IC/72/86

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

PCAC ANOMALIES AND GRAVITATION

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INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1972 MIRAMARE-TRIESTE

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PCAC ANOMALIES AND GRAVITATION *

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MIRAMARE - TRIESTE
August 1972

* To be submitted for publication.

ABSTRACT

The Feynman rules of electron-graviton interaction are used to calculate the graviton self-energy and the $\pi^0 \to 2g$ matrix element, the direct analogues of the photon self-energy and $\pi^0 \to 2\gamma$. The lowest-order perturbation result leads to the modified PCAC relation,

$$\partial_\alpha j_{\alpha 5} = 2mj_5 + e^2 \epsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta} F_{\gamma\delta}/16\pi^2 + \epsilon_{\kappa\lambda\mu\nu} R_{\kappa\lambda\rho\sigma} R_{\mu\nu\rho\sigma}/768\pi^2$$

where the anomalous terms on the right are the lowest-order non-generally covariant contributions of infinite-mass regulator fermion loops. The important question of whether the gravitational anomaly should be generally covariant and (if higher-order calculations show that this is not the case) whether compensating leptons are an absolute must, is, unfortunately, left open to the order we have worked.

I. INTRODUCTION

Anomalous Ward-Takahashi identities have had an important impact [1] on at least two branches of elementary particle physics. In the first place, they have proved relevant to the determination of pseudoscalar meson lifetimes and, in the second place, they have placed strong restrictions on the acceptability of renormalizable models [2] based on spontaneous symmetry breaking [3] which unify weak and electromagnetic interactions [4]. This paper generalizes the triangle loop anomaly to gravitation. We find that in lowest-order perturbation theory the PCAC relation is modified to

$$\partial_\alpha j_{\alpha 5} = 2mj_5 + e^2\epsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta}/16\pi^2 + \epsilon_{\kappa\lambda\mu\nu}R_{\kappa\lambda\rho\sigma}R_{\mu\nu\rho\sigma}/768\pi^2 \qquad \text{(A)}$$

where the corrections on the right are associated with infinite-mass regulator loops.

Since the Riemann tensor $R_{\alpha\beta\gamma\delta}$ plays the analogous role in gravity theory to the electromagnetic tensor $F_{\alpha\beta}$, it is interesting that the form of the gravitational anomaly for $\pi^0 \to 2g$ parallels the form for the electromagnetic anomaly $\pi^0 \to 2\gamma$. An important problem, however, remains. Are there also anomalous terms in the $\pi^0 \to 3g$, $\pi^0 \to 4g$, ... processes of a form such that they all sum together to make the expression ϵRR manifestly generally covariant? If this does not happen, would there be a disaster for gravity theory - e.g. emergence of a non-zero graviton self-mass or eventually a breakdown of the equivalence principle (i.e. a clash between the predictions of the theory and the Eötvös experiment)? If this happened, the existence of compensating leptons to remove the anomaly would become a necessity, not just for reasons of renormalizability, but for deeper physical reasons connected with the equivalence principle.

Unfortunately the work presented in this paper does not include $\pi^0 \to 3g$, $\pi^0 \to 4g$, or higher processes, and we cannot make any definitive statement about the question raised above, though a few discussion remarks are presented in the last section. We do, however, consider another problem which concerns the general attitude about the interpretation of the anomalies.

While one cannot dispute the truth of this relation (A) - it is an algebraic identity once the $j_{\alpha 5}$ and j_5 elements are given [5] — one can still question the interpretations that have been given to it. To take a famous example, the photon self-energy $\Pi_{\alpha\beta}$ when calculated in lowest-order perturbation theory shows up a quadratic self-mass $\Lambda^2\eta_{\alpha\beta}$ which is just as anomalous and quite intolerable in view of the fact that a miraculous cancellation between bare and self mass is called for with little sign of gauge invariance at the level of the bare Lagrangian. Of course the satisfactory way out of the dilemma is well known: one must regularize the self-energy gauge invariantly [6] from Π to the correct value $\overline{\Pi}$ which then becomes divergenceless. But one must apply this procedure everywhere else for consistency. We wish to argue that the so-called anomalies under discussion possess the same status in basic theory as the self-mass of the photon, no more and no less. They represent short-comings of the conventional calculational method and do not represent a break-down of basic things like gauge invariance or W-T identities.

The plan of the paper is as follows. We illustrate the gauge-invariant method of regularization for vacuum charge and mass polarization in Sec.III after evolving the Feynman rules of calculation for the electron in Sec.II; we also show, so far as two-point functions are concerned, that such regularization is essential to establish the Borchers' equivalence between Lagrangians involving electron fields connected by a chiral transformation. In Sec.IV we carry the argument over the famous VVA triangle graph where the anomaly is connected with the regulation of the element [1] $\langle 2mj_5\rangle$ and one is able to assert that the "correct" elements do satisfy PCAC in the form $\langle \partial_\alpha \overline{j}_{\alpha 5}\rangle = \langle \overline{2mj_5}\rangle$. This is similar to a point of view argued also by Hagen [7]. Our contribution is to show that precisely the same viewpoint is necessary in order that the axiomatic Borchers' equivalence theorems [8] for chirally related Lagrangians continue to hold for the theory in question. Lastly, we deal with the gravity analogue in Sec.V and once again find that the $R\tilde{R}$ anomaly can be incorporated with j_5 into the correct pseudoscalar element $\overline{2mj_5}$ in accordance with Borchers' equivalence theorem.

There are two physical consequences when one adopts this point of view. First, one is no longer able to calculate the π^0 lifetime from smooth PCAC [9] because one is asserting that the perturbation result for the fermion loop is wrong and requires a regularization. Second, the problem of non-renormalizability remains, regardless of one's interpretation. For spontaneously broken gauge models of weak interactions one is led to consider compensation mechanisms [10] to guarantee renormalizability. For gravity, with its intrinsic non-polynomiality, compensation may not be necessary as finiteness can be achieved in other ways [11]. However, here the demand for general covariance may lead to the same conclusion, that new leptons are indeed needed, as discussed before.

II. FEYNMAN RULES IN A SPINOR MODEL

Anomalous Ward-Takahashi identities [12] are associated with fermion loops [1]. As a necessary preliminary to their study we shall set up the Lagrangian framework and ensuing Feynman rules for a charged massive fermion field which therefore undergoes electromagnetic and gravitational interactions. Later we shall describe the modifications needed for dealing with neutrinos and further pseudoscalar-pseudovector interactions.

Begin with a world of photons, gravitons and electrons, where the Lagrangian is known to be [11]

$$\begin{aligned} \mathcal{L}(A,g,\psi)|-g|^{\frac{1}{2}} = & -\tfrac{1}{4}\, g^{\kappa\lambda} g^{\mu\nu} F_{\kappa\mu}F_{\lambda\nu} + \tfrac{1}{8} f^{-2} g^{\mu\nu}(\Gamma^{\lambda}_{\mu\kappa}\Gamma^{\kappa}_{\nu\lambda} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\kappa}_{\kappa\lambda}) \\ & + |-g|^{-w} L^{n\mu}\, \bar{\psi}\gamma_n (\tfrac{1}{2} i \overleftrightarrow{\partial}_{\mu} + e A_{\mu})\psi \\ & - |-g|^{-w}\, \bar{\psi}(m + \tfrac{1}{4} i \epsilon^{k\ell mn} B_{k\ell m}\gamma_n\gamma_5)\psi \end{aligned} \tag{1}$$

w signifying the weight of the fermion field. As usual,

$$\begin{aligned} F_{\mu\nu} &= \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} \\ \Gamma^{\kappa}_{\mu\nu} &= g^{\kappa\lambda}\{\mu\nu,\lambda\} = \tfrac{1}{2} g^{\kappa\lambda}(\partial_{\nu} g_{\mu\lambda} + \partial_{\mu} g_{\nu\lambda} - \partial_{\lambda} g_{\mu\nu}) \\ B_{\ell mn} &= L^{\lambda}_{\ell} B_{\lambda mn} = L^{\lambda}_{\ell} L^{\mu}_{m}(L_{n\nu}\Gamma^{\nu}_{\lambda\mu} - \partial_{\lambda} L_{n\mu}) \end{aligned} \tag{2}$$

and B and Γ are the spinorial and vectorial connections needed for covariant differentiation. The symmetric tetrad field L^{μ}_{m} is defined by $L^{\mu}_{m} L^{\nu}_{n} \eta^{mn} = g^{\mu\nu}$, $L_{m\mu} = g_{\mu\nu} L^{\nu}_{m}$, $L^{n\mu} = \eta^{mn} L^{\mu}_{m}$, etc., η being the Minkowski metric (latin indices) and g the space-time metric (greek indices).

Because we shall be setting up a perturbation expansion in powers of the fine-structure constant $\alpha = e^2/4\pi$ and the Newtonian gravitational constant $G = f^2/8\pi$, it will be necessary to expand the metric about the local Minkowski value. The question of how to choose a suitable interpolating graviton field then arises. Now it is known that the awkward factors involving determinant of g can be absorbed for the pure gravitational Lagrangian when the Lagrangian is expressed in terms of the density[13)]

$$\tilde{g}^{\mu\nu} = g^{\mu\nu}|-g|^{-\frac{1}{2}} = \eta^{mn} L^{\mu}_{m} L^{\nu}_{n} |-L|^{-1} \quad . \tag{3}$$

Clearly then, if we attach weight $w = -\frac{1}{4}$ to the fermion field, the same can be done for the kinetic part of the graviton-spinor Lagrangian which reads:

$$\bar{\psi} \left[\tilde{L}^{n\mu} \gamma_n (\tfrac{1}{2} i \overleftrightarrow{\partial}_{\mu} + eA_{\mu}) - \tfrac{1}{4} i \epsilon^{k\ell mn} B_{k\ell m} \gamma_n \gamma_5 - m |-\tilde{L}|^{\frac{1}{2}} \right] \psi \tag{4}$$

where $\tilde{g}^{\mu\nu} = \eta^{mn} \tilde{L}^{\mu}_{m} \tilde{L}^{\nu}_{n}$, $\tilde{L}^{\mu}_{m} = L^{\mu}_{m}|-L|^{-1/2}$. In this form the non-polynomiality is carried by the photon—graviton interaction Lagrangian $\frac{1}{4} \tilde{g}^{\kappa\lambda} \tilde{g}^{\mu\nu} F_{\kappa\mu} F_{\lambda\nu} |-\tilde{g}|^{-1/2}$ and the mass and spin terms. Define the physical graviton field h_n through the relation

$$\tilde{L}^{\mu}_{n} = \delta^{\mu}_{n} + f h^{\mu}_{n} \quad . \tag{5}$$

The substitution of (5) into (1) will yield an expansion of $\mathcal{L}$ to any desired power of fh —— in fact for our discussion of the simplest anomalies the order f^2h^2 will prove sufficient. In this linearized version, all summations, etc., have meaning relative to the Minkowski metric of special relativity, so we shall revert to the commonly used notation of lower-case greek indices throughout the remainder of this paper. Inserting

$$\tilde{g}^{\mu\nu} \rightarrow \eta_{\mu\nu} + 2fh_{\mu\nu} + f^2h^2{}_{\mu\nu}$$

$$\tilde{L}_{n\mu} \rightarrow \eta_{\mu\nu} - fh_{\mu\nu} + f^2h^2{}_{\mu\nu} - \dots$$

$$\epsilon^{k\ell mn} B_{k\ell m} \gamma_n = \epsilon^{k\ell mn} L_k^{\rho} L_{m\sigma} \partial_\rho L_\ell^{\sigma} \gamma_n = \epsilon^{klmn} \tilde{L}_k^{\rho} \tilde{L}_{m\sigma} \partial_\rho \left[\tilde{L}_\ell^{\sigma} |-\tilde{L}|^{-\frac{1}{2}}\right] \gamma_n$$

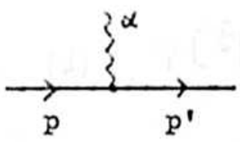
$$\rightarrow \tfrac{1}{2} f^2 \epsilon_{\kappa\lambda\mu\nu}\, h_{\lambda\rho} \overleftrightarrow{\partial}_\kappa h_{\rho\mu} \gamma_\nu + \dots$$

$$|-\tilde{L}|^{\frac{1}{2}} \rightarrow 1 + \tfrac{1}{2}fh_{\mu\mu} - \tfrac{1}{4}f^2h_{\mu\nu}h_{\mu\nu} + \tfrac{1}{8} f^2 h_{\mu\mu} h_{\nu\nu} - \dots$$

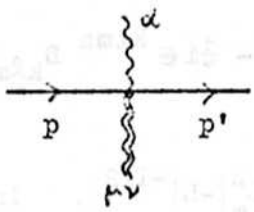
into $\mathcal{L}$ we obtain to order f^2,

$$\begin{aligned}\mathcal{L}(A,g,\psi) = {} & \mathcal{L}(A,h) + \bar{\psi}\left(\tfrac{1}{2}i\overleftrightarrow{\partial\!\!\!/} + e A\!\!\!/ - m\right)\psi + \\ & + fh_{\mu\nu}\bar{\psi}\left(\tfrac{1}{2}i\gamma_\mu \overleftrightarrow{\partial}_\nu + e\gamma_\mu A_\nu\right)\psi - \tfrac{1}{8}f^2\epsilon_{\kappa\lambda\mu\nu}\left(h_{\lambda\rho}\overleftrightarrow{\partial}_\kappa h_{\rho\mu}\right)\left(\bar{\psi}i\gamma_\nu\gamma_5\psi\right) \\ & - m\left(\tfrac{1}{2}fh_{\mu\mu} - \tfrac{1}{4}f^2h_{\mu\nu}h_{\mu\nu} + \tfrac{1}{8}f^2h_{\mu\mu}h_{\nu\nu}\right)\bar{\psi}\psi + \dots \end{aligned} \qquad (6)$$

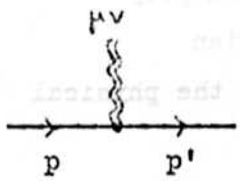
from which we can state the Feynman rules (Table I) for vertices giving emission of a photon ($\sim\!\sim\!\sim$) and up to gravitons ($\approx\approx$) from the electron.



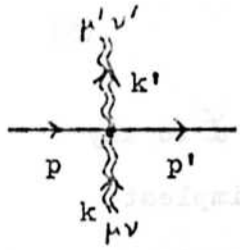
$e\gamma_\alpha$



$\tfrac{1}{2}ef\left[\gamma_\mu\eta_{\alpha\nu} + \gamma_\nu\eta_{\alpha\mu}\right]$



$\tfrac{1}{4}f\left[(p + p')_\mu \gamma_\nu + (p + p')_\nu \gamma_\mu - 2m\,\eta_{\mu\nu}\right]$



$$-\tfrac{1}{16} f^2\left[\eta_{\mu\mu'}\epsilon_{\nu\nu'\kappa\lambda} + \eta_{\nu\nu'}\epsilon_{\mu\mu'\kappa\lambda} + \eta_{\mu\nu'}\epsilon_{\nu\mu'\kappa\lambda} + \eta_{\nu\mu'}\epsilon_{\mu\nu'\kappa\lambda}\right](k+k')_\kappa\gamma_\lambda\gamma_5$$
$$+ \tfrac{1}{4}mf^2\left[\eta_{\mu\mu'}\eta_{\nu\nu'} + \eta_{\mu\nu'}\eta_{\nu\mu'} - \eta_{\mu\nu}\eta_{\mu'\nu'}\right]$$

TABLE I. Feynman rules for photon and graviton emission.

These rules should be supplemented by the photon-graviton and graviton-graviton vertices, as well as the fictitious particle vertices due to the choice of gauge and the canonical $\delta^4(0)$ vertices arising from the coupling of the graviton field to the kinetic matter Lagrangian. Fortunately they are not needed for the subsequent analysis of anomalies though they are relevant to the problem of the graviton self-energy.

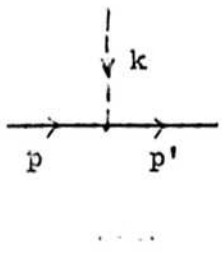
Next, suppose that we are concerned with neutrinos. Their left-handed character and the parity violation they cause is automatically incorporated in the Lagrangian by simply replacing γ_λ by $\frac{1}{2}\gamma_\lambda(1-i\gamma_5)$ everywhere in $\mathcal{L}$ since the spinorial affinity B commutes with the two-component projector $\frac{1}{2}(1-i\gamma_5)$. An equivalent formulation of the effect is to say that the stress tensor for neutrinos consists of equal and opposite normal ($\Theta_{\mu\nu} \sim 2^{++}$) and abnormal ($\Theta_{\mu\nu 5} \sim 2^{--}$) pieces in analogy to V-A theory..

We shall be scrutinizing the axial W-T identity for photon and graviton emission and considering processes like $\pi^0 \to 2g$. Introduce pseudoscalar mesons interacting axially (i.e. derivatively) with the fermions through the terms:

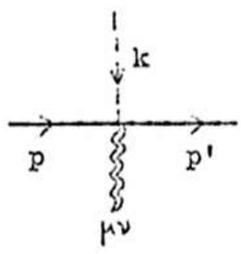
$$|-g|^{\frac{1}{2}}\mathcal{L}(\varphi,\psi,g) = \frac{1}{2}[g^{\mu\nu}(\partial_\mu\varphi\partial_\nu\varphi)-\mu^2\varphi^2] + i|-L|^{\frac{1}{2}}g\,L^{m\nu}\bar{\psi}\gamma_m\gamma_5\psi\partial_\nu\varphi, \text{ or}$$

$$\mathcal{L}(\varphi,\psi,g) = \mathcal{L}(\varphi,h) + ig\bar{\psi}\not{\partial}\varphi\gamma_5\psi + igf\,h_{\mu\nu}\bar{\psi}\gamma_\mu\gamma_5\psi\partial_\nu\varphi \qquad (7)$$

which provide the additional Feynman rules listed in Table II.



$g\not{k}\gamma_5$,



$\frac{1}{2}gf(\gamma_\mu k_\nu + \gamma_\nu k_\mu)\gamma_5$

TABLE II. Feynman rules for graviton emission from an axial vertex.

Here again there exist supplementary meson-graviton vertices which are irrelevant to the problem of anomalies.[14] If we perform a chiral gauge transformation on the fermion field, $\psi' = e^{\gamma_5 g\varphi}\psi$, the divergence of the axial current in (7) is converted into a non-polynomial pseudoscalar current:

$$\mathcal{L}_{int}(\varphi,\psi',g) = -m\,\bar{\psi}'\left[e^{-2g\gamma_5\varphi} - 1\right]\psi'|-\tilde{L}|^{\frac{1}{2}} \quad . \qquad (8)$$

As we shall see, the Borchers' equivalence of Lagrangian (8) with (7) will provide us with what we believe is a more reasonable interpretation of the anomaly.

At the final level of sophistication one can conceive of internal symmetries[15] and multiplets of fermions interacting with vector (V), axial vector (A) and tensor (T) meson multiplets. Although the generalization of the work is straightforward, we shall not go so far as to consider the possibility here because many of the relevant anomalies occur already at the U(1) gauge group level.

III. GAUGE-INVARIANT REGULARIZATION AND SELF-ENERGY PARTS

Let us try to draw some lessons from the problems of photon and graviton self-mass. In lowest-order perturbation theory the vacuum charge polarization is given by ($\langle\rangle$ stands for tracing)

$$\Pi_{\alpha\beta}(k,m^2) = ie^2\int dp \left\langle \gamma_\alpha (\not p - m)^{-1}\gamma_\beta(\not p + \not k - m)^{-1}\right\rangle \quad . \qquad (9)$$

The expression

$$k_\alpha k_\beta \Pi_{\alpha\beta} = ie^2\int dp \left\langle \left[(\not p - m)^{-1} - (\not p + \not k - m)^{-1}\right]\not k\right\rangle$$

would vanish were it not for the quadratic infinity sabotaging a shift of integration variable. However, the convergent second mass derivative satisfies

$$(\partial/\partial m^2)^2 k_\alpha k_\beta \Pi_{\alpha\beta} = 8ie^2\int dp \left\langle k.p\left[p^2-m^2\right]^{-3} - k.(p+k)\left[(p+k)^2-m^2\right]^{-3}\right\rangle = 0.$$

Let us therefore redefine the "correct" covariant self-energy $\overline{\overline{\Pi}}_{\alpha\beta}(k)$ to be the second integral of the second derivative,

$$\overline{\overline{\Pi}}_{\alpha\beta}(k,m^2) = \int_{-\infty}^{m^2} d\mu^2 \int_{-\infty}^{\mu^2} dM^2 \left(\frac{\partial}{\partial M^2}\right)^2 \Pi_{\alpha\beta}(k, M^2)$$

$$= \frac{e^2}{2\pi^2}(k_\alpha k_\beta - k^2\eta_{\alpha\beta})\int_0^1 dx\ x(1-x)\int_{-\infty}^{m^2} dM^2\ \ln\left[k^2x(1-x)-M^2\right] . \qquad (10)$$

Effectively, we are performing a gauge-invariant regularization of the expression (9) wherein we can recognize the logarithmic wave-function renormalization by the divergence of the mass integral at $-\infty$. In a renormalizable theory having quadratic infinities, at worst the operation $\int dm^2\,(\partial/\partial m^2)$ needs to be carried out twice for each charged loop and clearly has no effect on absolutely convergent integrals. It is the same as adding two regulator fields [6] with zero first moment.

Consider next the equally fundamental problem of the graviton self-energy due to the fermion. To order G, there are four proper graphs to be taken into account (Figure 1): one bubble graph, one seagull graph, and two tadpole graphs, all superficially quartically infinite.

In place of the self-energy $\Pi_{\kappa\lambda\mu\nu}$ we shall find it preferable to study the related vacuuum mass polarization

$$\Theta_{\kappa\lambda\mu\nu} = \Pi_{\kappa\lambda\mu\nu} - \tfrac{1}{2}\eta_{\kappa\lambda}\Pi_{\rho\rho\mu\nu} - \tfrac{1}{2}\Pi_{\kappa\lambda\sigma\sigma}\eta_{\mu\nu} + \tfrac{1}{4}\eta_{\kappa\lambda}\Pi_{\rho\rho\sigma\sigma}\eta_{\mu\nu} \quad (11)$$

since it is more closely connected to the stress tensor expectation value $\langle\theta_{\kappa\lambda}\,\theta_{\mu\nu}\rangle$. We shall also fix the graviton gauge so that the sum of tadpole and seagull graphs is proportional to the tensor $(2\eta_{\kappa\lambda}\eta_{\mu\nu} - \eta_{\kappa\mu}\eta_{\lambda\nu} - \eta_{\kappa\nu}\eta_{\lambda\mu})$ apart from the spin contribution. Explicitly,

$$\begin{aligned}\Theta_{\kappa\lambda\mu\nu}(k,m^2) = {} & \frac{if^2}{16}\int dp \Big\langle \big[(2p+k)_\kappa\gamma_\lambda - \eta_{\kappa\lambda}(2\not p+\not k-2m)\big](\not p-m)^{-1}\cdot \\ & \cdot\big[(2p+k)_\mu\gamma_\nu - \eta_{\mu\nu}(2\not p+\not k-2m)\big](\not p+\not k-m)^{-1}\Big\rangle \\ & + \frac{if^2}{16}\int dp\Big\langle\big(8\eta_{\kappa\mu}\epsilon_{\lambda\nu\rho\sigma}k_\rho\gamma_\sigma\gamma_5 + m\eta_{\kappa\lambda}\eta_{\mu\nu} - m\eta_{\kappa\mu}\eta_{\lambda\nu}\big)(\not p-m)^{-1}\Big\rangle \\ & + (\kappa\leftrightarrow\lambda \text{ and } \mu\leftrightarrow\nu) \end{aligned} \quad (12)$$

showing that

$$k_\kappa k_\lambda k_\mu k_\nu\Theta_{\kappa\lambda\mu\nu} = if^2\int dp\Big\langle k^4 + \not k\big[(k.p)^2(\not p-m)^{-1} - (k.p+k^2)^2(\not p+\not k-m)^{-1}\big]\Big\rangle$$

is nonzero because of the presence of a real quartic infinity quite apart from the difficulty with the shift of integration variable. Nevertheless,

$$\begin{aligned}(\partial/\partial m^2)^3 k_\kappa k_\lambda k_\mu k_\nu\Theta_{\kappa\lambda\mu\nu} &= 24if^2\int dp\Big\langle(k.p)^3[p^2-m^2]^{-4} - (k.p+k^2)^3[(p+k)^2-m^2]^{-4}\Big\rangle \\ &= 0\end{aligned}$$

Therefore, in exactly the same way as before, we regularize by calling the "correct" self-energy $\bar{\Theta}_{\kappa\lambda\mu\nu}$ to be the third integral of the third mass derivative; this ensures that Θ has a vanishing four-divergence in agreement with intuitive ideas about graviton self-mass and the commutation rules of the stress tensor. The result of a long but straightforward calculation can be summarized in the form

$$\left(\frac{\partial}{\partial m^2}\right)^3 \Theta_{\kappa\lambda\mu\nu} = \frac{f^2}{16\pi^2}\int \frac{dx \quad (1-2x)^2}{[k^2x(1-x)-m^2]^2}\left[\begin{array}{l} k^2(2d_{\kappa\lambda}d_{\mu\nu} - d_{\kappa\mu}d_{\lambda\nu} - d_{\kappa\nu}d_{\lambda\mu}) \\ -\dfrac{4k^4x(1-x)}{k^2x(1-x)-m^2}(d_{\kappa\mu}d_{\lambda\nu} + d_{\kappa\nu}d_{\lambda\mu}) \end{array}\right]$$

with $d_{\mu\nu} = \eta_{\mu\nu} - k_\mu k_\nu/k^2$, (13)

and $\overline{\Theta}$ following by mass integration. Although gauge invariant, the answer carries a quadratic as well as a logarithmic infinity which has bearing on the renormalization programme for gravity.

We can now turn to the implications of the regularization procedure for other self-energy parts and for PCAC. The lowest-order contribution to the axial self-energy,

$$\Pi_{\alpha 5\beta 5}(k,m^2) = -ig^2\int dp \left\langle \gamma_\alpha\gamma_5(\not{p} - m)^{-1}\gamma_\beta\gamma_5(\not{p} + \not{k} - m)^{-1}\right\rangle \qquad (14)$$

differs from the vector self-energy by a term of order $m^2\eta_{\alpha\beta}$ times a logarithmic infinity. We would be tempted to say that when $m \to 0$ the result is (chiral) gauge invariant. However the infinity precludes such a statement. Indeed the regularization needed to make the entire self-energy calculation consistent gives a non-gauge-invariant answer — this conflict between finiteness, gauge invariance and chiral gauge invariance is usually highlighted with reference to the three-point V V A vertex, but in fact the repercussions already enter at the level of the two-point A A vertex compared to the V V vertex. Since

$$\left(\frac{\partial}{\partial m^2}\right)^2 \Pi_{\alpha 5\beta 5} = \frac{g^2}{2\pi^2}\int dx\left[k^2x(1-x)-m^2\right]^{-2}\left[(k_\alpha k_\beta + \eta_{\alpha\beta})x(1-x) - m^2\eta_{\alpha\beta}\right] \quad (15)$$

we conclude that the correct self-energy $\overline{\Pi}_{\alpha 5\beta 5}$ has to be renormalized in both its transverse and longitudinal components.

At this stage it is worth commenting on the validity of W-T identities. Intuitively, one would expect from the fermion bilinear commutators $[j_{05}, j_{05}] = 0$ and $[j_{05}, j_5] = j\delta$, that

$$\partial_\alpha \partial_\beta \langle T\,(j_{\alpha 5}\; j_{\beta 5})\rangle = 4m^2 \langle T(j_5\; j_5)\rangle + 4m \langle j \rangle \tag{16}$$

where j is scalar, j_5 is pseudoscalar and $j_{\alpha 5}$ is pseudovector. An equivalent way of restating the identity in a Lagrangian context is to ask that the propagators in theories (7) and (8) be the same, at least on the mass shell, viz., that to order g^2 the bubble graph of model (7) should equal the sum of bubble and seagull graphs of model (8). As to whether this assertion is right or wrong, explicit perturbation calculation gives

$$-k_\alpha k_\beta \Pi_{\alpha 5 \beta 5} = 4m^2 i \int dp \langle \gamma_5 (\not{p}-m)^{-1} \gamma_5 (\not{p}+\not{k}-m)^{-1} \rangle - 4mi \int dp \langle (\not{p}-m)^{-1} \rangle$$
$$- i \int dp \langle (\not{k}+2m) \left[(\not{p}+\not{k}-m)^{-1} - (\not{p}-m)^{-1} \right] \rangle$$

so the self-energies apparently do not match even at $p^2 = \mu^2$. The cause is clearly visible and has to do with shifting the integration variable; the same reason why the photon appears to have self-mass. The cure is to regularize consistently via the operation $\int dm^2 (\partial / \partial m^2)$, when we find that the correct propagators do agree,

$$- k_\alpha k_\beta \overline{\Pi}_{\alpha 5 \beta 5} = \overline{4m^2 \Pi}_{55} + \overline{4m \Pi}\,. \tag{17}$$

It may appear from all this discussion that we are being somewhat pedantic with our treatment of perturbation integrals, integrals which are intrinsically badly defined and which in any case give much latitude to other possible treatments. However, when we turn to the vertex functions in the following sections, we shall find that this care will help us preserve the W-T identities in spite of their apparent violation by naive lowest-order perturbation calculations.

IV. VERTEX ANOMALIES AND ELECTRODYNAMICS

The traditional triangle anomaly[1] is the statement that, in the lowest relevant order, a perturbation calculation of the potentially divergent (but actually convergent) integral $\Gamma_{\alpha_1\alpha_2\alpha_3 5}$ for the graph involving two vector currents and one axial current does not satisfy the expected W-T identity relating it to the integral $\Gamma_{\alpha_1\alpha_2 5}$ involving two vectors and one pseudoscalar. Thus in perturbation theory,

$$-ik_{3\alpha_3}\Gamma_{\alpha_1\alpha_2\alpha_3 5}(k_1k_2k_3,m^2) = 2m\,\Gamma_{\alpha_1\alpha_2 5}(k_1k_2k_3,m^2) + \Delta^{-1}_{\alpha_1\alpha_2 5}(k_1,m) - \Delta^{-1}_{\alpha_1\alpha_2 5}(-k_2,-m) \tag{18}$$

where

$$\Delta^{-1}_{\alpha\beta 5}(k,m) = 2\int dp \left\langle \gamma_\alpha(\not p - m)^{-1}\gamma_\beta\gamma_5(\not p + \not k - m)^{-1}\right\rangle . \tag{19}$$

Vertices 1 and 2 are V, vertex 3 is A or P and their associated incoming momenta are k_1, k_2 and k_3. Formally one expects the difference $\Delta_1^{-1} - \Delta_2^{-1}$ on the right of (18) to vanish because it is connected to the commutator of uncharged V and A currents. Had (19) not been divergent, the difference, and indeed both Δ^{-1} individually, would have vanished. The disaster is that, far from vanishing, an actual computation gives $\Delta_1^{-1} - \Delta_2^{-1}$ as finite and nonzero, viz.

$$-ik_{3\alpha_3}\Gamma_{\alpha_1\alpha_2\alpha_3 5}(k,m^2) = 2m\,\Gamma_{\alpha_1\alpha_2 5}(k,m^2) + ge^2\epsilon_{\alpha_1\alpha_2\rho\sigma}\,k_{1\rho}k_{2\sigma}/2\pi^2, \tag{20}$$

the second term on the right being the so-called anomaly. Specifically, on the photon mass shell, one arrives at the integrals[5],

$$2m\,\Gamma_{\alpha_1\alpha_2 5} = \epsilon_{\alpha_1\alpha_2\rho\sigma}\,k_{1\rho}k_{2\sigma}\cdot\frac{m^2ge^2}{\pi^2}\int_0^1 \frac{\theta(1-x-y)\;dxdy}{\left[k_3^2xy - m^2\right]} \tag{21}$$

$$\Gamma_{\alpha_1\alpha_2\alpha_3 5} = k_{3\alpha_3}\epsilon_{\alpha_1\alpha_2\rho\sigma}\,k_{1\rho}k_{2\sigma}\cdot\frac{ige^2}{\pi^2}\int_0^1 \frac{\theta(1-x-y)\;dxdy.xy}{\left[k_3^2xy - m^2\right]} \tag{22}$$

From the point of view of gauge models of weak interactions, another more relevant way of stating the anomaly is to say that the equivalence between the Lagrangian models (7) and (8) appears to be destroyed, in spite of the fact that the chiral transformation connecting them is localizable [17]. Thus, according to equivalence theorems [8], the mass-shell matrix element of the process $\phi \to 2\gamma$ should be independent of whether we use ψ or ψ' as the basic field variables since they possess the same one-particle elements. On the other hand, the perturbation calculation does not bear out the assertion of the theorem since $k_\alpha \Gamma_{\alpha 5} \neq 2m\Gamma_5$.

Following the procedures of Sec.II, one may, however, observe that while $\Delta_1^{-1} - \Delta_2^{-1} \neq 0$, it is nevertheless true that $(\partial/\partial m^2)(\Delta_1^{-1} - \Delta_2^{-1})$ vanishes so that $(\partial/\partial m^2)\, k_\alpha \Gamma_{\alpha 5} = (\partial/\partial m^2)(2mi\,\Gamma_5)$. Starting from this vantage point and defining the "<u>correctly regularized</u>" vertices through a mass integration, one recovers the W-T identity in the form

$$2mi\,\Gamma_5(m) - 2Mi\,\Gamma_5(M) = k_\alpha\left[\Gamma_{\alpha 5}(m) - \Gamma_{\alpha 5}(M)\right] .$$

The important remark is that while $\lim_{M\to\infty} \Gamma_{\alpha 5}(M) = 0$,

$$\lim_{M\to\infty} 2M\,\Gamma_5(M) = ge^2 \epsilon\, k_1 k_2 / 2\pi^2 , \qquad (23)$$

a fact that was recognized by Adler and Hagen [1),7)].

One can now give two interpretations to the anomaly. The Adler interpretation is that the conventional calculations [5)] of potentially divergent integrals like Γ_5 and $\Gamma_{\alpha 5}$ give the right matrix elements to this order as they stand and need no subtraction since they happen to be finite (after gauge invariance is imposed in the axial case). Such an interpretation violates the W-T identity and, much worse, the equivalence theorem for chirally related Lagrangians. Alternatively, we choose to believe, with Hagen [7)], that calculations of potentially divergent but superficially convergent integrals like Γ_5 can be quite misleading and may require a regularization to accord with other theorems of field theory, like, for example, the Borchers' equivalence theorem, which asserts equivalence of Lagrangians (7) and (8). Indeed, if we follow our treatment of the photon and gravition self-energies, we should define the <u>correct</u> value of the pseudoscalar vertex to be

$$\overline{2m\Gamma_{\alpha_1\alpha_2 5}} = \int_{-\infty}^{m^2} dM^2 \frac{\partial}{\partial M^2}(2M\Gamma_{\alpha_1\alpha_2 5}) = \epsilon_{\alpha_1\alpha_2\rho\sigma} k_{1\rho} k_{2\sigma} \frac{ge^2}{\pi^2}\int_0^1 \frac{\theta(1-x-y)\,dxdy\;k_3^2 xy}{[k_3^2 xy - m^2]} \tag{24}$$

and likewise the <u>correct</u> value of the axial vertex as

$$\overline{\Gamma}_{\alpha_1\alpha_2\alpha_3 5} = \int_{-\infty}^{m^2} dM^2 \frac{\partial}{\partial M^2}\Gamma_{\alpha_1\alpha_2\alpha_3 5} = \Gamma_{\alpha_1\alpha_2\alpha_3 5}. \tag{25}$$

In this way we achieve regularization of ambiguous graphs and carry out all appropriate subtractions in a manner which respects equivalence theorems. From this point of view then it is the traditional evaluation of the potentially divergent quantity which is at fault with its resulting modification of PCAC to

$$\partial_\alpha j_{\alpha 5} = 2mj_5 + e^2\epsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta}F_{\gamma\delta}/16\pi^2 . \tag{26}$$

The correct evaluation of Γ_5 being given by (24) respects the W-T identity

$$\partial_\alpha \overline{j}_{\alpha 5} = \overline{2mj_5} . \tag{27}$$

This means that we are abandoning any chance of calculating the $\pi^0 \to 2\gamma$ lifetime via smooth PCAC.

To support this viewpoint, one can compute the potentially divergent but actually finite vector-vector-scalar vertex $\Gamma_{\alpha_1\alpha_2}$ in lowest order, where from gauge invariance one expects $k_{1\alpha_1}\Gamma_{\alpha_1\alpha_2} = k_{2\alpha_2}\Gamma_{\alpha_1\alpha_2} = 0$.

A calculation completely analogous to Steinberger's give a <u>nongauge-invariant</u> result even on the photon mass shell:

$$\frac{\Gamma_{\alpha_1\alpha_2}}{m} = -\frac{e^2}{4\pi^2}\eta_{\alpha_1\alpha_2} + \frac{e^2}{2\pi^2}\int_c^1 \frac{\theta(1-x-y)\,(1-4xy)}{[k_3^2 xy - m^2]}\,dxdy \quad , \tag{28}$$

showing once again that the traditional perturbation evaluation of an ostensibly <u>finite</u> quantity can be quite misleading. A gauge-invariant answer using

$$\overline{m^{-1}\Gamma_{\alpha_1\alpha_2}} = \int_{-\infty}^{m^2} dM^2 \, \frac{\partial}{\partial M^2}\,(M^{-1}\Gamma_{\alpha_1\alpha_2})$$

automatically gets rid of the troublesome $\eta_{\alpha_1\alpha_2}$ term.

If one takes over the method to the A A A vertex, one finds that

$$-i\, k_{3\alpha_3}\overline{\Gamma}_{\alpha_1 5\alpha_2 5\alpha_3 5} = \overline{2m\Gamma_{\alpha_1 5\alpha_2 5}} \tag{29}$$

where perturbation theory gives

$$-i\, k_{3\alpha_3}\Gamma_{\alpha_1 5\alpha_2 5\alpha_3 5} = 2m\Gamma_{\alpha_1 5\alpha_2 5} + ge^2\epsilon_{\alpha_1\alpha_2\rho\sigma}k_{1\rho}k_{2\sigma}/6\pi^2, \tag{30}$$

the anomaly being shared equally among the three legs as expected by symmetry. (The relevant off-shell integrals are all exhibited in the appendix). Finally one can extend the discussion to other triangle vertices such as A A S, etc. In no instance is there any contradiction between the <u>correctly</u> regularized functions $\overline{\Gamma}$ which satisfy their appropriate W-T identities. From the present point of view, we may note that when neutrinos are involved, one should take an $m \neq 0$ situation first, regulate, and then take the limit $m \to 0$ at the end. In this way, a $\phi\to 2V$ transition can occur even through zero-mass fermion loops, when we have

$$-i\, k_{3\alpha_3}\overline{\Gamma}_{\alpha_1\alpha_2\alpha_3 5} = 2g\alpha\epsilon_{\alpha_1\alpha_2\rho\sigma}k_{1\rho}k_{2\sigma}/3\pi \quad ,$$

a result which is consistent with analyticity because the axial vertex carries a $k_{3\alpha}/k_3^2$ threshold singularity in this limit.

V. VERTEX ANOMALIES AND GRAVITATION

The analogue of $\pi^o \to 2\gamma$ in electrodynamics is $\pi^o \to 2g$ in gravitation. Let us see how PCAC applies here, with reference to the spinor loop. What makes the calculation of the two-graviton mode so similar to the two-photon mode is gauge invariance. On general principles, we expect that with physical gravitons (i.e. upon contraction over the graviton polarization tensors) the T T P vertex should have the kinematic structure,

$$\Gamma_{\mu\nu_1\mu_2\nu_2 5} = \epsilon_{\mu_1\mu_2\rho\sigma} k_{1\rho} k_{2\sigma} (\eta_{\nu_1\nu_2} k_1 \cdot k_2 - k_{2\nu_1} k_{1\nu_2}) G_P \cdot + \mu \leftrightarrow \nu \text{ perms} \tag{31}$$

These extra four powers of momentum are sufficient to make the invariant function G_P converge. Similar considerations apply to the T T A vertex.

In performing the perturbation calculations for gravity one must be very careful to include all graphs to the given order, and there are very many more diagrams than in the corresponding situation for electrodynamics. For $\pi^o \to 2g$ we need to consider the whole set of diagrams shown in Figure 2. However as the pseudoscalar-graviton transition must vanish and the graviton is on its mass shell, all graphs but the first three can be dropped. A straightforward, if tedious, calculation gives the non-gauge-invariant answer,

$$2m\,\Gamma_{\mu_1\nu_1\mu_2\nu_2 5} = \epsilon_{\mu_1\mu_2\rho\sigma} k_{1\rho} k_{2\sigma} \frac{m^2 f^2 g}{8\pi^2} \int_0^1 dxdy\ \theta(1-x-y) \left[\frac{4xy(k_1 \cdot k_2 \eta_{\nu_1\nu_2} - k_{1\nu_2} k_{2\nu_1})}{[k_3^2\, xy - m^2]} - \eta_{\nu_1\nu_2} \right] \cdot + (\mu \leftrightarrow \nu \text{ perms}) \tag{32}$$

If one proceeds as for the V V S vertex by regularizing the ratio $\Gamma_{\mu_1\nu_1\mu_2\nu_2 5}/m$, then one will simply drop the offending $\eta_{\nu_1\nu_2}$ terms in (32) to get the naive gauge-invariant perturbation result G_P for the T T P vertex. However to be perfectly consistent, one should instead regularize the product $2m\Gamma$ via the operation $\int dm^2(\partial/\partial m^2)$ to obtain

$$\overline{2mG_P} = \frac{f^2 g}{2\pi^2}\int_0^1 \frac{\theta(1-x-y)\, k_3^2 x^2 y^2\, dxdy}{[k_3^2\, xy - m^2]} = 2mG_P - \frac{f^2 g}{48\pi^2} \quad . \tag{33}$$

The difference $\overline{G}_P - G_P$ is the anomalous contribution arising from the infinite-mass regulator loop.

A similar calculation based on the diagrams of Figure 2 can be performed for the two-graviton mode of the axial current. Here the kinematic structure reads

$$\Gamma_{\mu_1\nu_1\mu_2\nu_2\alpha 5} = k_{3\alpha}\epsilon_{\mu_1\mu_2\rho\sigma}k_{1\rho}k_{2\sigma}(\eta_{\nu_1\nu_2}k_1\cdot k_2 - k_{2\nu_1}k_{1\nu_2})\, G_A + \mu\leftrightarrow\nu \text{ perms} \tag{34}$$

and can be deduced from the fact that a 1^+ object cannot couple to two gravitons, so that Γ_α must vanish when contracted over $\epsilon_\alpha(k_3)$, or else by using the commutation rules $[\theta,\theta] = \partial\theta\delta$, $[\theta, A] = \partial A\delta$ and $\langle T(\theta\, A)\rangle = 0$. We can straightforwardly extract the gauge-invariant result

$$\overline{G}_A = \frac{if^2 g}{2\pi^2}\int_0^1 \frac{\theta(1-x-y)\, x^2 y^2\, dxdy}{[k_3^2\, xy - m^2]} = G_A \tag{35}$$

and observe that the correctly regularized vertices do satisfy the W-T identity,

$$-ik_{3\alpha}\overline{\Gamma}_{\mu_1\nu_1\mu_2\nu_2\alpha 5} = \overline{2m\Gamma}_{\mu_1\nu_1\mu_2\nu_2 5} \quad \text{or} \quad -ik_3^2\,\overline{G}_A = \overline{G}_P \tag{36}$$

whereas the perturbation values :

$$-ik_3^2\, G_A = 2mG_P - gG/6\pi \tag{37}$$

do not match owing to the gravitational anomaly. Relation (37) can be restated as an apparent gravitational violation of PCAC,

$$\partial_\alpha j_{\alpha 5} = 2mj_5 + \epsilon_{\kappa\lambda\mu\nu} R_{\kappa\lambda\rho\sigma} R_{\mu\nu\rho\sigma}/768\pi^2 \tag{38}$$

to this order in G, if one recasts the structure of the T T P vertex in terms of an effective Lagrangian:

$$\begin{aligned} &\epsilon_{\mu_1\mu_2\rho\sigma}\left[\partial_{\nu_1}\partial_\rho g_{\mu_1\nu_1}\partial_{\nu_2}\partial_\sigma g_{\mu_2\nu_2} - \partial_\lambda\partial_\rho g_{\mu_1\nu_1}\partial_\lambda\partial_\sigma g_{\mu_2\nu_2}\right] \\ &= \epsilon_{\mu_1\rho\mu_2\sigma}\left[\partial_\lambda\{\mu_1\nu_1,\rho\}\partial_\lambda\{\mu_2\nu_1,\sigma\} - \partial_{\nu_2}\{\mu_1\nu_1,\rho\}\partial_{\nu_1}\{\mu_2\nu_2,\sigma\}\right] \\ &= \tfrac{1}{2}\epsilon_{\kappa\lambda\mu\nu}R_{\kappa\lambda\rho\sigma}R_{\mu\nu\rho\sigma} . \end{aligned} \tag{39}$$

The analogy between (26) and (38) is complete and demonstrates the similarity between the Maxwell tensor of electrodynamics and to this order the Riemann tensor of gravitation.

One may generalize to neutrino loop anomalies (which do not occur in q.e.d. but can arise in weak interaction models) by making the usual $\frac{1}{2}(1 - i\gamma_5)$ projection of the stress tensor, a normal piece $\theta_{\mu\nu}$ and an abnormal piece $\theta_{\mu\nu 5}$. The overall abnormal structure of the $\frac{1}{8}(j-j_5)$ $(\theta - \theta_5)$ $(\theta - \theta_5)$ vertex, which bears the mark of an anomaly, can be calculated by taking the $m \to 0$ limit of a mass m fermion loop. For $m \neq 0$ one finds

$$\begin{aligned} 2m\,\Gamma^{ab}_{\mu_1\nu_1\mu_2\nu_2} = \tfrac{1}{4}\,2m\,\Gamma_{\mu_1\nu_1\mu_2\nu_2 5} + 8m^2f^2 i\int dp\, \frac{\theta(1-x-y)(1-x-y)\,dxdy}{[p^2 + k_3^2xy - m^2]^3} \,\cdot \\ \cdot\left[\epsilon_{\mu_1\mu_2\rho\sigma}k_{1\rho}k_{2\sigma}(\tfrac{1}{4}p^2\eta_{\nu_1\nu_2} - xyk_{1\nu_2}k_{2\nu_1}) + \text{perms}\right] \end{aligned} \tag{40}$$

$$\begin{aligned} \Gamma^{ab}_{\mu_1\nu_1\mu_2\nu_2\alpha} = \tfrac{1}{4}\Gamma_{\mu_1\nu_1\mu_2\nu_2\alpha 5} + 4m^2f^2\int dp\, \frac{\theta(1-x-y)(1-x-y)\,dxdy}{[p^2 + k_3^2xy - m^2]^3} \,\cdot \\ \cdot\left[\epsilon_{\mu_1\mu_2\alpha\rho}(k_1-k_2)_\rho(\tfrac{1}{4}p^2\eta_{\nu_1\nu_2} - xyk_{1\nu_2}k_{2\nu_1}) + \text{perms}\right] \end{aligned} \tag{41}$$

which provide corrections m^2 multiplying a logarithmic infinity to the previous results (33) and (35). So far as the overall abnormal part of the vertex is concerned, the PCAC rule in its original and corrected forms, (37) and (36), respectively, is unchanged and in the zero-mass limit we get

$$ik_{3\alpha}\Gamma_{\mu_1\nu_1\mu_2\nu_2\alpha 5} = gG(k_{2\nu_1}k_{1\nu_2} - k_1\cdot k_2\eta_{\nu_1\nu_2})\epsilon_{\mu_1\mu_2\rho\sigma}k_{1\rho}k_{2\sigma}/6\pi + \mu\leftrightarrow\nu\,\text{perms} \quad . \tag{42}$$

Of course we do not anticipate any problems for the overall normal part of the vertex.

Finally, we can inquire into the possibility that (38) can be made generally covariant. We have only some preliminary remarks to offer in this direction as this is a difficult problem which is intimately tied in with the renormalization of gravity theory itself. First, one needs to work out the G^N corrections of the basic fermion loop at which level we shall suppose that gravity itself is unquantized, and we can confine ourselves to external h lines emanating from the loop in tree approximation. From the form of (4) and (6) we see that multigraviton emission vertices each carry at most one field derivative. In particular, it is only the one-graviton emission vertex associated with L which carries the momentum of the spinor line, while the multigraviton complements associated with the spin connection (B-type) and the mass (M-type) carry a single power of momentum of gravitons or a mass.

The information embodied in the single (L-type) graviton vertex is actually quite strong[18]. Thus from the W-T identity for the canonical stress tensor,

$$(\not{p}+\not{k}-m)^{-1}k_\mu\Gamma_{\mu\nu}(p+k,p)(\not{p}-m)^{-1} = \left[p_\nu + \tfrac{1}{4}i\sigma_{\mu\nu}k_\mu\right](\not{p}-m)^{-1} - (\not{p}+\not{k}-m)^{-1}\left[(p+k)_\nu + \tfrac{1}{4}i\sigma_{\mu\nu}k_\mu\right] \tag{43}$$

one derives that, to order k^2,

$$(\not p+\not k-m)^{-1}\Gamma_{\mu\nu}(p+k,p)(\not p-m)^{-1} = -\tfrac{1}{2}\left[\frac{d}{dp_\mu}p_\nu+\frac{d}{dp_\nu}p_\mu\right](\not p-m)^{-1} - \tfrac{1}{2}k_\lambda\left[\begin{array}{l}\tfrac{1}{2}\left(\frac{d}{dp_\mu}p_\lambda\frac{d}{dp_\nu}+\frac{d}{dp_\nu}p_\lambda\frac{d}{dp_\mu}\right)+\\ \eta_{\mu\nu}\frac{d}{dp_\lambda}+\tfrac{1}{4}i\left(\frac{d}{dp_\nu}\sigma_{\lambda\mu}+\frac{d}{dp_\mu}\sigma_{\lambda\nu}\right)\end{array}\right](\not p-m)^{-1} + O(k^2) \tag{44}$$

in analogy to the electrodynamics vertex

$$(\not p+\not k-m)^{-1}\Gamma_\mu(p+k,p)(\not p-m)^{-1} = -\frac{d}{dp_\mu}(\not p-m)^{-1} + O(k) \quad .$$

Therefore, for a properly renormalized closed fermion loop diagram which emits an L-graviton with momentum k, we obtain by partial integration that the matrix element $T_{\mu\nu}(k,\ldots)$ vanishes as k^2, and this applies to each soft graviton — it is of course entirely analogous to the closed loop photon result[19] that $T(k,\ldots)$ vanishes as k. Thus a closed loop emitting a number n of L-gravitons and a pseudo-scalar meson can be expressed in the form

$$2m\Gamma_5^n \sim \epsilon(fk^2)^n L_n(k) = \epsilon(fk^2)^n m^2\int dz\, P_n^{-1}(k^2,z,m^2)$$

where $P_n \sim (\sum k^2 zz' - m^2)^n$ is a polynomial of degree n in k^2, m^2 and the Feynman parameters z. Thus in spite of appearances, the integrals for L-graviton emission, when regularized, must converge (just as photon-photon scattering converges because of gauge-invariant regularization). The same applies to emission of L-gravitons and an axial meson.

To proceed with the argument, we need to study the remaining diagrams involving B-type gravitons and M-type gravitons (either from internal lines or the P vertex) which follow from the interaction

$$\bar\psi\left[i\epsilon_{\kappa\lambda\mu\nu}B_{\kappa\lambda\mu}\gamma_\nu\gamma_5 - m(|-\tilde L|^{\frac{1}{2}}-1) + 2mg\gamma_5|-\tilde L|^{\frac{1}{2}}\right]\psi \quad .$$

The B-graphs become more convergent as the number of B-vertices is increased and, from that point of view, present no particular problem for higher orders. One could say the same for the M-graphs except for one fact: the proliferation of mass factors in conjunction with propagators can lead to nonzero contributions, i.e. anomalies, in arbitrarily high orders. Thus a diagram which sprouts n M-line vertices ($n \geqslant 4$ necessarily), on dimensional grounds, is expressible as $2m\Gamma_5^n = f^n \epsilon k^4 m^{n+2} \int dz\, P^{-1}_{\frac{1}{2}n+1}(k^2,z,m^2)$, and while it is true that the Γ_5 integral converges, it is likely that a nonzero subtraction term survives in the limit $m \to \infty$. In fact it is perfectly conceivable that when the whole collection of graphs is added to give the perturbation value of $2m\Gamma_5^n$ and is compared to $-ik_{3\alpha}\Gamma^n_{\alpha 5}$, then although the latter obviously vanishes as $k_3 \to 0$, the former may not, unless one adds successions of anomalous terms of order f^n which correspond to infinite-mass gauge-invariant regulator contributions. However, one might expect that the correct vertices $\overline{2m\Gamma_5}$ and $\overline{\Gamma}_{\alpha 5}$ do obey a generally covariant W-T identity, in contrast to the perturbation values $2m\Gamma_5$ and $\Gamma_{\alpha 5}$ which need supplementary higher-order anomalous terms. Needless to say, this argument is only meant to be suggestive, and a much more rigorous study is needed before one can categorically say whether there exists a manifest generally covariant W-T identity.

VI. DISCUSSION

In this paper we have tried to give several illustrations to stress the importance of correctly regularizing Feynman integrals in order that the basic tenets of field theory be preserved: gauge invariance, equivalence theorems, PCAC,... Thus, it is not the validity of the relations (26) and (38) which is in doubt but their interpretation [1),7)]. But whether or not the reader is sympathetic to the point of view which we have taken, there is no question that the anomalous terms $F\tilde{F}$ or $R\tilde{R}$ behave badly as $k\to\infty$ and lead to a theory that is nonrenormalizable even if it seemed renormalizable to begin with; for instance, in a theory involving π^0 mesons and photons, one can easily check that when PCAC is used in one form or another, the photon self-energy due to a $\pi^0\gamma$ intermediate state is quadratically divergent with triangle vertices, and that the situation deteriorates in higher orders. In this light, a compensation mechanism is a prerequisite for an acceptable finite theory of weak interactions.

In the case of gravity, with its inbuilt non-polynomiality and the consequent cut-off G^{-1}, the necessity for compensating fields and interactions is not so clearcut, so far as infinity suppression is concerned. However, if one can show that the lack of manifest general covariance of the "anomaly" leads to physical effects - like non-zero graviton self-mass - then a compensation mechanism or a postulation of new leptons to cancel the "anomaly" would become a necessity, not for securing renormalization, but for deeper reasons, connected (however distantly) with the equivalence principle of general relativity and the results of the Eötvös experiment.

APPENDIX

OFF-SHELL VERTEX FUNCTIONS

We give here a quasisymmetric expression for the VVA vertex, and then state the result for the symmetric AAA vertex. Expand the vertex in the form

$$\begin{aligned}\Gamma_{\alpha_1\alpha_2\alpha_3 5} &= \left[F_1\, k_{1\mu} + F_2\, k_{2\mu} + F_3\, k_{3\mu}\right] \epsilon_{\mu\alpha_1\alpha_2\alpha_3} \\ &+\left[G_1\, k_{1\alpha_1} + H_1 (k_2 - k_3)_{\alpha_1}\right] k_{2\mu} k_{3\nu}\, \epsilon_{\mu\nu\alpha_2\alpha_3} \\ &+\left[G_2\, k_{2\alpha_2} + H_2 (k_3 - k_1)_{\alpha_2}\right] k_{3\mu} k_{1\nu}\, \epsilon_{\mu\nu\alpha_3\alpha_1} \\ &+\left[G_3\, k_{3\alpha_3} + H_3 (k_1 - k_2)_{\alpha_3}\right] k_{1\mu} k_{2\nu}\, \epsilon_{\mu\nu\alpha_1\alpha_2} \quad .\end{aligned}$$

Although nine form factors have been used above, only six kinematic terms are needed for the expansion. For example, the Rosenberg[5)] amplitudes are given by

$$\begin{aligned}A_1 &= F_1 - F_3 + k_2\cdot k_3 G_3 + k_2\cdot(k_1 - k_2) H_3 \;, \\ A_2 &= F_2 - F_3 - k_1\cdot k_3 G_3 - k_1\cdot(k_1 - k_2) H_3 \;, \\ A_3 &= -G_3 + 2H_2 + H_3 \;, \\ A_6 &= G_3 + 2H_1 + H_3 \;, \\ A_4 &= -G_2 - G_3 + H_2 - H_3 \;, \\ A_5 &= G_1 + G_3 + H_1 - H_3 \;.\end{aligned}$$

With our form factors, the divergence property reads,

$$k_{3\alpha_3} \Gamma_{\alpha_1\alpha_2\alpha_3 5} = \left[F_2 - F_1 + k_3^2 G_3 + (k_2^2 - k_1^2) H_3\right] k_{1\mu} k_{2\nu}\, \epsilon_{\mu\nu\alpha_1\alpha_2} \;, \text{etc.}$$

The basic integral for the triangle can be reduced to

$$\Gamma_{\alpha_1\alpha_2\alpha_3 5} = -i \iint dp \, \frac{dx\,dy\,dz \;\; \delta(1-x-y-z)}{\left[p^2 + K^2 - m^2\right]^3} \left\langle (\not p + \not q_2 + m)\gamma_{\alpha_1} (\not p + \not q_3 + m) \gamma_{\alpha_2} (\not p + \not q_1 + m) \gamma_{\alpha_3} \gamma_5 \right\rangle$$

$$+ \text{ surface term,}$$

where $K^2 = k_1^2 yz + k_2^2 zx + k_3^2 xy$, $q_1 = k_3 y - k_2 z$, $q_2 = k_1 z - k_3 x$, $q_3 = k_2 x - k_1 y$. The surface term arises as a consequence of shifts of the integration variable and could be dropped if the integral were no more than logarithmically divergent. For $m \neq 0$ we list the off-shell form factors:

$$F_1 = \frac{1}{6\pi^2}\int\frac{dxdydz\ \delta(1-x-y-z)}{[K^2 - m^2]}\cdot\begin{bmatrix}-m^2(y-z) + k_3^2 y(1-y) - k_2^2 z(1-z)\\ -2(K^2 - m^2) - 3m^2(1-y)\end{bmatrix}$$

$$F_2 = \text{"}\quad\int\quad\text{"}\quad\cdot\begin{bmatrix}-m^2(z-x) + k_1^2 z(1-z) - k_3^2 x(1-x)\\ +2(K^2 - m^2) + 3m^2(1-x)\end{bmatrix}$$

$$F_3 = \text{"}\quad\int\quad\text{"}\quad\left[-m^2(x-y) + k_2^2 x(1-x) - k_1^2 y(1-y)\right]$$

$$G_1 = \text{"}\int\quad\text{"}\cdot\left[4yz + x(1-x)\right]$$

$$H_1 = \text{"}\quad\int\quad\text{"}\quad\cdot\left[\ x(y-z)\ \right]$$

with G_2, G_3 and H_2, H_3 obtained by cyclic permutation from G_1 and H_1, respectively.

The AAA vertex is identical to the above except that the square bracket in F_1 is replaced by $\left[-4m^2(y-z) + k_3^2 y(1-y) - k_2^2 z(1-z)\right]$ with F_2 and F_3 obtainable by cyclic permutation. It is obvious that these form factors possess the requisite symmetries,

$$F_1(1,2,3) = -F_1(1,3,2) = -F_2(2,1,3)$$
$$G_1(1,2,3) = G_1(1,3,2) = G_2(2,1,3)$$
$$H_1(1,2,3) = -H_1(1,3,2) = -H_2(2,1,3)\text{ , etc.}$$

Similarly we may write the VVP and AAP vertices in the respective forms

$$2m\,\Gamma_{\alpha_1\alpha_2 5} = \frac{m^2}{\pi^2}\int\frac{dxdydz\ \delta(1-x-y-z)}{[K^2 - m^2]}\ k_{1\mu}k_{2\nu}\epsilon_{\mu\nu\alpha_1\alpha_2}$$

and

$$2m\,\Gamma_{\alpha_5\alpha_2 55} = \frac{m^2}{\pi^2}\int\frac{dxdydz\ \delta(1-x-y-z)\ (1-2z)}{[K^2 - m^2]}\ k_{1\mu}k_{2\nu}\epsilon_{\mu\nu\alpha_1\alpha_2}\ .$$

One can check that, as they stand, the integrals above do not satisfy the W-T identities, but that upon differentiation with respect to m^2 they do.

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FIGURE CAPTIONS

Figure 1. Graviton self-energy contributions to order G.

Figure 2. The TTP or TTA graphs to order G.

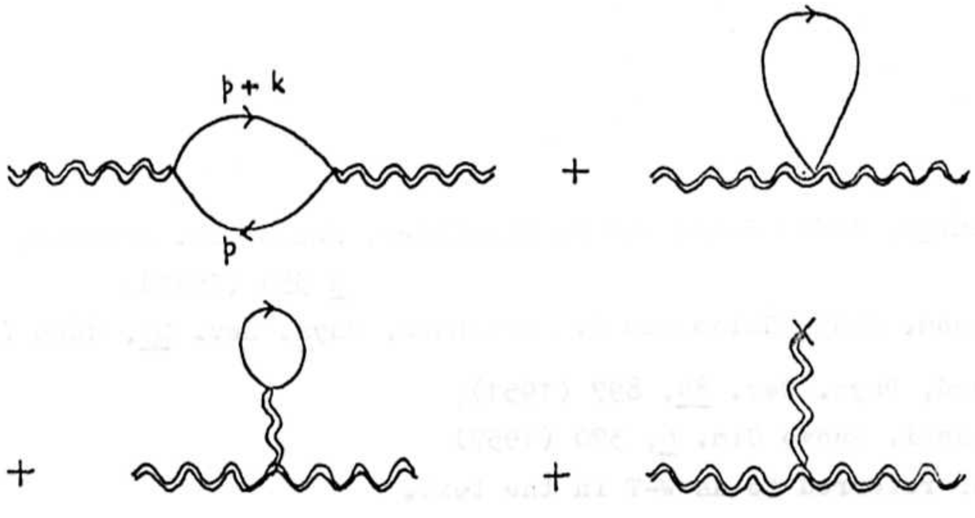

FIGURE 1

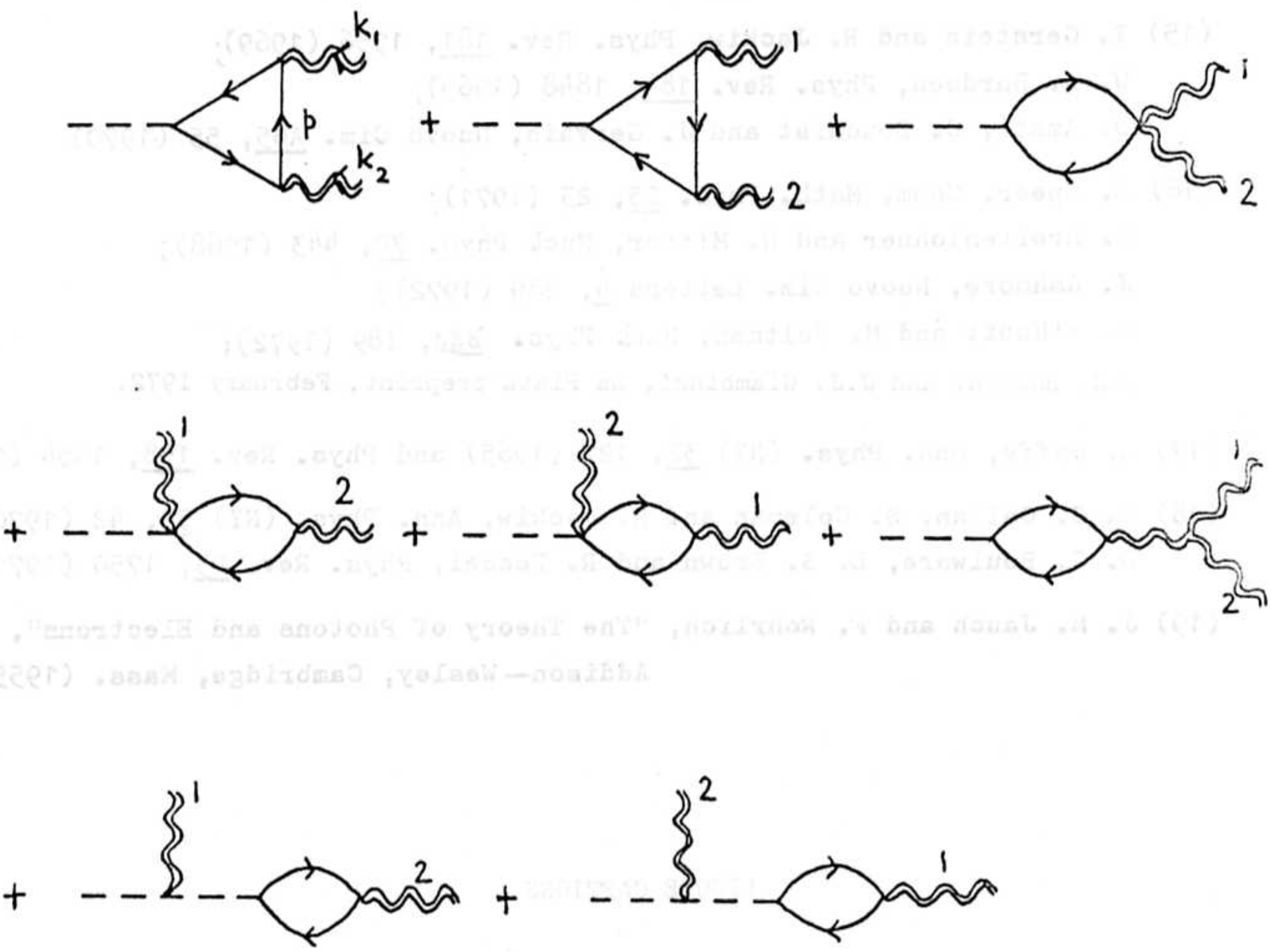

FIGURE 2