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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

ON SL(6,C) GAUGE INVARIANCE

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ABSTRACT

The Einstein-Cartan-Weyl theory of spin 2^+ particles is reviewed and its SL(2,C) gauge-invariant character brought out. The theory generalizes to describe nonets of 2^+ and 2^- particles when SL(2,C) invariance is extended to spin and unitary-spin-containing group SL(6,C) × SL(6,C). The conserved currents which close on the algebra of SL(6,C) × SL(6,C) are the sources of a spin-unitary-spin-torsional tensor which generalizes Cartan's tensor of spin-torsion.

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I. INTRODUCTION

Wigner's spin-isotopic-spin symmetry SU(4) was extended to SU(6) in 1964 by Gürsey, Radicati and Sakita. Ever since, we have had a problem:

a) Is there something deep in this unification of intrinsic spin with internal symmetry?

b) Do there exist Lagrangians with both kinetic energy and interaction terms invariant under SL(6,C) or U(6,6)? (SL(6,C) is the relativistic generalization of SU(6) and U(6,6) the relativistic generalization of U(6) × U(6).)

c) Equivalently, do there exist conserved currents which close on the algebra of SL(6,C)? If there do, what is their physical significance?

A parallel problem was posed by the theory of spin 2^+ strongly interacting mesons. In 1970 Isham, Strathdee and Salam and, independently, Wess and Zumino suggested that the strongly interacting massive spin 2^+ f^0 meson may be described by an equation similar to Einstein's with the field $f^{\mu\nu}$ replacing the graviton field $g^{\mu\nu}$. (One would of course also change the coupling parameter in Einstein's equation from the Newtonian constant G_N to the nuclear force constant G_F ($G_F \approx 10^{38} G_N$) and also supplement Einstein's equation with a mass term.) The problem with this suggestion was: how does one incorporate SU(3) (necessary to describe the known nonet of 2^+ particles) with the closely-knit space-time structure of Einstein's equation?

Clearly, both these problems are related to each other in that both require for their resolution a unification of internal symmetries with some sort of space-time structure.

I wish to report to-day some work that Isham, Strathdee and I have recently done in this direction. I shall essentially supply a critique of two notes we have written on the subject (Lettere al Nuovo Cimento 5, 969 (1972) and ICTP, Trieste, preprint IC/72/155).

Briefly, what we have discovered is that Weyl had already shown in 1929 that the Einstein-Cartan gravitational Lagrangian of 1922 possesses gauge invariance under the non-compact symmetry SL(2,C). When we rewrite the Einstein-Cartan-Weyl theory in a Dirac γ basis, we find that this permits of an instant generalization of SL(2,C) to SL(6,C) or SL(6,C) × SL(6,C) and, in particular, of Cartan's equation for spin-torsion, which generalizes to include what one may call internal-spin-torsion. One obtains thus an elegant unification of intrinsic spin with internal symmetries.

I am hoping very much that our approach may, at the least, make the beautiful geometrical ideas of Einstein, Weyl and Cartan accessible to particle physicists, though our motivation is to use these ideas for strong interaction physics. The literature on general relativity places so much emphasis on the GL(4,R) group of general co-ordinate transformations, that the other invariances of the theory, like SL(2,C) gauge invariance, so much more relevant to particle physicists' experience, tend to be ignored. We hope the balance gets somewhat redressed by our demonstration of the value of these other invariances of the Einstein-Cartan theory.

During the second part of my talk I shall be concerned with the problem of the particle spectrum given by the SL(6,C) × SL(6,C) gauge-invariant Lagrangian type. In particular we must ensure (like Einstein and Weyl had to) that, considered as a classical Lagrangian, ours permits of the excitation of only the positive frequencies. Here, we shall use, again and again the ideas of spontaneous symmetry breaking and non-zero expectation values of tensor fields $\left(\langle f^{\mu\nu}\rangle = \lambda\,\eta^{\mu\nu}\right)$ - amazingly enough, essentially first introduced by Einstein (though not using this language) and now so much in prominence in particle physics.

II. GAUGING THE NON-COMPACT GROUP SL(2,C). EINSTEIN-CARTAN-WEYL THEORY

2.1 The twin ideas of gauge-invariant Lagrangians and spontaneous symmetry breaking have played an important role recently in particle physics. As I have said before, both ideas go back fifty years to the work of Weyl and Einstein. Let us first take gauge Lagrangians.

DEVELOPMENT OF IDEAS IN GAUGE THEORIES

	Gauge group	Gauge particle(s)
1. (1918); recognition by Weyl that the Maxwell Lagrangian is a gauge Lagrangian.	U(1)	1^- photon
2. (1929); recognition by Weyl, Fock, Ivanenko that the Einstein-Cartan Lagrangian of 1922 is a gauge Lagrangian.	non-compact SL(2,C)	2^+ graviton
3. (1954); generalization by Yang, Mills and Shaw of Weyl's 1918 theory.	SU(2)	1^- triplet ρ^+,ρ^0,ρ^-
4. (1972); present generalization of Weyl's theory of 1929 .	SL(6,C)	nonet of 2^+ particles F,F',A_2,K^{**}
	SL(6,C)×SL(6,C)	nonets of 2^+ & 2^- particles

Let us consider the Einstein-Cartan theory of gravity in Weyl's formulation. As is well known, Einstein's gravity theory works with a ten-component symmetric tensor field $g^{\mu\nu}(x)$. In order to describe gravitational interaction of spin-$\frac{1}{2}$ particles, Weyl introduced the so-called vierbein 16-component fields $L^{\mu a}(x)$, which bear to $g^{\mu\nu}(x)$ the same relation as Dirac's γ matrices bear to the unit matrix, i.e. $L^{\mu a}$'s are essentially a square root of $g^{\mu\nu}$, in the sense

$$\eta_{ab}\, L^{\mu a}(x)\ \ L^{\nu b}(x) = g^{\mu\nu} ,$$

where

$$\eta_{ab} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} .$$

In the sense of interpolating fields, both fields, $g^{\mu\nu}$ and $L^{\mu a}$, are at par; both could describe the graviton equally well.

Now, one of the curses of relativity theory is the multiplicity of indices which, in the cluster of formalism, successfully obscure the real heart of the basic ideas. In order to minimize indices, I shall use the Dirac γ basis to write my formulae. For SL(2,C) I shall need the four γ_a's and the six generators σ_{ab}'s, with commutation relations of the type:

$$[\sigma , \sigma] = i\, \sigma$$

$$[\gamma , \sigma] = i\, \gamma .$$

In this Dirac basis the expression $L^{\mu}(x)$ will stand for the 4×4 matrix combination of Weyl fields $L^{\mu a}\, \gamma_a$. In particular, $g^{\mu\nu} = \frac{1}{4}\, \mathrm{Tr}(L^{\mu} L^{\nu})$.

2.2 Consider a spinor $\psi(x)$ which, for the SL(2,C) index transformations, transforms as

$$\psi(x) \rightarrow \psi' = \Omega\, \psi(x) ,$$

where

$$\Omega = \exp\ i\left(\sigma_{ab}\, \epsilon^{ab}\right) .$$

Note that there is no x transformation implied at this stage. The only role the x co-ordinate will play will come about when we consider, in the standard gauge fashion, the parameters ϵ^{ab} to be functions of x. If $\epsilon^{ab} = \epsilon^{ab}(x)$, clearly

$$\partial_\mu \psi \nrightarrow \Omega \partial_\mu \psi(x) \quad .$$

The ordinary derivative does not transform in a simple manner. To "correct" this, introduce in the standard gauge fashion, the 24-component gauge field $B_\mu(x) = B_\mu^{ab} \sigma_{ab}$ ($B_\mu^{ab} = -B_\mu^{ba}$) — also called the "Weyl connection". Provided that B_μ transforms as

$$B_\mu \rightarrow \Omega B_\mu \Omega^{-1} - i \Omega \partial_\mu \Omega^{-1} ,$$

the "Weyl covariant" derivative,

$$\nabla_\mu \psi = (\partial_\mu + iB_\mu) \psi ,$$

transforms "correctly" in the same manner as ψ itself; i.e.

$$\nabla_\mu \psi \rightarrow \Omega(x) \nabla_\mu \psi \quad .$$

Also, since

$$L^\mu \rightarrow \Omega(x) L^\mu(x) \Omega^{-1}(x) \quad ,$$

we may define

$$\nabla_\mu L^\nu = \partial_\mu L^\nu + i [B_\mu, L^\nu] ,$$

which will transform as:

$$\nabla_\mu L^\nu \rightarrow \Omega (\nabla_\mu L^\nu) \Omega^{-1} \quad .$$

Finally, from the transformation law of B_μ, one can easily verify that the "covariant curl"

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + i [B_\mu, B_\nu]$$

transforms as:

$$B_{\mu\nu} \rightarrow \Omega B_{\mu\nu} \Omega^{-1} \quad .$$

We are now ready to write SL(2,C) invariant Lagrangians. Summarizing the transformations:

$$
\left.\begin{array}{lll}
\psi \to \Omega\,\psi & , & \bar\psi \to \bar\psi\,\bar\Omega \quad (\bar\Omega = \Omega^{-1}) \\
L^\mu \to \Omega\, L^\mu\, \Omega^{-1} & , & B_{\mu\nu} \to \Omega\, B_{\mu\nu}\, \Omega^{-1} \\
\nabla_\mu \psi \to \Omega\, \nabla_\mu\, \psi & , & \nabla_\mu \psi = (\partial_\mu + iB_\mu)\,\psi
\end{array}\right\} \tag{1}
$$
,

we immediately see that

$\mathcal{L}_{\text{matter}} = i\bar\psi\,(L^\mu \nabla_\mu)\,\psi + m\bar\psi\psi$ is SL(2,C) gauge invariant ,

$\mathcal{L}_{\text{Weyl}} = -i\,\mathrm{Tr}\,[L^\mu, L^\nu]\,B_{\mu\nu}$ is SL(2,C) gauge invariant .

The amazing fact is that the beautifully simple expression $\mathcal{L}_{\text{Weyl}}$ will turn out to be identical to the well-known Einstein Lagrangian for gravity, when no matter is present. When spin-$\frac{1}{2}$ matter is present, $\mathcal{L}_{\text{Weyl}} + \mathcal{L}_{\text{matter}}$ gives Cartan's generalization (1922) of Einstein's theory. I shall demonstrate this equivalence in a heuristic fashion presently. Before doing this, however, let us consider this Lagrangian and the field equations following from it in some more detail.

2.3 Consider

$$
\mathcal{L} = \mathrm{Tr}\left(-i\,[L^\mu, L^\nu]\left\{\partial_\mu B_\nu - \partial_\nu B_\mu + i\,[B_\mu, B_\nu]\right\}\right)
$$

$$
+ i\bar\psi\, L^\mu\, (\partial_\mu + iB_\mu)\psi + m\bar\psi\psi + \text{h.c.} \tag{2}
$$

The field equations are:

$$
\boxed{\begin{array}{lll}
i\,[L^\mu, B_{\mu\nu}] = T_\nu & & \text{Einstein's curvature equation} \quad (3) \\
\nabla_\mu\, [L^\mu, L^\nu] = S^\nu & & \text{Cartan's torsional equation .} \quad (4)
\end{array}}
$$

Here T_ν and S^ν are the matter stress-tensor density and the matter intrinsic-spin densities, respectively. ($T_\nu = i\nabla_\nu\,\psi\bar\psi$, $S^\nu = i\psi\bar\psi L^\nu$.)

Operating on the second equation by ∇_ν and using the first equation we obtain

$$
\boxed{\nabla_\nu\, S^\nu = i\,[L^\nu, T_\nu] \qquad \underline{\text{Tetrode identity}} \quad .} \tag{5}
$$

Cartan's equation (4) written out in detail reads:

$$i\left[B_\mu,[L^\mu,L^\nu]\right] = S^\nu_{\text{matter}} - \partial_\mu [L^\mu,L^\nu] \qquad . \tag{6}$$

Essentially this equation tells us that B_μ can be solved in terms of $\partial_\mu L^\nu$, L^ν and matter spin density. When this solution for B_μ is substituted into Einstein's equation, we obtain a second-order equation for L^μ. Equivalently, one may substitute for B_μ into the Lagrangian (2) and recover a second-order Lagrangian for L^ν which is identical to the Einstein Lagrangian for $g^{\mu\nu} = \frac{1}{4}\,\mathrm{Tr.}\ L^\mu L^\nu$ plus a contact term proportional to the square of the spin density $\left((S^\nu)^2\right)$.

If we define the currents J^ν :

$$\boxed{J^\nu = S^\nu_{\text{matter}} - i\left[B_\mu,[L^\mu,L^\nu]\right]} \tag{7}$$

we see that

$$\partial_\nu J^\nu = \partial_\nu\partial_\mu [L^\mu,L^\nu] \equiv 0 \qquad . \tag{8}$$

It is easy to show that this conserved set of currents J^ν are indeed the appropriate Noerther currents for (2) and close on the algebra of SL(2,C) (see ICTP, Trieste, preprint IC/72/155, Sec.V).

2.4 As I just said, the equivalence of the theory described above with Einstein's gravitational theory can be shown algebraically by solving (6) for B_μ and eliminating this variable from (2). However, an elegant geometrical proof can be given and I shall sketch it for the case when no matter is present. First note an algebraic identity due to Möller:

$$\mathcal{L} = \mathrm{Tr}\left[-\,i[L^\mu,L^\nu]\,B_{\mu\nu}\right] \equiv \mathrm{Tr}\left\{(\nabla_\mu L^\nu)(\nabla_\nu L^\mu) - (\nabla_\mu L^\mu)(\nabla_\nu L^\nu)\right\} + \text{a surface term.} \tag{9}$$

So far we have ignored space-time transformations of $L_\mu(x)$ and $B_\mu(x)$. Let us assume with Einstein that these transform as standard contra- and covariant quantities:

$$B_\mu(x) \longrightarrow B'_\mu(x) = \frac{\partial x^\nu}{\partial x'^\mu}\, B_\nu(x) \tag{10}$$

$$L^\mu(x) \longrightarrow L'^\mu(x) = \frac{\partial x'^\mu}{\partial x^\nu}\, L^\nu(x) \quad . \tag{11}$$

The remarkable thing about the expression $-\,\mathrm{Tr}\; i[L^\mu,L^\nu]\,B_{\mu\nu}$ is that it transforms like a scalar for the general co-ordinate transformations, in Einstein's sense, the reason for this being that $B_{\mu\nu}$ has the character of a "curl". We may now use

$$g^{\mu\nu} = \tfrac{1}{4}\,\mathrm{Tr}\,L^\mu\,L^\nu \tag{12}$$

to raise the lower indices. To link with Cartan, we may define the "Cartan covariant derivative" (denoted with a double stroke $\|$) which must take account both of the general co-ordinate transformations and the Weyl SL(2,C) transformations. Quite generally the linearity of this "connection" requires that:

$$L^\nu_{\|\mu} = \partial_\mu L^\nu + i[B_\mu, L^\nu] + \begin{pmatrix}\nu\\ \mu\rho\end{pmatrix} L^\rho \quad .$$

Here $\begin{pmatrix}\nu\\ \mu\rho\end{pmatrix}$ is a Christoffel-like (asymmetric) connection, for which we shall demand that the Cartan-derivative of L^μ (or equivalently of $g^{\mu\nu}$) vanishes. Thus $\begin{pmatrix}\nu\\ \mu\rho\end{pmatrix}$ is defined by the relation:

$$0 = L^\nu_{\|\mu} = \nabla_\mu L^\nu + \begin{pmatrix}\nu\\ \mu\rho\end{pmatrix} L^\rho \quad . \tag{13}$$

Using (13), clearly, the Möller form of Weyl's Lagrangian,

$$(\nabla_\mu L^\nu)(\nabla_\nu L^\mu) - (\nabla_\mu L^\mu)(\nabla_\nu L^\nu) ,$$

reduces to a form familiar from Einstein (when no matter is present):

$$\mathcal{L}_{\text{Einstein}} = g^{\rho\rho'} \left(\begin{pmatrix} \mu \\ \nu\rho \end{pmatrix} \begin{pmatrix} \nu \\ \mu\rho' \end{pmatrix} - \begin{pmatrix} \mu \\ \mu\rho \end{pmatrix} \begin{pmatrix} \nu \\ \nu\rho' \end{pmatrix} \right) .$$

(For completeness, one should remark that one must divide $\mathcal{L}_{\text{Einstein}}$ (or $\mathcal{L}_{\text{Weyl}}$) by the factor $\sqrt{-\det g^{\mu\nu}} = \sqrt{-\det (\frac{1}{4} \operatorname{Tr} L^\mu L^\nu)}$ in order that $\int \mathcal{L}\, d^4x$ transforms as a scalar. Since in what follows we do not wish to worry about general co-ordinate transformations but only about the Poincaré set of space-time transformations, so far as strong interaction physics is concerned, this refinement can for the present be ignored.)

Note, in passing, that the Cartan equation (4),

$$\nabla_\mu [L^\mu, L^\nu] = S^\nu ,$$

reads, in terms of the generalized "Christoffel" connection:

$$\left[L^\nu \begin{pmatrix} \mu \\ \mu\rho \end{pmatrix} - L^\mu \begin{pmatrix} \nu \\ \mu\rho \end{pmatrix}, L^\rho \right] = S^\nu .$$

It is easy to see that the equation relates the antisymmetric part of $\begin{pmatrix} \nu \\ \mu\rho \end{pmatrix}$ in μ,ρ indices, i.e. the torsion tensor, to the spin density.

2.5 As will be seen later, when we generalize $SL(2,C)$ to $SL(6,C)$, it is the Cartan equation and the spin-density S^ν which get generalized to include not only spin but also internal spin. But before we exhibit this generalization, consider: what is the effect of the spin-torsion terms of Cartan in gravitational theory? Kibble has shown that, to the first order in the newtonian constant G_N, the gravitational potential between the two spin-$\frac{1}{2}$ particles acquires an extra repulsive contact term proportional to $G_N(\bar{\psi} \gamma_5 \gamma_\mu \psi)^2$, which, in the non-relativistic limit, reduces to a repulsive contact potential proportional to the square of the spin-density. The important point about this repulsive contact potential is that it is gravitational in origin and comes about on account of the torsional characteristics of space-time structure.

Following from this, recently Trautman has argued that the singularities of gravitational collapse and cosmology may be prevented by the direct influence of spin on the geometry of space-time, in virtue of Cartan's equation above. Trautman considers a universe filled with spinning dust, with spins all aligned along one direction - due presumably to the influence of some cosmic magnetic field. The Einstein and Cartan equations are compatible with a Robertson-Walker line element,

$$(ds)^2 = (dt)^2 - (R(t))^2 \left((dx)^2 + (dy)^2 + (dz)^2\right) \quad .$$

For small R, the spin density on the right-hand side of Cartan's equation plays the role of a "repulsive potential", which counteracts the universal "attractive" gravitational force. A universe consisting of 10^{80} neutrons would attain R_{min} of the order of 1 cm and collapse no further.

I am mentioning this because later, when we have generalized the Einstein-Cartan-Weyl Lagrangian to an SL(6,C) invariant form, we may find some speculative reasons why spin alignments and isotopic-spin alignments should occur together in regions of extreme spin-isotopic-spin density.

III. GENERALIZATION OF EINSTEIN-CARTAN-WEYL THEORY TO SL(6,C) GAUGE INVARIANCE

The generators of the SL(6,C) group are given by $\sigma_{ab}\,\lambda^i$, $\gamma_5\,\lambda^i$, λ^i , while $\gamma_a\,\lambda^i$, $i\,\gamma_a\,\gamma_5\,\lambda^i$ give the appropriate SL(6,C) generalization of the "ideal" γ_a .

Generalize L^μ and B_μ to contain (4×72) components each (rather than 4×4 components); thus

$$L^\mu = L^{\mu a i}\left(\gamma_a\,\frac{\lambda^i}{2}\right) + L^{\mu a i 5}\left(i\,\gamma_a\,\gamma_5\,\frac{\lambda^i}{2}\right) \tag{14}$$

$$B_\mu = B_\mu^{abi}\left(\frac{\sigma_{ab}\lambda^i}{4}\right) + B_\mu^i\,\frac{\lambda^i}{2} + B_\mu^{5i}\left(\frac{\lambda^i}{2}\,\gamma_5\right) , \tag{15}$$

and now adopt the same expression (2) as the Lagrangian for strong interactions exhibiting SL(6,C) gauge invariance. (Here λ^i are the nine 3×3 Gell-Mann U(3) matrices.) It is a triumph of the Dirac basis for the SL(2,C) case that the formalism carries over directly from SL(2,C) to SL(6,C).

Later we shall see that we need to add some more terms to the Lagrangian (2), particularly in order that the particles described by the Lagrangian possess mass. However, at this stage, remark that the Einstein's curvature equation (3), Cartan's torsional equation (4), the tetrode identity (5) and the definition of conserved currents (6) (which now close on the algebra of SL(6,C)) carry over directly without change from the SL(2,C) case, except that we are now dealing with 72-beins rather than vierbeins. Also remark that if the internal symmetry group were not a unitary group, but some other variety of Lie group, the generalization of SL(2,C) to include internal symmetries may have presented difficulties.

IV. THE PARTICLE SPECTRUM

As I said earlier, so far as strong interaction physics is concerned, we shall not worry, for the present, with general co-ordinate transformations. The symmetry group we shall specialize to, has the structure of a semi-direct product

$$P \odot SL(6,C) ,$$

where P denotes the Poincaré group. (The distinction of upper and lower indices is now trivial $(L^{\mu} = \eta^{\mu\nu} L_{\nu})$.)

Before considering the complicated SL(6,C) case, let us examine the meaning of SL(2,C) gauge invariance for the Einstein-Weyl-Cartan theory and introduce with Einstein the ideas of spontaneous symmetry breaking.

4.1 The Einstein-Weyl Lagrangian (and also the spin-$\frac{1}{2}$ Lagrangian in the limit $m = 0$) possesses no terms bilinear in field variables. If we assume with Einstein that

$$L^{\mu a}(x) = \eta^{\mu a} + \kappa \, \phi^{\mu a}(x) , \qquad (16)$$

where κ is the (strong gravity) coupling constant and $\phi^{\mu a}(x)$ is the (quantized) field variable with zero expectation value, then

$$\langle L^{\mu} \rangle = \gamma^{\mu} . \qquad (17)$$

The symmetry-breaking implied by $\langle L^{\mu a} \rangle = \eta^{\mu a}$ provides a bridge, through an identification of Greek (μ) and Latin (a) indices, between the Poincaré transformations and the index transformations SL(2,C). If we now set

$$L^\mu = \gamma^\mu + \kappa\phi^\mu$$

in the Einstein-Weyl-Dirac Lagrangian, we do recover a set of bilinear terms and with them a particle spectrum. In fact, symbolically, the structure of our Lagrangian now looks like:

$$\mathcal{L}_{\text{Weyl}} = L^2\, \partial B + L^2 B^2 + \overline{\psi}\, L\, \partial\, \psi$$

$$= \left\{ \begin{matrix} \phi\, \partial\, B + B^2 \\ + \overline{\psi}\, \gamma\, \partial\, \psi \end{matrix} \right\} + \left\{ \begin{matrix} \phi^2\, \partial\, B + \phi^2\, B^2 \\ \overline{\psi}\, \phi\, \partial\, \phi \end{matrix} \right\} .$$

Approximating $\mathcal{L}$ by its bilinears, we recover the field equations,

$$B = \partial\phi \quad , \quad \partial B = 0 \Longrightarrow \partial^2\phi = 0 \quad ;$$

telling us that we are dealing with a massless field ϕ (the graviton).

Before proceeding, let us add to $\mathcal{L}_{\text{Weyl}}$ further SL(2,C) gauge-invariant terms which give the particles mass, through the familiar intrinsic symmetry-breaking mechanism. Write

$$\mathcal{L}_{\text{mass}} = \text{Tr}\left(\beta_1\, L^\mu L_\mu + \beta_2 (L^\mu L_\mu)^2 + \beta_3 (L^\mu L^\nu)(L_\mu L_\nu)\right) . \tag{18}$$

We shall see later that this term indeed gives rise to a mass M for 2^+ particles, provided $\beta_1 + 8\beta_2 - 4\beta_3 = 0$ and $\beta_1 = -(3M^2/2\kappa^2)$, $\beta_2 = -\beta_3 = (M^2/3\kappa^2)$ consistent with the spontaneous symmetry-breaking ansatz $\langle L^\mu \rangle = \gamma^\mu$, in a manner very familiar nowadays from the Higgs-Kibble theory. (Freund and Maheshwari have remarked that in a GL(4,R) theory, which is invariant for general co-ordinate transformations, $\text{Tr}\, L^\mu L_\mu$ is a constant and we would not have been able to obtain mass from a symmetry-breaking formalism.)

4.2 Consider now the meaning of SL(2,C) gauge invariance. Infinitesimally,

$$L^\mu \longrightarrow L^\mu + i\left[\epsilon^{ab}\,\sigma_{ab}\, ,\, L^\mu\right] .$$

<u>Clearly, the gauge transformation affects only the antisymmetric parts of $L^{\mu a}$</u> $\left(L^{[\mu a]} = \tfrac{1}{2}(L^{\mu a} - L^{a\mu})\right)$ <u>infinitesimally.</u>

In fact, SL(2,C) gauge invariance of the Lagrangian is simply the statement that the antisymmetric components of $L^{\mu a}$ do not represent dynamical degrees of freedom and we can specialize to a gauge where these can be set equal to zero. The bilinear part of the Weyl Lagrangian now reads:

$$\mathcal{L}_{(2)} = -\frac{2}{\kappa^2} B_\mu^{\nu\alpha} (\varphi_{\mu\alpha,\nu} - \eta_{\mu\nu}\varphi_{\lambda\alpha,\lambda})$$

$$- \frac{1}{\kappa^2} (B_\mu^{\nu\alpha} B_\nu^{\mu\alpha} - B_\mu^{\mu\alpha} B_\nu^{\nu\alpha})$$

$$+ 8\beta_3 \varphi_{\mu\nu}\varphi_{\mu\nu} + 4(\beta_2 - \beta_3)\varphi_{\mu\mu}\varphi_{\nu\nu} \ .$$

This is the well-known Pauli-Fierz Lagrangian where the only particle excitations correspond to those of a mass M and spin 2^+ particles, provided

$$\beta_1 = -\frac{3M^2}{2\kappa^2} \quad , \qquad \beta_2 = -\beta_3 = \frac{M^2}{8\kappa^2} \ . \tag{19}$$

V. PARTICLE SPECTRA FOR SL(6,C) AND SL(6,C) × SL(6,C) GAUGE LAGRANGIANS

5.1 Consider now the Weyl Lagrangian,

$$L = \mathrm{Tr}\ i\ [L^\mu, L^\nu]\ B_{\mu\nu} + \mathcal{L}_{mass} \text{ (given by Eq.(18))} , \tag{20}$$

generalized to SL(6,C), and once again consider the bilinears obtained by setting

$$L^\mu = \gamma^\mu + \kappa\phi^\mu \ .$$

As one can see from (14) and (15), in addition to nonets of conjugate fields $L^{\mu a(k)}$ and $B_\mu^{ab(k)}$, the SL(6,C) theory needs the introduction of the following extra fields: $L^{\mu a5(k)}$, $B_\mu^{(k)}$, $B_\mu^{5(k)}$. If one examines the bilinear part of (20), one finds that the conjugate set $L^{\mu a(k)}$, $B_\mu^{ab(k)}$ correctly gives the propagation of a massive 2^+ nonet. The extra fields $L^{\mu a5(k)}$, $B_\mu^{(k)}$ and $B_\mu^{5(k)}$, however, make their appearance only in one place among the kinetic energy bilinears in a term which reads:

$$L^{\mu\nu 5(k)} \left(B_{\nu,\mu}^{5(k)} - B_{\mu,\nu}^{5(k)} \right) \ .$$

These fields do appear in trilinear and quadrilinear parts of the Lagrangian, but the bilinear terms give no clue as to their propagation character. This implies that either we should devise methods by which we can infer the particle spectrum corresponding to these fields from the trilinear and quadrilinear parts of the Lagrangian, or we should supplement the Weyl Lagrangian (20) by additional SL(2,C) invariant terms. These should be such as to give new sets of bilinears which should guarantee the (positive metric) propagation properties for the extra fields.

In the following I shall illustrate both approaches.

5.2 Before going on to consider the problem posed above, let me return for a moment to Weyl's SL(2,C) gauge-invariant Lagrangian and try to bring out the significance of its gauge invariance in a slightly different manner. The remarks I shall make will be relevant to the problem of propagation of the extra fields $L^{\mu a5}$, B_μ and B_μ^5.

Given a 16-component field quantity L^μ (with a non-zero expectation value $\langle L^\mu \rangle = \gamma^\mu$), one has a mathematical theorem - the so-called "polar decomposition theorem" - which states that one can write L^μ uniquely in the form:

$$L^\mu = S \, \ell^\mu \, S^{-1} , \qquad (21)$$

where ℓ^μ is symmetric in the sense $\ell^{\mu a} = \ell^{a \mu}$ and S has the form:

$$S = \exp i P \quad ; \quad P = P^{ab} \sigma_{ab} .$$

To prove this result, set up an iteration system,

$$L^\mu = \gamma^\mu + L_1^\mu + L_2^\mu + \cdots , \quad \ell^\mu = \gamma^\mu + \ell_1^\mu + \ell_2^\mu + \cdots ,$$

$$P = P_1 + P_2 + \cdots$$

The relation (21) is equivalent to the set of equations

$$\left. \begin{aligned} \ell_1^\mu &= L_1^\mu - i \, [P_1, \gamma^\mu] \\ \ell_2^\mu &= L_2^\mu - i \, [P_1, L_1^\mu] - i \, [P_2, \gamma^\mu] \end{aligned} \right\} . \qquad (22)$$

Clearly, the postulated symmetry of ℓ_1^μ implies that ℓ_1^μ and P_1 are respectively given by the symmetrical and antisymmetrical parts of L_1^μ and so on. Clearly, also in terms of the "polar decomposition" above, we can understand the transformation

$$L^{\mu} \to \Omega L^{\mu} \Omega^{-1} \tag{23}$$

as equivalent to the transformation

$$S \to \Omega S , \tag{24}$$

with ℓ^{μ} as a scalar quantity so far as the SL(2,C) transformations are concerned.

Given S as a functional of L^{μ}, define now a lower-case quantity b_{μ} through the relation

$$B_{\mu} = S b_{\mu} S^{-1} - i S \partial_{\mu} S^{-1} . \tag{25}$$

Combined with the transformation law for B_{μ}, viz.,

$$B_{\mu} \to \Omega B_{\mu} \Omega^{-1} - i \Omega \partial_{\mu} \Omega^{-1} , \tag{26}$$

the relation (25) guarantees that b_{μ} is also a scalar. The invariance of

$$\mathcal{L} = \mathrm{Tr}\ i\ [L^{\mu},L^{\nu}]\ B_{\mu\nu} \tag{27}$$

is now simply the statement that $\mathcal{L}$ identically equals

$$\mathcal{L} = \mathrm{Tr}\ i\ [\ell^{\mu},\ell^{\nu}]\ b_{\mu\nu} , \tag{28}$$

and that there is no S-dependence of the Lagrangian.

5.3 Let me now return to the problem posed earlier: the problem of propagation of the fields $L^{\mu a 5(k)}$, $B_{\mu}^{(k)}$ and $B_{\mu}^{5(k)}$.

So far as the variables $L^{\mu a 5(k)}$ are concerned, it appears that a simple "classical completion" of the Lagrangian is provided by an extension of the gauge group from SL(6,C) to SL(6,C) × SL(6,C). The details are worked out in Sec.IV of ICTP, Trieste, preprint IC/72/155. Here I wish to illustrate one important new idea which we had to introduce to cut down the multiplicity of fields and to guarantee a positive metric for the particles described by $L^{\mu a 5(k)}$. This is the idea of (SL(6,C) × SL(6,C)) <u>covariant constraints</u>.

Consider gauge transformations of the type

$$\Omega = \exp(i\beta)\,\exp(\gamma), \tag{29}$$

where β and γ contain 72 parameters each, corresponding to $SL(6,C) \times SL(6,C)$. The theory would possess two gauge fields now:

$$\left.\begin{aligned} (B + iC)_\mu &\to \Omega(B + iC)_\mu\, \Omega^{-1} - i\, \Omega\, \partial_\mu\, \Omega^{-1} \\ (B - iC)_\mu &\to (\bar{\Omega})^{-1}\, (B - iC)_\mu\, \bar{\Omega} - i\, \bar{\Omega}^{-1}\, \partial_\mu\, \bar{\Omega} \end{aligned}\right\} \quad . \tag{30}$$

So far, everything is straightforward. But now comes the subtle new feature of the theory. In general we should work with two distinct L-type fields, with the transformation characters,

$$L_1^\mu \to \Omega\, L_1^\mu\, \bar{\Omega} \tag{31}$$

$$L_2^\mu \to \bar{\Omega}^{-1}\, L_2^\mu\, \Omega^{-1} \quad . \tag{32}$$

In order, however, to reduce the independent degrees of freedom, we can tie these two fields together, defining one as a non-linear functional of the other, consistent with the transformation laws (31) and (32). To see this, assume we are given L_1^μ . One can show that a "polar decomposition" exists which permits us to write

$$L_1^\mu = S\left(\ell^{\mu a(k)}\, \gamma_q \lambda^{(k)} + \ell^{\mu a5(k)}\, i\, \gamma_a \gamma_5\, \lambda^{(k)}\right) \bar{S} \;, \tag{33}$$

where S has the form

$$S = \exp i\, P \exp Q \quad , \tag{34}$$

$\ell^{\mu a(k)} = \ell^{a\mu(k)}$ and $\ell^{\mu a5(k)} = \ell^{a\mu5(k)}$.

Like for the $SL(2,C)$ case, it is easy to see that (31) and (33) are consistent with the transformations

$$S \to \Omega\, S \,, \tag{35}$$

and with the symmetric quantities $\ell^{\mu a}$ and $\ell^{\mu a5}$ transforming as scalars.

Construct now the field quantity

$$\bar{S}^{-1} \left[\ell^{\mu a(k)} \gamma_a \lambda^{(k)} - \ell^{\mu a5(k)} i \gamma_a \gamma_5 \lambda^{(k)} \right] S^{-1} \quad . \tag{36}$$

Clearly this field is a non-linear functional of L_1^μ . The important point is that (on account of the crucial minus sign in front of $\ell^{\mu a5(k)}$ in (36)) it provides a representation of $L_{(2)}^\mu$ with the correct transformation law (32).

We are now in a position to write a simple SL(6,C) × SL(6,C) generalization of Weyl's Lagrangian. Consider

$$\mathcal{L} = \mathrm{Tr}\, i L_1^\mu L_2^\nu (B_{\mu\nu} + i C_{\mu\nu}) + \mathrm{h.c.} + \mathcal{L}_{mass} \quad . \tag{37}$$

The gauge invariance of the Lagrangian implies that the only (physical) fields which occur in this expression are the fields $\ell^{\mu a(k)}$, $\ell^{\mu a5(k)}$ and the fields b_μ and c_μ defined by the relation

$$B_\mu + i C_\mu = S (b + ic)_\mu S^{-1} - i S \partial_\mu S^{-1} \quad . \tag{38}$$

It is easy to verify that the bilinears obtained from (37) by the spontaneous symmetry-breaking ansatz describe the propagation of two nonets (both with positive definite metrics): a 2^+ nonet, described by the fields $\ell^{\mu a(k)}$ and $b_\mu^{ab(k)}$, and a 2^- nonet, described by $\ell^{\mu a5(k)}$ and $c_\mu^{ab(k)}$. There are no bilinear terms for the fields b_μ , c_μ , b_μ^5 and c_μ^5 .

5.4 Finally, now, we are confronted with the problem of the propagation character of the fields $b_\mu^{(k)}$, $b_\mu^{5(k)}$, $c_\mu^{(k)}$ and $c_\mu^{5(k)}$, which occur only among the trilinear and quadrilinear terms in the SL(6,C) × SL(6,C) Lagrangian proposed in (37).

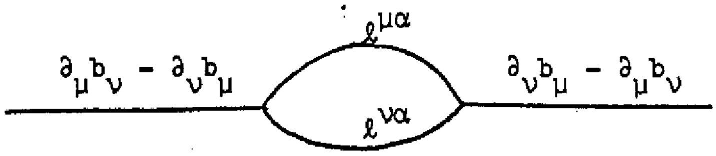
In the paper IC/72/155 I have been referring to, we have suggested the addition of one further gauge-invariant expression to the Weyl form of the Lagrangian which, so far as the bilinears are concerned, has the characteristic of adding mass-like terms $(b_\mu^k)^2$, $(c_\mu^k)^2$, etc. Thus our final Lagrangian is (37) plus one new gauge-invariant term to cope with the problem of the fields b_μ^k , c_μ^k ,... As a consequence of this new term, the field equations express these fields as algebraic functions of the dynamical fields $\ell^{a\mu}$, $\ell^{\mu a5}$, b_μ^{ab} and c_μ^{ab} . To-day I should like to discuss a different procedure which I believe is more general and likely, in the long run, to be more important for dealing with degrees of freedom which make their appearance only among trilinear and quadrilinear terms of quantum Lagrangians. We call this procedure "quantum completion".

Consider as an illustration the field $b_\mu^{(k)}$. The $SL(6,C) \times SL(6,C)$ Weyl Lagrangian (37) contains, among the trilinear terms, the expression

$$\left(b_{\nu,\mu} - b_{\mu,\nu} + b_\mu \times b_\nu\right) \cdot \left(\ell^{\mu\alpha} \times \ell^{\nu\alpha} - \ell^{\mu\alpha 5} \times \ell^{\nu\alpha 5}\right) .$$

(Here $\ell^{\mu\alpha} \times \ell^{\nu\alpha}$ is short for $f^{ijk}\, \ell^{\mu\alpha(j)}\, \ell^{\nu\alpha(k)}$.)

In a classical sense, this Lagrangian does not tell us much about the propagation of the b_μ field. In a quantum sense, however, a second-order iteration of this term will give rise to a loop diagram:



Since the propagators of the ℓ fields are known, this diagram can be computed, and immediately leads to an effective Lagrangian for the b_μ field. We have actually computed the loop diagram; its leading (most divergent) contribution is of the form:

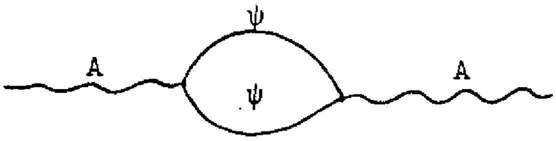
$$\mathcal{L}_{eff} = - \left(\partial_\mu b_\nu - \partial_\nu b_\mu + b_\mu \times b_\nu\right)^2 \frac{\delta^4(0)}{M^4} ,$$

with the <u>correct</u> metric for the positive frequency propagation of the b_μ field (provided we assume that $\delta^4(0) > 0$ and, in some renormalization sense, represents a finite number).

This idea of "quantum completion" is not new. It is closely linked with the old idea of using conditions like $Z = Z(\kappa^2, M^2) = 0$ to give propagation character to composite fields (which in fact the fields h_μ^k, c_μ^k, ... etc., really are) since the Lagrangian, as it stands, gives no propagation equations for these. In fact, Sacharov and Zeldovitch and Durr and Heisenberg have gone so far as to suggest that one may recover the Maxwell field equations by starting with the Lagrangian

$$\mathcal{L} = - \bar{\psi}\, \gamma(\partial + ieA)\psi - m\, \bar{\psi}\psi ,$$

with propagation terms for the ψ field only. One recovers the propagation character for the photon by considering the loop:



which, for special values of e determined by $Z(e^2) = 0$, would give rise to the effective Maxwell term $F_{\mu\nu} F_{\mu\nu}$. We are not going as far as Sacharov and Zeldovitch, since we have a simpler problem. Our Lagrangian already contains a first-order derivative for the b_μ field. A second-order iteration gives rise to an effective Lagrangian guaranteeing the propagation of b_μ apparently for all values of the coupling parameter.

To summarize, one may suggest (as an alternative to adding more terms to (37)), using the Weyl Lagrangian as it stands, to give a propagation character to a 2^+ and a 2^- nonet. In ensuring that both nonets possess positive definite metrics, we employed a non-linear realization of the L_2^μ field in terms of L_1^μ . The spin-1 fields B_μ , C_μ ,... may acquire a propagation character through the process of "quantum completion". Their effective coupling parameters, however, will have no definite relation to the coupling parameters of the 2^+ and 2^- particles. If one does not like the idea of a "quantum completion", there is always the possibility of adding to the Weyl Lagrangian one extra "classical" term, which guarantees that these fields can be eliminated from the theory as algebraic constraints. Of course the "classical completion" and the "quantum completion" give rise to different theories. Before closing with this part of my talk, I wish to make two remarks. One is that Gürsey (Contemporary Physics (IAEA, Vienna 1969) p.211) has discussed an SL(6,C) invariant - but not an SL(6,C) gauge-invariant theory, where, in our notation, a field L^μ is introduced through the definition $L^\mu = S \gamma^\mu S^{-1}$ with $S = \exp i P$. The 72-fields P are the basic fields and not just gauge degrees of freedom as in our formulation. There are no B_μ fields, no Einstein-like Lagrangian and naturally no spin-2 gauge particles. The fields P must be zero-mass Goldstone fields in the symmetry limit. It seems to us that the relationship between Gürsey's and our theory is roughly the same as that between a (non-linearly realized) Goldstone theory and a Higgs-Kibble type of gauge theory.

The second point I wish to make is that, in our opinion, it is not one particular Lagrangian versus another which is likely to be important in strong-interaction physics in the long run. Arguing with Gell-Mann, one must learn to "abstract" the basic truth from the outer wrappings of formalism. We believe the truth in this instance lies in the deep geometrical ideas of Cartan regarding torsion and its connection not only with spin but also with internal symmetries.

VI. TOWARDS SU(6)

I started this lecture by wanting to solve two problems: i) to find a possible origin for spin-internal-spin combination and thereby motivate SU(6); ii) to find an elegant generalization of Einstein's Lagrangian, describing spin 2^+ particles, so as to include SU(3).

What we have accomplished is to generalize Cartan's geometrical notion of torsion to include internal symmetries. We have also succeeded in finding an elegant theory of 2^+ (and 2^-) nonets. But have we succeeded in recovering SU(6) ?

Two years ago, lecturing here at Miami, I remarked on Kerr's exact solution to Einstein's equation; the remark was that in the Kerr solution for a charged spinning particle (mass m, charge Q, spin J), the charge Q always occurs in the combination

$$Q^2 + \frac{J^2}{m^2 G_N} .$$

I conjectured that once SU(2) or SU(3) is incorporated into the structure of the Einstein equation, Q^2 will generalize to an expression like $I(I + 1)$, where I is the isotopic spin. I suggested that SU(2) or SU(3) containing Einstein-like equations, when solved exactly (as for the Kerr case), may provide the dynamical basis for the emergence of SU(4)- or SU(6)-like combinations,

$$I(I + 1) + \frac{J(J + 1)}{m^2 G_F} ,$$

where G_F is the strong-gravity coupling parameter which replaces the newtonian constant G_N. The situation with SU(6) would then be similar to the situation for the hydrogen atom, where the observed SO(4) symmetry of hydrogen energy levels is a dynamical (and unexpected) consequence of the $1/r$ potential. Now that we have successfully incorporated SU(3) (or indeed SU(3) × SU(3)) into the Einstein-Cartan-Weyl-like equations, with (a non-linearly realized) SL(6,C) × SL(6,C) as our starting point and with an algebra of conserved currents, it seems eminently reasonable that SU(6) symmetry (with its attendant manifestations in terms of collinear and coplanar subgroups) should emerge dynamically from different approximations of the theory we have constructed. So far as matter is concerned, presumably we would consider writing the quark-antiquark Lagrangian describing the 35-plet of SU(3) in the generalized Bergmann-Wigner form

$$L = \frac{i}{2} \mathrm{Tr} \left(\Phi [L^{\mu} , \nabla_{\mu} \Phi] - m \Phi \Phi \right) ,$$

where Φ is the second-rank multi-spinor. Likewise for the third-rank multi-spinor describing the 56-plet of the baryons. In addition we may have direct SL(6,C)-invariant couplings of the type $\mathrm{Tr}\ \Phi\Phi\Phi$. The 2^+ (and 2^-) nonets described by the fields $L^{\mu}(x)$ will act as gluons. (As stated earlier, these particles constitute incomplete multiplets of the old quark-based phenomenological SU(6) . The situation for these multiplets is presumably completely analogous to non-linearly realized chiral theories, which also display incomplete multiplets of the larger symmetry group.) The true dynamics will be a complicated interplay of exactly symmetric vertices, like $\mathrm{Tr}\ \Phi\Phi\Phi$, of the spontaneous symmetry-breaking mechanism $\langle L^{\mu} \rangle = \gamma^{\mu}$ and of the covariant constraints, like those typified in (36). The spin 2^+ mesons described by the ϕ fields in $L^{\mu} = \gamma^{\mu} + \kappa\phi^{\mu}$ will break the SL(6,C) chain (SU(6) rest symmetry, SU(3) × SU(3) collinear symmetry) in a specified manner.