$ becomes a fixed set of numbers and we see that the operators generate a Clifford algebra. Since this algebra has just sixteen independent members, its one and only finite-dimensional irreducible representation is in terms of 4×4 matrices [3]. As we shall see, these matrices are hermitian (in the sense of (2)) when p_μ is timelike. In this case the super-gauge transformations serve to resolve the manifold of states with fixed p_μ into 4-dimensional invariant subspaces. If p_μ is lightlike, these 4-dimensional spaces involve two states of positive norm and two orthogonal states of zero norm.

One can contemplate a complete classification of the unitary irreducible representations according to whether the 4-momentum is timelike,

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*) For consistency with the Majorana constraint the factor i is necessary.

lightlike, spacelike or null. In all but the first case one would in general meet infinite-dimensional representations of the Clifford algebra *). Here we shall be dealing exclusively with timelike representations.

In addition to the operators S_α there are, of course, the well-known rotations of Wigner's little group which leave invariant the manifold of states with fixed p_μ. The generators of these rotations, taken together with the Clifford elements, S_α, $S_{[\alpha} S_{\beta]}$, etc., span an algebra which might be called the little algebra. The basic problem is to find the unitary irreducible representations of this algebra. These representations are in one-one correspondence with the unitary irreducible representations of the full group. Similar considerations apply to the generalization (4), (5) where the irreducible representations of the timelike Clifford algebra are 16-dimensional.

In the rest frame, $p_0 = M, \vec{p} = 0$, the little algebra is generated by the angular momentum operators $\vec{J}$ and the S_α which here obey the rule

$$\{S_\alpha , S_\beta\} = -M(\gamma_0 C)_{\alpha\beta} . \tag{6}$$

In a basis where γ_0 is diagonal and C is real, the Majorana constraint (2) implies $S_4 = -S_1^+$, $S_3 = S_2^+$ and the anticommutators (6) can be expressed in the suggestive form:

$$\{S_a , S_b\} = 0 , \quad \{S_a^+ , S_b^+\} = 0 ,$$
$$\{S_a , S_b^+\} = M \delta_a{}^b , \tag{7}$$

where S_a is a 2-component spinor under space rotations and transforms according to

$$S_a \to iS_a \tag{8}$$

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*) Such representations are relevant for group theoretical analysis in the crossed channels. See for example Ref.4.

under reflections. Viewed as creation and annihilation operators, the spinors S_a and S_a^+ can be used in the familiar way to set up a 4-dimensional "Fock space" with positive metric. The procedure is as follows.

Choose a set of $2\mathcal{J}+1$ vectors $|\mathcal{J},\mathcal{J}_3\rangle$, $-\mathcal{J}\leq\mathcal{J}_3\leq\mathcal{J}$, to represent the states of a particle at rest with mass M and spin $\mathcal{J}$. Let these states constitute the "Clifford vacuum", i.e.

$$S_a^+|\mathcal{J},\mathcal{J}_3\rangle = 0 \quad . \tag{9}$$

Define the orthonormal vectors

$$|\mathcal{J}\mathcal{J}_3 n_1 n_2\rangle = M^{-\frac{1}{2}(n_1+n_2)} S_1^{n_1} S_2^{n_2}|\mathcal{J}\mathcal{J}_3\rangle \tag{10}$$

with the pair (n_1,n_2) taking the values (0,0), (0,1), (1,0) and (1,1). These states span a $4(2\mathcal{J}+1)$-dimensional irreducible representation of the little algebra.*) The $(\text{spin})^{\text{parity}}$ content of the multiplet is $(\mathcal{J}-\frac{1}{2})^{\eta}$, $\mathcal{J}^{i\eta}$, $\mathcal{J}^{-i\eta}$, $(\mathcal{J}+\frac{1}{2})^{\eta}$, where η takes one of the values $\pm i$ (for integer $\mathcal{J}$) or ± 1 (for half-integer $\mathcal{J}$).

Basis vectors for the representation of the full group are obtained in the usual way by applying Lorentz boosts to the rest frame states (10),

$$|p\mathcal{J}\mathcal{J}_3 n_1 n_2\rangle = U(L_p)\ |\mathcal{J}\mathcal{J}_3 n_1 n_2\rangle \quad , \tag{11}$$

where L_p denotes a 3-parameter boost which carries the 4-vector $(M,\vec{0})$ into p_μ. The behaviour of these states under super-gauge transformations is easily obtained. Apply S_α on both sides of (11) and take it through the operator $U(L_p)$ using the knowledge that it transforms as a Dirac spinor under the Lorentz group. One finds

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*) The 3-vector operator $\vec{\mathcal{J}} = \vec{J} - (2M)^{-1}\ S^+\vec{\sigma}S$ commutes with the Clifford elements and satisfies the commutation rules of SU(2). It coincides in the rest frame with the transverse part of the generalized Pauli-Lubanski vector, K_μ, introduced in Ref.2. The singlet $(\vec{\mathcal{J}})^2$ is a Casimir operator. Its eigenvalues $\mathcal{J}(\mathcal{J}+1)$ serve to label the irreducible representations.

$$S_\alpha |p\xi\rangle = \sum_{\xi'} |p\xi'\rangle \langle \xi' | S_\alpha(p) | \xi \rangle \quad , \tag{12}$$

where ξ denotes the set of labels in (10) and

$$\langle \xi' | S_\alpha(p) | \xi \rangle = a_\alpha{}^\beta(L_p) \langle \xi' | S_\beta | \xi \rangle \quad , \tag{13}$$

where $a_\alpha{}^\beta(L_p)$ denotes the spinor representation of the Lorentz transformation L_p. The p-dependence of the $4(2\mathcal{J}+1)$-dimensional matrix $S_\alpha(p)$ is thereby given explicitly.

In defining the action of S_α on 2-particle states there is one subtle point which must not be overlooked. Suppose we take $S_\alpha = S_\alpha^{(1)} + S_\beta^{(2)}$ where $\{S_\alpha^{(1)}, S_\beta^{(1)}\} = -(\gamma_\mu C)_{\alpha\beta}\, p_{1\mu}$, and likewise for $S^{(2)}$. Then, in order to have

$$\{S_\alpha^{(1)} + S_\alpha^{(2)} , S_\beta^{(1)} + S_\beta^{(2)}\} = -(\gamma_\mu C)_{\alpha\beta} (p_1+p_2)_\mu \quad , \tag{14}$$

it is essential that $S^{(1)}$ should <u>anti</u>commute with $S^{(2)}$. The correct definition of S_α on the product state $|12\rangle$ is therefore,

$$S_\alpha |12\rangle = |1'2\rangle \langle 1' | S_\alpha(p_1) | 1 \rangle \pm |12'\rangle \langle 2' | S_\alpha(p_2) | 2 \rangle \; , \tag{15}$$

where the negative sign is used when the state $|1\rangle$ is fermionic.

The symmetry discussed here is a very potent one. To analyse 2-body amplitudes, for example, one would resolve the in- and out-states into irreducible representations of the (common) centre-of-mass algebra (14). In general this algebra is smaller than the little algebra of 1-particle states, because the presence of relative momentum vectors in the 2-particle states forbids the inclusion of $\vec{J}$. In general one can supplement (14) only by the discrete transformation corresponding to reflection in the interaction plane. (Forward amplitudes have the additional symmetry against rotations about the common direction of the relative momenta.) Consider, for example, the elastic scattering of two $\mathcal{J} = 0$ multiplets. The sixteen in-states must resolve into four quartets of the algebra (15). Two of these will be even under the discrete reflection and two odd. Likewise for the out-states. It follows that a total of eight amplitudes (four of which vanish in the forward direction) describe the various processes. This count would be further reduced if two or more of the multiplets are identical.

We now consider briefly the application of these ideas to the isospin-containing symmetry characterized by (4) and (5). In this case the Clifford algebra contains 256 independent members, and its fundamental representation is in terms of 16 x 16 matrices. As before, we express the rest-frame anti-commutation rules in the form

$$\{S_{ai}\ ,\ S_{bj}\} = 0 \quad , \quad \{S^{+}_{ai}\ ,\ S^{+}_{bj}\} = 0 \quad ,$$
$$\{S_{ai}\ ,\ S^{+}_{bj}\} = M\ \delta_{a}^{\ b}\ \delta_{i}^{\ j} \quad , \tag{16}$$

where the 4-component object S_{ai} is a spinor under both space and isospace rotations. Since the little algebra now includes $\vec{I}$ as well as $\vec{J}$, we construct its irreducible representations by starting with a multiplet of $(2\mathcal{I} + 1)\ (2\mathcal{J} + 1)$ states, $|\mathcal{I}\mathcal{I}_3, \mathcal{J}\mathcal{J}_3\rangle$, and applying the creation operator S_{ai} repeatedly. The resulting states span a space of $16(2\mathcal{I} + 1)\ (2\mathcal{J} + 1)$ dimensions with positive metric. These states can then be boosted to span an irreducible unitary representation of the full group. The representations made in this way are seen to be characterized by three numbers, $\mathcal{I}$, $\mathcal{J}$ and M , in addition to parity type.*)

In the rest frame the normal parity representation with $\mathcal{J} = \mathcal{I} = 0$ has the $(I,J)^P$ content

$$(0,0)^{+} + (\tfrac{1}{2},\tfrac{1}{2})^{i} + (1,0)^{-} + (0,1)^{-} + (\tfrac{1}{2},\tfrac{1}{2})^{-i} + (0,0)^{+} \quad . \tag{17}$$

The content of any other irreducible representation is obtainable from this by vector multiplication with the SU(2) x SU(2) multiplet $(\mathcal{J},\mathcal{I})$. It will be noticed that the rest frame states (17) can be grouped into multiplets of

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*) The 3-vector $\vec{\mathcal{J}} = \vec{I} - (2M)^{-1}\ S^{+}\vec{\tau}S$ commutes with the Clifford elements and with $\vec{\mathcal{I}}$. Its square is a Casimir operator.

Wigner's SU(4), viz, $1 + 4 + 6 + \overline{4} + 1$. This is because the rest frame Clifford algebra contains the matrices $(1/2)\ [S_{ai}\ ,\ S^{+}_{bj}]$ which obey the SU(4) commutation rules. *)

To analyse 2-body amplitudes one would resolve the in- and out-states into irreducible representations of the centre-of-mass algebra generated by $S_{\alpha i}$ and $\vec{I}$. For example, in the scattering of two $\mathcal{J} = \mathcal{I} = 0$ multiplets the 256 in-states are found to comprise the following representations (denoted $\mathcal{J}^P$) of the centre-of-mass algebra: 1^- , $(\frac{1}{2}^{i})^2$, $(\frac{1}{2}^{-i})^2$, $(0^+)^2$, $(0^-)^3$ (where repetitions are indicated by a superscript). One finds a total of twenty-two amplitudes (ten of which vanish in the forward direction) to describe the various processes. At threshold, where the relative momenta vanish, only eight amplitudes survive. Thus, one sees that although the particle multiplets are rather large their scattering appears to be controlled by a relatively manageable number of amplitudes.

A detailed exposition of the ideas sketched here and their application to the analysis of amplitudes is in preparation.

NOTE ADDED

After completion of this note, we have been informed by Professor B. Zumino that the extension of the theory presented here to include isotopic spin has also been completed by himself and Professor J. Wess, in the context of fields and Lagrangians, in a forthcoming CERN preprint.

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*) If we had widened the algebra to include three spinors S_{ai} (i = 1,2,3) and their adjoints, then the algebra of SU(6) would be contained. This programme is not straightforward, however, since the Majorana constraint (5) does not generalize. One may also consider inventing algebras where the anticommutators among the S_{ai} , representing Fermi statistics, are replaced by a more complicated system corresponding to para-statistics (or colour).

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