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IC/71/156

INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

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G. Lazarides
A. A. Patani
and
Q. Shafi

INTERNATIONAL ATOMIC ENERGY AGENCY

UNITED NATIONS EDUCATIONAL, SCIENTIFIC AND CULTURAL ORGANIZATION

1971 MIRAMARE-TRIESTE

IC/71/156

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and
United Nations Educational Scientific and Cultural Organization

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G. Lazarides
International Centre for Theoretical Physics, Trieste, Italy,
and
Imperial College, London, England,

and

A.A. Patani and Q. Shafi
International Centre for Theoretical Physics, Trieste, Italy.

MIRAMARE - TRIESTE
December 1971

* To be submitted for publication

ABSTRACT

A lagrangian field theory described by the interaction $\mathcal{L}_{int} = \frac{1}{2}\lambda\varphi^2(e^{\kappa\theta} - 1)$ is considered where φ and θ are scalar fields with masses μ and zero respectively and λ, κ are the coupling constants. It is shown that the sum of a particular class of ladder graphs for $\varphi\varphi \to \varphi\varphi$ scattering is polynomially bounded in s, the square of the incoming centre-of-mass energy. This is true although an individual ladder with n rungs grows like $\exp[\sigma_n s^{1/3n}]$ for increasing s.

I. INTRODUCTION

It has been shown for the exponential interactions [1)-4)] using the non-polynomial perturbation-theoretic methods that amplitudes can be constructed to all orders in the major coupling constant so that they are finite, unitary and contain no arbitrary parameters. The problem of the high-energy behaviour of this and all other non-polynomial theories has, however, remained untouched[5)].

For most theories [6)] the amplitudes evaluated to second order in the major coupling constant increase like $\exp[\sigma s^{\alpha}]$ for increasing energy. For certain theories this increase lies below the Jaffe bound [7)] for localizability. It has been demonstrated by Taylor [3)] that the exponential interactions are localizable in every order of perturbation theory and there are indications that this may be so for other theories localizable in second order. On this basis the Jaffe criterion has been employed to classify theories as localizable or non-localizable depending on their high-energy behaviour in second order. So far there has been no analysis of the high-energy behaviour of sums of graphs of different orders in the major coupling constant which could provide a more realistic estimate of the high-energy behaviour of these theories and accordingly a more realistic basis for classification into localizable and non-localizable types.

Since the work of Epstein, Glaser and Martin [8)], which showed polynomial boundedness of the two-particle scattering amplitude in the framework of the theory of local observables of Araki and Haag, it has always been hoped that some of the theories which appear to be localizable in the second order would prove to be polynomially bounded. In particular [9)], the belief has been expressed that the summation over the major coupling constant should dramatically alter the high-energy behaviour of these theories.

In this paper we should like to show, by considering a particular class of ladder graphs for the exponential interactions, that although each of the ladders with n rungs grows like $\exp[\sigma_n s^{1/3n}]$ the summation over these ladders yields a polynomially bounded result. The details are presented as follows.

In Sec.II we discuss the definition and the high-energy behaviour of the second-order scattering amplitude (the superpropagator). In Sec.III we first obtain an approximation for large s to the ladder amplitude with n rungs. This approximation is summed and shown to have a polynomial bound. Our analysis is based on the euclidicity postulate by which we first define all our amplitudes in the space-like region and then continue to other directions in the complex s plane. In order to display the actual bound and therefore the corresponding Regge trajectory, it is necessary to solve a transcendental equation. Coupled with the fact that one has arbitrary coupling constants and mass, this cannot be achieved analytically. In the Appendix we carry out an analysis for the $\lambda\phi^3$ theory in an analogous manner displaying the well-known results [10].

II. THE SUPERPROPAGATOR

In the theory with

$$L_{int} = \frac{1}{2}\lambda\phi^2(e^{\kappa\theta} - 1) \tag{2.1}$$

where ϕ and θ are scalar fields with mass μ and zero respectively, and λ and κ are the so-called major and minor coupling constants respectively, the superpropagator $\Sigma(s)$ given in Fig.1 is well known [4] and for $s > 0$ is equal to

$$\Sigma(s) = \frac{1}{2}\left[\frac{\kappa^2}{4\pi}\right]^2 \left\{G^{20}_{03}\left(\frac{\kappa^2 s}{16\pi^2}\,\middle|\,-2,0,-1\right) + G^{20}_{03}\left(\frac{\kappa^2 s\; e^{-2\pi i}}{16\pi^2}\,\middle|\,-2,0,-1\right)\right\} \tag{2.2}$$

where $G^{mn}_{pq}(\alpha|^{a}_{b})$ is the Meijer function [11].

This answer for the superpropagator is usually evaluated in the following way. The amplitude is first defined for space-like s and $\kappa^2 < 0$ for which one obtains the result

$$\left(\frac{|\kappa^2|}{4\pi}\right)^2 \quad G^{20}_{03}\left(\frac{|s|\,|\kappa^2|}{16\pi^2}\,\middle|\, -2,0,-1\right) \quad . \tag{2.3}$$

Continuation of $|s|$ to time-like s is performed by $|s| \to s\, e^{-i\pi}$, whereas that of κ^2 is in principle arbitrary with an associated ambiguity. Unitary amplitude is obtained by taking the average of the continuations $|\kappa^2| \to \kappa^2\, e^{\pm i\pi}$. By this procedure the ambiguity is also put equal to zero, in accordance with Lehmann's ansatz [1]. It is claimed [3] that these continuations of s and κ^2 also yield unitary results for the amplitudes in higher orders. Therefore we shall, in the next section, choose these continuations for defining the ladder amplitudes.

The high-energy $(s \to +\infty)$ behaviour of the superpropagator $\Sigma(s)$ is given by [11)]

$$\frac{2}{\sqrt{3}} \left(\frac{\kappa^2 s}{16\pi^2}\right)^{-\frac{1}{3}} \sin\left(\frac{3\sqrt{3}}{2}\left[\frac{\kappa^2 s}{16\pi^2}\right]^{\frac{1}{3}} + \frac{\pi}{3}\right) \exp\left[-\frac{3}{2}\left(\frac{\kappa^2 s}{16\pi^2}\right)^{\frac{1}{3}}\right] \left(1 + 0\left(\frac{1}{s^{\frac{1}{3}}}\right)\right) +$$

$$+ \frac{i}{\sqrt{3}} \left(\frac{\kappa^2 s}{16\pi^2}\right)^{-\frac{1}{3}} \exp\left[3\left(\frac{\kappa^2 s}{16\pi^2}\right)^{\frac{1}{3}}\right] \left(1 + 0\left(\frac{1}{s^{\frac{1}{3}}}\right)\right) \quad . \tag{2.4}$$

It is thus seen that the imaginary part of the amplitude has an order of growth 1/3 and the real part tends to zero rapidly. It should be noticed that the Mellin transform method is not applicable for finding the high-energy behaviour of this amplitude or, as we shall see in the next section, the behaviour of the individual ladder graphs. It is simply because the Mellin transform of these amplitudes does not allow the contour to be collapsed on the left.

III. THE LADDER GRAPHS

We shall consider the amplitude, as given in Fig.2, for the ladder graph with $n + 1$ rungs for $s < 0$ and $\kappa^2 < 0$.

Each of the rungs of the ladder is a superpropagator $\Sigma(p^2)$ given by Eq.(2.3) or by the representation

$$\Sigma(p^2) = \frac{1}{2\pi i} \int_{\substack{c-i\infty \\ 0<c<1}}^{c+i\infty} dz \frac{\Gamma(-z)\,|\kappa^2|^z}{(16\pi^2)^{z-1}\,\Gamma(z)\Gamma(z)} \int_0^\infty d\mu^2 \frac{(\mu^2)^{z-1}}{(p^2+\mu^2)^2} \tag{3.1}$$

where $p^2 = p_0^2 + \vec{p}^2$.

Using Feynman parametrization [12)] as shown in Fig.2 for all the momentum denominators and doing the loop integration, one obtains for $s < 0$, $t < 0$ and $\kappa^2 < 0$,

$$A_n(s,t) = \left(\frac{1}{2\pi i}\right)^{n+1} \int_{c_0-i\infty}^{c_0+i\infty} \cdots \int_{c_n-i\infty}^{c_n+i\infty} \prod_{i=0}^{n} \frac{dz_i \Gamma(-z_i) |\kappa^2|^{z_i}}{(16\pi^2)^{z_i-1}\Gamma(z_i)\Gamma(z_i)} \int_0^\infty \cdots$$

$$0<c_0 \cdots c_n<1 \qquad \cdots \int_0^\infty \prod_{j=0}^{n} d\mu_j^2 \, (\mu_j^2)^{z_j-1} \, \Xi(\mu_0^2 \cdots \mu_n^2; s,t) \tag{3.2}$$

where [12]

$$\Xi(\mu_0^2 \cdots \mu_n^2; s,t) =$$

$$= \Gamma(2n+2) \int_0^1 \cdots \int_0^1 \prod_{i=0}^{n} \alpha_i \, d\alpha_i \prod_{j=1}^{n} d\beta_j \, d\gamma_j \, \delta\left[\sum_{i=0}^{n} \alpha_i + \sum_{j=1}^{n} (\beta_j + \gamma_j) - 1\right] C^{2n}(\alpha,\beta,\gamma)$$

$$\otimes \left\{ \prod_{i=0}^{n} \alpha_i \, |s| + d(\alpha,\beta,\gamma) + \mu^2 \, C(\alpha,\beta,\gamma) \sum_{j=1}^{n} (\beta_j + \gamma_j) + \right.$$

$$\left. + C(\alpha,\beta,\gamma) \sum_{i=0}^{n} \alpha_i \, \mu_i^2 \right\}^{-2(n+1)} . \tag{3.3}$$

$C(\alpha,\beta,\gamma)$ is a homogeneous function of order n in α,β and γ and $C(\alpha,\beta,\gamma)\Big|_{\alpha_i=0} = C(0) = \prod_{j=1}^{n} (\beta_j + \gamma_j)$. $d(\alpha,\beta,\gamma)$ is a linear homogeneous function of all the relativistic invariants appropriate to the amplitude with coefficients which are homogeneous functions of order $n+1$ in α,β and γ and polynomials of order n in α_i . For $\alpha_i = 0$,

$$d(\alpha,\beta,\gamma)\Big|_{\alpha_i=0} = d(0) = C(0) \sum_{j=1}^{n} \frac{\beta_j \gamma_j}{(\beta_j + \gamma_j)} |t| .$$

In Eq.(3.3), for $|s| \to \infty$ in any direction, the main contribution to the integral comes from the region near $\alpha_i = 0$. Since both $C(\alpha,\beta,\gamma)$ and $d(\alpha,\beta,\gamma)$ are slowly varying in this region we can obtain a high-energy approximation $\Xi'(\mu_0^2 \cdots \mu_n^2; s,t)$ of $\Xi(\mu_0^2 \cdots \mu_n^2; s,t)$ by setting $\alpha_i = 0$ in

$c(\alpha,\beta,\gamma)$ and $d(\alpha,\beta,\gamma)$. Substituting this approximation of $\Xi(\mu_0^2\cdots\mu_n^2;s,t)$ in Eq.(3.2), we obtain a high-energy approximation $A_n'(s,t)$ of the amplitude $A_n(s,t)$ which is valid for all arguments of s. For space-like s, $A_n'(s,t)$ can be evaluated and the result can be continued to the values of the energy in the physical region.

From Eq.(3.3) we obtain for space-like s

$$\Xi'(\mu_0^2\cdots\mu_n^2;s,t) =$$

$$= \int_0^\infty \cdots \int_0^\infty \prod_{i=0}^{n} \alpha_i \, d\alpha_i \exp[-\alpha_i \, \mu_i^2] \prod_{j=1}^{n} d\beta_j \, d\gamma_j \exp\left[-\mu^2[\beta_j + \gamma_j]\right] \times$$

$$\times \; c^{-2}(0) \exp\left[- \prod_{i=0}^{n} \alpha_i \; |s| \; c^{-1}(0) - d(0) \; c^{-1}(0)\right] \quad . \tag{3.4}$$

Substituting $\Xi'(\mu_0^2\cdots\mu_n^2;s,t)$ in Eq.(3.2), one can interchange the α-integrals with both the μ_j^2 and the z-integrals [13]. Performing the μ_j^2-integrals followed by the z-integrals, we obtain

$$A_n'(s,t) = |\kappa^2|^{n+1} \int_0^\infty \cdots \int_0^\infty \prod_{i=0}^{n} d\alpha_i \; G_{20}^{01}\left(\frac{16\pi^2\alpha_i}{|\kappa^2|} \,\middle|\, 2\;1\right) \prod_{j=1}^{n} d\beta_j \; d\gamma_j \exp\left[-\mu^2(\beta_j + \gamma_j)\right] \times$$

$$\times \; c^{-2}(0) \exp\left[- \prod_{i=0}^{n} \alpha_i \; |s| \; c^{-1}(0) - d(0) \; c^{-1}(0)\right] \quad . \tag{3.5}$$

The α-integrals can be performed successively using the relations [13]

$$\int_0^\infty d\alpha \; G_{20}^{01}(\alpha B|2,1) \; G_{0\;\,2m-1}^{m0}\left(\alpha A \,\middle|\, \underbrace{-2\cdots-2}_{m-1},0,-1\cdots-1\right) =$$

$$= \frac{1}{B} \, G_{0\;\,2m+1}^{m+1\;0}\left(A/B \,\middle|\, \underbrace{-2\cdots-2}_{m},0,-1\cdots-1\right) \qquad \text{for } m = 1,2\ldots$$

and

$$G_{01}^{10}(\alpha A|0) = \exp[-\alpha A] \quad . \tag{3.6}$$

One then obtains

$$A_n'(s,t) = \left(\frac{|\kappa^2|}{4\pi}\right)^{2(n+1)} \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{n} d\beta_j\, d\gamma_j\, C^{-2}(0) \exp\left[-\mu^2(\beta_j+\gamma_j) - d(0)C^{-1}(0)\right] \times$$

$$\times\, G_{0\ \ 2n+3}^{n+2\ 0}\left(\left[\frac{|\kappa^2|}{16\pi^2}\right]^{n+1} \frac{|s|}{C(0)} \left| \underbrace{-2\cdots-2}_{n+1},0,-1\cdots-1\right.\right) \quad . \tag{3.7}$$

In order to perform the β and γ integrals we express the Meijer function as an inverse Mellin transform and so obtain

$$A_n'(s,t) = \frac{1}{2\pi i}\left(\frac{|\kappa^2|}{4\pi}\right)^{2(n+1)} \int_{\substack{c-i\infty \\ c<-1}}^{c+i\infty} dz\, \Gamma(-z) \left[\frac{\Gamma(-2-z)}{\Gamma(2+z)}\right]^{n+1} \left(\frac{|\kappa^2|}{16\pi^2}\right)^{z(n+1)} |s|^z\, \omega(z) \tag{3.8}$$

where

$$\omega(z) = \int_0^\infty \cdots \int_0^\infty \prod_{j=1}^{n} d\beta_j\, d\gamma_j\, C^{-2-z}(0) \exp\left[-\mu^2(\beta_j+\gamma_j) - d(0)C^{-1}(0)\right]$$

$$= \left[\int_0^\infty\int_0^\infty d\beta\, d\gamma\, (\beta+\gamma)^{-2-z} \exp\left[-\mu^2(\beta+\gamma) - \frac{\beta\gamma|t|}{\beta+\gamma}\right]\right]^n$$

$$= \left[(\mu^2)^z\, \Gamma(-z)\, {}_2F_1\left(1,-z;3/2;-\frac{|t|}{4\mu^2}\right)\right]^n \qquad \text{for} \quad \text{Re } z < 0 \quad . \tag{3.9}$$

Hence

$$A_n'(s,t) = \frac{1}{2\pi i}\left(\frac{|\kappa^2|}{4\pi}\right)^2 \int_{\substack{c-i\infty \\ c<-1}}^{c+i\infty} dz\, \Gamma(-z) \frac{\Gamma(-2-z)}{\Gamma(2+z)} \left(\frac{|s||\kappa^2|}{16\pi^2}\right)^z \times$$

$$\times \left[\left(\frac{\mu^2|\kappa^2|}{16\pi^2}\right)^z \left(\frac{|\kappa^2|}{4\pi}\right)^2 \frac{\Gamma(-z)\Gamma(-2-z)}{\Gamma(2+z)}\, {}_2F_1\left(1,-z;3/2;-\frac{|t|}{4\mu^2}\right)\right]^n . \tag{3.10}$$

In the forward direction

$$A_n'(s,t=0) = \left(\frac{|\kappa^2|}{4\pi}\right)^{2(n+1)} G_{0\ \ 3n+3}^{2n+2\ 0}\left(\left[\frac{|\kappa^2|}{16\pi^2}\right]^{n+1} |s|\,(\mu^2)^n \left| \underbrace{-2\cdots-2}_{n+1},\underbrace{0\cdots0}_{n+1},-1\cdots-1\right.\right) \quad . \tag{3.11}$$

Continuing to $s > 0$ and $\kappa^2 > 0$, the leading asymptotic behaviour of $A'_n(s,t=0)$ can be found [11] to be

$$\frac{1}{2}\left(\frac{\kappa^2}{4\pi}\right)^{2(n+1)} \frac{(2\pi)^{n/2}}{(3n+3)^{\frac{1}{2}}}\ \xi^{-\frac{(8+9n)}{6(n+1)}}\ i^{-(n+1)}\ \times$$

$$\times\left\{\exp\left[-3(n+1)\ \xi^{\frac{1}{3(n+1)}}\exp\left[\frac{i\pi(2n+1)}{3(n+1)}\right]\right]\exp\left[-i\pi\ \frac{(2n+1)(8+9n)}{6(n+1)}\right]\ +\right.$$

$$\left.+\ (-)^{n+1}\exp\left[-3(n+1)\ \xi^{\frac{1}{3(n+1)}}\exp\left[-i\pi\ \frac{(2n+3)}{3(n+1)}\right]\right]\exp\left[i\pi\ \frac{(2n+3)(8+9n)}{6(n+1)}\right]\right\} \tag{3.12}$$

where $\xi = \frac{s}{\mu^2}\left(\frac{\kappa^2\mu^2}{16\pi^2}\right)^{n+1}$.

Thus each of the ladder graphs has a growth $\sim \exp\left[\sigma_n\ s^{1/3n}\right]$ in the forward direction. It should be noted that, unlike the $\lambda\phi^3$ theory (see Appx.I), the contour in Eq.(3.10) cannot be collapsed on the left to pick up the leading asymptotic behaviour. This is reflected in the fact that each of the ladder graphs does not have a polynomial bound. In order to obtain such a bound we first perform the summation over the major coupling constant to get

$$A'(s,t) = \sum_{n=0}^{\infty} (\lambda^2)^{n+1}\ A'_n(s,t)$$

$$= \frac{1}{2\pi i}\left[\frac{\lambda|\kappa^2|}{4\pi}\right]^2 \int_{\substack{c-i\infty \\ c<-1}}^{c+i\infty} dz\ \frac{\Gamma(-z)\Gamma(-2-z)}{\Gamma(2+z)}\left(\frac{|s||\kappa^2|}{16\pi^2}\right)^z \times$$

$$\times\ \frac{1}{1-\left(\frac{\mu^2|\kappa^2|}{16\pi^2}\right)^z\left(\frac{\lambda|\kappa^2|}{4\pi}\right)^2\frac{\Gamma(-z)\Gamma(-2-z)}{\Gamma(2+z)}\ {}_2F_1\left(1,-z;3/2;-\frac{|t|}{4\mu^2}\right)} \tag{3.13}$$

provided

$$|\nu(z)| < 1 \tag{3.14}$$

on the contour of integration. Here

$$\nu(z) = \left(\frac{\mu^2|\kappa^2|}{16\pi^2}\right)^z \left(\frac{\lambda|\kappa^2|}{4\pi}\right)^2 \frac{\Gamma(-z)\Gamma(-2-z)}{\Gamma(2+z)}\ {}_2F_1\left(1,-z;3/2;\ -\frac{|t|}{4\mu^2}\right) \quad . \tag{3.15}$$

For every Re z, as Im $z \to \pm\infty$, $\nu(z)$ goes to zero exponentially and therefore one can always choose a contour parallel to the imaginary axis such that $\nu(z)$ is bounded. That is to say, one can always find restrictions on the coupling constants λ and κ and the mass μ so that condition (3.14) is satisfied. For example, if for forward scattering the contour is placed between -2 and -1, we obtain a condition like $\lambda^2(\kappa^2)^{1-\varepsilon}/(\mu^2)^{1-\varepsilon} <$ some constant, where $1 > \varepsilon > 0$. This can easily be satisfied by choosing small coupling constants and/or large mass μ. Evidently the contour should not be placed too far to the left so that reasonable constraints can be placed on these constants in order to achieve condition (3.14).

It should be noted that, as in the case of the ladder graph with n rungs, the contour can be collapsed only on the right. Now, however, the pole structure of the integrand is completely altered and, in particular, the unbounded set of poles at $z = -1,0,1\ldots$ no longer appears. The only possible poles are located at the solutions of the equation

$$\nu(z) = 1 \quad . \tag{3.16}$$

Along any ray which is not parallel either to the real or the imaginary axis, $\nu(z) \sim \exp[-(\mathrm{Re}\ z)\ \ell n\ z]$. Therefore, for a ray directed in the right half plane, $\nu(z)$ decreases rapidly, and along a ray in the left half plane it increases very fast. Hence along these rays Eq.(3.16) has no asymptotic solutions. The same can be shown for the rays which are parallel to the imaginary axis.

The contour in Eq.(3.13) can now be collapsed on the right as shown in Fig.3 so that the new contour Γ encloses all the poles that were to the right of the contour in Eq.(3.13). These poles lie within a circle of finite radius except for possible ones on the real axis which may lie outside it.

One can now make the continuation of $A'(s,t)$ to $s > 0$ and $\kappa^2 > 0$ to obtain

$$A'(s,t) = \frac{1}{4\pi i}\left(\frac{\lambda\kappa^2}{4\pi}\right)\left\{\int_{\Gamma'} dz\ \frac{\Gamma(-z)\Gamma(-2-z)}{\Gamma(2+z)}\left(\frac{s\kappa^2}{16\pi^2}\right)^z \frac{1}{1-e^{i\pi z}\,\nu'(z)}\right. \quad +$$

$$+ \int_{\Gamma''} dz \, \frac{\Gamma(-z)\Gamma(-2-z)}{\Gamma(2+z)} \left(\frac{s\kappa^2 \, e^{-2\pi i}}{16\pi^2}\right)^z \frac{1}{1 - e^{-i\pi z} \, \nu'(z)}$$

where $$\nu'(z) = \left[\frac{\mu^2\kappa^2}{16\pi^2}\right]^z \left[\frac{\lambda\kappa^2}{4\pi}\right]^2 \frac{\Gamma(-z)\Gamma(-2-z)}{\Gamma(2+z)} \, {}_2F_1(1,-z;3/2;t/4\mu^2), \; t \leq 0 \tag{3.17}$$

and the contours Γ' and Γ'' are obtained by continuously distorting the contour Γ so as not to cross any poles. In the case that the contour gets pinched by certain sets of poles, one should add the additional contribution to Eq.(3.17).

We now consider the solutions of the equation

$$1 = e^{i\theta z} \, \nu'(z) \quad . \tag{3.18}$$

As before, one can again show that along any ray through the origin, except for the real and imaginary axis, there are no asymptotic solutions. Along the imaginary axis, except for $\theta = \pm\pi/2$, $e^{i\theta z} \nu'(z)$ either decreases or increases exponentially. For $\theta = \pm\pi/2$ it decreases exponentially along one of the directions and like a power along the opposite direction. Therefore, again along the imaginary axis, there are no asymptotic solutions. On the real axis for $\theta \neq 0$, the equations may have a solution only for $z = n\pi/\theta$, for otherwise the right-hand side of the equation has either an imaginary part or is zero. For $\theta = \pm\pi$, however, the solutions may be only at the integers. For $z = n = 0,1,2\ldots$

$${}_2F_1(1,-n;3/2;t/4\mu^2) = \sum_{\ell=0}^{n} \frac{\Gamma(n)\Gamma(3/2)}{\Gamma(n-\ell)\Gamma(3/2+\ell)} (-t/4\mu^2) \geq 1 \quad \text{for } t \leq 0 \quad .$$

Therefore, at these values, since $\Gamma(-z)$ and $\Gamma(-2-z)$ have poles, the equation again has no solutions. For $z = -3,-4 \ldots$ and $|t/4\mu^2| < 1$, $\nu'(z)$ is equal to zero and, therefore, for $\theta = \pm\pi$ on the real axis there could be a solution only for the values $z = -2$ and $z = -1$ and only for appropriate values of the coupling constant, mass μ and t.

What we have therefore shown is that for $\theta = \pm\pi$, Eq.(3.18) has no asymptotic solutions. For $-\pi < \theta < \pi$, if there are any asymptotic solutions of the equation then they all lie on the real axis. This means that in distorting contour Γ to Γ' and Γ'' there is no possibility of pinching the contour asymptotically. Therefore, the pinch contribution that may have to be added to Eq.(3.17) is polynomially bounded. Also the contours Γ, Γ' and Γ'' will coincide asymptotically. Since for $\theta = \pm\pi$ there are no asymptotic solutions of Eq.(3.18), the contours Γ' and Γ'' in Eq.(3.17) can be cut off as in Fig.4. From this it automatically follows that the amplitude $A(s,t)$ is polynomially bounded. It appears from Eq.(3.17) that this is so in any direction in the s plane.

To determine the exact Regge type behaviour one has obviously to solve a transcendental equation for z in terms of the parameters λ, κ, μ and t. This, of course, is very difficult to do analytically and one necessarily has to resort to numerical methods. From the behaviour of the Γ-functions one can easily determine where the asymptotic formulae one has used to analyse the transcendental equations become valid. This is for $\mathrm{Re}\, z \approx 15$ implying that the bound would be well below this value. Whether it is interesting to attempt to find the exact bound for this model theory is questionable.

Finally we would like to point out that a similar analysis yielding a polynomial bound can be carried out for rational non-polynomial theories like $L_{int} = \frac{1}{2} \lambda \phi^2 \, (1/(1+\kappa\theta))$ if the questions of ambiguity are ignored.

IV. CONCLUSION

We have shown explicitly that for the exponential interaction with interaction Lagrangian $L_{int} = \frac{1}{2} \lambda \phi^2 [e^{\kappa\theta} - 1]$ the summation over a particular class of ladder graphs yields an amplitude which is polynomially bounded in the complex s plane, even though the ladders in every order of perturbation theory grow like $\exp[\sigma_n \, s^{1/3n}]$ as $s \to \infty$. This bound cannot be explicitly calculated because of extreme technical difficulties of solving analytically the transcendental equation. Our result clearly substantiates the hope that the amplitudes in non-polynomial theories after summation over the major coupling constant will be polynomially bounded.

ACKNOWLEDGMENTS

The authors are grateful to Professor Abdus Salam for continued encouragement while this work was in progress. They wish to thank Professors Abdus Salam and P. Budini, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

APPENDIX I

For the $(1/3!)\lambda\phi^3$ theory, the amplitude $A_n(s,t)$ corresponding to the ladder graph with $n+1$ rungs is given by

$$A_n(s,t) = (\lambda^2)^{n+1}\,\Gamma(2n+2)\int_0^1 \cdots$$

$$\cdots\int_0^1 \prod_{j=0}^{n} d\alpha_j \prod_{i=1}^{n} d\beta_i\, d\gamma_i\; \delta\left[\sum_{j=0}^{n}\alpha_j + \sum_{i=1}^{n}(\beta_i+\gamma_i) - 1\right] \times$$

$$\times\, C^{2n}(\alpha,\beta,\gamma)\left\{\prod_{i=0}^{n}\alpha_i\,|s| + d(\alpha,\beta,\gamma) + \right.$$

$$\left. + \mu^2\, C(\alpha,\beta,\gamma)\left[\sum_{j=1}^{n}(\beta_j+\gamma_j) + \sum_{i=0}^{n}\alpha_i\right]\right\}^{-2(n+1)} \tag{AI.1}$$

where $C(\alpha,\beta,\gamma)$ and $d(\alpha,\beta,\gamma)$ are defined in the text.

Repeating the same argument as before, one can get an approximation $A'_n(s,t)$ of the amplitude $A_n(s,t)$ for large s. For $s<0$ this can be written as

$$A_n(s,t) \approx A'_n(s,t) =$$

$$= (\lambda^2)^{n+1}\int_0^\infty\cdots\int_0^\infty \prod_{i=0}^{n} d\alpha_i \prod_{j=1}^{n} d\beta_j\, d\gamma_j\; C^{-2}(0)\exp\left[-d(0)\,C^{-1}(0) - \mu^2\sum_{j=1}^{n}(\beta_j+\gamma_j)\right]\times$$

$$\times\exp\left[-C^{-1}(0)\,|s|\prod_{i=0}^{n}\alpha_i - \mu^2\sum_{i=0}^{n}\alpha_i\right] =$$

$$= (\lambda^2)^{n+1}\frac{1}{2\pi i}\int_{\substack{c-i\infty\\ c<0}}^{c+i\infty} dz\,\Gamma(-z)\int_0^\infty\cdots\int_0^\infty \prod_{i=0}^{n} d\alpha_i \prod_{j=1}^{n} d\beta_j\, d\gamma_j\,|s|^z\, C^{-2-z}(0)\left(\prod_{i=0}^{n}\alpha_i\right)^z \times$$

$$\times\exp\left[-C^{-1}(0)\,d(0) - \mu^2\left[\sum_{K=0}^{n}\alpha_K + \sum_{m=1}^{n}(\beta_m+\gamma_m)\right]\right] . \tag{AI.2}$$

If c is restricted between -1 and 0, the α,β and γ integrations can be done explicitly giving

$$A_n'(s,t) = \frac{\lambda^2}{2\pi i} \int_{\substack{c-i\infty \\ -1<c<0}}^{c+i\infty} dz\ |s|^z\ \Gamma(-z)\Gamma(z+1)(\mu^2)^{-1-z} \times$$

$$\times \left[\frac{\lambda^2}{\mu^2}\Gamma(-z)\Gamma(z+1)\ {}_2F_1(1,-z;3/2;-|t|/4\mu^2)\right]^n \quad . \tag{AI.3}$$

The summation over the ladders can be carried out to give

$$A'(s,t) = \sum_{n=0}^{\infty} A_n'(s,t)$$

$$= \frac{\lambda^2}{2\pi i} \int_{\substack{c-i\infty \\ -1<c<0}}^{c+i\infty} dz \frac{|s|^z\ \Gamma(-z)\Gamma(z+1)(\mu^2)^{-1-z}}{1 - \frac{\lambda^2}{\mu^2}\Gamma(-z)\Gamma(z+1)\ {}_2F_1(1,-z;3/2;-|t|/4\mu^2)} \tag{AI.4}$$

provided there is a c between -1 and 0 such that

$$|\omega(z)| < 1 \tag{AI.5}$$

for z on the contour. Here

$$\omega(z) = \frac{\lambda^2}{\mu^2}\Gamma(-z)\Gamma(z+1)\ {}_2F_1(1,-z;3/2;-|t|/4\mu^2) \quad . \tag{AI.6}$$

For forward scattering it can be shown that the maximum of $|\omega(z)|$ on the contour of integration occurs on the real axis. Thus condition (AI.5) becomes

$$\left| \frac{\lambda^2\pi}{\mu^2\ \sin\pi c} \right| < 1 \quad . \tag{AI.7}$$

Therefore, in order to obtain the biggest possible range for λ^2/μ^2 satisfying this inequality we choose $c = -\frac{1}{2}$. For any value t , $|\omega(z)|$ falls off rapidly as $y \to \pm\infty$ and thus the inequality (AI.5) can still be maintained with possibly a more restricted range of λ^2/μ^2 .

In contrast to the interactions considered in the text, the contour in this case must be collapsed to the left for large $|s|$. Thus after the continuation to $s > 0$ we obtain

$$A'(s,t) = \frac{\lambda^2}{2\pi i} \int_\gamma dz \frac{s^z\, e^{-i\pi z}\, \Gamma(-z)\Gamma(z+1)(\mu^2)^{-1-z}}{1 - \frac{\lambda^2}{\mu^2}\Gamma(-z)\Gamma(z+1)\ {}_2F_1(1,-z;3/2;t/4\mu^2)} \tag{AI.8}$$

where γ is given in Fig.5.

The leading contribution to the integral comes from the right-most pole under γ which occurs at

$$z = -1 + \frac{\lambda^2}{\mu^2} + \frac{\lambda^2}{6\mu^4} t + \cdots \qquad \text{for small } t \ .$$

This reproduces the well-known result of the ϕ^3 theory [10].

APPENDIX II

a) The representation of the superpropagator [Eq.(3.1)]

$$\Sigma(p^2) = \frac{1}{2\pi i} \int_{\substack{c-i\infty \\ 0<c<1}}^{c+i\infty} dz \, \frac{\Gamma(-z)(\kappa^2)^z}{(16\pi^2)^{z-1}\,\Gamma(z)\Gamma(z)} \int_0^\infty d\mu^2 \, \frac{(\mu^2)^{z-1}}{(p^2+\mu^2)^2} \tag{AII.1}$$

converges well.

We first consider the μ^2-integral. By α-parametrizing the momentum denominator

$$\frac{1}{(p^2+\mu^2)^2} = \int_0^\infty d\alpha \, \alpha \, \exp\left[-\alpha(\mu^2+p^2)\right] \tag{AII.2}$$

and taking the μ^2 integral through,one obtains

$$\int_0^\infty d\alpha \, \alpha \, e^{-\alpha p^2} \int_0^\infty d\mu^2 (\mu^2)^{z-1} \, e^{-\alpha\mu^2} = \int_0^\infty d\alpha \, \Gamma(z) \, \alpha^{1-z} \, e^{-\alpha p^2} \qquad \text{for} \quad \mathrm{Re}\, z > 0 \, . \tag{AII.3}$$

Now, exchanging the z-integral with α-integral we get

$$\int_0^\infty d\alpha \, e^{-\alpha p^2} \frac{1}{2\pi i} \int_{\substack{c-i\infty \\ 0<c<1}}^{c+i\infty} dz \, \frac{\Gamma(-z)}{\Gamma(z)} \, \alpha^{1-z} \left(\frac{16\pi^2}{\kappa^2}\right)^{1-z} \, , \tag{AII.4}$$

which by changing variables $1-z = \zeta$ can be written as

$$\int_0^\infty d\alpha \, e^{-\alpha p^2} \frac{1}{2\pi i} \int_{\substack{\xi-i\infty \\ 0<\xi<1}}^{\xi+i\infty} d\zeta \, \frac{\Gamma(\zeta-1)}{\Gamma(1-\zeta)} \, \alpha^\zeta \left(\frac{16\pi^2}{\kappa^2}\right)^\zeta \, . \tag{AII.5}$$

For $|\mathrm{Im}\,\zeta| \to \infty$, the integrand of the ζ-integral behaves like $|\mathrm{Im}\,\zeta|^{2\mathrm{Re}\,\zeta - 2}$. The integral therefore converges for $\mathrm{Re}\,\zeta < \frac{1}{2}$ and ξ must therefore be restricted between 0 and $\frac{1}{2}$.

$$\therefore \frac{1}{2\pi i} \int_{\substack{\xi-i\infty \\ 0<\xi<\frac{1}{2}}}^{\xi+i\infty} d\zeta \, \frac{\Gamma(\zeta-1)}{\Gamma(1-\zeta)} \, \alpha^\zeta \left(\frac{16\pi^2}{\kappa^2}\right)^\zeta = G^{01}_{20}\left(\frac{\alpha 16\pi^2}{\kappa^2} \,\middle|\, 2 \; 1\right) \tag{AII.6}$$

converges and

$$\Sigma(p^2) = \int_0^\infty d\alpha\; e^{-\alpha p^2}\; G^{01}_{20}\left(\frac{\alpha 16\pi^2}{\kappa^2}\,\middle|\; 2\; 1\right) \tag{AII.7}$$

and therefore the above representation could have been used to arrive at Eq. (3.5) for space-like s .

b) The left-hand side of the formula (3.6) does not converge as it stands and therefore has to be defined as

$$\int_0^\infty d\alpha\; G^{01}_{20}(\alpha B|2,1)\; G^{m0}_{0\;2m-1}(\alpha A|\underbrace{-2\cdots-2}_{m-1},0,-1\cdots-1) =$$

$$= \lim_{\sigma\to 0}\int_0^\infty d\alpha\;\alpha^\sigma\; G^{01}_{20}(\alpha B|2,1)\; G^{m0}_{0\;2m-1}(\alpha A|-2\cdots-2,0,-1\cdots-1)$$

$$= \lim_{\sigma\to 0}\int_0^\infty d\alpha\,\alpha^\sigma \frac{d\alpha}{2\pi i}\int_{\substack{\xi-i\infty\\0<\xi<\frac{1}{2}}}^{\xi+i\infty} d\zeta\;\frac{\Gamma(\zeta-1)}{\Gamma(1-\zeta)}\,\alpha^\zeta\, B^\zeta\; G^{m0}_{0\;2m-1}(\alpha A|-2\cdots-2,0,-1\cdots-1)$$

$$= \lim_{\sigma\to 0}\frac{1}{2\pi i}\int_{\substack{\xi-i\infty\\0<\xi<\frac{1}{2}}}^{\xi+i\infty} d\zeta\;\frac{\Gamma(\zeta-1)}{\Gamma(1-\zeta)}\,B^\zeta\int_0^\infty d\alpha\;\alpha^{\sigma+\zeta}\; G^{m0}_{0\;2m-1}(\alpha A|-2\cdots-2,0,-1\cdots-1)\;. \tag{AII.8}$$

The α-integral converges for $1-\sigma<\zeta$ to the Mellin transform of $G^{m0}_{0\;2m-1}(\alpha A)$ as a function of $\sigma+\zeta+1$ since $G^{m0}_{0\;2m-1}(\alpha A)$ goes to zero exponentially as $\alpha\to\infty$ and like α^{-2} as $\alpha\to 0$. Therefore, for $\sigma>\frac{1}{2}$ the double integral is defined and equals

$$\frac{1}{2\pi i}\int_{\substack{\xi-i\infty\\1-\sigma,0<\xi<\frac{1}{2}}}^{\xi+i\infty} d\zeta\;\frac{\Gamma(\zeta-1)}{\Gamma(1-\zeta)}\,B^\zeta\left[\frac{\Gamma(-1+\zeta+\sigma)}{\Gamma(1-\zeta-\sigma)}\right]^{m-1}\Gamma(\sigma+s+1)\;A^{-\sigma-\zeta-1} \tag{AII.9}$$

which converges to

$$\frac{A^{-\sigma}}{B}\,G^{m+1\;0}_{0\;\;2m+1}(A/B|-2,-2+\sigma\;\cdots\;-2+\sigma,\sigma,-1+\sigma\;\cdots\;-1+\sigma,-1)\;.$$

The limit of this expression exists and equals

$$\frac{1}{B}\,G^{m+1\;0}_{0\;\;2m+1}(A/B|-2\cdots-2,0,-1\cdots-1)\;. \tag{AII.10}$$

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which by changing variables $1-z = \zeta$ can be written as

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$$= \lim_{\sigma\to 0}\int_0^\infty d\alpha\, \alpha^\sigma\, G^{01}_{20}(\alpha B|2,1)\, G^{m0}_{0\ 2m-1}(\alpha A|-2\cdots-2,0,-1\cdots-1)$$

$$= \lim_{\sigma\to 0}\int_0^\infty d\alpha\, \alpha^\sigma \frac{d\alpha}{2\pi i}\int_{\substack{\xi-i\infty\\ 0<\xi<\frac{1}{2}}}^{\xi+i\infty} d\zeta\, \frac{\Gamma(\zeta-1)}{\Gamma(1-\zeta)}\,\alpha^\zeta\, B^\zeta\, G^{m0}_{0\ 2m-1}(\alpha A|-2\cdots-2,0,-1\cdots-1)$$

$$= \lim_{\sigma\to 0}\frac{1}{2\pi i}\int_{\substack{\xi-i\infty\\ 0<\xi<\frac{1}{2}}}^{\xi+i\infty} d\zeta\, \frac{\Gamma(\zeta-1)}{\Gamma(1-\zeta)}\, B^\zeta \int_0^\infty d\alpha\, \alpha^{\sigma+\zeta}\, G^{m0}_{0\ 2m-1}(\alpha A|-2\cdots-2,0,-1\cdots-1)\ . \tag{AII.8}$$

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which converges to

$$\frac{A^{-\sigma}}{B}\, G^{m+1\ 0}_{0\ \ 2m+1}(A/B|-2,-2+\sigma\ \cdots\ -2+\sigma,\sigma,-1+\sigma\ \cdots\ -1+\sigma,-1)\ .$$

The limit of this expression exists and equals

$$\frac{1}{B}\, G^{m+1\ 0}_{0\ \ 2m+1}(A/B|-2\cdots-2,0,-1\cdots-1)\ . \tag{AII.10}$$

FOOTNOTES AND REFERENCES

1) H. Lehmann and K. Pohlmeyer, Commun. Math. Phys. 20, 101 (1971).

2) Abdus Salam and J. Strathdee, Phys. Rev. D2, 2869 (1970).

3) J.G. Taylor, Univ. of Southampton preprint, April 1971.

4) M.K. Volkov, Commun. Math. Phys. 7, 289 (1968).

5) Recently some work on these questions has been done by Z. Horváth and G. Pócsik in an Eötvös University preprint from Budapest. Their approach to non-polynomial theories appears to be different from the usual ones and we have not yet managed to understand it.

6) In Ref.3 Professor Taylor has claimed these statements for the exponential interactions to all orders.

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10) R.J. Eden *et al.*, *The Analytic S-Matrix* (Cambridge University Press 1966).

11) C.S. Meijer, Proc. K. Neder. Weten. 49, 227 (1846).

12) Details are well known and are contained in Ref. 10.

13) See Appendix II.

FIGURE CAPTIONS

Fig.1

The superpropagator $\Sigma(s)$.

Fig.2

The ladder amplitude with n rungs showing the Feynman parameters corresponding to each of the propagators.

Fig.3

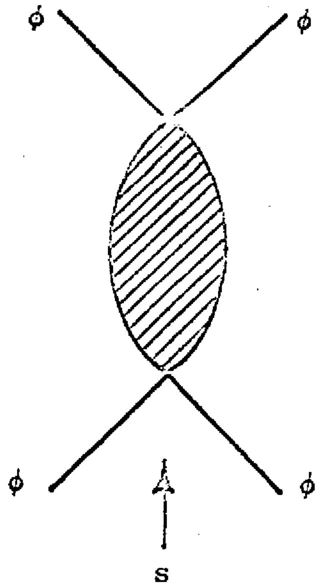
The contour Γ. All the poles of the integrand lie only in the shaded area, and asymptotically only along the real axis.

Fig.4

The contours Γ' and Γ''. All the poles are again in the shaded region.

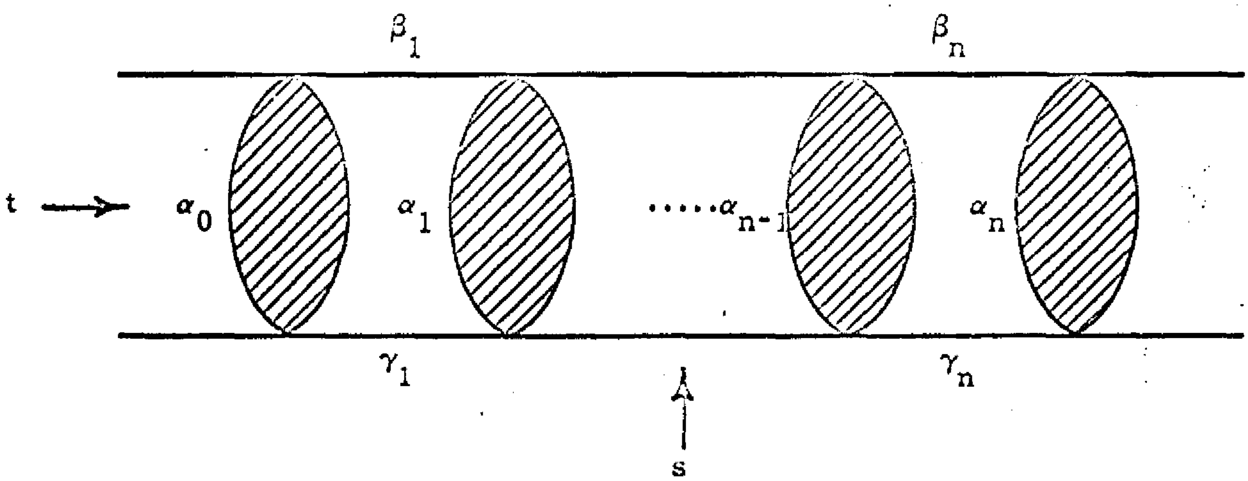
Fig.5

The contour γ.



ϕ
ϕ
ϕ
ϕ
s

Fig. 1



β_1
β_n
t
α_0
α_1
$\cdots\cdots\alpha_{n-1}$
α_n
γ_1
γ_n
s

Fig. 2

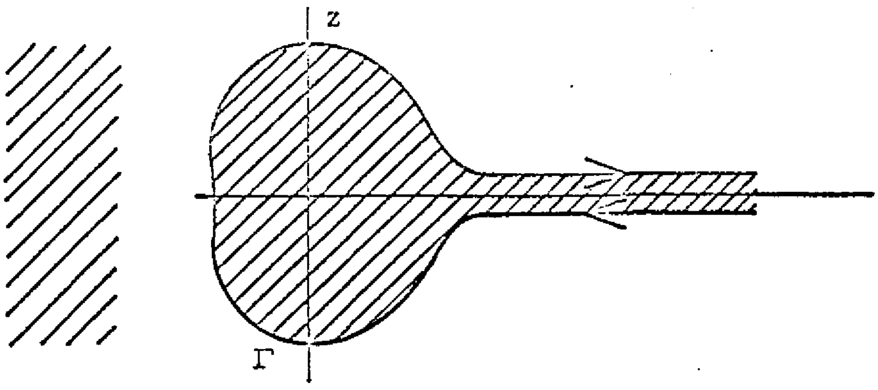


Fig. 3

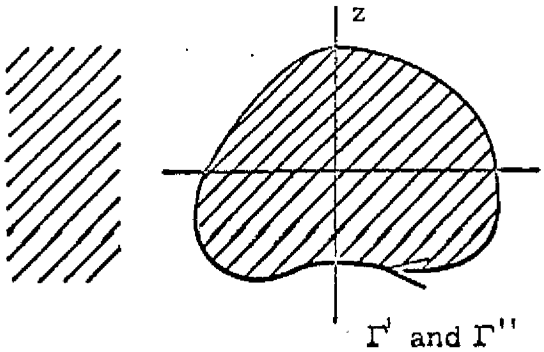


Fig. 4

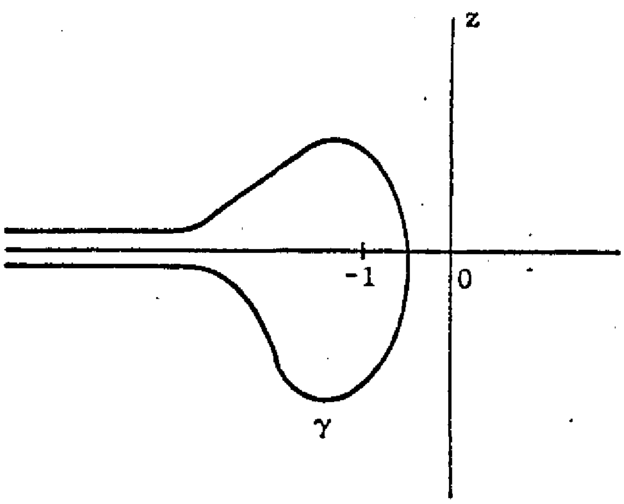


Fig. 5