THE HYDROGEN ATOM AS A RELATIVISTIC ELEMENTARY PARTICLE - II:
RELATIVISTIC SCATTERING PROBLEMS AND PHOTO-EFFECT

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The framework developed in Part I is applied to scattering problems. Relativistic continuum-continuum transitions are calculated as an overlap of scattering states by group-theoretical techniques. We further calculate relativistic bound continuum transitions (photo-effect) by the same methods and derive non-relativistic reductions.
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I. INTRODUCTION

In Part I we have established an infinite-component relativistic wave equation for a charged system which has a hydrogen-like mass spectrum. In this part we shall extend the bound state treatment developed there to include the continuum states. This will allow us to treat "Coulomb" scattering, photo-effect and other processes involving the dissociation of the "H atom".

The paper is divided into several sections. In Sec. II we establish the S matrix by an extension of the corresponding non-relativistic formulation. In Sec. III the analytic continuation from bound to continuum states is discussed, the scattering states are defined and their normalization given. In Sec. IV we find the Coulomb cross-section and give its Klein-Gordon and non-relativistic limits. In Sec. V we derive the photo-effect T matrix and give its non-relativistic reduction. We conclude this paper with a general discussion of the results.

To our knowledge, the method of Coulomb scattering presented here is the first instance where the Coulomb amplitude is evaluated actually as an overlap integral of the form \( \langle \psi^- | \psi^+ \rangle \). Both Coulomb and photo-effect amplitudes have also been considered recently, using a different method, by Fronsdal and Lundberg (1970).

II. THE S MATRIX AND SCATTERING STATES

1. Kinematics

The observable quantities in relativistic interaction are the properties of free particles such as momenta and polarization (Berestetskil et al. 1971). The scattering matrix \( S \) is the unitary matrix that connects the initial free states \( |i\rangle \) at \( t \to -\infty \) to the final free states \( |f\rangle \) at \( t \to +\infty \). Its matrix elements are written as

\[
S_{fi} = S_{fi} + i(2\pi)^4 \frac{h}{\hbar} \delta^4(P_f - P_i) T_{fi} ,
\]  

(2.1)

where the actual scattered part \( (T_{fi}) \) has been separated out along with an overall four-momentum delta function expressing the conservation of total momentum. We normalize the single-particle states to "one particle per volume \( V \)". This simply means that the delta function normalization is changed into
For a two-body scattering process, \( 1 + 2 \rightarrow 3 + 4 \), the differential cross-section for the scattering of one of the final free particles into the solid angle \( d\Omega \) in the centre-of-momentum frame is given by
\[
\frac{d\sigma}{d\Omega} = \frac{V^4}{(2\pi)^2} |T_{fi}|^2 \frac{|\vec{p}_f|}{|\vec{p}_i|} \frac{E_3 E_4}{\sqrt{s}} .
\]

Similarly the differential cross-section for the scattering of an initial free particle with a photon, resulting in the two final free particles 3 and 4, is
\[
\frac{d\sigma}{d\Omega} = \frac{V^4}{2\pi} |T_{fi}|^2 |\vec{p}_f| \frac{E_3 E_4}{\sqrt{s}} .
\]

Here \( \vec{p}_i \) and \( \vec{p}_f \) indicate the relative centre-of-momentum momenta of the initial and final configurations, respectively, and \( s \) is the invariant mass squared.

In order to define a relativistic \( S \) matrix for Coulomb scattering in the dynamical group formulation, we need to define the scattering states which we obtain by a direct generalization of the non-relativistic time-independent incoming and outgoing scattering states.

In the interaction representation, the \( S \) matrix for scattering can be written in terms of the asymptotic time-independent free states \( \phi_i \) and \( \phi_f \), with the help of the time development operator \( U(t_2, t_1) \),

\[
S_{fi} = \lim_{t_1 \to -\infty} \lim_{t_2 \to +\infty} \langle \phi_f | U(t_2, t_1) | \phi_i \rangle \\
= \lim_{t_1 \to -\infty} \lim_{t_2 \to +\infty} \left\{ \langle \phi_f | U(t_2, 0) | \{ U(0, t) | \phi_i \} \right\} \\
= \langle \psi_i^+ | \psi_f^- \rangle .
\]

Here \( \psi_i^+ \) and \( \psi_f^- \) are the "in" and "out" scattering states. In non-relativistic two-body scattering, the "in" and "out" states can be written as
after the usual separation of the centre of momentum and relative co-ordinates. Here $P$ refers to the total momentum and $p$ to the momentum conjugate to the relative co-ordinate $\vec{r}$. The functions $\psi^\pm(r)$ satisfy the asymptotic free particle conditions

$$\lim_{r \to \infty} [\psi^\pm(\vec{r})] = \frac{1}{v^2} \left[ e^{ik \cdot \vec{r}} + f^\pm(0) \frac{e^{ikr}}{r} \right]. \quad (2.7)$$

The "in" and "out" scattering states can be found in closed form for scattering in a Coulomb potential by looking at the asymptotic behaviour of the continuum parabolic states and selecting the quantum numbers $n_1$, $n_2$ and $m$ to fit condition (2.7). Our main hypothesis in this section is that the relativistic infinite-component scattering states are relativistic continuum states with the same group structure, i.e. quantum numbers, as the corresponding non-relativistic scattering states. In the next section, we first consider the continuum states in general and then derive the quantum numbers and normalization of the "in" and "out" scattering states.

2. Analytic continuation into scattering states

The states $\psi^\pm(r)$ of the Coulomb scattering problem are special cases of the continuum states of the $H$ atom. We first discuss the group properties of the continuum states in the angular momentum basis. This basis will also be used in the Appx.B to find the normalization constants of the $\psi^\pm$ group states.

a) Continuum group states in the $(n|m)$ basis

The continuum group states, as discussed in Part I, are found by diagonalizing the non-compact generator $L_{46}$ with continuous spectrum rather than the compact generator $L_{56}$ with discrete spectrum. We have

$$L_{46} |\lambda, \ell m \rangle = \lambda |\lambda, \ell m \rangle, \quad (2.8)$$

where $\lambda$ is real and continuous. Rather than directly diagonalizing $L_{46}$ by completely changing the basis of our representation, we would like to treat the discrete and continuum group states in the same basis. We chose basis states for which $L_{56} \equiv \Gamma_0$ and $L_{12} \equiv L_3$ are diagonal and which are
simply related to the states defined by (2.8). Let us consider the group states $|nm\rangle$ and continue the quantum number $n$ analytically to $-i\lambda$, $\lambda$ is real and continuous,

$$L_{56} \begin{pmatrix} -i\lambda, \lambda m \end{pmatrix} = -i\lambda \begin{pmatrix} -i\lambda, \lambda m \end{pmatrix} . \quad (2.9)$$

A tilt operation on these states mixes $L_{56}$ and $L_{46}$,

$$\begin{align*}
\exp \left( i\theta L_{45} \right) L_{56} e^{-i\theta L_{45}} \left\{ e^{i\theta L_{45}} \begin{pmatrix} -i\lambda, \lambda m \end{pmatrix} \right\} \\
= \left\{ L_{56} \cosh \theta - L_{46} \sinh \theta \right\} e^{i\theta L_{45}} \begin{pmatrix} -i\lambda, \lambda m \end{pmatrix} \\
= -i\lambda \left\{ e^{i\theta L_{45}} \begin{pmatrix} -i\lambda, \lambda m \end{pmatrix} \right\} .
\end{align*}$$

$L_{46}$ can now be diagonalized and $L_{56}$ eliminated by taking the tilting angle $\theta$ to be $\theta = \frac{\pi}{2} = \ln i$, so that $\cosh \theta = \cos \frac{\pi}{2} = 0$ and $\sinh \theta = i \sin \frac{\pi}{2} = i$. We find

$$L_{46} \left\{ e^{-\frac{\pi}{2} L_{45}} \begin{pmatrix} -i\lambda, \lambda m \end{pmatrix} \right\} = \lambda \left\{ e^{-\frac{\pi}{2} L_{45}} \begin{pmatrix} -i\lambda, \lambda m \end{pmatrix} \right\} . \quad (2.10)$$

Comparing with relation (2.8), we can then identify $|\lambda, \lambda m\rangle$ with

$$\exp \left( -\frac{\pi}{2} L_{45} \right) \begin{pmatrix} -i\lambda, \lambda m \end{pmatrix} .$$

The normalization of the continued and then tiled group states is given by

$$\begin{align*}
\left( \begin{pmatrix} -i\lambda', \lambda' m' \end{pmatrix} e^{-\frac{\pi}{2} L_{45}} \right) \left( \begin{pmatrix} -\frac{\pi}{2} L_{45} \end{pmatrix} \begin{pmatrix} -i\lambda, \lambda m \end{pmatrix} \right) \\
= \left( \begin{pmatrix} -i\lambda', \lambda' m' \end{pmatrix} e^{-\frac{\pi}{2} L_{45}} \begin{pmatrix} -i\lambda, \lambda m \end{pmatrix} \right) \\
= \left[ \begin{array}{c}
\delta_{n', n} \quad \delta_{\lambda', \lambda} \\
\frac{\Gamma(n + 1 + i\lambda)}{\Gamma(-i\lambda - \lambda)} \\
\Gamma(n + 1 + i\lambda) \Gamma(-i\lambda - \lambda) \\
\Gamma(n + 1 + i\lambda) \Gamma(-i\lambda - \lambda) \\
\left[ \frac{\Gamma(n + 1 + i\lambda)}{\Gamma(-i\lambda - \lambda)} \Gamma(n + 1 + i\lambda) \Gamma(-i\lambda - \lambda) \Gamma(n + 1 + i\lambda) \right]^{1/2} \end{array} \right] \delta_{\lambda', \lambda} \delta_{m', m} . \quad (2.11)
\end{align*}$$

The normalized group states are then
\[ |\lambda,\ell,m\rangle = \frac{1}{\sqrt{2\pi}} e^{\frac{\pi \lambda}{2}} \left[ \Gamma(\ell + 1 + i\lambda) \Gamma(-i\lambda - \ell) \right]^{1/2} \times \]
\[ \times e^{\frac{\pi}{2} L_4^5} \left| -i\lambda,\ell,m \right\rangle , \]
\[ \langle \lambda,\ell m | \lambda',\ell' m' \rangle = \delta(\lambda' - \lambda) \delta_{\ell,\ell'} \delta_{m,m'} . \quad (2.12) \]

The D functions are given in Appx.B. The orthogonality is proved using the procedure given in Appx.B. To see clearly what has been achieved by analytic continuation let us look at the group states \(|\lambda m\rangle\) in the \(r\) representation. The group states \(|n^\ell m\rangle\) in the \(r\) representation are (see Part I)

\[ \psi^G_{n^\ell m}(x) = \frac{2}{(2\ell + 1)!} \left[ \frac{(n + \ell)!}{(n - \ell - 1)!} \right]^\frac{1}{2} e^{-r(2\ell + 1)} F(- (n-\ell-1), 2\ell+2, 2\ell) Y_{\ell m}(\theta, \varphi) . \quad (2.13a) \]

Continuing \(n\) to \(-i\lambda\), applying the tilt operation,

\[ \frac{\pi}{e^{2}} L_4^5 e^{i \ln(i)} \left[ -i \frac{x}{\ell} \cdot \frac{\partial}{\partial x} - i \right] \]
\[ = i e^{i \ln(i)} \frac{x}{\ell} \cdot \frac{\partial}{\partial x} \]

and including the new normalization factors (2.13a), we find

\[ \psi^G_{-i\lambda,\ell m}(x) = \sqrt{\frac{2}{\pi}} e^{\frac{\pi}{2}} \left| \frac{\Gamma(\ell + 1 + i\lambda)}{(2\ell + 1)!} \right|^\frac{1}{2} \times \]
\[ \times e^{-r(2i\ell + 1)} F(\ell + 1 + i\lambda, 2\ell + 2, 2i\ell) Y_{\ell m}(\theta, \varphi) . \quad (2.13b) \]

If we apply the further tilt \(e^{\theta \lambda L_4^5}\), \(\theta = \ln \frac{n_0}{\lambda}\), arising from the diagonalization of \(L_4^5\) in the non-relativistic infinite component H-atom wave equation (Eq.(3.18) of Part I) we obtain, except for a difference in the normalization arising from the \(\frac{1}{r}\) metric, the usual continuum physical Coulomb wave function:

\[ \psi^S_{-i\lambda,\ell m} \left( \frac{i}{a_0 \lambda} \right) = \psi^S_{\lambda \ell m}(i\kappa r) , \]

where \(\kappa = \frac{1}{a_0 \lambda}\) (Bethe and Salpeter 1957).
b) Non-relativistic Coulomb scattering states

The "in" and "out" non-relativistic physical scattering states $\psi^\pm_k(r)$, where the asymptotic relative momentum $k$ points along the z axis, are easily found as special forms of the continuum physical parabolic states (Bethe and Salpeter 1957). The physical continuum parabolic states are reached from the discrete parabolic states through the same procedure employed for the $(nlm)$ states: continue the principal quantum number $n$ (the eigenvalue of $L^2$) to $-i\lambda$, tilt with $L_z$ through $\Theta = i \frac{\pi}{2}$ and then go from the group continuum states to the physical states by tilting with $\Theta_A = ln \frac{1}{\sigma_0 \lambda}$. The parabolic quantum numbers $n_1$, $n_2$ and $m$ of the $\psi^\pm_k$ physical scattering states are then determined by matching the asymptotic form of the continuum parabolic state with the asymptotic form of $\psi^\pm_k(2.7)$. Because we have a central potential, there is no $\phi$ dependence, i.e. $m = 0$. The asymptotic form of the continuum parabolic states is given by

$$\psi^S_{n_1n_20} \propto e^{ik(\xi+\eta)} F(-n_1, 1, ik\xi) F(-n_2, 1, ik\eta)$$

where we have taken $k = \frac{1}{d_0 \lambda}$. Eq.(2.14) simply follows from (2.3) and (2.6) of Part I. In order to realize the asymptotic incoming plane wave and the outgoing radial wave of $\psi^+$, we need to choose the quantum numbers $n_1$, $n_2$, $m$ to be (Barut and Rasmussen 1971):

for $\psi^+_k$: $n_1 = -1$, $n_2 = -i\lambda$; $m = 0$. (2.15a)
Similarly, for \( \psi^- \) we need

\[
\psi^-_{k2} : n_1 = -i\lambda - 1, \; n_2 = 0, \; m = 0 . \tag{2.15b}
\]

Here we have also used the condition that

\[
n + -i\lambda = n_1 + n_2 + |m| + 1 = n_1 + n_2 + 1
\]

and Eq. (2.23) of Part I

\[
E_{CM} = \frac{\alpha^2 m}{2\lambda^2} = \frac{1}{2m a_0^2} = \frac{k^2}{2m} .
\]

The general "in" or "out" scattering states can be found by rotating the states \( \psi^\pm_{k2} \) so that \( k_2 \) points in an arbitrary direction \( \vec{k} \equiv (k_1, \theta, \varphi) \):

\[
\psi^\pm_{k2}(\theta, \varphi) = R(\varphi, \theta) \psi^\pm_{k2} = e^{i\theta L_3} e^{i\theta L_2} \psi^\pm_{k2} . \tag{2.16}
\]

Thus the group states corresponding to the physical "in" and "out" scattering states are

\[
\psi^G(\theta, \varphi) \equiv R(\varphi, \theta) C^\pm e^{-\frac{\pi}{2} L_4} \left\{ \begin{array}{c}
\psi^G_{-1, -i\lambda, 0} \\
\psi^G_{-i\lambda - 1, 0, 0}
\end{array} \right\} , \tag{2.17a}
\]

where \( C^\pm \) are the normalization constants,

\[
C^\pm = \frac{1}{2\sqrt{\pi}} e^{\frac{\pm \lambda}{2}} \Gamma(1 \mp i\lambda) . \tag{2.17b}
\]

chosen so that the group states are normalized to the delta function

\[
\langle \psi^G_1(\varphi', \theta'), (\lambda'), \psi^G_2(\lambda) \rangle = \delta(\lambda - \lambda') \delta(1 - \cos \theta) , \tag{2.18}
\]

as shown in Appx. B.

c) Definition of relativistic scattering states

We shall now determine the physical scattering states corresponding to the relativistic particle described by the infinite component wave equation (3.26) of Part I. To obtain the continuum states, we diagonalize \( S = L_{46} \) in the rest system (3.18) of Part I and find for the tilting angle,
\[ \tanh \theta_\lambda = - \frac{\alpha \gamma_1 \omega}{\alpha_3 \gamma_2 + \beta} = - \frac{2m_2 \sqrt{s}}{s + m_2^2 - m_1^2}. \]  

(2.19)

Here \( S \) is the total initial or final four momentum squared of the H-atom system,

\[ s \equiv (p_1 + p_2)^2 = M_\lambda^2. \]

The definition of the physical "in" and "out" scattering states now proceeds in the same way as for the parabolic bound states. We take a group state at rest, tilt it, boost it and then normalize it,

\[ \psi^\pm_{\lambda}(\lambda, \vec{q}) = C_{\lambda}(\lambda, \vec{q}) e^{i\vec{p} \cdot \vec{M}} e^{i\vec{p} \cdot \vec{x}} \times \]

\[ \times \left\{ \begin{array}{c}
\frac{1}{e} \left(\begin{array}{c}
\psi^G_{\lambda} \\
\psi^G_{\lambda} \end{array}\right)
\end{array}\right\}, \]

(2.20)

where \( \vec{q} \) is the asymptotic relative momentum \( \vec{q} = \vec{p}_1 - \vec{p}_2 \) in the rest system, \( \vec{M} \) is the internal boost, \( \vec{x} \) is the CM co-ordinate and \( |\psi^G_{\lambda}(\varphi, \theta)\rangle \) are the "in" and "out" group states. Here \( \vec{q} \) points in the \((\varphi, \theta)\) direction in the rest frame. The magnitude of \( \vec{q} \) is related, through the wave equation (3.24) and the mass spectrum (3.27) of Part I, to \( \lambda (= -iN) \). In the CM frame (Appx. A)

\[ q^2 = (\vec{p}_1 - \vec{p}_2)^2 = \frac{1}{4s} \left\{ \left( s + m_2^2 - m_1^2 \right)^2 - 4m_2^2 s \right\} \]

\[ = - \frac{4m_2^2}{4M^2} \left[ M^2 - \left( \frac{m^2_2 + M^2 - m^2_1}{2m_2} \right)^2 \right], \]

(2.21)

so that, by using (3.24) of Part I with \( N^2 = \lambda^2 \), we find

\[ q^2 = \frac{\alpha_2}{\lambda^2} \left[ \frac{m_2^2}{M^2} \right] \left[ \frac{M^2 - (m_1^2 + m_2^2)^2}{2m_2} \right] \]

\[ = \frac{\alpha_2}{\lambda^2} \left[ \frac{M^2 - (m_1^2 + m_2^2)}{2M} \right]^2. \]  

(2.22a)

Now,

\[ M^2 = m_1^2 + m_2^2 + 2m_1m_2 \left[ 1 - \frac{\alpha_2}{\lambda^2} \right]^{-\frac{1}{2}}, \]

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so that

$$q = \frac{\alpha}{\lambda} \left[ \frac{m_1 m_2}{M} \right] \left[ 1 - \frac{\alpha^2}{\lambda^2} \right]^{-\frac{1}{2}}. \quad (2.22b)$$

The normalization that we want is no longer the one-particle normalization, but the asymptotic two-free-particle normalization (2.2), i.e. one particle per volume $V$ for each of the particles. To obtain this we first go to a $\delta^3(q' - q)$ normalization and then convert to the two asymptotic free-particle normalization (Appx.E). The final states thus normalized are:

$$\psi^\pm_{q_0(\lambda, \frac{P}{V})} = \left[ \frac{M}{2\pi^2\alpha m_1 m_2} \right]^{\frac{1}{2}} e^{i\frac{P}{V} \cdot \mathbf{X}} \times \left[ \frac{(2\pi)^3}{V} \right]^{\frac{1}{2}} e^{i\frac{P}{V} \cdot \mathbf{M}} e^{i\lambda \mathcal{L}_{45}} \psi^\pm_{\lambda(\theta, \phi)} \quad (2.23).$$

III. RELATIVISTIC COULOMB SCATTERING AMPLITUDE

1. $S$ matrix as a scalar product

After these rather lengthy preparations, we are finally ready to solve the first problem, the scattering of two spinless particles described by a relativistic infinite component wave equation. The physical situation we should like to describe is the scattering of particles 1 and 2 in the CM system with relative momentum $\mathbf{q}$ parallel to the $z$ axis, scattering into free particles 3 and 4 with final relative momentum $\mathbf{q}'$ in the direction $(\theta', \phi')$. The $S$ matrix for this process is (using (2.5) and (2.23))

$$S_{f1} = \langle \psi^{G}_{\lambda'(\theta', \phi')} | \left. \mathcal{J}_o \psi^{+G}_{\lambda(\theta, \phi)} \right. \rangle = \frac{(2\pi)^3}{V^2} \delta^{3}(\mathbf{P} - \mathbf{P}') \left( \frac{2\pi}{\lambda m_1 m_2} \right)^2 \left| \frac{2\lambda}{\delta q} \right|^2 \times \left[ \psi^{G}_{\lambda(\theta', \phi')} | \mathcal{J}_o e^{i\theta_\lambda \mathcal{L}_{45}} | \psi^{+G}_{\lambda(\theta, \phi)} \right. \right> \quad (3.1)$$

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where we have used the current conservation condition to set $\lambda' = \lambda$ in the tilting angle, since states with different energy, i.e. $\lambda$, are orthogonal. The Coulomb $S$ matrix (not the $T$ matrix) is thus essentially reduced to the overlap of the two asymptotic group states

$$G \equiv \langle \psi_{\lambda'}^{G}(q',\theta') | \psi_{\lambda}^{G}(0,0) \rangle$$

$$= \frac{1}{4\pi} e^{\frac{\pi}{2}(\lambda'-\lambda)} \Gamma(1-i\lambda') \Gamma(1-i\lambda) \langle \psi_{\lambda}^{G} | R_{2}(-\theta') R_{3}(-\phi') e^{-\frac{\pi L_{45}}{2}} | -1, -i\lambda, 0 \rangle$$

$$= \frac{1}{4\pi} \left[ \Gamma(1-i\lambda) \right]^{2} \langle \psi_{\lambda}^{G} \right|_{-1, -i\lambda, 0} R_{2}(-\theta) e^{-\frac{\pi L_{45}}{2}} | -1, -i\lambda, 0 \rangle,$$

(3.2)

where we have simply used (2.17a) and the fact that $R_{3}(-\phi')$ acting to the left is 1,

$$e^{-i\phi' L_{3}} | -1, -i\lambda, 0 \rangle = | -1, -i\lambda, 0 \rangle.$$

We evaluate the overlap integral (3.2) by first calculating the corresponding bound state expression and then continuing the quantum numbers and the tilting angles to their continuum values. In subsection 2 we evaluate

$$F_{n' n} = \langle n'-1, 0, 0 | R_{2}(-\theta) e^{i\phi L_{45}} | -1, n, 0 \rangle,$$

then make the continuation

$$n' \rightarrow -i\lambda'$$

$$n \rightarrow -i\lambda$$

(3.3)

In subsection 3 we then take the limit

$$\phi' \rightarrow i\pi.$$

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2. Evaluation of the group matrix element

The group matrix element $F_{n'n}$ is evaluated by treating the states initially as if they were "bound states and then continuing their quantum numbers to their actual values. We are presenting this case in some detail since we use the same operations in the photo-effect calculation. We write $F_{n'n}$ (3.3) as

$$F_{n'n} = \langle n'_1, n'_2, 0 | R_2^+(\theta) e^{-i\phi L_4} | n_1, n_2, 0 \rangle$$

(3.4)

and put in a complete set of states to split the rotation and tilt operations

$$F_{n'n} = \sum_{n_1', n_2'} \langle n'_1, n'_2, 0 | R_2^+(\theta) | n''_1, n''_2, 0 \rangle \langle n''_1, n''_2, 0 | e^{-i\phi L_4} | n_1, n_2, 0 \rangle$$

(3.5)

We first look at the rotation matrix element $R_{n''_1 n''_2}$. Because the group generator $L_{56}$ commutes with the rotation operator, its eigenvalue must be the same for the $n'$ and $n''$ states,

$$n' \equiv n'_1 + n'_2 + 1 = n''_1 + n''_2 + 1 \equiv n''$$

(3.6)

We can simplify the sum by defining

$$n''_1 \equiv M = 0, 1, 2, ...$$

(3.7)

and then eliminating $n''_2$ from (3.6),

$$n''_2 = -(n' + 1) - M$$

The rotation matrix (3.5) becomes

$$R_{n'_1 n'_2} = R(M) = \langle n'_1, n'_2, 0 | R_2^+(\theta) | M, -(n' + 1) - M, 0 \rangle$$

(3.8)

Rotation matrices can easily be evaluated in the oscillator realization of the group (given in the Appendix of Part I) by factorizing the generators and the basis into two rotation subgroups, one for the creation operator $a^+$ and one for the $b^+$. We can write the group element
$$R_2^*(-	heta) = e^{\frac{-i\theta}{2} [a^+ a + b^+ b]}$$

$$= e^{\frac{-i\theta}{2} [a^+ a]} e^{\frac{-i\theta}{2} [b^+ b]}$$

$$= [R_2(-\theta)]_a [R_2(-\theta)]_b$$  \hspace{1cm} (3.9)$$

Correspondingly, the group basis (A.3) of Part I can be written as \((m \geq 0)\)

$$| n_1, n_2, m \rangle = \left[ \left( n_2 + |m| \right) ! \left| n_1 ! \right] \right]^{-(n_2 + |m|)} a_1^{+ (n_2 + |m|)} a_2^{+ n_1} |0\rangle$$

$$\times \left[ \left( n_1 + |m| \right) ! \left| n_2 ! \right] \right]^{-(n_1 + |m|)} b_1^{+ (n_1 + |m|)} b_2^{+ n_2} |0\rangle$$

$$\equiv N_a (\phi, m_a) a_1^{+ (\phi + m_a)} a_2^{+ (\phi - m_a)} |0\rangle$$

$$\times N_b (\phi, m_b) b_1^{+ (\phi + m_b)} b_2^{+ (\phi - m_b)} |0\rangle$$

$$\equiv | \phi, m_a \rangle_a | \phi, m_b \rangle_b \hspace{1cm} (3.10)$$

where we make the identifications

$$\phi = \phi_a = \phi_b = \frac{1}{2} (n_1 + n_2 + |m|) = \frac{1}{2} (n - 1)$$

$$m_a = \frac{1}{2} (n_2 + \left| m \right| - n_1)$$

$$m_b = \frac{1}{2} (n_1 + \left| m \right| - n_2) = -m_a + |m| \hspace{1cm} (3.11)$$

The rotation matrix (3.9) becomes

$$R(M) = \langle m'_a, \phi | R_2(-\theta) | \phi, m'_a \rangle \cdot \langle m'_b, \phi | R_2(-\theta) | \phi, m'_b \rangle \hspace{1cm} (3.12)$$

Here, we recall that \(n'_2 = 0\), \(m = 0\) so that we have, from (3.11),

$$m'_a = -\phi \leq m''_a = \phi - M$$

$$m'_b = \phi \leq m''_b = M - \phi \hspace{1cm} (3.13)$$

In terms of the rotation matrices (Appx. C), (3.12) becomes

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\[ R(M) = D^\phi_{m_1^a m_1^b}(-S) D^\phi_{m_2^a m_2^b} \]

\[ = \frac{(n'-1)!}{M!(n' - 1 - M)!} (-1)^{n'-1-M} (\beta_R \beta_R)_{n'-1-M} (1 - \beta_R \beta_R)^M, \quad (3.14) \]

where

\[ \beta_R = -\sin \frac{\theta}{2}. \quad (3.15) \]

The tilt can also be factorized in a similar fashion, but on the product basis of two \( O(2,1) \) algebras, one for \( n_1 \) and one for \( n_2 \), called the transition basis. We again write down the group basis (A.3) of Part I, but this time we factorize the basis into a product basis of \( n_1 \) and \( n_2 \) subgroups:

\[ |n_1 n_2^m\rangle = \frac{1}{[n_1! (n_1 + |m|)!]^{1/2}} a_2^+ b_1^+|m|^{+n_1} |0\rangle \times \]

\[ x \frac{1}{[n_2! (n_1 + |m|)!]^{1/2}} a_1^+ b_2^+|m|^{+n_2} |0\rangle \]

\[ = |k,m\rangle_{n_1} |k,m\rangle_{n_2}. \quad (3.16) \]

The raising and lowering operators for \( n_1 \) and \( n_2 \) are easily constructed. From these, we define the generators of the groups \( O(2,1) \) as

\[ N_1^+ = a_2^+ b_1^+ \quad ; \quad N_1^- = \frac{1}{2}(a_2^+ b_1^+ + a_2^+ b_1^+ + 1) \]

\[ N_1^0 = a_2 b_1 \quad ; \quad N_2^0 = \frac{1}{2}(a_2^+ b_1^+ - a_2^+ b_1^+) \]

\[ N_1^3 = \frac{1}{2}(a_2^+ a_2 + b_1^+ b_1^+ + 1). \quad (3.17) \]

The generators for \( n_2 \) are found simply by replacing \( a_2 \) by \( b_2 \) and \( b_1 \) by \( a_1 \). The eigenvalue of the Casimir operators of these groups is

\[ k = \frac{|m| + 1}{2}, \quad (3.18) \]

and the spectra of \( N_1^3 \) and \( N_2^3 \) are given by...
We thus have the direct product of two discrete unitary representations $(D^+_k)$. We can make the $O(4,2)$ generator identifications (A.2) of Part I and (3.19):

\[ L_{34} = N_1^3 - N_2^3, \quad L_{36} = N_1^2 + N_2^2, \]

\[ L_{35} = - (N_1^1 + N_2^1), \quad L_{46} = - N_1^1 + N_2^1. \]

\[ L_{45} = - N_1^2 + N_2^2, \quad L_{56} = N_1^3 + N_2^3. \]

The tilt matrix $T_{n_1 n_2}^{n_1' n_2'}$ can now be split into two $O(2,1)$ matrix elements. We have, using (3.7), (3.11) and (3.20),

\[
T_{n_1 n_2}^{n_1' n_2'} = T(M) = \langle n_1'' + \frac{1}{2}, \frac{1}{2} | e^{i \phi N_1^2} | \frac{1}{2}, n_1 + \frac{1}{2} \rangle \langle n_2'' + \frac{1}{2}, \frac{1}{2} | e^{-i \phi N_2^2} | \frac{1}{2}, n_2 + \frac{1}{2} \rangle = T_{n_1}^{n_1'} T_{n_2}^{n_2'}. \]

To evaluate these we need to specify, as we did for the rotation matrices, the relative values of $n_1$ to $n_1''$ and $n_2$ to $n_2''$. To obtain a non-zero and finite answer, we take

\[ n_1 > n_1'' \quad \text{and} \quad n_2 > n_2''. \]

We can then immediately evaluate the $O(2,1)$ "rotation" matrices (Appx. C). Substituting the value of $n_1 = -1$ and $n_2 = n$, we find

\[
T_{n_1}^{n_1'} = D^{\frac{1}{2}}_{M+\frac{1}{2}, -\frac{1}{2}}(W) = (\alpha_1)^M (\beta_1)^{-(M+1)} F(0, -M, 1, \frac{1}{\alpha_1}).
\]

\[
= (\alpha_1)^M (\beta_1)^{-(M+1)}. \]

(3.23)
and

\[ T_n^m = \frac{\text{D}_{n-1-M+\frac{1}{2}}}{\sqrt{2}}, n+\frac{1}{2} (W_2) \]

\[ = (-1)^n (\alpha_2)^{-M} (\beta_2^+)^{(\alpha_2)^+} (\alpha_2^{-M})^{-1} (\beta_2^+)^{(\alpha_2)^+} \times F(n+1, -n'+1+M, 1, 1/\alpha \alpha') \]  

(3.24)

where

\[ \alpha_1 = \alpha_2 = \alpha = \cosh \frac{\Phi}{2} \quad \beta = \beta_1 = -\beta_2 = \sinh \frac{\Phi}{2} \]  

(3.25)

Thus

\[ T(M) = (-1)^{-n'+M}(\alpha)^{-2} \left( \frac{\beta}{\alpha} \right)^{n-n'} \times \]

\[ \times F(n+1, -n' + 1 + M, 1, \frac{1}{\alpha \alpha'}) \]  

(3.26)

We recombine the rotation and tilt matrices to obtain, finally, for (3.5)

\[ F_n^{\nu} = \sum_M R(M) T(M) \]

\[ = \sum_M \frac{(n'-1)!}{M! [(n'-1) - M]!} \left( \beta \beta_R \right)^{(n'-1)-M} (1-\beta_R \beta_R)^{M} \times \]

\[ \times F(n+1, -(n'-1)+M, 1, 1/\alpha \alpha') \].

(3.27)

This sum can be performed using the formula given in Appx. D,

\[ \sum_{m=0}^{\infty} \frac{R!}{m! (R-m)!} \mu^m (1-\mu)^{R-M} F(-R+m, \beta; \alpha', \mu) = F(-R, \beta; \alpha'; \mu). \]

(3.28)

If in the expression we identify \( R \) with \( n'-1 \) in (3.27), we find

...
We can make a further convenient transformation using

\[ F(a, b; c; z) = (1-z)^{c-a-b} F(c-a, c-b; c; z) \]

and then obtain

\[
F_{n'n} = \frac{1}{a^2} \left( \frac{\beta}{\alpha} \right)^{n-n'} \left( \begin{array}{c} n-n' + 1 \\ -n-n' \end{array} \right) \times \\
\times F\left( +n', -n, 1; \frac{\sin^2 \frac{\theta}{2}}{\alpha} \right) .
\]

(3.30)

3. The limits

The matrix element \( F_{n'n} \) above is now in a form suitable for analytic continuation indicated in (3.3)

\[
\lim_{n \to i\lambda} \left| \frac{-i(\lambda + \lambda') - i(\lambda + \lambda')}{\alpha - \sin^2 \frac{\theta}{2}} \right| = \left( \frac{\alpha}{\alpha} \right)^{-1} \times F\left( i\lambda, i\lambda', 1; \frac{\sin^2 \frac{\theta}{2}}{\alpha^2} \right).
\]

(3.31)

We must then take the limit

\[ \phi \to -i\pi . \]

This limit can be changed to a limit on \( \beta \) and \( \alpha \) by virtue of Eq.(3.25)

\[ \alpha = \cosh \frac{\phi}{2} , \quad \beta = \sinh \frac{\phi}{2} . \]

Thus

\[
\lim_{\phi \to i\pi} \left( \beta \right) = \lim_{\beta \to -i} \left( \alpha \to \alpha' \right) .
\]

(3.32)

Applying (3.32) to \( F_{i\lambda', -i\lambda} \), we find
\[ \lim_{\alpha \to 0} F_{i\lambda',i\lambda} = \lim_{\alpha \to 0} \left\{ \frac{-i(\lambda' + \lambda)}{\alpha \sin^2 \frac{\theta}{2}} \frac{\Gamma(i(\lambda - \lambda'))}{\Gamma(i\lambda)\Gamma(i\lambda')} \cdot \frac{-i(\lambda' + \lambda)}{\alpha \sin^2 \frac{\theta}{2}} \cdot \frac{i(\lambda + \lambda)}{\alpha (\lambda' + \lambda)} \right\} \]

\[ \times \left[ \frac{-\sin^2 \theta/2}{\alpha \lambda'} \frac{i\lambda'}{\alpha \lambda} \frac{\Gamma(-i(\lambda - \lambda))}{\Gamma(i\lambda')\Gamma(i\lambda')} \right] F\left(i\lambda', i\lambda', 1 + i(\lambda' - \lambda); \frac{\alpha \lambda' / \sin^2 \frac{\theta}{2}}{\alpha \lambda / \sin^2 \frac{\theta}{2}} \right) \]

\[ + \left( \frac{-\sin^2 \theta/2}{\alpha \lambda} \right)^{i\lambda} \frac{\Gamma(i(\lambda - \lambda))}{\Gamma(i\lambda)\Gamma(i\lambda')} \cdot \frac{\alpha \lambda' / \sin^2 \frac{\theta}{2}}{\alpha \lambda / \sin^2 \frac{\theta}{2}} \right] \}

\[ = \frac{(-\sin^2 \theta/2)^{i\lambda - 1}}{\Gamma(i\lambda)\Gamma(i\lambda')} \lim_{\alpha \to 0} \left[ \alpha^{-i(\lambda - \lambda')} \Gamma(i(\lambda - \lambda')) \right] \]

\[ + \alpha^{-i(\lambda - \lambda')} \Gamma(-i(\lambda - \lambda')) \]

\[ = \frac{(-\sin^2 \theta/2)^{-1 + i\lambda}}{\Gamma(i\lambda)\Gamma(i\lambda')} (2\pi)^{-1} \delta(\lambda - \lambda'), \quad (3.33) \]

where we have first used a relation from Appx. D to invert the argument of the hypergeometric function, and then used the relation (Barut and Phillips 1968)

\[ 2\pi \delta(x) = \lim_{\eta \to 0} \left[ \eta^i x \Gamma(i\eta) + \eta^{-i x} \Gamma(-i\eta) \right] \quad (3.34) \]
It is a simple matter now to convert the principal quantum number delta function into an energy delta function by using the mass spectrum (2.22a):

\[
\delta(\lambda - \lambda') = \frac{\alpha^2 m_1 m_2}{\lambda^3 \left[1 - \frac{\alpha^2}{\lambda^2}\right]^{3/2}} \frac{1}{M_\lambda} \delta(P_0 - P_0') \]

\[\sqrt{S} = M = P_0. \]  

(3.35)

We can now go back to the matrix element (3.2) and from there to the S matrix \( S_{\mu \iota} \) (3.1). We find

\[
S_{\mu \iota} = (2\pi)^4 \delta^3(P - P') \frac{\pi}{q^2} \frac{1}{\sqrt{2}} \left[ \frac{1}{4\pi} \left( \frac{\Gamma(1-i\lambda)}{\Gamma(1+i\lambda)} \right) \right] \]

\[
\left[ \frac{(2\pi)(\sin^2 \theta/2)}{\Gamma(1-i\lambda) \Gamma(1+i\lambda)} \right] \delta^2 m_1 m_2 \frac{1}{M_\lambda} \left[1 - \frac{\alpha^2}{\lambda^2}\right]^{3/2} \delta(P_0 - P_0') \]

\[
= i (2\pi)^4 \delta^4(P - P') \frac{\pi}{q^2} \frac{1}{\sqrt{2}} \left| \frac{\lambda}{\sqrt{q^2}} \right| \frac{\Gamma(1-i\lambda)}{\Gamma(1+i\lambda)} \]

\[
\times \left[ \sin^2 \theta/2 \right]^{-1+i\lambda} \frac{\alpha^2 m_1 m_2}{M_\lambda} \left[1 - \frac{\alpha^2}{\lambda^2}\right]^{3/2}. \]  

\[\text{(3.36)}\]

4. Results

The transition matrix \( T_{\mu \iota} \) can now be identified immediately by comparing (3.36) with (2.1). If we also make the substitution for \( \frac{\partial \lambda}{\partial q} \) from the mass spectrum, we find

\[
T_{\mu \iota} = \frac{\pi}{\sqrt{2}} \frac{\Gamma(1-i\lambda)}{\Gamma(1+i\lambda)} \frac{1}{\sin^2 \theta/2} \left[ \frac{1}{1 - \frac{\alpha^2}{\lambda^2} \frac{m_1 m_2}{M_\lambda^2 (1-\alpha^2/\lambda^2)^{1/2}}} \right]^{-1} \]

\[\text{(3.37)}\]
We can replace $\theta$, $q$ and $\lambda$ by $s$ and $t$ using the easily derived relationships

$$S = (p_1 + p_2)^2 = m_1^2 + m_2^2 + 2m_1m_2 \left[1 - \frac{\alpha^3}{\lambda^2}\right]^{1/2}$$

$$t = (p_1 - p_2)^2 = -4q^2 \sin^2 \theta/2$$

$$q_t^2 = \frac{i}{4S} \left[ (s-m_1^2)(s-m_2^2) \right] \equiv \Delta(s)$$

$$\lambda = \alpha(q_t) = \alpha \frac{s-(m_1^2 + m_2^2)}{\left[(s-m_1^2)(s-m_2^2)\right]^{1/2}},$$

(3.38)

and obtain

$$T_{fi} = \alpha \frac{4\pi}{\sqrt{s}} \frac{1}{t} \frac{\Gamma(1-i\alpha(q))}{\Gamma(1+i\alpha(q))} \left( -\frac{t}{2\Delta(s)} \right) \frac{2s(s-(m_1^2 + m_2^2))}{s - m_1^2 m_2^2}.\quad (3.39)$$

We briefly note that $T_{fi}$ now has the correct analyticity properties of the usual Coulomb amplitude. The poles in $T_{fi}$ occur when the trajectory function $\alpha(q)$ takes the value $-iN$, $N$ being a positive integer. This corresponds, of course, to the usual bound states.

In the non-relativistic limit we find, using the normalizations given by (2.3) and

$$\frac{d\sigma}{d\omega} = |f(\theta)|^2 = \left| \frac{\nu^2}{2\pi} \frac{E_1 E_2}{K} T_{fi} \right|^2,$$

(3.40)

the usual non-relativistic amplitude

$$f(\theta) = \alpha \frac{m_1 m_2}{m_1 + m_2} \frac{\Gamma(1-i\alpha_0/k)}{\Gamma(1+i\alpha_0/k)} \frac{i}{2k^2 \sin^2 \theta/2} \exp \left[ \frac{i\alpha_0}{2k} \ln \left( \frac{s m_1^2}{m_2^2} \right) \right],$$

(3.41)

where $\alpha_0$ is the Bohr radius and $k$ is the non-relativistic counterpart of $q$ (Sommerfeld 1960).

We can also compare (3.37) or (3.39) with the amplitude for the scattering of a scalar meson in a fixed Coulomb field in Born approximation by taking the static limit of (3.37), i.e. $\lim m_2 \to \infty$. Using (3.40) again, we find
\[
\int \frac{KG}{(\theta)} = \alpha E' \frac{\Gamma'(i-\lambda')}{\Gamma(i+i\lambda)} \frac{1}{2q' \sin^2 \theta/2} \exp \left(i \lambda' \ln \left(\sin^2 \frac{\theta}{2}\right)\right)
\]

\[
= \alpha \frac{m_1}{vq' \sin^2 \frac{\theta}{2}} \frac{\Gamma'(i-\lambda')}{\Gamma(i+i\lambda)} \exp \left(i \lambda' \ln \left(\sin^2 \frac{\theta}{2}\right)\right).
\]

Here \( q', v \) and \( m_1 \) refer to the momentum, velocity and mass of the electron. Except for the Coulomb phase, which does not appear in the Born amplitude, we have the usual result (Bjorken and Drell 1964).

IV. THE RELATIVISTIC PHOTO-EFFECT

The photo-effect in the non-relativistic hydrogen atom is the process of ionization of the atom through a single photon collision (cf. Sommerfeld 1960). Because the scattering states are included in the solutions of our infinite-component wave equation, the process of photo-ionization of the relativistic charged particle can also be reduced to the calculation of a group matrix element.

1. The S matrix

The S matrix for this process is

\[
S_{fi} = -i e \int d^4 x \left[ j_\mu(x) A^\mu(x) \right].
\]

The calculation is simpler if we take the initial particle to be in the ground state. The final ionized particle state is \( \psi^{-} ((2.5),(2.20)) \). For the photon field we shall take ((3.10) of Part I)

\[
A^\mu = \frac{2m}{qV} \gamma^\mu e^{-iqx}
\]

so as to have the normalization of "one-photon" per volume V. Putting in the energy factor, we have
The T is given by

\[ S_{fi} = -ie \int dt \int d^3x \left( \frac{M_f}{p_f^0 V} \right)^{1/2} e^{-i \vec{p}_f \cdot \vec{X}} \]

\[ \times \left\langle \Psi_{k_f, \vec{p}_f}^- \right| J_{\mu} \mid \Psi_{N=0, \vec{p}_i}^+ \right\rangle e^{i \frac{\vec{p}_i \cdot \vec{X}}{2}} \left( \frac{M_i}{p_i^0 V} \right)^{1/2} \]

\[ \times \epsilon^\mu \left( \frac{2\pi}{q_i V} \right)^{1/2} e^{i \frac{\vec{q}_i \cdot \vec{X}}{2}} \]

\[ = i (2\pi)^4 \delta^4(P_f - P_i - q_f) T_{fi} \]

The \( T_{fi} \) is given by

\[ T_{fi} = e \left( \frac{M_f}{p_f^0 V} \frac{M_i}{p_i^0 V} \frac{2\pi}{q_i V} \right)^{1/2} \epsilon^\mu \left\langle \Psi_{k_f, \vec{p}_f}^- \right| J_{\mu} \mid \Psi_{N=0, \vec{p}_i}^+ \right\rangle \] (4.3)

The current operator is given by

\[ J_{\mu} = -\frac{1}{m_i} \left( \gamma_{\mu} + (P_f + P_i)_{\mu} \right) \frac{1}{2m_2} (S - \alpha) \] (4.4)

For convenience, we again work in the centre-of-momentum frame and choose a
co-ordinate system such that the photon comes in along the positive z axis,
with its polarization \( \epsilon^\mu \) along the x axis. We shall take the final free
particle relative momentum \( \vec{q} \) to point in the \((\theta, \psi)\) direction (see Fig.1).
With these definitions, the states become (3.30) of Part I, (2.23), (2.16)
and (2.17), i.e.

\[ \left| \Psi_{N=0, \vec{p}_i}^+ \right> = \left[ \frac{\alpha m_i m_2}{M_i (1 + \alpha^2)} \right]^{1/2} e^{i \frac{\vec{q}}{2} \cdot \vec{L}_{35} i \Theta \cdot L_{45}} \left| N_i \right> \]

\[ (N_i = 0) \]
\[
|\psi_{-1}^{*}\rangle = \left(\frac{(2\pi)^3}{V}\right)^{1/2} \left| \frac{M_{\lambda}}{2\pi i m_1 m_2} \frac{\partial\lambda}{\partial k} \right|^{1/2} e^{-\pi\lambda^2/2} \Gamma(1+i\lambda) \times e^{i\theta L_5} e^{-\frac{\pi}{2} L_4} e^{i\theta L_3} e^{i\theta L_2} |\psi_{N=1}^{*}\rangle.
\]

Hence from (4.3)

\[
T_{fi} = \frac{e}{m_1} \left(\frac{M_{\lambda}}{p_0 q}\right)^{1/2} \frac{2\pi}{\sqrt{2}} \left(\frac{1}{2(1+q^2)} \frac{\partial\lambda}{\partial k}\right)^{1/2} e^{-\pi\lambda^2/2} \Gamma(1-i\lambda) M_{fi},
\]

where

\[
M_{fi} = \langle \psi_{-1}^{*}\psi_{1}\rangle = \left< \psi_{-1}^{*}\theta L_2 -i\theta L_3 -i\theta L_4 \right| e^{i\frac{\pi}{2} L_4} e^{-i\theta L_5} e^{i\theta L_3} -i\theta L_5 e^{i\theta L_5} \rangle
\]

Here we used the transverse gauge condition \( q \cdot \xi = 0 \) to eliminate the \((P_f + P_i)_\mu \epsilon^\mu \) part of the current.

In the next subsection we shall calculate \( M_{fi} \). In subsection 3 we shall evaluate \( T_{fi} \) and then, in subsection 4 discuss the non-relativistic reduction of the photo-effect amplitude.

2. Calculation of group matrix elements

The calculation of \( M_{fi} \) is similar to the calculation of \( F_n' \) in (3.4). We shall again work in the oscillator realization of the group representation, split the radial and angular parts and then finally go to the continuum limit.

The operator \( \Gamma^4 \equiv L_{16} \) commutes with the boost \( L_{35} \) and tilt operator \( L_{45} \) in (4.7), so that it can be commuted to the right. We find...
\[ L_{16}^{(000)} = \frac{1}{2} \left\{ |001\rangle - |00 - 1\rangle \right\} . \quad (4.8) \]

We now insert a complete set of group states into \( M_{fi} \) (\( n \equiv i\lambda \)):

\[
M_{fi} = \frac{i}{2} \sum_{n' n_2 m'} <\psi_G | e^{-i\theta L_2 - i\xi L_3} | n' n_2 m' \rangle \langle n' n_2 m' | x e^{-i\theta L_4} e^{-i\pi L_5} x e^{-i\xi q L_3} e^{-i\theta_1 L_4} \{ |001\rangle - |00 - 1\rangle \} .
\]

Now \( L_{12} = L_3 \) commutes with \( L_4 \) and \( L_5 \) so that \( m' \) takes only the two values \( \pm 1 \). Furthermore, \( L_4 \) and \( L_5 \) can be expressed by the elements of the algebras \( O(2,1) \), \( N_1 \) and \( N_2 \) (3.20).

The Casimir operators (3.18) of these subalgebras is given by
\[ k = \frac{|m| + 1}{2} \] . The transition matrix elements then depend only on the absolute value of \( m \), up to a phase which in the continuum case turns out to be one, and we can write

\[
M_{fi} = \sum_{n' n_2} \frac{i}{2} <\psi_G | e^{-i\theta L_2 - i\xi L_3} \{ |n' n_2 + 1\rangle - |n' n_2 - 1\rangle \} \times \langle n' n_2' | e^{-i\theta L_4} e^{-i\pi L_5} e^{-i\xi q L_3} e^{-i\theta_1 L_4} |001\rangle \]
\[
= \sum_{n' n_2} R_{n_2} \left[ T_{n' n_2} \right] R_{n_2} \left[ T_{n' n_2} \right] , \quad (4.10)
\]

with the obvious identifications. We shall then evaluate the energy transition part \( T_{n_2} \). For this purpose we define

\[
\theta_r = (\theta_\lambda - i \frac{\pi}{2}) \equiv \theta_r - \theta_i \quad (4.11)
\]

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and combine the two tilts,

\[ T_{1}^{n_{1} n_{2}'} = \langle n_{1} n_{2}' | e^{-i \theta_{f} L_{45}} e^{i \theta_{1} L_{45}} | 001 \rangle \]

\[ = \langle n_{1} n_{2}' | e^{i \Theta_{f} L_{45} \cosh \Theta_{f} + L_{45} \sinh \Theta_{f}} e^{-i \theta_{1} L_{45}} | 001 \rangle. \]

(4.12)

The matrix elements \( T_{1}^{n_{1} n_{2}'} \) can be factorized using the \( O(2,1) \) transition subgroups \( N_{1} \) and \( N_{2} \) mentioned before (cf. (3.20)). We find

\[ T_{1}^{n_{1} n_{2}'} = D_{1}^{1} D_{n_{1} + 1,1}^{1} W_{1}^{2} D_{n_{2} + 1,1}^{1} W_{2}^{2}, \]

(4.13)

where \( D_{m_{1}, m_{2}}^{k} (W) \) are the \( O(2,1) \) hyperbolic rotation matrices (Appx.C). Here

\[ W_{1,2} = \begin{pmatrix} \alpha_{1,2} & \beta_{1,2} \\ \bar{\beta}_{1,2} & \bar{\alpha}_{1,2} \end{pmatrix} \]

(4.14a)

labels a group element such that

\[ \alpha_{1,2} = (\cosh \frac{\xi}{2} \pm i \sinh \Theta_{f} \sinh \frac{\xi}{2}) \cosh \frac{\Theta_{-}}{2} \mp i \sinh \frac{\Theta_{-}}{2} \cosh \Theta_{f} \sinh \frac{\xi}{2} \]

\[ \beta_{1,2} = \pm \sinh \frac{\Theta_{-}}{2} (\cosh \frac{\xi}{2} \pm i \sinh \Theta_{f} \sinh \frac{\xi}{2}) - i \cosh \frac{\Theta_{-}}{2} \cosh \Theta_{f} \sinh \frac{\xi}{2}, \]

\[ \bar{\alpha}_{1,2} = \Theta_{f} \pm \Theta_{i} \]

(4.14b)

Eq. (4.13) then becomes
As in the case of the Coulomb scattering, we shall treat everything as we
would a bound state problem, and only in the end shall we continue \( n' \) and
\( \theta_f \) to their actual (complex) values. From (4.14) we note that
\[
\alpha_1 = \bar{a}_2 \\
\beta_1 = -\bar{b}_2 \\
n' = n_1' + n_2' + 2
\]
so that we can write
\[
\left( \frac{\alpha_2 \beta_1}{\bar{\alpha}_1 \beta_2} \right)^{n'} \equiv -e^{2i\eta}, \quad \bar{\alpha}_2 \beta_1 \equiv e^{i\eta} \left| \frac{\alpha_2 \beta_1}{\bar{\alpha}_1 \beta_2} \right|.
\]
Then (4.15) becomes
\[
\mathcal{T}_{n_1' n_2'} = \left[ \left( \frac{\alpha_2}{\alpha_1} \right)^{n'} \left( \frac{\bar{\alpha}_2}{\bar{\alpha}_1} \right)^{-n_2} \right] \left[ \left( n_1' + n_2' \right) \right]^{1/2} e^{2i(n\frac{\eta}{2})}.
\]
Next, we consider the rotation matrices \( R_{n_1'n_2'} \). Here we shall take a slightly
more general assignment of the quantum numbers, i.e. \( n_{1f} = n_1, \ n_{2f} = n_2; \)
\( m_f = 0 \). We then have
We now split each of the rotation matrices into two separate \( O(3) \) rotation matrices using the factorization into the product bases defined earlier in Eq. (3.9). Defining

\[
\phi = \frac{1}{2} (n - 1)
\]
\[
m' = \frac{1}{2} (n_2 - n_1) ; \quad n_1 = \phi - m ,
\]
\[
m = \frac{1}{2} (n_2 - n_1 + 1) ; \quad n' = n - 2\phi + 1 ,
\]
we find

\[
R_{n',n_2} = \frac{i}{2} \left\{ D_{m',m}^{\phi}(-\theta) D_{m',-m+1}^{\phi}(-\theta) e^{-i\varphi} - D_{m',m-1}^{\phi}(-\theta) D_{m',-m}^{\phi}(-\theta) e^{i\varphi} \right\}
\]

Further, using the symmetry properties

\[
D_{m',m}^{\phi}(\theta) = (-1)^{m-m'} D_{-m',-m}^{\phi}(-\theta) , \quad D_{m',-m}^{\phi}(-\theta) = (-1)^{m-m'} D_{-m',m}^{\phi}(-\theta) ,
\]
we obtain

\[
R_{n',n_2} = -i \cos \varphi \left\{ D_{m',m}^{\phi}(-\theta) D_{m',-m+1}^{\phi}(-\theta) e^{-i\varphi} \right\}
\]
\[
= (-1)^m \left\langle m',\phi \right| e^{-i\theta L_2} \left| \phi,m \right\rangle \left\langle m-1,\phi \right| e^{i\theta L_2} \left| \phi,m' \right\rangle
\]

(4.20)
The matrix elements $T_{n' n_2}^{i n}$ can also be rewritten in terms of the parameters (4.18):

$$T_{n' n_2}^{i n} = (\alpha_1')^2 (\alpha_2')^{n' - 2} e^{2i(\eta - \frac{\pi}{2})\phi} \left[ (\phi + m)(\phi - m) \right]^{\frac{1}{2}} e^{-2i(\eta - \frac{\pi}{2})m}$$

(4.21)

The expression inside the square root is the coefficient of an angular momentum raising operator applied to the state $|\phi, m-1\rangle$,

$$L_+ |\phi, m-1\rangle = [(\phi + m)(\phi - m + 1)]^\frac{1}{2} |\phi, m\rangle .$$

(4.22)

We can now recombine the matrices $T_{n' n}^{i n}$ and $R_{n' n_2}$ to form $M_{fi}$ according to (4.10),

$$M_{fi} = \left\{ -i \cos \varphi (-i)^{m'} \left[ (\alpha_1')^2 (\alpha_2')^{n' - 2} \right] e^{2i(\eta - \frac{\pi}{2})\phi} \right\} \chi_i \sum_{m} \left[ [(\phi + m)(\phi - m + 1)]^{\frac{1}{2}} e^{-2i(\eta - \frac{\pi}{2})m} \right. \left. \times \langle m', \phi | e^{-i\theta L_2} | \phi, m \rangle \langle m-1, \phi | e^{i\theta L_2} | \phi, m' \rangle \right\}

= \left\{ \chi_i \sum_{m} \langle m', \phi | e^{-i\theta L_2} e^{-2i\eta L_3} L_+ | \phi, m-1 \rangle \langle \phi, m-1 | e^{i\theta L_2} | \phi, m' \rangle \right\}

= \left\{ \chi_i \sum_{m} \langle m', \phi | e^{-i\theta L_2} e^{-2i\eta L_3} L_+ e^{i\theta L_2} | \phi, m' \rangle \right\} .

(4.23)

Here we used Eqs. (4.20)-(4.22) and the completeness of the set of angular momentum eigenstates

$$\sum |\phi, m - 1\rangle \langle m - 1, \phi| = 1 ,$$

which is true by virtue of $L_+ |\phi, \phi\rangle = 0$. Commuting the rotation operators
and inserting a new complete set into $X_{\mathbf{i}}$, we have

$$M_{\mathbf{i}} = \left\{ \begin{array}{c} \left\{ \sum_{m} \langle m', \phi | e^{i\theta_{L_2}} e^{-2i\eta L_3} e^{i\theta_{L_2}} | \phi, m \rangle \right. \\
\left. \times \langle m, \phi | e^{-i\theta_{L_2}} L^+ e^{i\theta_{L_2}} | \phi, m' \rangle \right\} \right\}
$$

$$= \left\{ \begin{array}{c} \left\{ \sum_{m} \langle m', \phi | e^{-2i\eta} (L_3 \cos \theta + L_1 \sin \theta) | \phi, m \rangle \\
\times \langle m, \phi | L^+ \frac{\cos \theta + 1}{2} + L^- \frac{\cos \theta - 1}{2} - L_3 \sin \theta | \phi, m \rangle \right\} \right\}
$$

$$= \left\{ \begin{array}{c} \left\{ \mathcal{D}_{m', m'+1}^{m, \phi} (W) \left[ \frac{\cos \theta + 1}{2} \left[ (\phi - m')(\phi + m' + 1) \right]^{1/2} \right. \\
+ \mathcal{D}_{m', m}^{m, \phi} (W) \left( -m' \sin \theta \right) \\
+ \mathcal{D}_{m', m'-1}^{m, \phi} (W) \left( \frac{\cos \theta + 1}{2} \left[ (\phi - m')(\phi + m' + 1) \right]^{1/2} \right) \right\} \right\}
$$

(4.24)

where

$$W = \begin{pmatrix} \alpha & \beta \\ -\beta & \bar{\alpha} \end{pmatrix} \quad \alpha = \cos \eta + i \cos \theta \sin \eta \\ \beta = -i \sin \eta - i \cos \theta \cos \eta \quad \bar{\alpha} = +i \sin \theta \sin \eta
$$

(4.25)

Now we go back to (4.18) and, recalling that for $(\psi^+_{-i\lambda})^*$

$$n \to +i \lambda \quad n_1 = n - 1 \quad n_2 = 0, m = 0$$

we find
The $\alpha$'s and $\eta$ are defined in (4.14b) and (4.16b). Inserting these, we obtain for $M_{f1}$ the expression

\[
M_{f1} = \left[ -i \cos \varphi \left( -\frac{1}{2} \right)^{n+2} (\alpha_1) (\alpha_2) \right] (\beta_2) \left( n-2 \right) \left( 2i \eta \right) \left( \frac{n-1}{2} \right)
\]

\[
\times \left( \frac{n-1}{2} \right) \left( \cos \eta \cos \theta \sin \eta \right)^{n-2} \left( \sin \theta \right) \left( i \eta \right)
\]

The $\alpha$'s and $\eta$ are defined in (4.14b) and (4.16b). Inserting these, we obtain for $M_{f1}$ the expression

\[
M_{f1} = -i \left( \frac{N+1}{2} \right) \left( \cos \varphi \sin \theta \right) \left( -1 \right)^N
\]

\[
\left[ \cosh \gamma \cosh \theta_1 \cosh \theta_2 \sinh \theta_1 \sinh \theta_2 + 1 \right]^N
\]

\[
\left( \cosh \gamma \sinh \theta_1 \cosh \theta_2 - \sinh \theta_1 \cosh \theta_2 \right)
\]

\[
- i \cos \theta \left( \sinh \gamma \cosh \theta_1 \right)^{N-2}
\]

Here we have set

\[
N \equiv n^w = -i \lambda,
\]

so that we can keep close to the notation employed by Sommerfeld (1960) in the non-relativistic treatment. In order to express $M_{f1}$ in a more useful form, we use (4.11) and (2.19),

\[
\theta_\lambda = \theta_\lambda - i \frac{\pi}{2}
\]

\[
\tanh \theta_\lambda = -2m_2 M_\lambda \left( M^2 + m_2^2 - m_1^2 \right),
\]

and find
Furthermore, the rapidity $\xi_q$ of the ground state $H$ atom and its tilting angle $\theta_1$ are given by

$$\sinh \xi_q = - \frac{q}{m_1}, \quad \sinh \theta_1 = - \frac{M_i^2 + m_1^2 - m_i^2}{A_i}$$

$$\cosh \xi_q = \frac{p_0}{m_1}, \quad \cosh \theta_1 = \frac{2m_1M_i}{A_i}$$

$$A_i = \left[ (2m_1M_i)^2 - (M_i^2 + m_1^2 - m_i^2)^2 \right]^{1/2} \quad (4.28)$$

If (4.27) and (4.28) are inserted into (4.26), then $M_{fi}$ can be written, after a little manipulation, as

$$M_{fi} = 16i N \frac{\alpha^2 m_i^2 m_2^2}{(1 + \alpha^2)^2} \frac{N+1}{2} \cos \varphi \sin \varphi \left( \frac{|c|}{m_i^2 - t} \right)^N e^{i \tau N} \frac{i}{(t - m_i^2)^2} \quad (4.29)$$

where

$$|c| e^{i \tau} \equiv \frac{(M_i^2 + m_1^2)(m_1^2 + m_2^2) - M_i^2 M_{fi}^2 - (m_1^2 - m_i^2)^2}{2m_1M_{fi}^2} - i \frac{2k \alpha}{(1 + \alpha^2)^{1/2}}$$

$$\tau = \left( p_\perp - p_\perp \right)^2 = (E_i - q)^2 - (\vec{k} - \vec{q})^2 = m_1^2 - 2qE + 2kq \cos \Theta$$

$$E_i = (m_1^2 + k_2^2)^{1/2} = \frac{1}{2M_{fi}^2} \left[ M_{fi}^2 + m_i^2 - m_2^2 \right]$$

In the next subsection we substitute (4.29) into the expression for $T_{fi}$ (4.6).
The transition matrix $T_{fi}$

Inserting for $M_{fi}$ in (4.6) our result (4.29), we obtain for the $T$ matrix

$$
T_{fi} = \frac{e}{m_i} \frac{1}{\sqrt{2}} \left[ \frac{M_f}{p_{f_i}^0} \frac{2\pi^2}{(i+\alpha^2)^3} \left( \frac{\partial\lambda}{\partial k} \right)^2 \right]^{1/2} 8 k^2 \frac{N+1}{2} \frac{m_i^2 m_f^2}{(m_i^2 - t)^3} \Gamma(1+n) \left( \frac{1}{m_i^2 - t} \right)^N e^{i \left( \tau + \frac{\pi}{2} \right) N} \frac{\sin \theta \cos \varphi}{(t - m_i^2)^2}.
$$

In the cross-section formula (2.4), the factor $\left[ \frac{|c|}{m_i^2 - t} \right]^N$ does not appear because $N$ is pure imaginary. The term $\exp \left[ i \left( \tau + \frac{\pi}{2} \right) N \right]$ is similar to a term found in the non-relativistic case and is thus not unexpected. The angular dependence of $T_{fi}$ is basically simple.

The factor $[\sin \theta \cos \varphi]$ is the usual dipole term

$$
k \cdot \hat{e} = k \sin \theta \cos \varphi.
$$

Furthermore

$$
t - m_i^2 = -2q E_1 \left[ 1 - \frac{k}{E_1} \cos \theta \right]
$$

$$
= -2q E_1 \left[ 1 - \beta \cos \theta \right],
$$

where

$$
\beta = \frac{V_1}{c},
$$

and $V_1$ is the velocity of the asymptotically free electron. The more interesting angular dependence of $T$ is thus

$$
T \propto \frac{\hat{e} \cdot \vec{k}}{[1 - \beta \cos \theta]^2}
$$

or

$$
\frac{\partial \sigma}{\partial \Omega} \propto \frac{(\hat{c} \cdot \vec{k})^2}{[1 - \beta \cos \theta]^4}.
$$
We note the comment of Sommerfeld (1960) (p. 494) that the factor \([1 - \beta \cos \theta]\)
is the correct relativistic generalization of the corresponding non-relativistic
factor \([1 - \beta \cos \theta + \gamma]\).

### 4. Non-relativistic reduction

Our relativistic result reduces to the corresponding non-relativistic
expression derived by Sommerfeld in the limit of small photon energy and to
lowest order in \(\alpha^2\). It is easiest to begin the reduction by starting
from Eqs. (4.26) - (4.28). If we expand the masses \(M_i\) or \(M_f\) (3.27) of
Part I or (2.22) in powers of \(\alpha^2\), we can write

\[ M_{i,f} = m_i + m_2 - B_{i,f}, \]

where, to lowest order,

\[ B_{i,f} = \frac{m_im_2}{m_1+m_2} \frac{\alpha^2}{2N_{i,f}^2}, \]

\[ N_i = 1, \quad N_f = N = -i\lambda. \]

Furthermore,

\[ \sinh \xi_q = -\frac{\alpha}{M_1}, \]

so that

\[ \cosh \xi_q = \left[1 + \sinh^2 \xi_q\right]^{1/2} \approx 1 + \frac{1}{2} \frac{\alpha^2}{M_1^2}. \]

The relationship between \(\lambda\) and \(\lambda\) relative momentum \(K\) in the non-relativistic
limit becomes (2.22a)

\[ K = \frac{\alpha}{\lambda} \frac{m_1m_2}{M} \left(1 - \frac{\alpha^2}{\lambda^2}\right)^{-1/2} \approx \frac{\alpha}{\lambda} \frac{m_1m_2}{m_1+m_2} = -i \frac{m_1m_2}{m_1+m_2}. \]

If we put (4.34a,b,c,d,e) into (4.26) and (4.6), we find, after some
calculation,
\[ T_{fi} = \frac{e}{\sqrt{2}} \left( 2\pi^2 \frac{\alpha}{(q \cdot m_1)} \right)^{1/2} \Gamma(i+N) 8i^{(N+1)} \frac{k \cdot \hat{E}}{a_0^2 (c^r_b)^2} \]
\[ \times \left( \frac{|c|}{c^r_b} \right)^N e^{i(\tau + N/2)N} , \]

where

\[ c' = \left| c \right| e^{i\tau} = \left( q \cdot \frac{\mu}{m_1} \right)^2 k^2 + \frac{1}{a_o^2} - 2i \frac{k}{a_o} \]
\[ c^r_b = k^2 + \frac{1}{a_o^2} + (q \cdot \frac{\mu}{m_1})^2 - 2k \left( q \cdot \frac{\mu}{m_1} \right) \cos \theta \]
\[ \mu = \frac{m_1 m_2}{m_1 + m_2} , \quad a_o = \frac{1}{\alpha \mu} \]

This expression can be compared with Eqs. (5) and (20) of Chapter VI.4 (p.456) of Sommerfeld (1960). Except for the difference between our volume normalization and his delta function normalization and for the factor relating \( T_{fi} \) to \( f(\theta) \), the results are identical.

V. CONCLUSION AND DISCUSSION

The formalism outlined in this paper can be used to treat all hydrogen atom problems wherein the photon, except for its interaction with the electron, is treated as a neutral particle and only the electron has charge (as far as external electromagnetic interactions are concerned). We are then actually considering a hydrogen-like particle with the charge of an electron, but having the structure of a hydrogen atom. Thus, strictly speaking, an extension of the formalism is required to the case where the proton is also charged so that the system as a whole is neutral. This problem has not yet been elegantly solved.

A further development of the dynamical group theory will have to account for the breaking of the O(4) symmetry of the hydrogen atom. This problem is not trivial.
Mathematically it is equivalent to the problem of combining the known $O(2,1)$ group of the radial equation with the rotation group in a sufficiently large algebraic structure to allow for the calculation of physical form factors and current matrix elements.

One of the aesthetically most pleasing points in the relativistic infinite-component wave equation treatment is the demonstration that a "hydrogen" atom equation can be written in the form of a Dirac equation. That both the Dirac equation and the infinite-component wave equation are based on the representations of the conformal group $SO(4,2)$ is surely not accidental; it reflects the minimum algebraic structure needed for a realistic particle model with many mass states.

The techniques developed here for the treatment of Coulomb scattering and photo-effect are being applied to dyonium systems, an atom formed out of two particles having both electric and magnetic charges.

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APPENDIX A

KINEMATICS

System 1: Centre of momentum

The four vectors for two free particles, 1 and 2, are:

\[ p_1 = \begin{bmatrix} E_1 \\ +q \end{bmatrix}; \quad p_2 = \begin{bmatrix} E_2 \\ -q \end{bmatrix}, \text{ where } E_1 = m_1^2 + q^2, \quad E_2 = m_2^2 + q^2 \]

then

\[ s = (p_1 + p_2)^2 = \left[ (m_1^2 + q^2)^{1/2} + (m_2^2 + q^2)^{1/2} \right]^2 \]

\[ = m_1^2 + m_2^2 + 2q^2 + 2 \left[ (m_1^2 + q^2)(m_2^2 + q^2) \right]^{1/2} \]

Solving for \( q^2 \), we find

\[ 4q^2 s = \left[ s - (m_1^2 + m_2^2) \right]^2 - 4m_1^2 m_2^2 \]

\[ = \left[ s - (m_1^2 - m_2^2) \right]^2 - 4m_2^2 s \]

\[ = \left[ s - (m_2^2 - m_1^2) \right]^2 - 4m_1^2 s \]

\[ = \left[ s - (m_1 - m_2)^2 \right] \left[ s - (m_1 + m_2)^2 \right] \]

Solving for \( E_1^2 \), we find

\[ E_1^2 = m_1^2 + q^2 \]

\[ = \frac{1}{4s} \left[ s - (m_2^2 - m_1^2) \right]^2 \]

\[ E_2^2 = \frac{1}{4s} \left[ s + m_1^2 - m_2^2 \right]^2 \].
System 2: Particle 2 at rest

The four vectors for two free particles, 1 and 2, with 2 at rest, are:

\[ p_1 = \begin{bmatrix} E_1 \\ -k \end{bmatrix}, \quad p_2 = \begin{bmatrix} m_2 \\ 0 \end{bmatrix}, \text{ where } e_1 = m_1 + k^2, \]

\[ e_2 = m_2 \]

then

\[ s = [p_1 + p_2]^2 = \left[ \left( m_1^2 + k^2 \right)^{\frac{1}{2}} + m_2 \right]^2 - k^2 \]

\[ = m_1^2 + m_2^2 + 2m_1 m_2 E_1 \]

If we make the choice \( E_1 = m_1 \left[ 1 + \frac{a^2}{N^2} \right]^{-\frac{1}{2}} \), we find

\[ s = m_1^2 + m_2^2 + 2m_1 m_2 \left[ 1 + \frac{a^2}{N^2} \right]^{-\frac{1}{2}} \]

Next,

\[ E_1 = \frac{s - (m_1^2 + m_2^2)}{2m_2} \]

Also,

\[ k^2 = E_1^2 - m_1^2 \]

\[ = \frac{1}{4m_2^2} \left\{ \left[ s - (m_1^2 + m_2^2) \right]^2 - 4m_1^2 m_2^2 \right\} \]

\[ h m_2 \cdot k^2 = \left[ s - (m_2^2 - m_1^2) \right]^2 - 4sm_2 \]

\[ = \left[ s - (m_1^2 - m_2^2) \right]^2 - 4sm_2 \]
APPENDIX B

NORMALIZATION OF GROUP STATES

We carry out the normalization procedure in the "r" space realization. This allows us to make use of the partial-wave analysis procedure developed for the non-relativistic hydrogen atom (Landau and Lifshitz 1958, Messiah 1962).

In the "r" space realization taking \( (\psi^G)^+ \), for example, we have

\[
(\psi^G)^+ = c^+ e^{-i\frac{L}{2}} \sum_{\eta_1, \eta_2, m} \frac{\psi^G}{r^{\eta_1} r^{\eta_2} m} \frac{\eta_1 = -1}{\eta_2 = -i \lambda} \frac{m = 0}{\lambda} \\
= c^+ (i\sqrt{\pi}) e^{i\frac{1}{2}(\xi + \eta)} F(1, i, \xi) F(i\lambda, 1, i\eta) \\
= c^+ \frac{i}{\sqrt{\pi}} e^{i\xi} F(i\lambda, 1, i(r-z))
\]

Using a contour integral representation for the coefficient hypergeometric function (Abramowitz and Stegun)

\[
F(a, b, z) = (1 - e^{-2\pi i \alpha})^{-1} \frac{\Gamma(b)}{\Gamma(a) \Gamma(b-a)} \int e^{zt} t^{a-1} (1-t)^{b-a-1} dt
\]

we find

\[
[\psi^S]^+ = c^+ \frac{i}{\sqrt{\pi}} \int e^{i(r-z) t} i^{(r-z) t} i^{(1-t) (1-t)} dt \\
\times (1 - e^{-2\pi i (i\lambda)})^{-1} \frac{t}{\Gamma(i\lambda) \Gamma(i-i\lambda)}
\]

-38-
Next, we use the standard radial decomposition of $e^{i z}$ and expression for the spherical Bessel function

$$e^{i z} = \sum_{l=0}^{\infty} (2l+1) i^l j_l(r) P_l(\cos \theta)$$

$$j_l(\xi) = \frac{2^l l!}{(2l+1)!} \xi^l e^{i \xi} \frac{\Gamma(l+1, 2l+2, -2i \xi)}{\Gamma(l+1)}$$

$$= 2^l \xi^l e^{i \xi} \sum_{p=0}^{\infty} \frac{(l+p)!}{(2l+1+p)!} \frac{(-2i \xi)^p}{p!}$$

and find

$$\left[ \Psi_\xi^G \right]^+ = \frac{ic^+}{\sqrt{\pi}} (1 - e^{2\pi i \lambda})^{-1} \frac{1}{\Gamma(i \lambda) \Gamma(1-i \lambda)} 2^l (2l+1) i^l P_l(\cos \theta)$$

$$\times \sum_{p} \frac{(l+p)!}{(2l+1+p)!} \frac{(-2i r)^p}{p!} \int e^{i r \xi} \int t^{i \lambda-1} (1-t)^{l+p-i \lambda} dt$$

The integral is now in a standard form of the beta function,

$$\int t^{i \lambda-1} (1-t)^{l+p-i \lambda} dt = (1 - e^{2\pi i (l+p-i \lambda)}) \frac{\Gamma(i \lambda) \Gamma(1-i \lambda) \Gamma(l+p-i \lambda)}{\Gamma(l+p+1)}$$

If we re-insert this into the sum, we find that we have a confluent hyper-

geometric function.

$$\left[ \Psi_\xi^G \right]^+ = \frac{c^+ e^{-(i \xi) \lambda}}{\sqrt{\pi} \Gamma(1-i \lambda)} \sum_{\xi} (2l+1) i^l e^{i \xi} P_l(\cos \theta) \times$$

-39-
where

\[ \sigma_{\ell} = \arg \left( \Gamma(l+1-i\lambda) \right). \]

\[ [e^{iZ} F(l+1-i\lambda, 2l+2, -2iZ) ] \text{ is real} \]

We have made a partial-wave analysis of \( [\psi^G_{\ell}]^+ \). The general state \( [\psi^G_{\ell}]^+ \)
is found by rotating \( k^2 \) in the direction \((\theta, \phi)\). Let us take the incoming direction to be \( \hat{k} \). Then \( \cos \theta = \frac{k \cdot \hat{x}}{r} \) and we find

\[ \left[ \psi^G_{\ell} \right]^+ = \frac{c^+}{\sqrt{2}} \frac{e^{-i\lambda \frac{\pi}{2}}}{\Gamma(i-\lambda)} \sum_{\ell} (2l+1) \frac{i\ell}{r} e^{i\sigma_{\ell}} R^G_{\lambda,\ell} P_\ell \left( \frac{k \cdot \hat{x}}{r} \right) \]

Then, to find the normalization constants, we consider

\[ \langle \left[ \psi^G_{\ell, \hat{k}} \right]^+ | \left[ \psi^G_{\ell', \hat{k}} \right]^+ \rangle = \frac{1}{2} \frac{e^{-i(\lambda+i\chi)}}{\Gamma(l+i\chi)\Gamma(l-i\chi)} (c^+_\lambda)^* c^+_\lambda \]

\[ \times \sum_{t, t'} (2l+1)(2l'+1) e^{i(\sigma_{\ell} - \sigma_{\ell'})} \langle R^G_{\lambda', \ell'} | R^G_{\lambda, \ell} \rangle \int \frac{P_\ell \left( \frac{k' \cdot \hat{r}}{r} \right) P_\ell \left( \frac{k \cdot \hat{r}}{r} \right) \sin \theta'}{r} d\Omega. \]

For simplicity, we take \( k = \hat{z} \) and \( \hat{k}' \) in the x-z plane. Then we have \( \frac{k' \cdot \hat{r}}{r} = \cos \gamma = \cos \theta \cos \theta' + \sin \theta' \sin \theta \cos \phi \). The angles are shown in Fig.2. From the spherical harmonic addition formula we find
\[ P_L(\cos \theta') = \sum_{m=1}^l \frac{(l-m)!}{(l+m)!} P_L^m(\cos \theta) P_L^m(\cos \theta') \cos [m(\phi - \phi')]. \]

If we put this into the angular integral, the \( \phi \) integral gives \( \delta_{m0} \), and the remaining term gives

\[ \int P_L(\cos \theta) P_L(\cos \theta') d\Omega = \frac{4\pi}{(2l+1)} \delta_{ll'} \delta_{mm'} P_L(\cos \theta') \]

The radial integral is a delta function on \( \lambda \) since \( l' = l \). Then the "phase" shifts are equal, and we finally have

\[ \langle \psi^+ | \psi^+ \rangle = \frac{2\pi}{\Gamma(1+i\lambda) \Gamma(1-i\lambda)} \sum_L (2l+1) P_L(\cos \theta') \delta(\lambda' - \lambda) \]

\[ = \frac{4\pi}{\Gamma(1+i\lambda) \Gamma(1-i\lambda)} \delta(\lambda' - \lambda) \delta(1 - \cos \theta') \]

For the sum over \( l \) we simply used the completeness relation for spherical harmonics. If we now choose

\[ C^+ = \frac{1}{\sqrt{4\pi}} e^{\pi \lambda/2} \Gamma(1-i\lambda) \]

we have finally

\[ \langle (\psi^G_{\lambda', \hat{k}(\theta', \phi')}^+) | (\psi^G_{\lambda, \hat{\lambda}}^+) \rangle = \delta(\lambda' - \lambda) \delta(1 - \cos \theta), \]

i.e. Eq. (2.18).
APPENDIX C

0(3) AND O(2,1) MATRIX ELEMENTS

1. **Rotations in 0(3) (Barut and Phillips 1968)**

The spinor representation of 0(3) is

\[
W = \begin{bmatrix}
\alpha & \beta \\
-\beta & \alpha
\end{bmatrix} ; \quad \text{det } W = 1
\]

We use: \( l_1 = \frac{\sigma_1}{2} \); \( l_2 = \frac{\sigma_2}{2} \), \( l_3 = \frac{\sigma_3}{2} \). The rotation matrix elements are

\[
D_{m'm}^\phi (W) = \langle m', \phi | R(W) | \phi, m \rangle = \frac{1}{(m'-m)!} \left[ \frac{(\phi+m')!(\phi-m)!}{(\phi-m')!(\phi+m')!} \right]^{1/2} \\
\times (\alpha')^{-m-m'} \langle \beta \rangle F(-\phi-m, \phi-m+1, m'-m+1; \beta \bar{\beta}) , \\
(m' \geq m)
\]

and for \( m' \leq m \)

\[
D_{m'm}^\phi (W) = \frac{1}{(m'-m)!} \left[ \frac{(\phi+m')!(\phi-m')!}{(\phi-m)!(\phi+m')!} \right]^{1/2} (\alpha')^{-(m'+m)} \langle \beta \rangle^{-m-m'} \\
\times F(-\phi-m', 1+\phi-m', m'-m+1; \beta \bar{\beta}) .
\]

(C.1)

(C.2)

Here \( \phi \) is the usual \( J \), \( m \) is the usual \( J_3 \).

2. **Hyperbolic rotations in 0(2,1)**

The spinor representation of SO(2,1) is

\[
W = \begin{bmatrix}
\alpha & \beta \\
-\bar{\beta} & \bar{\alpha}
\end{bmatrix} ; \quad \text{det } W = 1
\]
We use: \( \lambda_1 = \frac{i \sigma_3}{2} \); \( \lambda_2 = -\frac{i \sigma_1}{2} \); \( \lambda_3 = \frac{C_3}{2} \). The rotation matrix elements are

\[ D_{m'm'}^{k}(W) = \langle m',k|W(t)|k,m \rangle. \]

**Case 1: \( m' \geq m \); \( D^+ \) representations, \( k \geq 0 \)

\[
D_{m'm'}^{k}(W) = \frac{1}{(m'-m)!} \left[ \frac{\Gamma(k+m') \Gamma(m'-k+1)}{\Gamma(k+m) \Gamma(m-k+1)} \right]^\frac{1}{2} \begin{pmatrix} \alpha \cr \beta \end{pmatrix}^{-(m'+m)} \begin{pmatrix} \alpha \cr \beta \end{pmatrix}^{m'-m}
\]

\[ \times \sqrt{F(k-m, 1-k-m; m'-m+1; \beta, \beta)}. \quad (C.3) \]

**Case 2: \( m' \leq m \)

\[
D_{m'm'}^{k}(W) = \frac{1}{(m-m')!} \left[ \frac{\Gamma(k+m) \Gamma(m-k+1)}{\Gamma(k+m) \Gamma(m'-k+1)} \right]^\frac{1}{2} \begin{pmatrix} \alpha \cr \beta \end{pmatrix}^{-(m'+m)} \begin{pmatrix} \alpha \cr \beta \end{pmatrix}^{m'-m'}
\]

\[ \times \sqrt{F(k-m', 1-k-m'; m-m'+1; \beta, \beta)}. \quad (C.4) \]

Using (D.2) we find

\[
D_{m'm'}^{k}(W) = (-1)^{m-k} \left[ \frac{\Gamma(k+m) \Gamma(k+m')}{\Gamma(m+1-k) \Gamma(m'+1-k)} \right]^\frac{1}{2}
\]

\[ \times \alpha^{m'+m} (\alpha \beta)^{-(m+k)} \beta^{m-m'} \]

\[ \times \sqrt{F(m+k, k-m', 2k, \frac{1}{\alpha \beta})}. \quad (C.5) \]
APPENDIX D

USEFUL FORMULAE (Abramowitz and Stegun)

\[ F(a, b, c, z) = (1-z)^{c-a-b} F(c-a, c-b; c; z) \]  
(D.1)

\[ F(a, b, c; z) = \frac{\Gamma(c) \Gamma(b-a)}{\Gamma(b) \Gamma(c-a)} (-z)^a F(a, 1-c+a, 1-b+a; \frac{1}{z}) + \frac{\Gamma(c) \Gamma(a-b)}{\Gamma(a) \Gamma(c-b)} (-z)^b F(b, 1-c+b, 1-a+b; \frac{1}{z}). \]  
(D.2)

Asymptotic expansion, large \(|z|\) (a, b fixed),

\[ \frac{F(a, b, z)}{\Gamma(b)} = \frac{e^{\pm i\pi a}}{\Gamma(b-a)} \left\{ \sum_{n=0}^{R-1} \frac{(a)_n (1+a-b)_n}{n!} (-z)^n + O(|z|^R) \right\} \]
\[ + \frac{z}{\Gamma(a)} \left\{ \sum_{n=0}^{S-1} \frac{(b-a)_n (1-a)_n}{n!} z^n + O(|z|^S) \right\}, \]

(D.3)

upper sign if \(-\frac{1}{2} \pi < \arg z \leq \frac{3}{2} \pi\); lower sign if \(-\frac{3}{2} \pi < \arg z \leq \frac{1}{2} \pi\).

\[ F(a, b, c, z) = (1-z)^{-a} \frac{\Gamma(c) \Gamma(b-a)}{\Gamma(b) \Gamma(c-a)} F(a, c-b; a-b+1; \frac{1}{1-z}) + (1-z)^{-b} \frac{\Gamma(c) \Gamma(a-b)}{\Gamma(a) \Gamma(c-b)} F(b, c-a; b-a+1; \frac{1}{1-z}) \]  
(D.4)

\[ \binom{n+\alpha}{n} (x) = \frac{(n+\alpha)!}{n! \alpha!} F(-n, \alpha+1, x) = \frac{(n+\alpha)!}{n! \alpha!} F(-n, \alpha+1, x). \]  
(D.5)
Sum formula

\[
\sum_{m=0}^{n} \binom{n+m}{m} \mu^{-m} (1-\mu)^{m} L_{n-m}(x) = L_{n}(\mu x),
\]

or, using (D.5),

\[
\sum_{m=0}^{n} \frac{n!}{m! (n-m)!} \mu^{-m} (1-\mu)^{m} F(-n+m, \alpha+1, x) = F(-n, \alpha+1, \mu x),
\]

or, expanding the confluent hypergeometric functions,

\[
\sum_{m=0}^{n} \frac{n!}{m! (n-m)!} \mu^{-m} (1-\mu)^{m} \frac{\Gamma(-n+m+M) \Gamma(\alpha+1)}{\Gamma(-n+m) \Gamma(\alpha+1+M)} \frac{x^{M}}{M!} = \sum_{M'} \frac{\Gamma(-n+M')}{\Gamma(-n)} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1+M')} \frac{x^{M'/M}}{M'!}.
\]

The proof of this identity depends only on the fact that for each power of \(x\), the coefficients on both sides must be the same. Thus one can just add another index to both sides (\(\alpha\)). Then

\[
\sum_{m=0}^{n} \frac{n!}{m! (n-m)!} \mu^{-m} (1-\mu)^{m} F(-n+m, \beta, \alpha+1, x) = F(-n, \beta, \alpha+1, \mu x).
\]

(D.6)
We determine the normalization constant $C$ of the scattering states (2.20) by the condition that

$$\langle \psi_{\tilde{\xi}(\eta), \tilde{p}}^{\pm} | J_{0} | \psi_{\tilde{\xi}(\eta), \tilde{p}}^{\pm} \rangle = \delta_{\tilde{p} \tilde{p}'} \delta_{\tilde{\xi} \tilde{\xi}'}$$

The left-hand side is equal to

$$\begin{align*}
C_{\tilde{p}(\xi), \tilde{p}'} C_{\tilde{p}(\xi), \tilde{p}} \int \frac{d^{3}X e^{i(\tilde{p} - \tilde{p}') \cdot \tilde{X}}}{V} \\
\times \langle \psi_{\chi(\theta', \phi')}^{\pm G} | e^{-i\theta_{\chi} \cdot L_{45}} \left\{ e^{-i(\tilde{q} \cdot \tilde{p} - \tilde{q}' \cdot \tilde{p}')} \right\} e^{i\theta_{\lambda} \cdot L_{45}} | \psi_{\chi(\theta, \phi)}^{\pm G} \rangle \\
= C^{2} \frac{\delta_{\tilde{p} \tilde{p}'} V \left(-\frac{1}{m_{1}}\right) \cosh \xi}{c_{0}} \\
\times \langle \psi_{\chi(\theta', \phi')}^{\pm G} | e^{-i\theta_{\chi} \cdot L_{45}} \left[ c_{0} - \frac{M_{\lambda}}{m_{2}} \alpha + \frac{M_{\lambda}}{m_{2}} S \right] \\
\times e^{i\theta_{\lambda} \cdot L_{45}} | \psi_{\chi(\theta, \phi)}^{\pm G} \rangle
\end{align*}$$
\[= C^* C \cdot \frac{\delta(p,p')}{\overleftrightarrow{p}} \left( -\frac{1}{m_1} \right) \cosh \xi \left\langle \psi_{\pm G}^\dagger \chi(\theta,\varphi) \right\rangle \]

\[\times \left\{ \Gamma_0 \left( \cosh \theta_{\lambda} + \frac{H_{\lambda}}{m_2} \sinh \theta_{\lambda} \right) + \frac{m_1}{m_2} \left( \cosh \theta_{\lambda} + \sinh \theta_{\lambda} \right) - \frac{H_{\lambda}}{m_2} \right\} \times \left\langle \psi_{\pm G}^\dagger \chi(\theta,\varphi) \middle| \psi_{\pm G} \chi(\theta,\varphi) \right\rangle \]

\[= C^* C \cdot \frac{\delta(p,p')}{\overleftrightarrow{p}} \left( -\frac{1}{m_1} \right) \cosh \xi \left\{ \frac{\lambda}{m_2} \left[ \frac{H_{\lambda}}{m_2} \cosh \theta_{\lambda} + \sinh \theta_{\lambda} \right] - \frac{H_{\lambda}}{m_2} \right\} \times \left\langle \psi_{\pm G}^\dagger \chi(\theta,\varphi) \middle| \psi_{\pm G} \chi(\theta,\varphi) \right\rangle .\]

Choosing \( \mathbf{q}' \) along the \( z \) axis and \( \mathbf{q} = (q, \theta, \varphi) \) we have from (2.18)

\[\delta^3(\mathbf{q}' - \mathbf{q}) = \left[ C^* C \cdot \frac{\delta(p,p')}{\overleftrightarrow{p}} \left( -\frac{1}{m_2} \right) \cosh \xi \frac{m_1 m_2}{M_{\lambda}} \frac{1}{q^2} \right] \delta(1 - \cos \theta) \delta(\chi - \lambda) .\]

To evaluate the normalization constant, we integrate over \( d^3q \). Then

\[1 = -C^* C \cdot \frac{\delta(p,p')}{\overleftrightarrow{p}} \frac{m_1 m_2}{M_{\lambda}} \frac{1}{q^2} \int \left( 1 - \cos \theta \right) \frac{\delta(\mathbf{q}' - \mathbf{q})}{|\partial \lambda | \partial q |} \frac{q^2 dq d\theta (\sin \theta) d\phi}{q^2} .\]

\[= -C^* C \cdot \frac{\delta(p,p')}{\overleftrightarrow{p}} \frac{m_1 m_2}{M_{\lambda}} \frac{1}{q^2} \frac{2\pi}{|\partial \lambda | \partial q |} q^2 .\]
Thus we find

$$\mathcal{C}^\xi \mathcal{C} = \left[ -\mathcal{V} \frac{E_\alpha}{M_\lambda} \left( \frac{m_1 m_2}{M_\lambda} \frac{1}{q^2} \right) \left( \frac{2\pi q^2}{|\partial\lambda/\partial q|} \right)^{-1} \right].$$

Using (2.22b) and the mass spectrum, we find

$$\frac{\partial \lambda}{\partial q} = i \frac{\alpha}{\kappa \lambda} \frac{m_1 m_2}{M_\lambda} \frac{1}{q^2} \left\{ \left( 1 - \frac{\lambda^2}{\lambda^2} \right)^{1/2} \left[ 1 - \frac{\lambda^2}{\lambda^2} \frac{m_1 m_2}{M_\lambda^2 (1 - \frac{\lambda^2}{\lambda^2})^{1/2}} \right] \right\}.$$ 

In order to change over to the asymptotic two-free-particle normalization, we multiply the wave function by \( \left[ \frac{2\pi}{V} \right]^{1/2} \). This converts the normalization from a \( \delta_{\vec{p}', \vec{p}} \delta^3(q' - q) \) to a \( \delta_{\vec{p}', \vec{p}} \delta_{q', q} \) normalization appropriate to a one particle per volume \( V \) for each of the two free particles. Finally we have the result (2.23).
REFERENCES TO PART II


Salamó, S. (1972), Bound state and scattering problems for particles with electric and magnetic charges, University of Colorado, thesis.

Kinematics of the photo-effect.

Fig. 1

The angles in the calculation of the normalization of continuum states.

Fig. 2