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ABSTRACT

The ultraviolet behaviour of Green's functions in field theories with a local gauge symmetry is, to some extent, gauge dependent. This is due to the intervention of ghost-like gauge degrees of freedom. We argue that it may be possible,in the case of non-polynomial gauge theories, to find a class of gauges in which all ultraviolet divergences and their attendant ambiguities are suppressed.

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I. INTRODUCTION

It is well known that the Green's functions of a field theory whose Lagrangian displays invariance with respect to a group of local transformations are not uniquely defined. Rather, they are gauge-dependent quantities which can be computed only relative to some choice of gauge. Moreover, as has become clear in recent years, the singularity structure and asymptotic behaviour of the Green's functions is dependent on the gauge. Two examples of this are:

a) In quantum electrodynamics, the electron wave function renormalization constant, Z_2 , a gauge-dependent parameter which measures the asymptotic behaviour of the renormalized propagator, is finite in a special gauge defined by Baker and Johnson [1]. In other gauges it diverges.

b) In Yang-Mills theories of massive vector particles, where the mass is induced by the Higgs-Kibble mechanism, there exists a class of gauges (of which 't Hooft's [2] is an example) in which the Green's functions behave as if the theory were renormalizable. In other gauges - such as the unitary gauge [3] - this is not so.

This mellowing of the asymptotic behaviour of Green's functions in certain gauges can be attributed to the contributions of <u>negative metric particles</u> (ghosts) in the intermediate states. It is a general feature of theories with a local symmetry that the spectrum of states which contribute to the Green's functions is itself gauge dependent. Although it is always possible to select a gauge in which such ghosts do not arise in the intermediate states (e.g. the unitary gauge in massive Yang-Mills theory or the radiation gauge in electrodynamics) one excludes thereby the possible benefits of their damping effect on the asymptotic behaviour of Green's functions.

It must be recognized that the computation of Green's functions is only a means to an end. Thus, from the Green's functions one proceeds to extract <u>gauge-independent</u> information: the spectrum of physical states, their S-matrix elements, etc. The ghosts discussed above, which are gauge-dependent phenomena, must decouple from the physical states. The use of ghosts in the intermediate stages of computation, however, can be advantageous in that it allows one to work with less singular quantities.

The purpose of this note is to suggest that a quantized theory of gravity - necessarily a non-polynomial gauge theory - may be free of ambiguity and entirely finite. Ordinary non-polynomial theories, like chiral theories, in which there is no local symmetry, are to some extent ambiguous on account of the exponentially increasing absorptive parts that are found at each order in the major coupling constant. Although it is possible to define the corresponding real parts by means of a minimal ansatz [4], this procedure has the character of an ad hoc prescription and fails, moreover, to remove all ambiguities. Even with this prescription, it is not possible to compute all physically relevant parameters. Our point is that in gravity theory it may be possible to find a class of gauges in which - owing to the ghost contributions - the absorptive parts are exponentially decreasing. In such gauges the real parts computed using standard superpropagator techniques would be completely free of ambiguities. This note is intended to provide an indication of the possible existence of such gauges and the way to choose them.

II. AMBIGUITIES IN NON-POLYNOMIAL THEORIES

In the following we shall work with localizable field theories. By this we mean that the Lagrangian is to be expressible as an entire function of the fields which parametrize it. (For example, $\mathcal{L} = (1/2)(\partial_\mu\phi)^2 - g(\exp[\kappa\phi]-1-\phi)$.) Green's functions are to be obtained by analytic continuation from the Symanzik region (roughly, the region where all external momenta are spacelike). The advantage of this is that the theory can be set up in euclidean space-time,

$$x^2 = -\underset{\sim}{x}^2 - x_4^2 < 0 \quad , \qquad x_4 = ix_0 \quad .$$

To see the origin of the ambiguities in non-polynomial field theories, consider the typical two-point function,

$$\left\langle T : \exp\big(\kappa\phi(x)\big) : \; : \exp\big(\kappa\phi(0)\big) : \right\rangle = \exp\left(-\frac{\kappa^2}{4\pi^2}\frac{1}{x^2}\right) , \qquad (1)$$

where $\phi(x)$ propagates as a free massless scalar field. The right-hand side, because of the strong singularity at $x^2 = 0$, is not generally a well-defined distribution. However, if $\mathrm{Re}\ \kappa^2 < 0$, we do have a well-defined distribution in euclidean space-time, $x^2 < 0$. In particular, it has a unique euclidean Fourier transform. Thus, if $\mathrm{Re}\ \kappa^2 < 0$, the right-hand side of (1) can be defined as a distribution in Minkowskian space-time by analytic continuation of the euclidean Fourier transform from the Symanzik region, $p^2 < 0$, to other regions of the complex p^2 plane. One can show, in particular, that if κ^2 is real and negative, the momentum space transform of (1) has an exponentially decreasing imaginary part on the positive p^2 axis.

Now in a typical field theory one expects the parameter κ to be real on account of the hermitian character of the Lagrangian, i.e. $\kappa^2 > 0$. In this case one cannot interpret the right-hand side of (1) so easily. The euclidean Fourier transform diverges. The simplest way to define this distribution is by means of analytic continuation in κ^2 . Thus, a momentum space amplitude which is real in the Symanzik region is obtained by continuing from the half plane, $\mathrm{Re}\ \kappa^2 < 0$, to positive real values of κ^2 and making a suitable average of the results - which depend upon the path followed in the continuation. It is in the phrase "suitable average" that all the ambiguities are sheltering. In effect we are simply giving a prescription for the subtraction constants, of which an infinite number are needed since, for $\kappa^2 > 0$, the absorptive part is exponentially increasing. An infinite number of subtraction constants go to make up an <u>entire function</u>, and this is the measure of the ambiguity in the distribution (1) for real κ .

The statements just made do not do full justice to the problem of defining (1), however. It can be shown that there is a unique preferred definition (i.e. choice of subtraction constants) which yields an amplitude whose asymptotic growth is "minimal". [4)] In particular, the real part of this amplitude tends to zero as $p^2 \to +\infty$. It has been shown by Lehmann and Pohlmeyer that such a minimality requirement serves to remove ambiguity - at least to the third order in the major coupling - from the Green's functions of <u>localizable</u> theories.

So much for non-polynomial theories of the ordinary kind. What we wish to suggest now is that, in theories with a local symmetry, <u>the gauge freedom can be so exploited that one encounters only distributions of the type (1) with $\kappa^2 < 0$</u> . In other words, the negative metric particles which appear in some gauges may serve to simulate the imaginary value of κ , which makes the distributions unambiguous. The best known example of a Lagrangian of physical interest with this type of gauge freedom is Einstein's. <u>In order to guarantee localizability,</u> the contravariant density is parametrized in the form

$$\hat{g}^{\mu\nu} \equiv (\sqrt{-g})^{\omega}\, g^{\mu\nu} = (\exp[\kappa\phi])^{\mu\nu} \quad ,$$

where the field $\hat{g}^{\mu\nu}$ has the weight ω,

$$\hat{g}^{\mu\nu}(x) \to \bar{\hat{g}}^{\mu\nu}(\bar{x}) = \left|\det \frac{\partial x}{\partial \bar{x}}\right|^{\omega} \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \hat{g}^{\alpha\beta}(x) .$$

Other such Lagrangians are the generally covariant versions of the recently proposed SL(6,C) invariant Lagrangians which describe the nonets of 2^{+} particles [5].

III. GREEN'S FUNCTIONS IN EINSTEIN'S THEORY

Consider Einstein's gravitational Lagrangian expressed in terms of $\hat{g}$

$$\mathcal{L} = \frac{1}{2\kappa^2} (-\det \hat{g})^{\Lambda} \left[\hat{g}_{\mu\lambda} \hat{g}_{\kappa\nu} \hat{g}^{\rho\sigma} - \frac{1}{2} \hat{g}_{\mu\kappa} \hat{g}_{\lambda\nu} \hat{g}^{\rho\sigma} - 2 \delta^{\sigma}_{\kappa} \delta^{\rho}_{\lambda} \hat{g}_{\mu\nu}\right] \times$$

$$\times \left[\hat{g}^{\mu\kappa}{}_{,\rho} + \Lambda \left(\ln(-\det \hat{g})\right)_{,\rho} \hat{g}^{\mu\kappa}\right] \times \left[\hat{g}^{\lambda\nu}{}_{,\sigma} + \Lambda\left(\ln(-\det \hat{g})\right)_{,\sigma} \hat{g}^{\lambda\nu}\right] . \quad (2)$$

The well-known Goldberg form of the Lagrangian is obtained by setting $\omega = 1$ in (2). Here

$$\Lambda = - \frac{\omega - 1}{2(2\omega - 1)} .$$

In order to fix the gauge, additional terms are needed:

$$\frac{1}{2\kappa^2} \left[\left[(-\det \hat{g})^{\Lambda} \hat{g}^{\mu\nu}\right]_{,\nu}\right]^2 + \mathcal{L}_{FP} , \quad (3)$$

where $\mathcal{L}_{FP}$ denotes the Faddeev-Popov contribution. This term involves a "fictitious particle" field described by anticommuting fields ξ and ζ with the Lagrangian

$$\mathcal{L}_{FP}(\xi,\zeta,\hat{g}) = (-\det \hat{g})^{\Lambda} \hat{g}^{\rho\nu} \left[\zeta^{\mu}{}_{,\nu} \xi^{\mu}{}_{,\rho} + \zeta^{\nu}{}_{,\mu} \xi^{\mu}{}_{,\rho} + \zeta^{\rho}{}_{,\nu\mu} \xi^{\mu}\right] ,$$

with

$$\langle T \xi^{\mu} \eta^{\nu} \rangle = \left(\eta_{\mu\nu} + 3 \frac{\partial_{\mu}\partial_{\nu}}{\partial^2}\right) D .$$

The bare graviton propagator is determined by the bilinear terms in (2) and (3),

$$\left\langle T\, \phi_{\kappa\lambda}\, \phi_{\mu\nu} \right\rangle = \frac{1}{2}\left[\eta_{\kappa\mu}\, \eta_{\lambda\nu} + \eta_{\kappa\nu}\, \eta_{\lambda\mu} - 2c\, \eta_{\kappa\lambda}\, \eta_{\mu\nu}\right] D(x) \,, \tag{4}$$

where $D(x)$ denotes the zero-mass scalar propagator and the gauge parameter $c = \omega(\omega-1) + \frac{1}{2}$. The case of $\omega = 1$ corresponds to the de-Donder gauge.

Since the Lagrangian (2) is a scalar density of unit weight while $\hat{g}^{\mu\nu}$ and its inverse $\hat{g}_{\mu\nu}$ are tensor densities of weight ω and $-\omega$, respectively, it follows that there must, of necessity, be a surplus of one power of the contravariant density in each term of (2). In the gauge-fixing term (3), which has weight two, there is a surplus of two. This suggests that expansions in powers of $\mathcal{L}_{\text{int}}$ will tend to behave like the amplitudes

$$\left\langle T\, \exp[\tfrac{\kappa}{\omega}\phi(1)]\, \exp[\tfrac{\kappa}{\omega}\phi(2)], \ldots \right\rangle \,, \tag{5}$$

in which the factor $\exp[\frac{\kappa}{\omega}\,\phi]$ appears repeatedly, but its inverse $\exp[-\frac{\kappa}{\omega}\,\phi]$ appears not at all. Assuming that this can be verified in detail, we substantiate the main point of this note by considering the explicit calculation of the two-point amplitude due to Ashmore and Delbourgo [6]:

$$\left\langle T(\exp \kappa\phi)^{\alpha\beta} \left(\exp \kappa\phi(0)\right)^{\gamma\delta} \right\rangle =$$

$$= \frac{2}{9}\left[\left(\eta^{\alpha\gamma}\, \eta^{\beta\delta} + \eta^{\alpha\delta}\, \eta^{\beta\gamma} - \frac{1}{2}\,\eta^{\alpha\beta}\, \eta^{\gamma\delta}\right) \frac{d}{d(\kappa^2 D)} + \frac{4c-1}{18}\left(\eta^{\alpha\gamma}\, \eta^{\beta\delta} + \eta^{\alpha\delta}\, \eta^{\beta\gamma}\right) + \right.$$

$$\left. + \frac{5-2c}{18}\, \eta^{\alpha\beta}\, \eta^{\gamma\delta}\right] a(D) \,, \tag{6}$$

where $a(D)$ is given by

$$a(D) = \left[\left(2-3z + \frac{1}{2}\,z^2\right) e^{-2cz} + \left(2 + 3z - z^2 + \frac{1}{2}\,z\left(z + \frac{1}{2}\right) 3\pi\, L_0(z) \right.\right.$$

$$\left.\left. - \frac{1}{2}\, z_1^2 \pi\, L_1(z)\right) e^{z(1-2c)}\right]_{z = \kappa^2 \frac{D}{2}} \,, \tag{7}$$

and L_0 and L_1 are Struve functions. The leading behaviour as $x^2 \to -0$ is given by

$$\left\langle T(\exp \kappa\phi)^{\alpha\beta}\, (\exp \kappa\phi(0))^{\gamma\delta} \right\rangle \sim \left(\eta^{\alpha\gamma}\, \eta^{\beta\delta} + \eta^{\alpha\delta}\, \eta^{\beta\gamma} + \eta^{\alpha\beta}\, \eta^{\gamma\delta}\right) \kappa D^{3/2} \exp[\kappa^2(1-c)D]. \tag{8}$$

It follows that the distribution (6) is finite and free of ambiguities provided

$$c > 1 \quad (\text{or} \quad \omega(\omega - 1) > \tfrac{1}{2}) \quad . \tag{9}$$

That is, the contributions of the ghosts in the class of gauges (9) indeed simulate an imaginary coupling constant (the operative combination is $\kappa^2(1-c)$ which appears in the exponent of (8)) and are sufficient to damp out the absorptive part of the two-point amplitude as $p^2 \to +\infty$.

What happens for the general amplitude

$$\langle T(\exp \kappa\phi(1))^{\alpha\beta} \quad (\exp \kappa\phi(2))^{\gamma\delta} \quad (\exp \kappa\phi(3))^{\rho\tau} \ ,\ldots \rangle \ ? \tag{10}$$

To evaluate (10) exactly is a task of extreme complexity. However, an indication of whether the amplitude is ambiguity-free may be provided by considering a very large value for c ($c \gg 1$). Since the right-hand side of (4) reduces to the scalar-ghost contribution $-c\eta_{\kappa\lambda}\,\eta_{\mu\nu}\,D(x)$, one may expect that for large c the amplitude (10) is likely to be dominated by the term

$$\exp\left[-\kappa^2 c \sum_{i<j} D(x_i - x_j)\right] ,$$

with $-\kappa^2 c$ in the exponent and should therefore be ambiguity-free.

An important point is the c independence of physical amplitudes. The precise form in which c (or ω) occurs in $\mathcal{L}_{\text{Einstein}}$ is given by (2). For matter fields, consider as an example a scalar field χ . The simplest form for its Lagrangian density (weight +1) is

$$\mathcal{L}_\chi = \left\{\frac{1}{2}\hat{g}^{\mu\nu}\,\nabla_\mu\chi\,\nabla_\nu\chi - m^2\,(-\det\hat{g})^{\omega/2(2\omega-1)}\,\chi^2\right\} ,$$

where the covariant derivative, $\nabla_\mu\chi$, is given by

$$\nabla_\mu\chi = \partial_\mu\chi - \frac{1}{4}\,\frac{(1-\omega)}{2\omega-1}\,\partial_\mu\,\ln(-\det\hat{g})\chi$$

since χ must have the weight $(1-\omega)/2$. Note now that the source tensor $T_{\mu\nu} = \delta\mathcal{L}_\chi/\delta\phi^{\mu\nu}$ has the expansion

$$\partial_\mu\chi\,\partial_\nu\chi - \eta_{\mu\nu}\,\frac{\omega}{2\omega-1}\,m^2\chi^2 + \eta_{\mu\nu}\,\frac{\omega}{4(2\omega-1)}\,\partial^2\chi^2 + O(\kappa) \quad .$$

By explicit calculation one can verify that the rather complicated ω dependence of $T_{\mu\nu}$ is just what is needed to ensure ω independence of tree diagrams on the mass shell, and the theory is indeed gauge independent to this approximation, and will remain so, up to the order in κ considered, when graviton lines are replaced by superpropagators.

To conclude, it appears likely that for calculations in gravity-modified electrodynamics for electron self-mass and self-charge, where amplitude (6) was all that was needed to obtain finite results to order [7] $\alpha \log (\kappa^2 m^2)$, the Lehmann-Pohlmeyer minimality ansatz is unnecessary and there are no ambiguities. In view of the generality and power of the mechanism discussed in this note, one may reasonably expect that the same is true for calculations in non-polynomial gravity [8] itself and other theories possessing gauge degrees of freedom, provided such calculations are carried out in an appropriate gauge. Thus non-polynomial gauge theories will be sharply distinguished from theories like non-linear chiral theory with no local symmetry.

REFERENCES AND FOOTNOTES

1) M. Baker and K. Johnson, Phys. Rev. D3, 2516 (1971).

2) G. 't Hooft, Nucl. Phys. B33, 173 (1971).

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4) H. Lehmann and K. Pohlmeyer in Nonpolynomial Lagrangian Renormalization and Gravity, 1971 Coral Gables Conference, Vol.1, p.60;
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5) C.J. Isham, Abdus Salam and J. Strathdee, ICTP, Trieste, preprint IC/72/155 (to appear in Phys. Rev.) and Int.Rep. IC/73/22.

6) J. Ashmore and R. Delbourgo, J. Math. Phys. 14, 176 (1973).

7) C.J. Isham, Abdus Salam and J. Strathdee, Phys. Rev. D5, 2548 (1972), see in particular p.2557, where it was noted that even in de-Donder gauge ($\omega = 1$, $c = \frac{1}{2}$) the scalar "ghost" contribution dominates the super-propagator

$$\left\langle T \exp\left(\kappa \operatorname{Tr} \phi(x)\right) \exp\left(\kappa \operatorname{Tr} \phi(0)\right)\right\rangle = \exp\left(-4\kappa^2 D(x)\right) \quad .$$

However, the importance of the circumstance that this makes the super-propagator ambiguity-free was not recognized.

8) We are indebted to Dr. R. Delbourgo for pointing out that the spin-zero ghost contributions survive even in the de-Donder gauge in the exact second-order calculation of graviton self-energy by Capper, Leibbrandt and Ramón Medrano (ICTP, Trieste, preprint IC/73/26, to appear in Phys. Rev.) and by M. Brown (Imperial College, London, preprint ICTP/72/13) where the Faddeev-Popov loops are properly included. This explicit calculation thus confirms the ideas of this paper.