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SL(6,C) GAUGE INVARIANCE OF EINSTEIN-LIKE LAGRANGIANS *

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1. Einstein's gravitational Lagrangian possesses two distinct invariances; the well-known GL(4,R) invariance of Einstein under co-ordinate transformations, and the less familiar SL(2,C) gauge-invariance of Weyl. The literature of general relativity places so much emphasis on the GL(4,R) group structure with its differential geometry connotations that the particle group theorist has tended to regard Einstein's beautiful Lagrangian as an entity foreign to his experience. In this note we wish to rewrite this Lagrangian [1] in a Dirac γ-matrix basis [2], emphasising its SL(2,C) gauge-invariance aspects. Our motivation is to show that the underlying SL(2,C) gauge symmetry can be immediately generalized to SL(6,C) or U(6,6) without increasing the dimensionality of space and time. In another note we shall show that such Einstein-like Lagrangians appear to offer a deep basis for marrying internal symmetries with space and time.

2. Consider SL(2,C) gauge transformations,

$$S(x) = \exp i\left[\varepsilon_{ab}(x) \frac{\sigma^{ab}}{2}\right] \tag{1}$$

under which a spin-$\frac{1}{2}$ field $\psi(x)$ transforms as

$$\psi(x) \rightarrow \psi'(x) = S(x)\,\psi(x) \tag{2}$$

together with a set of sixteen fields $L^{\mu}(x) = L^{\mu a}(x)\,\gamma_a$ which transform as:

$$L^{\mu}(x) \rightarrow L'^{\mu}(x) = \left(\bar{S}(x)\right)^{-1} L^{\mu}(x)\, S^{-1}(x) \quad . \tag{3}$$

Here

$$\bar{S} = \gamma_0\, S^{+}\, \gamma_0 \quad . \tag{4}$$

For the SL(2,C) gauge transformation (1), $\bar{S}(x) = S^{-1}(x)$, so that the mass term $\bar{\psi}\psi$ is an invariant but the kinetic-energy-like term $\bar{\psi}\, L^{\mu}\, \partial_{\mu}\, \psi$ is not. To correct this, introduce in the usual manner a "connection" B_{μ},

$$B_{\mu} = B_{\mu}^{ab}\, \frac{\sigma_{ab}}{2} \, , \tag{5}$$

consisting of 24 fields B_μ^{ab}, which transform as

$$B_\mu \to B_\mu' = S(x)\, B_\mu\, S^{-1}(x) - i\, S(x)\, \partial_\mu S^{-1}(x) \quad . \tag{6}$$

One easily verifies that the combination $(\partial_\mu + i\, B_\mu)\psi$ transforms "correctly" as

$$(\partial_\mu + i\, B_\mu)\psi \to S(x)\, (\partial_\mu + i\, B_\mu)\psi \quad , \tag{7}$$

so that finally

$$\bar{\psi}\, L^\mu(\partial_\mu + i\, B_\mu)\psi + m\bar{\psi}\psi \tag{8}$$

is an SL(2,C) scalar.

The sixteen fields $L^\mu(x)$ are the so-called vierbein fields which are connected with Einstein's metric field $g^{\mu\nu}(x)$ through the γ-matrix trace relation $g^{\mu\nu} = \frac{1}{4}\,\mathrm{Tr}\, L^\mu L^\nu$. As usual $g^{\mu\nu}$ can be used to raise and lower Greek indices so far as GL(4,R) co-ordinate transformations are concerned. However, from our present point of view the SL(2,C) scalar fields $g^{\mu\nu}$ are a complete irrelevance and the vierbein fields $L^\mu(x)$ are the essential physical entities. Our real problem, thus, is to provide equations of motion for the vierbein $L^\mu(x)$ and the connection $B_\mu(x)$. To do this, define the covariant "curl" of B_μ in the conventional manner

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + i\, [B_\mu, B_\nu] \quad . \tag{9}$$

From (6), $B_{\mu\nu}$ transforms as

$$B_{\mu\nu} \to B_{\mu\nu}' = S\, B_{\mu\nu}\, S^{-1} \quad . \tag{10}$$

The simplest SL(2,C) gauge-invariant Lagrangian for L^μ and B_μ is that given by the (Dirac trace) expression:

$$i\, \mathrm{Tr}\, [L^\mu, L^\nu]\, B_{\mu\nu} \quad . \tag{11}$$

Our claim is that this succinct expression when divided by the factor [3] $\sqrt{-\det \frac{1}{4}\, \mathrm{Tr}\, L^\mu L^\nu}$ is equivalent to the Palatini formulation of the Einstein Lagrangian. This can be shown as follows:

a) Vary B_μ and L^μ separately in $\mathcal{L}$, given by

$$\sqrt{-\det \tfrac{1}{4}\,\mathrm{Tr}\, L^\mu L^\nu}\;\; \mathcal{L} = i\, \mathrm{Tr}\;\; [L^\mu L^\nu]\, B_{\mu\nu} \,, \tag{12}$$

and obtain two sets of first-order equations of motion. One of these two sets can be solved algebraically and gives the usual expression of the connection B_μ in terms of L^μ and $\partial_\lambda L^\mu$.

b) When B_μ is eliminated from $\mathcal{L}$, we obtain an expression which reduces to the standard Einstein Lagrangian with $g^{\mu\nu} = \frac{1}{4}\,\mathrm{Tr}\;\;(L^\mu L^\nu)$.

It is worth remarking that apart from a four-divergence,(11) can be written in the Møller form [4)]

$$\mathrm{Tr}\;\left(L^\mu_{|\mu}\, L^\nu_{|\nu} - L^\nu_{|\mu}\, L^\mu_{|\nu}\right) \,, \tag{13}$$

where, in accordance with the discussion on covariant derivatives, we define

$$L^\mu_{|\nu} = \partial_\nu\, L^\mu + i\, [B_\nu, L^\mu] \quad . \tag{14}$$

For completeness, one may also remark that the relation between $\Gamma^\beta_{\mu\alpha}$ and B^{ab}_μ (the co-ordinate-based and vierbein-based affine connections, respectively) could be expressed in this γ-matrix language as:

$$B_\mu = \frac{i}{8}\left\{[L_\beta, \partial_\mu L^\beta] + \Gamma^\beta_{\mu\alpha}\,[L_\beta, L^\alpha]\right\} \,, \tag{15}$$

where $L_\mu = g_{\mu\nu}\, L^\nu$ and, inversely,

$$\Gamma^\beta_{\mu\alpha} = -\frac{1}{4}\,\mathrm{Tr}\;(L_\alpha\, L^\beta_{|\mu}) \quad . \tag{16}$$

This equation has the status of a definition for Γ. It is not the only possible one. It has, however, the merit that, using (16), the usual covariant derivative $(\partial + \Gamma)$ acting on $g^{\mu\nu}$ gives zero.

3. To summarize, the SL(2,C) gauge invariant (as well as GL(4,R) invariant) $\mathcal{L}_{\text{Einstein-Weyl}}$ is given by

$$\sqrt{\det - \frac{1}{4}\,\mathrm{Tr}\,(L^{\mu}L^{\nu})}\;\mathcal{L}_{\text{E.W.}} = i\,\mathrm{Tr}\,[L^{\mu}L^{\nu}]B_{\mu\nu} + i\bar{\psi}\,L^{\mu}(\partial_{\mu}+iB_{\mu})\psi + m\bar{\psi}\psi \quad , \tag{17}$$

where $B_{\mu\nu}$ is given by (9).
There is nothing in this expression, however, which cannot be generalized to SL(6,C). Thus, parametrize [5] L^{μ} and B_{μ} for SL(6,C) in the form:

$$L^{\mu} = L^{\mu ai}\left(\gamma_{a}\,\frac{\lambda^{i}}{2}\right) + L^{\mu ai5}\left(i\gamma_{a}\gamma_{5}\,\frac{\lambda^{i}}{2}\right) \tag{18}$$

$$B_{\mu} = B_{\mu}^{abi}\left(\frac{\sigma_{ab}\,\lambda^{i}}{4}\right) + B_{\mu}^{i}\left(\frac{\lambda^{i}}{2}\right) + B_{\mu}^{5i}\left(\frac{\lambda^{i}}{2}\,\gamma_{5}\right) , \tag{19}$$

where λ^{i}, $i = 0,\ldots,8$, are the Gell-Mann U(3) matrices. The field transformations, covariant derivatives, invariant Lagrangians, etc. all remain the same as above. The Lagrangian (17) is SL(6,C) gauge invariant and as such yields a set of SL(6,C) conserved currents which are given by $\partial\mathcal{L}_{int}/\partial B_{\mu}$. Clearly the same ideas can be applied to any extension of SL(6,C), for example, U(6,6) or SL(6,C) × SL(6,C). As an unnecessary luxury it transforms correctly for GL(4,R). The structure and the physical significance of the extra fields introduced in (18) and (19) will be discussed elsewhere.

ACKNOWLEDGMENT

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REFERENCES AND FOOTNOTES

1) After the pioneering work of H.Weyl (Z. Physik 56, 330 (1929)) and of V.Fock and D.Ivanenko (Compt. Rend. 189, 25 (1929)) the first modern exposition of the SL(2,C) gauge invariance point of view appears to be that of R. Utiyama (Phys. Rev. 101, 1547 (1956)). See also T.W.B. Kibble, J. Math. Phys. 2, 212 (1961) and J. Schwinger, Phys. Rev. 130, 1253 (1963).

2) See also M. Carmeli, Nucl. Phys. B38, 621 (1972), who gives a formulation similar in spirit but different in structure.

3) So far as SL(2,C) gauge invariance is concerned, the scalar factor $\sqrt{\det}$ is irrelevant. It is needed, however, if we additionally want the action $\int \mathcal{L}\, d^4x$ to transform as a scalar under GL(4,R).

4) C. Møller, Matematisk-fysiske Meddelelser 34, Nr.3 (1964).

5) We are indebted to Prof. F. Gürsey who pointed out to us after this note was written his considerations on vierbein generalizations to SL(6,C) (Contemporary Physics, Vol.II (IAEA, Vienna 1969), p.193). We differ from Gürsey in that we emphasise the gauge aspects of the theory.