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INTERNATIONAL CENTRE FOR
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COUPLING CONSTANTS ARE NOT ARBITRARY IN ASYMPTOTICALLY FREE THEORIES

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ADDENDUM TO IC/73/176

SUM RULES IN ASYMPTOTICALLY FREE THEORIES

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In connection with our note IC/73/176, Sucher and Woo (private communication) *) have suggested that the dispersion integrals for the coupling parameters in asymptotically free theories are likely to be identically satisfied - and thus have the status of sum rules. They have shown this by taking a special subset of chain-graphs for ϕ-ϕ scattering.

What we wish to remark is that, for their example, 1) the unsubtracted dispersion relation can be an identity only within a limited range of values of g; 2) both the relevance of the example they give and the fact that (within the range above) the relation may be identically satisfied appears to depend crucially on the fact that we are dealing with an asymptotically free theory. If the theory were not asymptotically free, either the unsubtracted relation would not hold, or it would not be an identity in the coupling parameter in the sense specified below.

Sucher and Woo have given the example of a sum of chain graphs from $g\phi^4$ theory:

$$\Gamma^{(4)}(s,g) = g[1 + gb\, F(s)]^{-1} , \tag{1}$$

where g is the value of $\Gamma^{(4)}$ at $s = 0$ and b is a positive number. The function $-g^2 b\, F(s)$ corresponds to the "bubble" diagram  which has the value

*) We appreciate their kind courtesy in sending us a copy of their remark prior to publication.

$$F(s) = s \int_{4m^2}^{\infty} \frac{dx}{x} \left|1 - \frac{4m^2}{x}\right|^{\frac{1}{2}} \frac{1}{x-s} \quad . \tag{2}$$

This function is negative on the semi-axis $-\infty < s < 0$, positive in the interval $0 < s < 4m^2$, and logarithmically increasing for $|s| \to \infty$.

The amplitude (1) decreases logarithmically and must satisfy an unsubtracted dispersion relation. However, in addition to the unitarity cut it may have a pole on the physical sheet at the point $s = \bar{s}$, where

$$F(\bar{s}) = -\frac{1}{bg} \quad . \tag{3}$$

If this is the case, then the dispersion integral evaluated at $s = 0$ will serve merely to define the coupling parameters of this "bound state" as a function of g.

If g is positive, then (3) has a tachyon solution: the famous ghost pole discussed by Redmond [1] and by Bogolubov et al. [2] However, if g is negative (in which case the theory is asymptotically free) there is no tachyon. In this case there will be a true bound state pole with $0 < \bar{s} < 4m^2$ if $g < -1/2b$. This pole will migrate into the second sheet if g is moved into the interval $-1/2b < g < 0$. For such values of g there will be no bound state or tachyon on the physical sheet and $\Gamma^{(4)}$ must be representable in the form

$$\Gamma^{(4)}(s,g) = \frac{1}{\pi} \int_{4m^2}^{\infty} dx \; \frac{\mathrm{Im}\Gamma^{(4)}(x,g)}{x-s} \quad . \tag{4}$$

In particular, the equation

$$g = \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{dx}{x} \, \mathrm{Im}\Gamma^{(4)}(x,g) \tag{5}$$

will be true for all g in the interval $-1/2b < g < 0$. Thus only for this restricted set of values of g, the unsubtracted dispersion relation in formula (4) holds identically. Otherwise there is an extra bound state contribution whose coupling parameter is completely determined as a function of g. In any case, relation (5) has the status of a sum rule and may be helpful in controlling approximate solutions for asymptotically free theories.

This example would appear to suggest that the ghost-pole phenomenon, the circumvention of which was central to the Redmond-Bogolubov approach to improving on lowest-order perturbation theory, does not occur in the case of asymptotically free models.

Speaking loosely, sums of chain diagrams for appropriate functions are likely to be ghost-free in asymptotically free theories for a range of physically interesting values of the coupling parameters. This would be an additional interesting feature of such theories.

REFERENCES

1) P.J. Redmond, Phys. Rev. 112, 1404 (1958).

2) N.N. Bogolubov, A.A. Logunov and D.V. Shirkov, Sov. Phys.-JETP 37 (10), 574 (1960).

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International Atomic Energy Agency
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COUPLING CONSTANTS ARE NOT ARBITRARY IN ASYMPTOTICALLY FREE THEORIES *

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ABSTRACT

It is remarked that vertex functions in asymptotically free theories may be representable by unsubtracted dispersion integrals. This would imply that the associated coupling constants are not free parameters.

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It appears likely that the renormalized coupling constants (and possibly **also the ratios of masses) are not free parameters in asymptotically free theories,** but instead must be computed self-consistently from unsubtracted dispersion equations of the eigenvalue variety. Our reasoning behind this observation is the following:

(1) The renormalization group equations show that, for asymptotically free theories, the vertex functions fall to zero (in contrast to their perturbation estimates) when momenta go to infinity in the deep Euclidean region.

(2) As a consequence of this rapid fall to zero, one may write unsubtracted dispersion relations for the vertex functions.

(3) Since the renormalized coupling constants are mass-shell or symmetry-point boundary values of these vertex functions, the unsubtracted dispersion relations provide eigenvalue equations for the coupling parameters which, therefore, are not free parameters.

To illustrate, consider the somewhat academic case of a self-interacting $g\phi^4$ theory, with $g < 0$. It is well known that this theory is asymptotically free. (This particular theory can be of no more than academic interest, since the vacuum state may not exist. However, the general analysis should apply also to non-Abelian asymptotically-free gauge theories (with spontaneous symmetry-breaking to avoid infra-red problems).)

Now Symanzik [1] has shown that in the deep Euclidean region the four-point vertex function has the behaviour:

$$\Gamma(\kappa p_1, \ldots, \kappa p_4 | g) \sim -\frac{2a^2(g)}{b_0 \ln \kappa^2}$$

for $\kappa \to \infty$. Here $a = \exp[2b_0^{-1} c_0 g + \cdots]$, $b_0 = \frac{3}{32\pi^2}$, $c_0 = \frac{1}{2^{11}\ 3\pi^4}$,

with the (negative) coupling parameter g defined by the relation:

$$\Gamma(p_1, \ldots, p_4 | g)\Big|_{\text{sym. point}} = -g$$

at the symmetry point $p_i p_j = \frac{\mu^2}{3}(4\delta_{ij} - 1)$ where μ^2 is a reference mass.

Consider now a once-subtracted parametric dispersion relation proposed by Nishijima: [2]

$$\mathrm{Re}\ \Gamma(\xi p_i p_j) = \Gamma(0) + \frac{\xi}{\pi} P \int_{-\infty}^{\infty} \frac{d\xi'}{\xi'} \varepsilon(\xi') \frac{\mathrm{Im}\Gamma(\xi' p_i p_j)}{\xi' - \xi} .$$

With momenta in the Euclidean region, $\Gamma(\xi p_i p_j)$ falls to zero like $-\frac{2a^2}{b_0 \ln \xi^2}$ for $\xi^2 \to \infty$. Thus, with some extrapolation, the dispersion relation may be rewritten in the unsubtracted form

$$\mathrm{Re}\ \Gamma(\xi p_i p_j) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{d\xi'}{\xi'-\xi}\ \epsilon(\xi')\ \mathrm{Im}\ \Gamma(\xi' p_i p_j) \quad .$$

From this we infer that

$$-g = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\xi'}{\xi'-1}\,\epsilon(\xi')\quad \mathrm{Im}\Gamma(\xi'\ p_i p_j | g)\Big|_{p_i p_j = \frac{\mu^2}{3}(4\delta_{ij} - 1)} \quad .$$

This eigenvalue equation - in the form of a sum rule - must be solved self-consistently for the coupling parameter g.

The basic idea of writing down infinity-free sets of eigenvalue equations for coupling parameters (and ratios of masses) - stated in the form $Z\ (g, \frac{m}{\mu}) = 0$ - is not new. [3] Indeed, the point has been forcefully made by Mack and Todorov [4] for conformal invariant theories (of which $g\phi^4$ theory is an example) in the context of Dyson-Schwinger equations. What is perhaps new is the remark that the equation [5] $Z_1 = 0$ can in principle be implemented for asymptotically free theories, when the high-energy behaviour of vertex functions given by the renormalization group analysis is used to write unsubtracted dispersion relations.

One must remark that the discontinuity function which appears in Nishijima's relation, on the right-hand side, pertains to an unphysical region. Whether any practical use can be made of the relations proposed we shall not discuss here.

FOOTNOTES AND REFERENCES

1) K. Symanzik, DESY preprint 72/73, December 1972.

2) K. Nishijima, Tokyo University preprint UT 218, November 1973.

3) There is an extensive bibliography on this subject. We believe that B. Jouvet's work is perhaps the first suggestion in this regard. J.C. Howard and B. Jouvet, Nuovo Cimento 18, 466 (1960).

4) G. Mack and I.T. Todorov, Phys. Rev. D8, (1973).

5) Abdus Salam, Nuovo Cimento 25, 224 (1962). The suggestion that $Z_1 = 0$ should be treated as an eigenvalue equation was first made in this paper. This was exploited to secure finiteness in the following papers.
Abdus Salam, Phys. Rev. 130, 1287 (1963);
R. Delbourgo and Abdus Salam, Phys. Rev. 135B, 1398 (1964);
J. Strathdee, Phys. Rev. 135B, 1428 (1964);
For an alternative approach to these problems see M. Baker and K. Johnson, Phys. Rev. D3, 2516, 2541 (1971) and earlier references contained in these papers.