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INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

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ADDENDA

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Page 6, line 6: After "superpropagators" and before the asterisk, add: "of the non-polynomial co-tensors".

Page 7, line 14: After "non-localizable" add a second footnote **)

The footnote is: **) This is on account of the co-tensors $L_{\mu a}$ being rational rather than entire functions of the physical field $\phi^{\mu a}$.

ABSTRACT

It is argued that the use of a visibly localizable parametrization of the gravitational interaction yields a number of advantages. Firstly, the question of ambiguities can be completely solved: according to a theorem of Lehmann and Pohlmeyer there exists in such theories a <u>unique</u> "minimally singular" solution which it is natural to adopt as the physical one. Secondly, it is possible to show that this solution satisfies the usual requirements of analyticity and unitarity in the sense of perturbation theory. Thirdly, it is possible to arrange the computations in a manifestly gauge independent manner. These points are briefly reviewed in this letter, the main object of which is to introduce a new technique for the treatment of those non-polynomial Lagrangians in which the interaction terms are intimately associated with the free part. The gravity-modified theories exemplify this type of Lagrangian: in such theories the zero-graviton approximant to any process is "cradled" in a sequence of graphs with arbitrarily large numbers of gravitons whose sum exists, is finite, free of ambiguities and gauge independent to any required order.

I. INTRODUCTION

In a recent paper, with essentially the same title as this one,[1] the authors showed by actual computation that when tensor-gravity effects are taken into account, the conventional logarithmically infinite expressions $\alpha \log \alpha$ appearing in electron self-charge and self-mass get realistically regularized to $\alpha \log(\kappa^2 m^2)$ where $(16\pi\kappa^2)$ is the Newtonian constant G_N. There were, however, a number of problems the computation left dark:

<u>Physically</u>

It was not clear whether it was <u>true tensor</u> gravity which was responsible for the finite computation of the renormalization constants or whether it could be some scalar version of it.

<u>Mathematically</u>

1) The results were not gauge invariant.

2) The role of field-theoretic equivalence transformations was not clear; the elimination of <u>all</u> infinities seems to necessitate the assumption that $g^{\mu\nu}(x)$ must be treated as the fundamental field and not the co-tensor $g_{\mu\nu}(x)$.

3) A divergent series was encountered in the calculation, and for this the Borel sum was adopted. Such a prescription is necessarily arbitrary and renders the result ambiguous.

Basically these shortcomings all stemmed from the same unanswered question: Since it is analytic continuation techniques which sharply distinguish non-polynomial Lagrangians like gravity from polynomial Lagrangians, under what conditions do these techniques define <u>good</u> field theories in the conventional sense?

An important advance has recently been made in answering this question by Lehmann and Pohlmeyer [2], who have shown that non-polynomial field theories with <u>localizable</u> (<u>microcausal</u>) interaction certainly do define <u>good</u> field theories in the conventional sense. This is because:

1) The superpropagators in localizable theories are entire functions in the variable $\Delta(x)$ and no Borel ambiguities ever arise for these as they do for non-localizable theories.*)

2) For localizable theories, there exists a perturbation solution in the major coupling constant which satisfies the usual requirements of analyticity and unitarity to all orders. (This has been demonstrated by Lehmann and Pohlmeyer for a particular Lagrangian [2], though their considerations are general. See also Volkov [3] and Filippov [4].)

3) Other solutions can be obtained from this one by adding to it in a specific manner entire functions (in momentum space) with the same order of exponential growth but which are otherwise arbitrary. Such modified solutions can be distinguished from the original one - in the case of localizable theories only - by their high-energy behaviour. The original solution (which Lehmann and Pohlmeyer call "minimally singular") is the only one which falls to zero in some direction in the complex energy plane.

---

*) By the term "superpropagator" is here meant the 2-point function $\langle T\mathcal{L}_{int}(\phi(x))\,\mathcal{L}_{int}(\phi(y))\rangle_0$ computed in the free field approximation - i.e. without the inclusion of any internal vertices. A theory is localizable and microcausal in Jaffe's sense if the Fourier transform of the imaginary part of the superpropagators increases no faster than $\exp \|p^2\|^a$ with $a < \frac{1}{2}$ for large p^2 ; it is non-localizable and non-microcausal if $a > \frac{1}{2}$. One can show that in x-space the superpropagator can be expressed as an entire function of the bare propagator, $\Delta(x-y) = \langle T\,\phi(x)\,\phi(y)\rangle_0$, for localizable theories. An example of this is the Lehmann-Pohlmeyer Lagrangian, $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + g : e^{\kappa\phi} :$, for which the superpropagator is given by the entire function $g^2 \exp \kappa^2\Delta(x-y)$. On the other hand, for non-localizable theories the superpropagator takes the form of a divergent series. For example, with the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + g : (1+\kappa^2\phi^2)^{-1} :$, the superpropagator is represented by the divergent series $g^2 \sum_n (2n)!(\kappa^2\Delta)^{2n}$.

4) For localizable, microcausal theories there exists the fundamental theorem of Glaser, Epstein and Martin [5] which asserts that such theories (presumably after a summation of the perturbation expansion in the major coupling constant) give mass-shell matrix elements which are Froissart bounded.

5) For localizable theories (and only for such theories) one may make the ansatz that the physically relevant solution is the minimally singular one in the sense of Lehmann and Pohlmeyer. This ansatz would eliminate all ambiguities (to all orders in the major coupling constant) in localizable theories. It is our conjecture that the attainment of the Glaser-Martin-Epstein upper bound is intimately connected with the choice of that perturbation solution which is minimally singular.

Summarizing, localizable, microcausal non-polynomial theories offer the prospect of ambiguity-free, analytic, unitary, Froissart-bounded solutions and are <u>good</u> field theories in the conventional sense.

In the present paper we use these crucial results of Lehmann and Pohlmeyer on localizable theories to sharpen the earlier computation and in particular to:

1) arrange the calculation so that the electromagnetic gauge invariance for S-matrix elements is manifest,

2) show that it is indeed <u>tensor gravity</u> and not a scalar variety of it which is responsible for gauge-independent infinity suppression (this would be in accordance with the physical expectation that infinity suppression must come from light-cone fluctuations which are peculiar to tensor (rather than scalar) gravity in its metrical aspects *)).

---

*) We are indebted to Prof. V. Weisskopf for emphasising this point (private communication).

II. LOCALIZABLE GRAVITY

To apply Lehmann and Pohlmeyer's result, we must work with a manifestly localizable version of gravity theory. One way to accomplish this is to parametrize the vierbein gravity field $L^{\mu a}$ through an entire function, an exponential for example. Thus, write *)

$$L^{\mu a} = \left[\exp\left(\frac{\kappa}{2}\ \phi\right)\right]^{\mu a} \qquad (2.1)$$

where $\phi^{\alpha\beta} = \phi^{\beta\alpha}$ are the basic interpolating fields which at $\pm\infty$ coincide with the outgoing and incoming graviton fields. Since, in gravity theory, one assumes that $\det L \neq 0$, it is clear that, like (2.1), the co-tensor $L_{\mu a}$, the inverse of $L^{\mu a}$, is also an entire function in the ϕ-plane.

---

*) Strictly, one should refer this formula to a Euclidean basis by means of the matrix $(\eta^{\frac{1}{2}})^{\alpha\beta} = \mathrm{diag}(1, i, i, i)$. Then it takes the matrix form

$$(\eta^{\frac{1}{2}}\, L\, \eta^{\frac{1}{2}})^{\alpha\beta} = \left[\exp \frac{\kappa}{2}\, \eta^{\frac{1}{2}}\, \phi\, \eta^{\frac{1}{2}}\right]^{\alpha\beta}$$

where ϕ denotes a symmetric matrix with elements $(\phi^{\alpha\beta})$. The exponential can be expanded to give the formula

$$\eta^{\frac{1}{2}} L\, \eta^{\frac{1}{2}} = 1 + \frac{\kappa}{2}\, \eta^{\frac{1}{2}}\, \phi\, \eta^{\frac{1}{2}} + \tfrac{1}{2}\left(\frac{\kappa}{2}\right)^2 \eta^{\frac{1}{2}}\, \phi\, \eta\, \phi\, \eta^{\frac{1}{2}} + \cdots$$

i.e.,

$$L = \eta + \frac{\kappa}{2}\,\phi + \tfrac{1}{2}\left(\frac{\kappa}{2}\right)^2 \phi\, \eta\, \phi + \cdots \quad .$$

In particular one finds

$$-\det L = \exp \frac{\kappa}{2}\, \mathrm{tr}(\eta\phi)$$

$$= \exp \frac{\kappa}{2}\, (\phi^{\alpha}_{\alpha}) \quad .$$

The parametrization (2.1) replaces the conventional parametrization which we used in Ref.1:

$$L^{\mu a} = \eta^{\mu a} + \frac{\kappa}{2}\phi^{\mu a} \quad , \qquad (2.2)$$

where the co-tensor $L_{\mu a}$ was a ratio of two polynomials in ϕ .

Now to the extent that one may neglect graviton-graviton (derivative-containing) self-interactions, one can easily check that superpropagators *) $\langle T : L_{\mu a}(1) : , : L_{\nu b}(2) : , : L_{\rho c}(3) : , \cdots \rangle_0$ for parametrization (2.1) are entire functions in the (multiple) Δ-plane and define a localizable theory, while those arising from (2.2) are divergent series and give a non-localizable theory.

Thus, either

1) The two parametrizations define two distinct quantum theories, one local and the other non-local.

Or

2) The inclusion of graviton-graviton interactions will produce vast cancellations for case (2.2), reducing the virulently singular behaviour of superpropagators to that for parametrization (2.1), rendering both versions localizable. (The reverse seems less likely.) Till this can be demonstrated we shall abandon (2.2) and work with a visibly localizable entire-function parametrization of gravity like the one given by (2.1).

Now comes the crucial point. Since localizable theories are microcausal, and microcausality is the basis of Borchers' theory of equivalence classes, we shall assume that those field transformations which transform one localizable theory into another do respect the field-theoretic equivalence theorem which asserts equality of mass-shell S-matrix elements. Such transformations will play an important role in exhibiting manifest gauge-invariance. Further, we can dispense completely with the distinction made in Ref.1 between the tensor $L^{\mu a}$ and its co-tensor $L_{\mu a}$, so far as infinity suppression is concerned.

---

*) We shall always discard the tadpoles in defining superpropagators, that is we take the normally ordered vierbein $: L^{\mu a} :$.

All matrix elements deduced from Lagrangians of the variety $g : \chi^r \exp(\kappa\phi) :$ are as finite as those from Lagrangians *) $g : \chi^r \exp(-\kappa\phi) :$.

To summarize, the essential message of this paper is that the basic reason for the shortcomings of Ref.1 was the use of a non-localizable parametrization of gravity theory. Once this is cured, the shortcomings disappear. It is important to emphasise once again that it is far too early to condemn the parametrization (2.2) in comparison with (2.1). It is perfectly possible - through the large number of cancellations associated with the derivative self-couplings of gravity which have not been taken into account in this paper - that the two parametrizations, in the end, give completely equivalent S-matrices. However, (2.1) is visibly localizable; (2.2) is visibly non-localizable, and, in view of the advantages of working with a localizable theory, we shall henceforth discard (2.2).

III. THE GRAVITY-MODIFIED LAGRANGIAN

The electron-photon part of the gravity-modified Lagrangian is given by

$$(-\det L)^{-2w_e - 1}\left[\frac{i}{2} L^{\mu a} (\bar{\psi}\gamma_a \psi_{;\mu} - \bar{\psi}_{;\mu}\gamma_a \psi) - m_0 \bar{\psi}\psi\right] +$$

$$+ (-\det L)^{-2w_e - 1} e_0 L^{\mu a} \bar{\psi}\gamma_a \psi A_\mu$$

$$- (-\det L)^{-1} \frac{1}{4} g^{\mu\kappa} g^{\nu\lambda} F_{\mu\nu} F_{\kappa\lambda} \tag{3.1}$$

*) Heuristically, a two-point superpropagator behaves like $g^2 (r!)(-1/x^2)^r \exp(-\kappa^2/x^2)$. The exponential is potent enough to regularize $(1/x^2)^r$ to zero provided we approach $x^2 \to 0$ along the appropriate direction in x-space, filling in for other directions by analytic continuation. (For detailed discussion see Lehmann and Pohlmeyer [2].)

where

$$\psi_{;\mu} = \partial_\mu \psi - \frac{i}{4} B_\mu{}^{ab} \sigma_{ab} \psi + w_e (\det L)_{,\mu} \psi \tag{3.2}$$

$$\det L = \det L^{\mu a} . \tag{3.3}$$

The parameter w_e denotes the weight of the electron field and $B_\mu{}^{ab}$ is a (for the present purposes irrelevant) field which involves the derivative of L. It is defined in Ref.1. The weight w_e can be changed at will by making field transformations and, if Borchers' theorem holds, it should not appear in the physical S-matrix.

It is difficult to make gauge-independent computations because of the awkward factor $L^{\mu a} (-\det L)^{-2w_e - 1}$ which multiplies the interaction term $e_0 \bar\psi \gamma_a \psi A_\mu$ in (3.1). This factor can be removed by making a suitable choice of the basic field variables. To this end, we choose to assign the weight $w_e = -\frac{1}{2}$ and to regard the combination $A^a = A_\mu L^{\mu a}$ as the photon field. If the equivalence theorem is valid then such a choice should not affect the values of S-matrix elements.

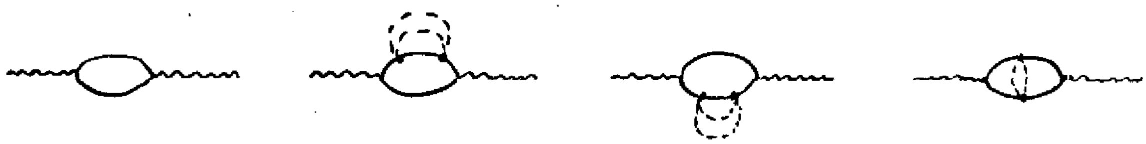
With these choices write the Lagrangian (3.1) in the form

$$\mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 \tag{3.4}$$

where

$$\left.\begin{aligned}
\mathcal{L}_0 &= -\frac{1}{4} : F_{ab}{}^2 : + : \bar\psi (i\gamma\cdot\partial - m_0)\psi : \\
\mathcal{L}_1 &= e_0 : \bar\psi \gamma_a \psi A_a : \\
\mathcal{L}_2 &= \frac{i}{2} : (L^{\mu a} - \eta^{\mu a})(\bar\psi \gamma_a \partial_\mu \psi - \partial_\mu \bar\psi \gamma_a \psi) : \\
\mathcal{L}_3 &= \frac{1}{4} : \left(\frac{g^{\mu\kappa} g^{\nu\lambda}}{\det L} + \eta^{\mu\kappa} \eta^{\nu\lambda} \right) F_{\mu\nu} F_{\kappa\lambda} : \\
\mathcal{L}_4 &= -\frac{1}{2} : L^{\mu a} \bar\psi \gamma_a \left(\frac{i}{2} B_\mu{}^{ab} \sigma_{ab} \right) \psi :
\end{aligned}\right\} \tag{3.5}$$

The terms $\mathcal{L}_2$, $\mathcal{L}_3$ and $\mathcal{L}_4$ all give rise to vertices where at least one graviton is emitted or absorbed. Consider the contributions of $\mathcal{L}_1$ and $\mathcal{L}_2$ to the photon self-energy. The graphs of second order in $\mathcal{L}_1$ and zeroth and second order in $\mathcal{L}_2$ are shown in Fig.1. This collection of graphs is gauge independent.



(a) (b) (c) (d)

Fig.1

In Sec.IV we shall show that although the graphs of Fig.1 are logarithmically divergent, their infinities compensate in the sense that they can all be incorporated in one "supergraph" whose contribution is finite. To prove this we shall need to introduce two complementary techniques, the "kinking" and "cradling" of graphs.

Before closing this section, we indicate why it is tensor gravity rather than a scalar version of it which is necessary for infinity suppression. Firstly, scalar (Nordström) gravity is recovered from the formalism of this paper by setting

$$L^{\mu a} = \exp\left(\frac{\kappa}{2}\ \phi\right)\eta^{\mu a} \ .$$

Choosing the weight $w_e = -3/8$ causes the Lagrangian (3.1) to reduce to the form

$$\frac{i}{2}(\bar{\psi}\gamma_\mu \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma_\mu \psi) - m_0 e^{-\frac{\kappa}{2}\phi}\ \bar{\psi}\psi$$
$$+ e_0 \bar{\psi}\gamma_\mu A_\mu \psi$$
$$- \frac{1}{4} F_{\mu\nu} F_{\mu\nu}$$

where the scalar graviton, ϕ, has decoupled from the photon and couples to the electron only through the mass term. Clearly scalar gravitons can have no regularizing role for massless electrons. However, even for massive electrons there is no regularization. The graphs of Fig.1, for example, fail to arrange themselves into a single finite supergraph in the manner to be discussed in the following section.[*)]

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*) The same remark applies to non-polynomial strong interactions which could modify the photon propagator and thus regularize Figs. 1(b), (c) and (d). However, the infinity in Fig. 1(a) would remain.

IV. "KINKED" GRAPHS AND THE "CRADLING" TECHNIQUE

As a prototype for $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ ((3.6) and (3.7)) consider a model Lagrangian:

$$\mathcal{L}'_1 = e_0 : \psi^2 A : \tag{4.1}$$

$$\mathcal{L}'_2 = :(e^{\kappa\phi} - 1)(\partial\psi)^2 : \tag{4.2}$$

where ψ , A and ϕ represent scalar massless electrons, photons and gravitons. Characteristically $\mathcal{L}'_1$ is a renormalizable polynomial Lagrangian while $\mathcal{L}'_2$ is non-polynomial which, operating by itself, would give finite matrix elements but for any possible infinities which its "kinking" part $-(\partial\psi)^2$ might produce.*) The "kinking" terms in $\mathcal{L}'_2$ are so called because inside any (ψ–) line the operation of $(\partial\psi)^2$ acts simply like a unit operator. In momentum space, for example, the (scalar) electron-line factor $1/p^2$ equals

$$\frac{1}{p^2} = \frac{1}{p^2} \cdot p^2 \cdot \frac{1}{p^2} = \frac{1}{p^2} \cdot p^2 \cdot \frac{1}{p^2} \cdot p^2 \cdot \frac{1}{p^2} = \ldots$$

corresponding to successively kinked lines shown graphically in Fig.2.

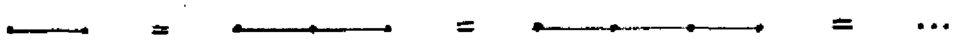

Fig.2

In what follows, we shall use "<u>kinking</u>" to represent zero graviton emission or absorption in graphs given by the operation of the polynomial Lagrangian $\mathcal{L}'_1$. This will then permit us to "<u>cradle</u>" these in corresponding graphs obtained by operation of $\mathcal{L}'_2$.

As an example, consider the computation of the propagator $\langle T\psi(x)\,\psi(y)\, e^{i\mathcal{L}'_2} \rangle$, graphically shown in Fig.3 up to second order in $\mathcal{L}'_2$.

*) We are here using the result that all matrix elements in a theory with $\mathcal{L}_{int} \sim : \chi^r e^{\kappa\phi} :$ are finite, but for $\mathcal{L}_{int} \sim : \chi^r (e^{\kappa\phi} - 1) :$, finiteness holds only when $r = 0, 1$. For second-order superpropagators the result is trivial to prove, as is also a heuristic proof for higher orders. Dr. J.G. Taylor (private communication) has kindly informed us of having completed a rigorous proof.

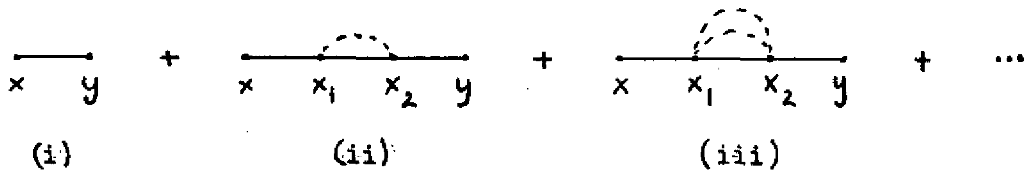

Fig.3

Fig.3(ii) represents the one-graviton modification, Fig.3(iii) the two-graviton modification, and so on. It is easy to see that if we kink 3(i) (see Fig.4),

Fig.4

it would represent zero-graviton exchange and it could be "cradled" in a supergraph (Fig.5)

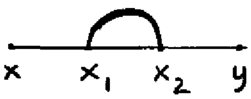

Fig.5

where the effective Lagrangian operating at x_1 and x_2 is the finite Lagrangian $(\partial\psi)^2\, e^{\kappa\phi}$. In the next section we shall show the precise manner of how this "cradling" is crucial to the regularization of the series of graphs in Fig.3. Here we simply remark that if one was considering terms <u>up to</u> third order in $\mathcal{L}_2'$ in $\langle T\psi\psi e^{i\mathcal{L}_2'} \rangle_0$, the graphs (i) and (ii) shown in Fig.6

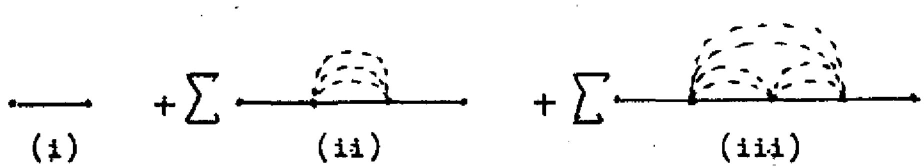

Fig.6

could be kinked as shown in Fig.7

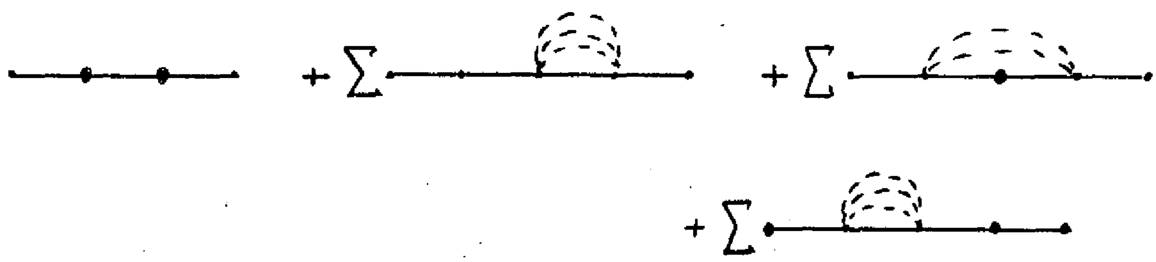

Fig.7

and then cradled into the single supergraph shown in Fig.8 which

Fig.8

corresponds to $\mathcal{L}_{eff} = e^{\kappa\phi}\,(\partial\psi)^2$. It is important to note that "kinking" and "cradling" are possible only when free Lagrangians are hewn out from a total Lagrangian which is finite, i.e. $((\partial\psi)^2$ separated out from $e^{\kappa\phi}\,(\partial\psi)^2)$. This is of course always the case for gravity theory where $\mathcal{L}_0$ for matter fields is obtained from $\mathcal{L}_{matter}$ by replacing $L^{\mu a}$ by $\eta^{\mu a}$.

To apply this graphical discussion for the realistic case of photon self-energy, let us go back to Fig.1. One can verify that the set of graphs shown in Fig.1 are all part of one supergraph

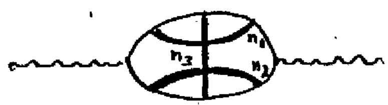

Fig.9

with n_1, n_2, n_3 (the numbers of gravitons exchanged in Fig.9). Taking the values 0,1,2,..., this supergraph includes graph (a) of Fig.1 in the kinked form shown in Fig.10.

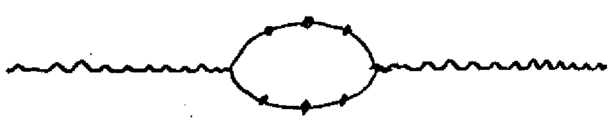

Fig.10

It is important to stress once again that it is always the one graph((a) of Fig.1) which in its multi-kinked form will find its appropriate cradle among sequences like that of Fig.9.

V. KINKING, CRADLING AND THE CALCULUS OF DERIVATIVES

Analytically, the graphs of Fig.3 or, equivalently, of Fig.5 correspond to the expression

$$\int dx_1\, dx_2\, \frac{\partial D(x-x_1)}{\partial x_\mu}\, F_{\mu\nu}(x_1-x_2)\, \frac{\partial D(x_2-y)}{\partial y_\nu} \tag{5.1}$$

where $F_{\mu\nu}$ is given by

$$F_{\mu\nu}(x) = e^{\kappa^2 D(x)}\, \partial_\mu \partial_\nu D(x) = \sum_0^\infty \frac{1}{n!}\, \kappa^{2n} [D(x)]^n\, \partial_\mu \partial_\nu D(x) \;. \tag{5.2}$$

The zero-mass causal propagator $D(x)$ is given by $(-4\pi^2 x^2)^{-1}$.

The problem is to define the Fourier transform of (5.2). This could be done by the method of Lehmann and Pohlmeyer [2] or by the following, less rigorous, method. Consider the integral

$$F_{\mu\nu}(x,\lambda) = \frac{1}{2\pi i} \int_C dz\, \Gamma(-z)\, (-\lambda)^z\, (\kappa^2 D(x))^z\, \partial_\mu \partial_\nu D(x) \tag{5.3}$$

where the contour C comes from positive infinity, encircles the origin in the clockwise sense and returns to infinity. This integral evidently reproduces the sum (5.2) if $\lambda = 1$. On the other hand, if $|\arg(-\lambda)| < \pi/2$ then it is possible to replace the contour C by one running parallel to the imaginary axis with $\mathrm{Re}\, z < 0$. Disregarding for the moment the problems caused by the derivatives in (5.3) one could follow the Gel'fand-Shilov [6] prescription for obtaining the Fourier transform of $D^z \partial_\mu \partial_\mu D$ since it is now possible to arrange the contour such that $0 < \mathrm{Re}\,(z+2) < 2$: a necessary condition for the convergence of the Fourier integral. It must be emphasised that if the kinked graph of Fig.4 had not been included in the sum then the contour would have been confined to the strip $0 < \mathrm{Re}\, z < 1$ and the Gel'fand-Shilov requirement could not have been met - signalling the presence of an unregularized infinity.

The derivative problem is dealt with in the following way. Firstly, combine the factors D^z and $\partial_\mu \partial_\nu D$ into the form

$$D(x)^z \partial_\mu \partial_\nu D(x) = \frac{2}{(z+1)(z+2)} \left(\partial_\mu \partial_\nu - \frac{1}{4} \eta_{\mu\nu} \partial^2 \right) D^{z+1} \tag{5.4}$$

which is an identity except in the neighbourhood of $x_\mu = 0$ where it becomes ambiguous. We shall adopt this formula as a definition for all x_μ except in the neighbourhood of z = 0 where it needs to be elaborated. It is clear that (5.4) cannot be a satisfactory definition at z = 0 since the left-hand side assumes the well defined form, $\partial_\mu \partial_\nu D(x)$, while the right-hand side assumes the equally well defined form, $\partial_\mu \partial_\nu D(x) + (i/4)\delta(x)$, which is different.

To meet this difficulty and also to render the formula useful for computing the Fourier transform of integrals like

$$\int_{\mathrm{Re}\, z < 0} dz \ f(z) [D(x)]^z \partial_\mu \partial_\nu D(x) \quad ,$$

where $f(z)$ has a pole of order $r \geqslant 1$ at $z = 0$, we shall adopt the definition

$$[D(x)]^z \partial_\mu \partial_\nu D(x) = \lim_{\epsilon \to 0} \frac{2}{(z+1)(z+2)} \left[\partial_\mu \partial_\nu - \frac{1}{4} \eta_{\mu\nu} \left(\frac{z}{z+\epsilon} \right)^N \partial^2 \right] [D(x)]^{z+1} \tag{5.5}$$

where $N \geqslant r$ is an integer and ϵ is a positive number. It is to be understood that the singularity at $z = -\epsilon$ lies to the left of the z-contour and that the limit $\epsilon \to 0$ is therefore to be taken after evaluating the Fourier transform and after translating the contour to the right of z = 0. In this way one obtains a definition which is consistent at z = 0 where (5.4) failed. For other values of z it coincides with (5.4).

Another feature of (5.5) may be noted. Contracting the indices μ, ν, one finds

$$[D(x)]^z \partial^2 D(x) = 0 , \qquad \text{for } z \neq 0 \quad .$$

This has the important consequence that all those tadpole-like graphs in the theory which arise from a consonance of terms like $D(x)^z \partial^2 D(x) = D(0)^z \delta(x)$ and which cannot be removed by the normal-ordering procedures, automatically vanish. Thus, in effect $D(0) = 0$ everywhere.

Using (5.5) and taking the Fourier transform of (5.3), one obtains

$$\tilde{F}_{\mu\nu}(p, \lambda) = -i \frac{p_\mu p_\nu}{p^2} + O(\kappa^2) \quad .$$

The higher-order terms will depend on the auxiliary parameter λ. By taking an average of the limits $\lambda \to -e^{i\pi}$ and $\lambda \to -e^{-i\pi}$ one obtains the minimally singular solution of Lehmann and Pohlmeyer.

VI. CONCLUDING REMARKS

Secs. IV and V were the dull preparations for a proof of the following theorem:

> Given a conventional renormalizable Lagrangian polynomial in fields $\psi_1, \psi_2, \ldots$, together with interaction terms of the type $(\partial\psi_1)^2 (e^{\kappa\phi} - 1)$, $(\partial\psi_2)^2 (e^{\kappa\phi} - 1)$, ..., the graphs made up from the purely polynomial part of the Lagrangian can be cradled (after suitable kinking) by those from the non-polynomial part, the two forming a complementary whole, with infinities realistically regularized.

A detailed proof will be given elsewhere, as also the detailed gauge-covariant calculation for electron and photon self-energy graphs which has been carried out in the manner described in this paper. The proof rests on noting that:

1) "Kinking" and "cradling" procedures convert effective Lagrangians from the potentially infinity-producing form $(e^{\kappa\phi} - 1)(\partial\psi)^2$ to the finite form $e^{\kappa\phi}(\partial\psi)^2$.

2) The more superpropagators there are in a graph with $\mathcal{L}_{eff}$ of the type $e^{\kappa\phi}(\partial\psi)^2$, the greater are its chances of being finite.*)

*) Note that $\mathcal{L}_4$ in (3.5) is of the form $(\partial e^{\kappa\phi})\, \bar{\psi}\psi$ and its inclusion causes no difficulties - neither for gauge invariance, nor for finiteness.

3) In Sec.V we computed the modified Feynman-Dyson propagator for the ψ field ($\psi\psi e^{i\mathcal{L}_2'}$). It is easy to see that this super-propagator behaves like -1 (rather than $1/x^2$). Even if the only modification in graphs made from the polynomial part, $\mathcal{L}_1'$, of the Lagrangian were to consist of such insertions, a renormalizable theory based on $\mathcal{L}_1'$ would have all its surviving infinities regularized.

As stated before, these general considerations need verifying by actual computations. A number of such computations have been carried out and will be published elsewhere. The gauge-invariant results for $\delta e/m$ and $\delta m/m$ are the same as obtained in Ref.1 up to terms of orders $\alpha \log (\kappa^2 m^2)$.

The following problems remain:

1) Inclusion of gravity-gravity interactions and, in particular, the verification of gravitational gauge-invariance. (A part of this investigation would be concerned with superpropagators of Feynman's auxiliary particles.)

2) Developments in the calculus of derivatives, presented in Sec.V; justification of formulae like (5.5) from more basic distribution theory.

3) The problem of renormalization of $D(0) = \lim_{x \to 0} 1/x^2$ to the value zero (generalized normal ordering).

4) Further checks of the validity of the scheme presented here by actual higher-order computations.

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FIGURE CAPTIONS

Fig.1 Contributions to the photon self-energy of order e_0^2. The graph (a) is due to $\mathcal{L}_1$ only while those of (b), (c) and (d) are the second-order contributions of $\mathcal{L}_2$. In each of them the number, n, of gravitons involved ranges over the values 1,2,3,... .

Fig.2 The kinking of a free propagator.

Fig.3 Graphs which contribute to the graviton-modified electron propagator.

Fig.4 The free electron propagator kinked so as to represent the zero-graviton modification.

Fig.5 The second-order supergraph for the electron propagator which incorporates all the graphs of Fig.3.

Fig.6 The graphs which contribute to the gravity-modified electron propagator including terms up to third order in $\mathcal{L}_2'$.

Fig.7 The graphs of Fig.6(i) and Fig.6(ii) represented with kinks.

Fig.8 The third-order supergraph for the electron propagator which incorporates all the graphs of Fig.6.

Fig.9 The supergraph which incorporates all the contributions to photon self-energy shown in Fig.1.

Fig.10 The graph of Fig.1(a) represented with six kinks.