IC/73/41
INTERNAL REPORT
(Limited distribution)

International Atomic Energy Agency
and
United Nations Educational Scientific and Cultural Organization

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON THE COMPUTATION OF EFFECTIVE POTENTIALS *

Abdus Salam
International Centre for Theoretical Physics, Trieste, Italy,
and
Imperial College, London, England,

and

J. Strathdee
International Centre for Theoretical Physics, Trieste, Italy.

ABSTRACT

The recent result of Coleman and Weinberg deriving the one-loop quantum corrected effective potential in a self-coupled scalar field theory is rederived using the Dyson-Schwinger equation.

MIRAMARE - TRIESTE
April 1973

* To be submitted for publication.

In a recent article Coleman and Weinberg [1] made the interesting point that the spontaneous violation of a Lagrangian symmetry could well be a purely quantum effect. They showed this by computing the one-loop corrections to the equations of motion in a simple model with quartic self-couplings of a set of massless scalar fields. The vehicle for this proof was an effective potential which they computed by summing all connected and one-particle irreducible one-loop graphs with zero-momentum external lines. It turns out that this quantum corrected potential can have a qualitatively different character from the purely classical component. For the case they consider, the new potential favours the emergence of a symmetry-breaking ground state in situations where the purely classical part does not.

The purpose of this note is to remark that the computation of effective potentials and of effective actions in general may be profitably undertaken within the formal framework of Dyson-Schwinger equations. This remark will come as no surprise to those who are familiar with functional methods but, it seems to us, the simplicity and formal power of the method deserves wider recognition [2]. The amusing thing is that, in the approximation studied by Coleman and Weinberg, the functional differential equation which defines the effective action reduces to an ordinary (albeit non-linear) differential equation. Although it may be exceedingly difficult to find exact solutions to this equation, it is certainly possible to develop a solution in powers of $\hbar$. The terms of this semiclassical development are obtained by simple quadratures.

Given a system of fields $\phi^i(x)$ whose dynamics is governed at the classical level by an action functional $S(\phi)$, the basic problem is to construct an effective action functional $W(\phi)$ which includes the quantum corrections. This effective action can be obtained, in principle, by solving the Dyson-Schwinger functional differential equations. Here we sketch very briefly the derivation of these equations.

Firstly, if the classical system is perturbed by the introduction of an external source term,

$$- \int dx \ J_i(x) \ \phi^i(x) \quad ,$$

into the action, the corresponding vacuum amplitude, $\exp(i/\hbar)Z(J)$, is given, for example, by the path integral

$$\exp(i/\hbar)\ Z(J) = \int (d\phi)\ \exp(i/\hbar) \left[S(\phi) - \int dx\ J_i(x)\ \phi^i(x)\right] . \quad (1)$$

The details of how this integral is to be defined are not important. One need suppose only that integration by parts is permissible, viz.,

$$0 = \int (d\phi)\ \frac{\delta}{\delta\phi(x')}\ \exp(i/\hbar) \left[S(\phi) - \int dx\ J_i(\phi)\ \phi^i(x)\right] .$$

It is then straightforward to deduce the functional differential equation,[3]

$$S_{,i}\left(- \delta Z/\delta J(x) - (\hbar/i)\ \delta/\delta J(x)\right) = J_i(x) , \quad (2)$$

where the operator $\delta/\delta J$ acts always to the right, and $S_{,i}(\phi)$ denotes the variational derivative $\delta S/\delta\phi^i(x)$. This equation can presumably be given a rigorous meaning if $S(\phi)$ is sufficiently regularized.

The solution of (2), subject to appropriate boundary conditions, should in principle yield the connected vacuum to vacuum transition amplitude $(i/\hbar)Z(J)$. An alternative version of this equation, in which the dependent variable $Z(J)$ is replaced by its Legendre transform, the effective action

$$W(\phi) = Z(J) + \int dx\ J_i(x)\ \phi^i(x) , \quad (3)$$

is more convenient, however. The new independent variable $\phi^i(x)$ is defined by

$$\phi^i(x) = - \delta Z(J)/\delta J_i(x) . \quad (4)$$

The inverse mapping is clearly given by

$$J_i(x) = \delta W(\phi)/\delta\phi^i(x) . \quad (5)$$

The connected vacuum amplitude $Z(J)$ is of course the generating functional for connected Green's functions. The effective action, $W(\phi)$, on the other hand, is the generating functional for one-particle irreducible vertices (see, for example, Jona-Lasinio [2]). In particular, since the Jacobian matrix $\delta\phi^i(x)/\delta J_j(x')$ is evidently the inverse of $\delta J_j(x')/\delta\phi^i(x)$, it follows that the inverse of the propagator,

$$G^{ij}(x,x'|J) = \frac{\delta^2 Z(J)}{\delta J_i(x)\, \delta J_j(x')} , \tag{6}$$

is given by

$$G^{-1}_{ij}(x,x'|\phi) = - \frac{\delta^2 W(\phi)}{\delta\phi^i(x)\, \delta\phi^j(x')} . \tag{7}$$

In passing from J_i to ϕ^i, as independent variable we can use the identity

$$\frac{\delta}{\delta J_i(x)} = - \int dx'\, G^{ij}(x,x'|\phi) \frac{\delta}{\delta\phi^j(x')} , \tag{8}$$

which follows from (4) and (6). It is necessary now to regard G^{ij} as a functional of ϕ rather than J. That is to say, we shall regard G^{ij} as being <u>defined by the inverse relation</u> (7).

The Dyson-Schwinger equation for $W(\phi)$ is now simply obtained from (2) by substituting from (4), (5) and (8),

$$\frac{\delta W(\phi)}{\delta\phi^i(x)} = S_{,i}\left\{\phi^j(x) + \frac{\hbar}{i} \int dx'\, G^{jk}(x,x'|\phi) \frac{\delta}{\delta\phi^k(x)}\right\} . \tag{9}$$

This equation, together with the definition(7) is to be regarded as the basic functional differential equation for the effective action $W(\phi)$.

Corresponding to the simple case of a single massive scalar field with quartic self-interactions,

$$S(\phi) = \int dx \left[\frac{1}{2} (\partial_\mu\phi)^2 - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4!} \phi^4\right] , \tag{10}$$

the Dyson-Schwinger equation takes the form

$$\frac{\delta W(\phi)}{\delta\phi(x)} = \frac{\delta S(\phi)}{\delta\phi(x)} - \frac{\hbar}{i} \frac{\lambda}{2} \phi(x)\, G(x,x|\phi) -$$

$$- \left(\frac{\hbar}{i}\right)^2 \frac{\lambda}{6} \int dy_1\, dy_2\, dy_3\, G(x,y_1|\phi)\, G(x,y_2|\phi)\, G(x,y_3|\phi) \frac{\delta^3 W}{\delta\phi(y_1)\delta\phi(y_2)\delta\phi(y_3)} . \tag{11}$$

This equation can be solved by iteration in $\hbar$. The coefficient of $\hbar^n$ in the solution would be represented by the set of n-loop irreducible vacuum graphs, in which the lines correspond to the classical propagator $G_0(x,x'|\phi)$ defined by

$$G_0^{-1}(x,x'|\phi) = \left(\partial^2 + m^2 + \frac{\lambda}{2}\phi(x)^2\right)\delta(x-x') ,$$

and the 3- and 4-leg vertices to the third and fourth order variational derivatives of S,

$$\Gamma_3(x_1,x_2,x_3|\phi) = -\lambda\,\phi(x_1)\,\delta(x_1-x_2)\,\delta(x_2-x_3)$$

$$\Gamma_4(x_1,x_2,x_3,x_4) = -\lambda\,\delta(x_1-x_2)\,\delta(x_2-x_3)\,\delta(x_3-x_4) .$$

Examples of such semiclassical developments of the effective action have been treated in great detail by DeWitt [2].

A method for dealing with (19), which is not semiclassical but which may, perhaps, be appropriate to the treatment of low-energy phenomena, is the Coleman-Weinberg scheme. The zeroth-order approximation here is assumed to be a local functional of $\phi(x)$ given by

$$W_0(\phi) = S(\phi) - \int dx\, V(\phi(x)) , \tag{12}$$

where V depends on $\phi(x)$ but <u>not its derivatives</u>. If this expression is substituted into (19) one obtains, <u>for constant ϕ</u>, the equation

$$\frac{dV}{d\phi} = \frac{\hbar}{i}\,\frac{\lambda}{2}\,\phi\, G(xx|\phi) - \left(\frac{\hbar}{i}\right)^2 \frac{\lambda}{6}\left(\lambda\phi + \frac{d^3V}{d\phi^3}\right)\int dy\, G(xy|\phi)^3 , \tag{13}$$

where $G(xy|\phi) = G(x-y|\phi)$, for constant ϕ, is defined by

$$G^{-1}(x-y|\phi) = \left(\partial^2 + m^2 + \frac{\lambda}{2}\,\phi^2 + \frac{d^2V}{d\phi^2}\right)\delta(x-y) \tag{14}$$

and is simply the free-field propagator with the effective mass $(m^2 + (\lambda/2)\,\phi^2 + d^2V/d\phi^2)^{1/2}$. The x dependence in (13) is spurious; since ϕ is a constant we can use translation invariance to set $x = 0$. Thus, it appears that $V(\phi)$ is to be obtained by solving a <u>second-order ordinary differential equation</u> for $dV/d\phi$.

The equation is highly non-linear and no doubt extremely difficult to solve exactly. Moreover, since the local approximation (12) is probably quite unrealistic for more than one loop approximation, such exact solutions may not have great relevance. Perhaps a better starting approximation could be made by allowing V to depend on $\partial_\mu\phi$ as well.

The one-loop contribution to V can be obtained from (13) by a simple quadrature. Thus, neglecting terms of order $\hbar^2$, one finds

$$\begin{aligned} V(\phi) &\approx \int_0^\phi d\phi \, \frac{\hbar\lambda}{2i} \phi \, G_0(xx|\phi) \\ &= \int_0^\phi d\phi \, \frac{\hbar\lambda}{2i} \phi \int \frac{dk}{(2\pi)^4} \left[-k^2 + m^2 + (\lambda/2)\,\phi^2\right]^{-1} \\ &= \frac{\hbar}{2i} \int \frac{dk}{(2\pi)^4} \, \ell n \left[\frac{-k^2 + m^2 + (\lambda/2)\,\phi^2}{-k^2 + m^2}\right] \\ &= \frac{\hbar}{2} \int \frac{dK}{(2\pi)^4} \, \ell n \left[\frac{K^2 + m^2 + (\lambda/2)\,\phi^2}{K^2 + m^2}\right] , \end{aligned} \tag{15}$$

after making a Wick rotation. This is essentially the result of Coleman and Weinberg. It remains to be seen whether or not the replacement of their summation procedure by a differential equation can lead to useful improvements in the approximation.

REFERENCES

1) S. Coleman and E. Weinberg, Phys. Rev. D7, 1888 (1973).

2) The ideas involved are all contained - at least implicitly - in the fundamental papers of Schwinger, Proc. Natl. Acad. Sci. U.S. 37, 452 (1951); 37, 455 (1951), and have been much developed by B.S. DeWitt, Dynamical Theory of Groups and Fields (Blackie, London 1965), Chapter 22, and G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964).

3) This derivation is due to D.G. Boulware and L.S. Brown, Phys. Rev. 172, 1628 (1968).

IC/73/1 R. DELBOURGO and ABDUS SALAM: The mass matrix for a leptonic U(3) model.

IC/73/2 ABDUS SALAM: Progress in renormalization theory since 1949.

IC/73/3 G.W. PARRY and P. ROTELLI: Application of the truncated gaussian to the inelastic pp charged multiplicity distribution.

IC/73/4 M. BABIKER: The gravitational red shift. A simple quantum field theoretical consideration ina curved space.

IC/73/5 AHMED ALI and M.S.K. RAZMI: $K_{\ell 3}$ form factors and the (8, 8) model of chiral symmetry breaking - unitarized version.

IC/73/6* INT.REP. WING-YIN YU: Bose distribution and the Lorentz contracted geometric model.

IC/73/7 C. FRONSDAL: Elementary particles in a curved space - II.

IC/73/8 L. FONDA, G.C. GHIRARDI, A. RIMINI and T. WEBER: On quantum foundations of the exponential decay law.

IC/73/9* INT.REP. M. KOCA: Inclusive reactions and the algebra of vertex strengths.

IC/73/10 A.O. BARUT and W. RASMUSSEN: Hydrogen atom as a relativistic elementary particle - I: The wave equation and mass formulae.

IC/73/11 A.O. BARUT and W. RASMUSSEN: Hydorgen atom as a relativistic elementary particle - II: Relativistic scattering problems and photo-effect.

IC/73/13 A.O. BARUT and H. BEKER: Classical relativistic scattering of two dyons.

IC/73/14 A.O. BARUT and A. SEVGEN: Alpha decay as a weak interaction.

IC/73/15* INT.REP. P. GENSINI: FESR for soft pion amplitudes, or the zero-energy theorems revisited.

IC/73/16* INT.REP. S.A. BARAN and A.O. BARUT: Cancellations of divergences in fourth-order μ-μ elastic scattering in the gauge model of weak interactions.

IC/73/17* INT.REP. H. SCHARFSTEIN: An algebraic classification of fields and particles.

IC/73/18* INT.REP. G.W. PARRY and P. ROTELLI: The normal distribution in two variables as a model for produced particle multiplicity distributions.

IC/73/19* INT.REP. A.N. TRIPATHY, K.C. MATHUR and S.K. JOSHI: On the polarization of the resonance lines of lithium and sodium atoms by electron impact.

IC/73/20* INT.REP. S.S. COHEN: Remarks on the central force approximation.

IC/73/21 M.J. DUFF: On the significance of perihelion shi calculations.

IC/73/22* INT.REP. C.J. ISHAM, ABDUS SALAM and J. STRATHDEE: Vector and tensor gauge particles in SL(6,c) theo

IC/73/23* INT.REP. K. HAYASHI: The A_1 and ρ mesons from the violation of local Lorentz invariance.

IC/73/24 A.O. BARUT: Conformal group → Schrödinger group → dynamical group. The maximal kinemat group of the massive Schrödinger particle.

IC/73/25* INT.REP. G. ALBERI, M. GREGORIO and Z. THOMÉ: On tl tail of the "spectator" in $K^+D \to K^0pp$.

IC/73/26 D.M. CAPPER, G. LEIBBRANDT and M. RAMÓN MEDRANO: Calculation of the graviton self-energy using dimensional regularization.

IC/73/27 E. IACOPINI: Remarks on some non-unitary representations of the Poincaré group.

IC/73/28 ABDUS SALAM: On SL(6,C) gauge invariance.

IC/73/30* INT.REP. S.M.M.R. CHOWDHURY and S.C. MUKHERJEE: Ionization in proton-hydrogen atom collisions.

IC/73/31* INT.REP. P. BANDYOPADHYAY and P. RAYCHAUDHURI: Weak interactions and self-consistent theory.

IC/73/33* INT.REP. A.O. BARUT: The impossibility of splitting the superposition of states of opposite magnetic charges in external electromagnetic fields - An anti-superselection rule.

IC/73/34 A.O. Barut: Magnetic charges and strings as buildir blocks of hadrons. Connection between electromagnetic and dual strings.

IC/73/35* INT.REP. A.M. HALPERN and J. LAW: K vacancy creation by high Z heavy ion impact.

IC/73/38 E.C. NJAU: The effect of winds on the electric field in the lower troposphere.

* Internal Report: Limited distribution.

THESE PREPRINTS ARE AVAILABLE FROM PUBLICATIONS OFFICE, ICTP, P.O. BOX 586, I-34100 TRIESTE, ITALY. IT IS NOT NECESSARY TO WRITE TO THE AUTHORS.