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GRAVITATIONAL WARD IDENTITIES

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GRAVITATIONAL WARD IDENTITIES

The gravitational field of general relativity constitutes an example of a gauge field. That is, the gravitational action is invariant with respect to an infinite-dimensional pseudogroup - the group of general co-ordinate transformations. Consequently the quantized version of this theory must possess a feature which is characteristic of all gauge theories: a set of conserved currents which are intimately connected with the source currents of the gauge quanta. In this case the gauge quanta are of course gravitons. However, in contrast with gauge theories of the internal kind such as electrodynamics or Yang-Mills theory, the connection between the two kinds of current - the conserved ones and the source currents - is more subtle. In electrodynamics, for example, these currents are identical: the photon source is precisely the electromagnetic conserved current, i.e., the photon couples to the electric charge which is conserved. In Yang-Mills theories, on the other hand, the source current differs from the conserved one by an additive term which, though generally different from zero, is a gauge-dependent quantity. This term vanishes in some gauges with the result that the Yang-Mills particle couples to the conserved isospin only in some gauges. The graviton source, however, never coincides with the conserved current - in this case the energy-momentum tensor. We give now a demonstration of this.

Consider the classical action functional $\mathscr{S}_c(g,\varphi)$ which governs the motion of the gravitational field, $g_{\mu\nu}$, and a set of (boson) matter fields, φ. Since this functional is invariant under infinitesimal co-ordinate transformations,

$$\left.\begin{aligned} \delta g_{\mu\nu} &= -\delta x^{\alpha}\, g_{\mu\nu,\alpha} - \delta x^{\alpha}{}_{,\mu}\, g_{\alpha\nu} - \delta x^{\alpha}{}_{,\nu}\, g_{\mu\alpha} \\ \delta\varphi &= -\delta x^{\alpha}\, \varphi_{,\alpha} + \delta x^{\alpha}{}_{,\beta}\, F^{\beta}_{\alpha}\, \varphi \end{aligned}\right\} \qquad (1)$$

where F^{β}_{α} denotes a set of numerical spin matrices, it follows that the equations of motion are underdetermined. Among these equations $\delta\mathscr{S}/\delta g_{\mu\nu} = 0$ and $\delta\mathscr{S}/\delta\varphi = 0$ there will subsist four identities. In order to obtain a unique solution from a given set of initial data it is necessary to impose four co-ordinate conditions which we shall express in the form

$$\Phi_{\alpha}(g) = 0 \; . \qquad (2)$$

These conditions must, of course, be non-covariant. This mathematical device for obtaining a well-determined set of equations of motion must be compensated by the requirement that the theory be used to compute only quantities which are covariant and therefore independent of Φ .

A technique for quantizing this theory has been developed by Faddeev and Popov.[1] It employs the method of Feynman for representing the quantized Green's functions by path-integrals. Thus, one writes for an n-point function

$$\langle T\, g_{\mu\nu}(1)\cdots\varphi(n)\rangle_{\Phi} = \int (dg\, d\varphi)\; g_{\mu\nu}(1)\cdots\varphi(n)\, \exp[i\mathscr{S}_c(g,\varphi) + i\mathscr{S}_{\Phi}(g)] \qquad (3)$$

where $\mathscr{S}(g,\varphi)$ denotes the classical action functional and $\mathscr{S}_{\Phi}(g)$ denotes a supplementary, non-invariant, action whose role is to incorporate the co-ordinate conditions (2). It can be shown that the quantum analogue of the classical requirement that generally covariant quantities be independent of the conditions (2), is given by the normalization condition

$$\int (d\Omega)\ \exp\,[i\mathscr{S}_\Phi(g^\Omega)\,] = 1 \tag{4}$$

identically in $g_{\mu\nu}$, where $g^{\Omega}_{\mu\nu}$ denotes the co-ordinate transform of $g_{\mu\nu}$ and the path-integral extends over all co-ordinate transformations Ω. The measure $(d\Omega)$ is invariant.

The gravitational Ward identities are obtained by examining the behaviour of the total action

$$\mathscr{S} = \mathscr{S}_c + \mathscr{S}_\Phi \tag{5}$$

under infinitesimal co-ordinate transformations. Quite generally one finds

$$\delta\mathscr{S} = \int dx \left[\frac{\delta\mathscr{S}}{\delta g_{\mu\nu}(x)}\ \delta g_{\mu\nu}(x) + \frac{\delta\mathscr{S}}{\delta\varphi(x)}\ \delta\varphi(x)\right]$$

$$= \int dx\ \delta x^\alpha \left[\partial_\beta \left\{2\,\frac{\delta\mathscr{S}}{\delta g_{\beta\nu}}\ g_{\alpha\nu} - \frac{\delta\mathscr{S}}{\delta\varphi}\ F^\beta_\alpha\ \varphi\right\} - \frac{\delta\mathscr{S}}{\delta g_{\mu\nu}}\, g_{\mu\nu,\alpha} - \frac{\delta\mathscr{S}}{\delta\varphi}\varphi_{,\alpha}\right]. \tag{6}$$

But, since $\mathscr{S}_c$ is invariant one must have $\delta\mathscr{S} = \delta\mathscr{S}_\Phi$ and one can write this in the form

$$\delta\mathscr{S}_\Phi = \int dx\ \delta x^\alpha{}_{,\beta}\ \vartheta^\beta_\alpha(g)$$

$$= -\int dx\ \delta x^\alpha\ \vartheta^\beta_{\alpha,\beta} \tag{7}$$

where $\vartheta^\beta_\alpha(g)$ denotes a function whose form depends upon the chosen co-ordinate conditions (2). Comparing (6) and (7) one obtains the identities

$$\partial_\beta \left\{ 2 \frac{\delta \mathcal{L}}{\delta g_{\beta\nu}} g_{\alpha\nu} - \frac{\delta \mathcal{L}}{\delta \varphi} F^\beta_\alpha \varphi + \vartheta^\beta_\alpha \right\} = \frac{\delta \mathcal{L}}{\delta g_{\mu\nu}} g_{\mu\nu,\alpha} + \frac{\delta \mathcal{L}}{\delta \varphi} \varphi_{,\alpha} . \qquad (8)$$

If one assumes that $\mathcal{L}$ contains no terms which are linear in the fields $g_{\mu\nu}$ and φ then $\delta\mathcal{L}/\delta g_{\mu\nu}$ and $\delta\mathcal{L}/\delta\varphi$ will contain no terms which are independent of the fields. Hence the linear terms must automatically disappear from (8). In view of this one can define the <u>conserved currents</u>, t^β_α, by

$$t^\beta_\alpha = 2 \frac{\delta \mathcal{L}}{\delta g_{\beta\nu}} g_{\alpha\nu} - \frac{\delta \mathcal{L}}{\delta \varphi} F^\beta_\alpha \varphi + \vartheta^\beta_\alpha - \text{(linear terms)} . \qquad (9)$$

The gravitational Ward identities are concerned with matrix elements of t^β_α. Consider for example the three-point function

$$\langle T\, t^\beta_\alpha(x)\, g_{\kappa\lambda}(x_1)\, g_{\sigma\tau}(x_2) \rangle = \int (dg\, d\varphi)\, t^\beta_\alpha(x)\, g_{\kappa\lambda}(x_1)\, g_{\sigma\tau}(x_2)\, e^{i\mathcal{L}} . \qquad (10)$$

Applying the operator $\partial/\partial x^\beta$ to both sides and using the identity (8) one obtains

$$\partial_\beta \langle T\, t^\beta_\alpha(x)\, g_{\kappa\lambda}(x_1)\, g_{\sigma\tau}(x_2) \rangle = \int (dg\, d\varphi)\, g_{\kappa\lambda}(x_1)\, g_{\sigma\tau}(x_2) \cdot$$

$$\cdot \left(2\, g_{\mu\nu,\alpha}(x) \frac{1}{i} \frac{\delta}{i\, \delta g_{\mu\nu}(x)} + \varphi_{,\alpha}(x) \frac{1}{i} \frac{\delta}{\delta\varphi(x)} \right) e^{i\mathcal{L}}$$

$$= i\, \delta(x - x_1) \langle T\, g_{\kappa\lambda,\alpha}(x_1)\, g_{\sigma\tau}(x_2) \rangle + i\, \delta(x - x_2) \langle T\, g_{\kappa\lambda}(x_1)\, g_{\sigma\tau,\alpha}(x_2) \rangle$$

$$+ \; i \left(\frac{\delta g_{\mu\nu,\alpha}(x)}{\delta g_{\mu\nu}(x)} + \frac{\delta \varphi_{,\alpha}(x)}{\delta \varphi(x)} \right) \qquad (11)$$

after integrating by parts with respect to $g_{\mu\nu}$ and $\varphi(x)$. This relation clearly has the form of a Ward identity apart from the last term. This last term can be eliminated by subtracting the vacuum expectation value from t_α^β , i.e., by defining <u>ordered</u> currents

$$T_\alpha^\beta = t_\alpha^\beta - \langle T\, t_\alpha^\beta \rangle \quad . \tag{12}$$

The <u>sources of the various quanta</u> are defined quite generally by subtracting the linear terms from the variational derivatives of the action,

$$\left.\begin{aligned} j^{\mu\nu}(x) &= \frac{\delta \mathcal{S}}{\delta g_{\mu\nu}} - \text{(linear terms)} \\ j(x) &= \frac{\delta \mathcal{S}}{\delta \varphi} - \text{(linear terms)} \end{aligned}\right\} \quad . \tag{13}$$

These equations correspond to the classical equations of motion if one sets the variational derivatives equal to zero. (For example, in spinor electrodynamics, in the Feynman gauge,

$$j_\mu(x) = \frac{\delta \mathcal{S}}{\delta A_\mu} + \Box A_\mu(x) = e \bar{\psi} \gamma_\mu \psi \, . \Big)$$

<u>Comparing (9) and (13) one sees that the conserved currents t_α^β take a very complicated form when expressed in terms of the sources.</u> In particular they receive contributions not only from the graviton source $j^{\mu\nu}$ but also from the source, j , of every spinning matter quantum ($F_\alpha^\beta \neq 0$). The statement that the energy-momentum tensor coincides with the graviton source is valid only in the weak-field limit $g_{\mu\nu} = \eta_{\mu\nu}$ in which internal gravitons are neglected and, moreover, when external matter particles are taken on-mass-shell ($\delta \mathcal{S}/\delta \varphi \approx 0$).

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